Contraction Analysis of Time-Delayed Communications and Group Cooperation

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***Abstract*— We study stability of interacting nonlinear systems with time-delayed communications, using contraction theory and a simplified wave variable design inspired by robotic teleoperation. We show that contraction is preserved through specific time-delayed feedback communications, and that this property is independent of the values of the delays. The approach is then applied to group cooperation with linear protocols, where it is shown that synchronization can be made robust to arbitrary delays.**

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I. Introduction

In many engineering applications, communications delays between subsystems cannot be neglected. Such an example is bilateral teleoperation, where signals can experience significant transmission delays between local and remote sites. Throughout the last decade, both internet and wireless technologies have vastly extended practical communication distances. Information exchange and cooperation can now occur in very widely distributed systems, making the effect of time delays even more central.

In the context of telerobotics, [2] proposed a control law for force-reflecting teleoperators which preserves passivity, and thus overcomes the instability caused by time delays. The idea was reformulated in [18] in terms of scattering or "wave" variables [6], [3]. Transmission of wave variables across communication channels ensures stability without knowledge of the time delay. Further extensions to internet-based applications were developed [16], [17], [4], in which communication delays are variable.

Recently, [14], [23] extended the application of wave variables to a more general context by performing a nonlinear contraction analysis [13], [14], [1], [7] of the effect of time-delayed communications between contracting systems. This paper modifies the design of the wave variables proposed in [14], [23]. Specifically, a simplified form provides an effective analysis tool for interacting nonlinear systems with time-delayed feedback communications. For appropriate coupling terms, contraction as a generalized stability property is preserved regardless of the delay values. This also sheds a new light on the well-known fact in bilateral teleoperation that even small time-delays in feedback Proportional-Derivative (PD) controllers may create stability problems for simple coupled second-order systems, which in turn motivated approaches based on passivity and wave variables [2], [18]. The approach is then applied to derive the paper's main results on the group cooperation problem with delayed communications. We show that synchronization with linear protocols [19], [15] is robust to time delays and network connectivity without requiring the delays to be known or equal in all links. In a leaderless network, all the coupled elements reach a common state which depends on the initial conditions and the time delays, while in a leader-followers network the group agreement point is fixed by the leader. The approach is suitable to study both continuous-time and discrete-time models.

II. Contraction Analysis of Time-Delayed Communications

This introductory section shows that a simplified form of the transmitted wave variables in [14] can be applied to analyze time-delayed feedback communications. The results which follow could be obtained using a variety of alternative techniques. For instance, they could be easily derived based on input-output analysis [31], although in an L^2 sense rather than exponentially. The use of modified wave variables provides a unifying framework which will be central in the derivation of the paper's main results in section III.

Consider two interacting systems of possibly different dimensions,

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t) + \mathbf{G}_{21}\boldsymbol{\tau}_{21} \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, t) + \mathbf{G}_{12}\boldsymbol{\tau}_{12} \end{cases} \tag{1}$$

where $\mathbf{x}_1 \in \mathbb{R}^{n_1}$, $\mathbf{x}_2 \in \mathbb{R}^{n_2}$, $\boldsymbol{\tau}_{12}, \boldsymbol{\tau}_{21} \in \mathbb{R}^n$, and $\mathbf{G}_{12} \in \mathbb{R}^{n_1 \times n}$, $\mathbf{G}_{21} \in \mathbb{R}^{n_2 \times n}$ are two constant matrices. Inputs $\boldsymbol{\tau}_{ij}$ are computed by transmitting between the two systems simplified "wave" variables, defined as

$$\mathbf{u}_{21} = \mathbf{G}_{21}^T\mathbf{x}_1 + k_{21}\boldsymbol{\tau}_{21} \qquad \mathbf{v}_{12} = \mathbf{G}_{21}^T\mathbf{x}_1$$
$$\mathbf{u}_{12} = \mathbf{G}_{12}^T\mathbf{x}_2 + k_{12}\boldsymbol{\tau}_{12} \qquad \mathbf{v}_{21} = \mathbf{G}_{12}^T\mathbf{x}_2$$

where k_{12} and k_{21} are two strictly positive constants. Because of time delays, one has

$$\mathbf{u}_{12}(t) = \mathbf{v}_{12}(t - T_{12}) \qquad \mathbf{u}_{21}(t) = \mathbf{v}_{21}(t - T_{21})$$

where T_{12} and T_{21} are two positive constants. Note that subscripts containing two numbers indicate the communication direction, e.g., subscript "12" refers to communication from node 1 to 2. This notation will be helpful in Section III, where results will be extended to groups of interacting subsystems.

Consider, similarly to [14], [23], the differential length

$$V = \frac{k_{21}}{2}\ \delta \mathbf{x}_1^T \delta \mathbf{x}_1 \ + \ \frac{k_{12}}{2}\ \delta \mathbf{x}_2^T \delta \mathbf{x}_2 \ + \ \frac{1}{2}\ V_{1,2}$$

where

$$V_{1,2} = \int_{t-T_{12}}^{t} \delta \mathbf{v}_{12}^T \delta \mathbf{v}_{12}\ d\epsilon \ + \ \int_{t-T_{21}}^{t} \delta \mathbf{v}_{21}^T \delta \mathbf{v}_{21}\ d\epsilon$$

This yields

$$\begin{aligned}\dot{V} &= k_{21} \delta \mathbf{x}_1^T \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1}\ \delta \mathbf{x}_1 \ + \ k_{12} \delta \mathbf{x}_2^T \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_2}\ \delta \mathbf{x}_2 \\ &- \ \frac{k_{21}^2}{2}\ \delta \boldsymbol{\tau}_{21}^T \delta \boldsymbol{\tau}_{21} \ - \ \frac{k_{12}^2}{2}\ \delta \boldsymbol{\tau}_{12}^T \delta \boldsymbol{\tau}_{12}\end{aligned}$$

If $\mathbf{f}_1$ and $\mathbf{f}_2$ are both contracting with identity metrics (i.e., if $\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1}$ and $\frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_2}$ are both uniformly negative definite), then $\dot{V} \leq 0$, and V is bounded and tends to a limit. Applying Barbalat's lemma [22] in turn shows that if $\ddot{V}$ is bounded, then $\dot{V}$ tends to zero asymptotically, which implies that $\delta \mathbf{x}_1$, $\delta \mathbf{x}_2$, $\delta \boldsymbol{\tau}_{12}$ and $\delta \boldsymbol{\tau}_{21}$ all tend to zero. Regardless of the values of the delays, all solutions of system (1) converge to a single trajectory, independent of the initial conditions. In the sequel we shall assume that $\ddot{V}$ can indeed be bounded as a consequence of the boundedness of V.

This result has a useful interpretation. Expanding system dynamics (1) yields

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t) + \frac{1}{k_{21}} \mathbf{G}_{21} (\mathbf{G}_{12}^T \mathbf{x}_2(t - T_{21}) - \mathbf{G}_{21}^T \mathbf{x}_1(t)) \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, t) + \frac{1}{k_{12}} \mathbf{G}_{12} (\mathbf{G}_{21}^T \mathbf{x}_1(t - T_{12}) - \mathbf{G}_{12}^T \mathbf{x}_2(t)) \end{cases}$$

If we assume further that $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same dimension, and choose $\mathbf{G}_{12} = \mathbf{G}_{21} = \mathbf{G}$, the whole system is actually equivalent to two diffusively coupled subsystems

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t) + \frac{1}{k_{21}}\ \mathbf{G}\mathbf{G}^T\ (\ \mathbf{x}_2(t - T_{21}) - \mathbf{x}_1(t)\ ) \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, t) + \frac{1}{k_{12}}\ \mathbf{G}\mathbf{G}^T\ (\ \mathbf{x}_1(t - T_{12}) - \mathbf{x}_2(t)\ ) \end{cases}$$

This implies that, for appropriate coupling terms, contraction as a generalized stability property will be preserved regardless of the time delays and the delay values.

This result does not contradict the well-known fact in teleoperation that even small time delays in bilateral PD controllers may create stability problems for coupled second-order systems [2], [18], [17], [4], which motivates approaches based on passivity and wave variables. In fact, a key condition for contraction to be preserved is that the coupling gains be symmetric positive semi-definite *in the same metric* as the subsystems.

Example 2.1 Consider two identical second-order systems coupled through time-delayed feedback PD controllers

$$\begin{cases} h_1 = k_d(\dot{x}_2(t - T_{21}) - \dot{x}_1(t)) + k_p(x_2(t - T_{21}) - x_1(t)) \\ h_2 = k_d(\dot{x}_1(t - T_{12}) - \dot{x}_2(t)) + k_p(x_1(t - T_{12}) - x_2(t)) \end{cases}$$

where $h_1 = \ddot{x}_1 + b\dot{x}_1 + \omega^2 x_1$, $h_2 = \ddot{x}_2 + b\dot{x}_2 + \omega^2 x_2$, and $b > 0$, $\omega > 0$. If $T_{12} = T_{21} = 0$, x_1 and x_2 converge together exponentially regardless of initial conditions, which makes the origin a stable equilibrium point. If $T_{12}, T_{21} > 0$, a simple coordinate transformation yields

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} &= \begin{bmatrix} \omega y_1 - b x_1 \\ -\omega x_1 \end{bmatrix} + \mathbf{K}(\begin{bmatrix} x_2(t - T_{21}) \\ y_2(t - T_{21}) \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}) \\ \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} &= \begin{bmatrix} \omega y_2 - b x_2 \\ -\omega x_2 \end{bmatrix} + \mathbf{K}(\begin{bmatrix} x_1(t - T_{12}) \\ y_1(t - T_{12}) \end{bmatrix} - \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}) \end{aligned}$$

where $\mathbf{f}_1 = \begin{bmatrix} \omega y_1 - b x_1 \\ -\omega x_1 \end{bmatrix}$ and $\mathbf{f}_2 = \begin{bmatrix} \omega y_2 - b x_2 \\ -\omega x_2 \end{bmatrix}$ are both (semi-)contracting with identity metric [29]. However, the transformed coupling gain $\mathbf{K} = \begin{bmatrix} k_d & 0 \\ \frac{k_p}{\omega} & 0 \end{bmatrix}$ is neither symmetric nor positive semi-definite for any $k_p \neq 0$. Contraction cannot be preserved in this case, and the coupled systems turn out to be unstable for large enough delays as the simulation result in Figure 1(a) illustrates. While in Figure 1(b), once we set $k_p = 0$, the overall system is therefore contracting. □

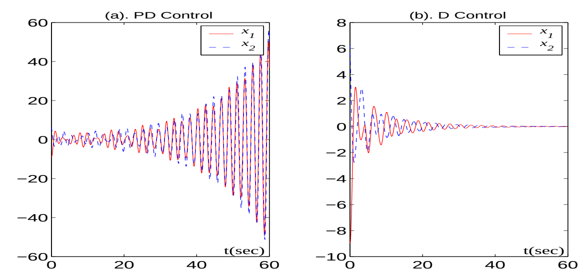

Fig. 1. Simulation results of two coupled mass-spring-damper systems with (a) PD control and (b) D control. Parameters are $b = 0.5$, $\omega^2 = 5$, $T_{12} = 2s$, $T_{21} = 4s$, $k_d = 1$, $k_p = 5$ in (a) and $k_p = 0$ in (b). Initial conditions, chosen randomly, are identical for the two plots.

The instability mechanism in the above example is actually very similar to that of the classical Smale model [26], [29] of spontaneous oscillation, in which two or more identical biological cells, inert by themselves, tend to self-excited oscillations through diffusion interactions. In both cases, the instability is caused by a non-identity metric, which makes the transformed coupling gains lose positive semi-definiteness. Note that the relative simplicity with which both phenomena can be interpreted makes fundamental use of the notion of a metric, central to contraction theory.

Finally, it is straightforward to show that the results here apply recursively to feedback hierarchies of such systems.

III. Group Cooperation with Time-Delayed Communications

We now present the paper's main results. Recently, synchronization or group agreement has been the object of extensive literature [11], [12], [19], [20], [21], [27], [28]. Understanding natural aggregate motions as in bird flocks, fish schools, or animal herds may help achieve desired collective behaviors in artificial multi-agent systems. In our previous work [24], [29], a synchronization condition was obtained for a group of coupled nonlinear systems, where the number of the elements can be arbitrary and the network structure can be very general. In this section, we study a simplified continuous-time model of schooling or flocking with time-delayed communications, and generalize recent results in the literature [19], [15]. In particular, we show that synchronization is robust to time delays both for the leaderless case and for the leader-followers case, without requiring the delays to be known or equal in all links. Similar results are then derived for discrete-time models.

A. Leaderless Group

We first investigate a flocking model without group leader. The dynamics of the ith element is given as

$$\dot{\mathbf{x}}_i \ = \ \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji} \ (\mathbf{x}_j - \mathbf{x}_i) \tag{2}$$

where $i = 1, \ldots, n$ and $\mathbf{x}_i \in \mathbb{R}^m$. $\mathcal{N}_i$ denotes the set of the active neighbors of element i, which for instance can be defined as the set of the nearest neighbors within a certain distance around i; and $\mathbf{K}_{ji}$ is the coupling gain, which is assumed to be symmetric and positive definite.

Theorem 1: Consider n coupled elements with linear protocol (2). The whole system will tend to reach a group agreement $\mathbf{x}_1(t) = \cdots = \mathbf{x}_n(t) = \frac{1}{n}(\mathbf{x}_1(0) + \cdots + \mathbf{x}_n(0))$ exponentially if the network is connected, and the coupling links are either bidirectional with $\mathbf{K}_{ji} = \mathbf{K}_{ij}$, or unidirectional but formed in closed rings with identical gains.

Theorem 1 is derived in [24], [29] based on partial contraction analysis, and the result can be extended further to time-varying couplings ($\mathbf{K}_{ji} = \mathbf{K}_{ji}(t)$), switching networks ($\mathcal{N}_i = \mathcal{N}_i(t)$) and looser connectivity conditions.

Assume now that time delays are non-negligible in communications. The dynamics of the ith element turns out to be

$$\dot{\mathbf{x}}_i \ = \ \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji} \ (\ \mathbf{x}_j(t - T_{ji}) - \mathbf{x}_i(t) \ ) \tag{3}$$

Theorem 2: Consider n coupled elements (3) with time-delayed communications. Regardless of the explicit values of the delays, the whole system will tend to reach a group agreement $\mathbf{x}_1(t) = \cdots = \mathbf{x}_n(t)$ asymptotically if the network is connected, and the coupling links are either bidirectional with $\mathbf{K}_{ji} = \mathbf{K}_{ij}$, or unidirectional but formed in closed rings with identical gains.

Proof: For notational simplicity, we first assume that all the links are bidirectional with $\mathbf{K}_{ji} = \mathbf{K}_{ij}$, but the time delays could be different along opposite directions, i.e., $T_{ji} \neq T_{ij}$. Thus, Equation (3) can be transformed to

$$\dot{\mathbf{x}}_i \ = \ \sum_{j \in \mathcal{N}_i} \mathbf{G}_{ji} \boldsymbol{\tau}_{ji}$$

where $\boldsymbol{\tau}_{ji}$ and correspondingly $\boldsymbol{\tau}_{ij}$ are defined through

$$\begin{aligned} \mathbf{u}_{ji} \ &= \ \mathbf{G}_{ji}^T \mathbf{x}_i \ + \ \boldsymbol{\tau}_{ji} \qquad & \mathbf{v}_{ij} = \mathbf{G}_{ji}^T \mathbf{x}_i \\ \mathbf{u}_{ij} \ &= \ \mathbf{G}_{ij}^T \mathbf{x}_j \ + \ \boldsymbol{\tau}_{ij} \qquad & \mathbf{v}_{ji} = \mathbf{G}_{ij}^T \mathbf{x}_j \end{aligned} \tag{4}$$

with $\mathbf{G}_{ij} = \mathbf{G}_{ji} > 0$ and $\mathbf{K}_{ji} = \mathbf{K}_{ij} = \mathbf{G}_{ij} \mathbf{G}_{ij}^T$. Define

$$V \ = \ \frac{1}{2} \sum_{i=1}^{n} \delta \mathbf{x}_i^T \delta \mathbf{x}_i \ + \ \frac{1}{2} \sum_{(i,j) \in \mathcal{N}} V_{i,j} \tag{5}$$

where $\mathcal{N} = \cup_{i=1}^n \mathcal{N}_i$ denotes the set of all active links, and $V_{i,j}$ is defined as in Section II for each link connecting two nodes i and j. Therefore

$$\dot{V} \ = \ -\frac{1}{2} \sum_{(i,j) \in \mathcal{N}} (\delta \boldsymbol{\tau}_{ji}^T \delta \boldsymbol{\tau}_{ji} + \delta \boldsymbol{\tau}_{ij}^T \delta \boldsymbol{\tau}_{ij})$$

Since $\dot{V}(t)$ is non-positive, $V(t)$ is bounded if initial states are bounded. One can easily show that $\delta \mathbf{x}_i(t)$ is bounded for any i. This implies that $\delta \boldsymbol{\tau}_{ij}(t)$ and $\delta \dot{\mathbf{x}}_i(t)$ are both bounded, and so is $\delta \dot{\boldsymbol{\tau}}_{ij}(t)$ since

$$\delta \dot{\boldsymbol{\tau}}_{ji}(t) \ = \ \mathbf{G}_{ij}^T \delta \dot{\mathbf{x}}_j(t - T_{ji}) \ - \ \mathbf{G}_{ji}^T \delta \dot{\mathbf{x}}_i$$

Thus we can say that $\ddot{V}(t)$ is bounded. According to Barbalat's lemma, $\dot{V}$ will then tend to zero asymptotically, which implies that, $\forall (i,j) \in \mathcal{N}$, $\delta \boldsymbol{\tau}_{ji}$ and $\delta \boldsymbol{\tau}_{ij}$ tend to zero asymptotically. Thus we know that $\forall i, \ \delta \dot{\mathbf{x}}_i$ tends to zero. Now in general, a vanishing $\delta \dot{\mathbf{x}}_i$ does not necessarily imply that $\delta \mathbf{x}_i$ is convergent. However it does in this case, because otherwise it would contradict the fact that $\delta \mathbf{x}_i$ tends to be periodic with constant period $T_{ji} + T_{ij}$,

$$\begin{aligned} \delta \mathbf{u}_{ji}(t) \ &= \ \mathbf{G}_{ji}^T \delta \mathbf{x}_i(t) + \delta \boldsymbol{\tau}_{ji}(t) \ = \ \mathbf{G}_{ij}^T \delta \mathbf{x}_j(t - T_{ji}) \\ \delta \mathbf{u}_{ij}(t) \ &= \ \mathbf{G}_{ij}^T \delta \mathbf{x}_j(t) + \delta \boldsymbol{\tau}_{ij}(t) \ = \ \mathbf{G}_{ji}^T \delta \mathbf{x}_i(t - T_{ij}) \end{aligned}$$

We can also conclude that, if $\forall i, \ \delta \mathbf{x}_i$ is convergent, they will tend to a steady state

$$\delta \mathbf{x}_1(t) = \cdots = \delta \mathbf{x}_n(t) = \mathbf{c}$$

where $\mathbf{c}$ is a constant vector whose value depends on the specific trajectories we analyze. Moreover, we notice that, in the state-space, any point inside the region $\mathbf{x}_1 = \cdots = \mathbf{x}_n$ is invariant to (3). By path integration this implies immediately that regardless of the delay values or the initial conditions, all solutions of system (3) will tend to reach a group agreement $\mathbf{x}_1 = \cdots = \mathbf{x}_n$ asymptotically.

In the case that coupling links are unidirectional but form closed rings with identical coupling gains in each ring, we set

$$V = \frac{1}{2} \sum_{i=1}^{n} \delta \mathbf{x}_i^T \delta \mathbf{x}_i + \frac{1}{2} \sum_{(j \to i) \in \mathcal{N}} \int_{t - T_{ji}}^{t} \delta \mathbf{v}_{ji}^T \delta \mathbf{v}_{ji} d\epsilon$$

and the rest of the proof is the same. The case when both types of links are involved is similar. □

Example 3.1 Compared with Theorem 1, the group agreement point in Theorem 2 generally does not equal the average value of the initial conditions, but depends on the values of the time delays.

Consider the cooperative group (3) with one-dimensional $\mathbf{x}_i$, $n = 6$, and a two-way chain structure

$$1 \longleftrightarrow 2 \longleftrightarrow 3 \longleftrightarrow 4 \longleftrightarrow 5 \longleftrightarrow 6$$

The coupling gains are set to be identical with $k = 5$. The delay values are different, and each is chosen randomly around 0.5 second. Simulation results are plotted in Figure 2. □

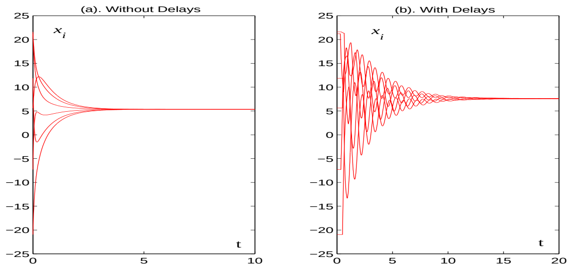

Fig. 2. Simulation results for Example 3.1 without delays and with delays. Initial conditions, chosen randomly, are the same for each simulation. Group agreement is reached in both cases, although the agreement value is different.

Note that the conditions on coupling gains can be relaxed. If the links are bidirectional, we do not have to require $\mathbf{K}_{ij} = \mathbf{K}_{ji}$. Instead, the dynamics of the ith element could be

$$\dot{\mathbf{x}}_i \;=\; \mathbf{K}_i \sum_{j \in \mathcal{N}_i} (\; \mathbf{x}_j(t - T_{ji}) - \mathbf{x}_i(t) \;)$$

where $\mathbf{K}_i = \frac{1}{k_i}\mathbf{G}\mathbf{G}^T$ and $\mathbf{G}$ is unique through the whole network. The proof is the same except that we incorporate k_i into the wave variables and the function V. Such a design brings more flexibility to cooperation-law design. The discrete-time model studied in Section III-C is in this spirit. A similar condition was derived in [5] for a delayless swarm model.

Model (2) with delayed communications was also studied in [19], but the result is limited by the assumptions that communication delays are equal in all links and that the self-response part in each coupling uses the same time delay. Recently, [15] independently analyzed system (3) in the scalar case with the assumption that delays are equal in all links.

B. Leader-Followers Group

Similar analysis can be applied to study coupled networks with group leaders. Consider such a model

$$\dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji}(\mathbf{x}_j(t - T_{ji}) - \mathbf{x}_i(t)) + \gamma_i \mathbf{K}_{0i}(\mathbf{x}_0 - \mathbf{x}_i) \quad (6)$$

where $i = 1, \ldots, n$; $\mathbf{x}_0$ is the state of the leader, which we first assume to be a constant; $\mathbf{x}_i$ are the states of the followers; $\mathcal{N}_i$ indicate the neighborship among the followers; and $\gamma_i = 0$ or 1 represents the unidirectional links from the leader to the corresponding followers. For each non-zero γ_i, the coupling gain $\mathbf{K}_{0i}$ is positive definite.

Theorem 3: Consider a leader-followers network (6) with time-delayed communications. Regardless of the explicit values of the delays, the whole system will tend to reach a group agreement $\mathbf{x}_1(t) = \cdots = \mathbf{x}_n(t) = \mathbf{x}_0$ asymptotically if the whole network is connected, and the coupling links among the followers are either bidirectional with $\mathbf{K}_{ji} = \mathbf{K}_{ij}$, or unidirectional but formed in closed rings with identical gains.

Proof: Exponential convergence of the leader-followers network (6) without delays has been shown in [24], [29] using contraction theory. If the communication delays are non-negligible, and assuming that all the links among the followers are bidirectional, we can transform the equation (6) to

$$\dot{\mathbf{x}}_i \;=\; \sum_{j \in \mathcal{N}_i} \mathbf{G}_{ji} \boldsymbol{\tau}_{ji} + \gamma_i \; \mathbf{K}_{0i} \; (\mathbf{x}_0 - \mathbf{x}_i)$$

where $\boldsymbol{\tau}_{ji}$ and $\boldsymbol{\tau}_{ij}$ are defined the same as those in (4). Considering the same Lyapunov function V as (5), we get

$$\dot{V} \;=\; -\sum_{i=1}^{n} \gamma_i \delta\mathbf{x}_i^T \mathbf{K}_{0i} \delta\mathbf{x}_i - \frac{1}{2} \sum_{(i,j) \in \mathcal{N}} (\delta\boldsymbol{\tau}_{ji}^T \delta\boldsymbol{\tau}_{ji} + \delta\boldsymbol{\tau}_{ij}^T \delta\boldsymbol{\tau}_{ij})$$

where $\mathcal{N} = \cup_{i=1}^{n} \mathcal{N}_i$ denotes the set of all active links among the followers. Applying Barbalat's lemma shows that $\dot{V}$ will tend to zero asymptotically. It implies that $\forall i$, if $\gamma_i = 1$, $\delta\mathbf{x}_i$ will tend to zero, as well as $\delta\boldsymbol{\tau}_{ji}$ and $\delta\boldsymbol{\tau}_{ij}$ $\forall (i,j) \in \mathcal{N}$. Moreover, since

$$\delta\boldsymbol{\tau}_{ji}(t) \;=\; \mathbf{G}_{ji}^T(\; \delta\mathbf{x}_j(t - T_{ji}) \;-\; \delta\mathbf{x}_i(t) \;)$$

we conclude that if the whole leader-followers network is connected, the virtual dynamics will converge to $\delta\mathbf{x}_1(t) = \cdots = \delta\mathbf{x}_n(t) = 0$ regardless of the initial conditions or the delay values, i.e., the whole system is asymptotically contracting. All solutions will converge to a particular one, which in this case is the point $\mathbf{x}_1(t) = \cdots = \mathbf{x}_n(t) = \mathbf{x}_0$. The proof is similar for unidirectional links in closed rings. □

Example 3.2 Consider a leader-followers network (6) with one-dimensional $\mathbf{x}_i$, $n = 6$, and structured by

$$0 \longrightarrow 1 \longleftrightarrow 2 \longleftrightarrow 3 \longleftrightarrow 4 \longleftrightarrow 5 \longleftrightarrow 6$$

The state of the leader is constant with value $\mathbf{x}_0 = 10$. All the coupling gains are set to be identical with $k = 5$. The delays are not equal, each of which is chosen randomly around 0.5 second. Simulation results are plotted in Fig. 3. □

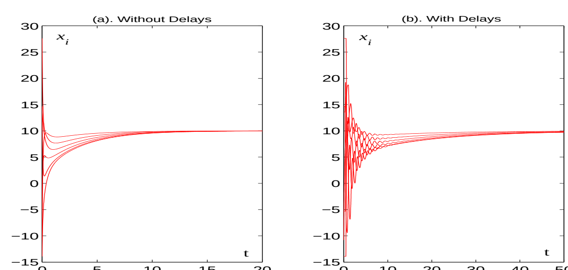

Fig. 3. Simulation results for Example 3.2 without delays and with delays. Initial conditions, chosen randomly, are the same for each simulation. In both cases, group agreement to the leader value $\mathbf{x}_0$ is reached.

Note that even if $\mathbf{x}_0$ is not a constant, i.e., the dynamics of the ith element is given as

$$\dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji} \; (\mathbf{x}_j(t - T_{ji}) - \mathbf{x}_i(t)) + \gamma_i \; \mathbf{K}_{0i} \; (\mathbf{x}_0(t - T_{0i}) - \mathbf{x}_i(t))$$

the whole system is still asymptotically contracting according to exactly the same proof. Regardless of the initial conditions, all solutions converge to a particular one, which in this case depends on the dynamics of $\mathbf{x}_0$ and the explicit values of the delays. Moreover, if $\mathbf{x}_0$ is periodic, as one of the main properties of contraction [13], all the followers' states $\mathbf{x}_i$ will tend to be periodic with the same period as $\mathbf{x}_0$.

C. Discrete-Time Models

Simplified wave variables can also be applied to study time-delayed communications in discrete-time models. Consider the model of flocking or schooling studied in [11], [28]:

$$x_i(t+1) \ = \ x_i(t) \ + \ \frac{1}{1+n_i} \sum_{j \in \mathcal{N}_i} (\ x_j(t) \ - \ x_i(t) \ )$$

where $i = 1, \ldots, n$, and n_i equals the number of the neighbors of element i. As proved in [11], the whole system will tend to reach a group agreement if the network is connected in a very loose sense.

Assume now that time delays are non-negligible in communications. The update law of the ith element changes to

$$x_i(t{+}1) \ = \ x_i(t) + \frac{1}{1+n_i} \sum_{j \in \mathcal{N}_i} (\ x_j(t{-}T_{ji}) - x_i(t) \ ) \quad (7)$$

where the delay value T_{ji} is an integer based on the number of updating steps. As in previous sections, T_{ji} could be different for different communication links, or even different along opposite directions on the same link.

We assume: all elements update their states synchronously; the time interval between any two updating steps is a constant; the network structure is connected and fixed, i.e., n_i is a positive integer $\forall i$; and the value of the neighborhood radius r is unique throughout the whole network, which leads to the fact that all interactions are bidirectional.

Theorem 4: Consider n coupled elements (7) with time-delayed communications. Regardless of the explicit values of the delays, the whole system will tend to reach a group agreement $x_1(t) = \cdots = x_n(t)$ asymptotically.

Proof: See [30], with $x_i(t+1) \ = \ x_i(t) \ + \ \sum_{j \in \mathcal{N}_i} \tau_{ji}(t)$ and wave variables $u_{ji} = x_i \ + \ (1+n_i) \ \tau_{ji} \ , \ \ v_{ij} = x_i$.

A similar analysis leads to the same result as Theorem 3 for a discrete-time model with a leader-followers structure, both of which can be applied to study group cooperation problem with asynchronous updating instants [8].

IV. Conclusion

This paper introduces modified wave variables in the context of contraction analysis, and shows that they yield effective and simple tools for analyzing interacting systems and group synchronization with time-delayed feedback communications. Future work includes coupled networks with switching topologies and time-varying time-delays.

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