Coherence patterns originating from incoherent volume sources

Kehan Tian and George Barbastathis

Department of Mechanical Engineering, Massachusetts Institute of Technology, Room 3-461c, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139

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We derive the complex degree of coherence that originates from generalized incoherent two- and three-dimensional sources. Further, we find the locus of maximum coherence and analyze the dependence of the decay of coherence on source thickness. © 2004 Optical Society of America

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Fast-spinning random phase masks are often used to make the behavior of light sources that are inherently spatially coherent (e.g., a monochromatic laser source) emulate spatially incoherent behavior. The coherence pattern generated by such a source can be well characterized by the Airy function on a plane perpendicular to the optical axis. Here we derive the off-plane behavior of the coherence function for two-dimensional (such as a spinning disk) and three-dimensional generalized incoherent sources. We find that maximum coherence off plane is obtained along radial lines that diverge from the source center, branching out at certain locations. The decay of coherence as a function of source thickness is also characterized.

Consider a field in the half-space originating from a quasi-monochromatic, spatially incoherent volume source, as shown in Fig. 1. Mutual intensity between any two sources is given by

\[ J(r_1', r_2') = I(r_1') \delta(r_2' - r_1'). \] (1)

The complex degree of coherence of the field at any pair of points \( P(r_1) \) and \( Q(r_2) \) in half-space \( z > 0 \) past the source is

\[ j(r_1, r_2) = \frac{1}{[I_0(r_1)][I_0(r_2)]^{1/2}} \iint_{\mathbb{V}} I(r_1') \times \frac{\exp[-j(2\pi/\lambda)|r_2 - r_1'|]}{-j\lambda|r_2 - r_1'|} \times \frac{\exp[-j(2\pi/\lambda)|r_1 - r_1'|]}{j\lambda|r_1 - r_1'|} \, d^3 r_1'. \] (2)

where

\[ I_0(r_i) = \frac{1}{\lambda^2} \iint_{\mathbb{V}} \frac{I(r_1')}{|r_i - r_1'|^2} \, d^3 r_1'. \] (3)

Consider the geometry in Fig. 1 again. We are interested in the degree of coherence between point \( P(r_1) \) at reference plane \( \Pi_1 \) located a distance \( z_1 \) from the coordinate origin and point \( Q(r_2) \) located at a different plane, \( \Pi_2 \). \( \Delta z \) is the distance between the two planes. Assuming that the paraxial approximation is valid for both \( P(r_1) \) and \( Q(r_2) \), Eq. (2) can be written as

\[ j(r_1, r_2) \approx \frac{\exp[-j(2\pi/\lambda)|\Delta z|]}{\lambda^2 z_1(\Delta z)} \iint_{\mathbb{V}} I(r_1') \times I(x_1', y_1', z_1') \exp\left[ j \pi A(z_1') \left( x_1'^2 + y_1'^2 \right) \right] \times \exp\left[ j 2\pi \left( B_z(z_1')x_1' + B_y(z_1')y_1' \right) \right] \times \exp\left[ j \pi C(z_1') \right] \, dx_1' \, dy_1' \, dz_1'. \] (4)

Coefficients \( A(z_1'), B_x(z_1'), B_y(z_1'), \) and \( C(z_1') \) are expressed as

\[ A(z_1') = \frac{1}{\lambda(z_1 - z_1')} - \frac{1}{\lambda(z_1 + \Delta z - z_1')}, \] (5)

\[ B_z(z_1') = -\frac{1}{\lambda} \left( \frac{x_1}{z_1 - z_1'} - \frac{x_2}{z_1 + \Delta z - z_1'} \right), \] (6)

\[ B_y(z_1') = -\frac{1}{\lambda} \left( \frac{y_1}{z_1 - z_1'} - \frac{y_2}{z_1 + \Delta z - z_1'} \right), \] (7)

\[ C(z_1') = \frac{x_1^2 + y_1^2}{\lambda(z_1 + \Delta z - z_1')} - \frac{x_2^2 + y_2^2}{\lambda(z_1 - z_1')} \] (8)

The special case when the source is a cylinder of radius \( R \) and thickness \( L \) centered at the coordinate origin with uniform intensity is shown in Fig. 2. We introduce Fresnel number \( N_F = R^2/\lambda z_1 \) (Ref. 3) and thickness/distance ratio \( \alpha = L/z_1 \) and express all

Fig. 1. Illustration of the coherence that originates from a quasi-monochromatic, spatially incoherent volume source.
the degree of coherence from the cylinder volume is the co-
from a disk source, Eq. (10) means that the complex

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expression (4) can be integrated explicitly in
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indicates how coherent the two points
expression as a function of $Q(\mathbf{r}_2)$. If $Q(\mathbf{r}_2)$ is on the
Pi$_1$ plane, the maximum coherence occurs at $P(\mathbf{r}_1)$ and
decays like an Airy pattern. If $Q(\mathbf{r}_2)$ is not on the


distances in dimensionless units as

$$
\eta_x = x_1'/R, \quad \eta_y = y_1'/R, \quad \xi = z_1'/z_1,
$$

$$
\delta = \Delta z/z_1, \quad v_{x_1} = \frac{2\pi R}{\lambda} z_1 x_1, \quad v_{y_1} = \frac{2\pi R}{\lambda} z_1 y_1,
$$

$$
\nu_1 = \sqrt{v_{x_1}^2 + v_{y_1}^2}, \quad \nu_2 = \frac{2\pi}{\lambda} z_1 x_2,
$$

$$
\nu_3 = \frac{2\pi}{\lambda} z_1 y_2, \quad \nu_4 = \sqrt{v_{x_2}^2 + v_{y_2}^2}.
$$

Then expression (4) can be integrated explicitly in
cylindrical coordinates, leading to

$$
\begin{align*}
 j(\mathbf{r}_1, \mathbf{r}_2) &= \exp[-j(2\pi/\lambda)\Delta z] \int_{-\alpha/2}^{\alpha/2} \exp[j\pi C(\xi)] \\
 &\quad \times \mathcal{L}[2\pi A(\xi), 2\pi B(\xi)]d\xi,
\end{align*}
$$

where $B(\xi) = [B_{1}(\xi)^2 + B_{2}(\xi)^2]^1/2$ and $A(\xi)$ is a
function that also describes the three-dimensional field dis-
tribution in the focal vicinity of an ideal lens.$^4$ The
derivation is similar to that which is used to obtain the
field diffracted from a volume hologram.$^5$ Modulus
$|j(\mathbf{r}_1, \mathbf{r}_2)|$ indicates how coherent the two points $P(\mathbf{r}_1)$
and $Q(\mathbf{r}_2)$ are with respect to each other. A straight-
forward deduction from Eq. (10) is in the limit $\alpha \to 0$,
i.e., a disk source that can be emulated by a laser
source and a spinning random phase mask in the labora-
tory.$^2$ The result is

$$
\begin{align*}
 j(\mathbf{r}_1, \mathbf{r}_2) &= \exp\left(-j\frac{2\pi}{\lambda}\Delta z\right) \exp[j\pi C(0)] \\
 &\quad \times \mathcal{L}[2\pi A(0), 2\pi B(0)].
\end{align*}
$$

Then Eq. (10) has an interesting interpretation. Be-
cause $\mathcal{L}(\cdot)$ describes the complex degree of coherence
from a disk source, Eq. (10) means that the complex
degree of coherence from the cylinder volume is the 
coherent superposition of many disk sources stacked in the
z direction.

For a disk source, with one reference point $P(\mathbf{r}_1)$
fixed as in Fig. 3 we can calculate the coherence dis-

Thus for $\delta \ll 1$ the upper limit of the coherence
between two planes is a monotonic decreasing function
of $N_F\delta$ or, equivalently, $(R/z_1)^2$. This result verifies
the intuitive notion that if we want the illumination
to be effectively incoherent we should use a large spinning disk or bring the illuminated object closer to the
spinning disk. The exact contour of the coherence in the cross section at plane $S$ is identical to
Fig. 8.41 of Ref. 4. When $\delta \ll 1$, the coherence distribu-
tion is mirror symmetric with respect to plane $\Pi_1$ and rotationally symmetric about the line defined
by Eqs. (12). From Fig. 3 we can also see that at
distance $\delta = 1.5219/(N_F - 1.5219)$ away from $\Pi_1$ the
locus branches into two segments. The reason for the
branching is the peak-null configuration of $|\mathcal{L}(\cdot)|$ (Fig. 8.41 of Ref. 4).

We now turn to the case of the thick cylinder, $\alpha > 0$.
To get the locus of maximum coherence we maximize
the slowly varying components $A(\xi)$ and $B(\xi)$ of the 
 integrand by setting the arguments to $\mathcal{L}(\cdot)$ equal to
zero, which gives the same result as Eqs. (12), and we
replace $C(\xi)$ with its Taylor expansion about $\xi = 0$:

$$
C(\xi) \approx \frac{1}{4\pi^2 N_F} \frac{v_1^2}{\delta} \left[-\delta + 0\xi + \frac{\delta}{(1+\delta)^2}\xi^2\right].
$$

Then $\delta_{\max}$ is given as

$$
\frac{\nu_{x_2}^2}{\nu_{y_2}^2} = \left(1 + \frac{\delta}{(1+\delta)^2}\xi_{\max}^2\right)^{-1}.
$$

Thus

$$
\nu_{x_2} = (1 + \delta)\nu_{x_1}, \quad \nu_{y_2} = (1 + \delta)\nu_{y_1},
$$

which we found by setting $B_z(0) = 0$ and $B_y(0) = 0$. We
see that near $\Pi_1$ the locus of points that are maxi-
mally coherent with $P(\mathbf{r}_1)$ is a line passing through the 
center of the disk source and through $P(\mathbf{r}_1)$ (Fig. 3).
The modulus of the degree of coherence along that
line is

$$
|j(\mathbf{r}_1, \mathbf{r}_2)|_{\max} = \sin\left(\frac{N_F}{2} \frac{\delta}{1 + \delta}\right).
$$

Fig. 2. Illustration of the coherence that originates from
a cylindrical, quasi-monochromatic, spatially incoherent
source: The parameters that we used to calculate the
locus of maximum coherence are $\lambda = 488 \text{ nm}$, $R = 5 \text{ mm}$,
$L = 10 \text{ mm}$, and $\mathbf{r}_1 = (10,10,100 \text{ mm})$, which led to
$N = 512.3$, $\alpha = 0.1$, $v_{x_1} = 6437.7$, and $v_{y_1} = 6437.7$. 

Fig. 3. Illustration of the coherence that originates from
a quasi-monochromatic, spatially incoherent disk
source: The parameters that we used to calculate the
locus of maximum coherence are $\lambda = 488 \text{ nm}$, $R = 5 \text{ mm}$,
and $\mathbf{r}_1 = (10,10,100 \text{ mm})$, which led to $N = 512.3$,
$v_{x_1} = 6437.7$, and $v_{y_1} = 6437.7$. 

Fig. 4. Modulus of the degree of coherence along the line defined by Eqs. (12): The degree of coherence along the line is calculated for a disk source and three cylindrical sources with different values of \(\alpha\). The parameters used in the simulation were \(N = 512.3\), \(v_x = 6437.7\), and \(v_y = 6437.7\).

The result is

\[
| j(r_1,r_2)|_{\text{max}} \approx \text{sinc}\left(\frac{NF}{2}\frac{\delta}{1 + \delta}\right) \times \frac{1}{a} \left| F\left(-\frac{a}{2}\right) - F\left(\frac{a}{2}\right) \right|,
\]

where

\[
a = a \left(\frac{1}{\alpha} \frac{v_1^2}{NF} \frac{\delta}{1 + \delta}\right)^{1/2} = \frac{L}{z_1^2} \left[ \frac{\pi(x_1^2 + y_1^2)\Delta z}{\lambda(1 + \delta)} \right]^{1/2}
\]

and \(F()\) is the Fresnel integral. Because of the weak dependence of the term \(1/a[F(-a/2) - F(a/2)]\) on \(a\) (Fig. 4, inset), we can conclude that the effect of thickness on the degree of coherence along the line defined by Eqs. (12) is minimal. The exact effect of thickness on the maximal degree of coherence is shown in Fig. 4.

Fig. 5. Contour map of the coherence that originates from a cylindrical source in the cross section at plane \(S\) along the line defined by Eqs. (12): The parameters used in the simulation are the same as for Fig. 2.

The exact contour of the coherence in the cross section at plane \(S\) for a relatively thick cylinder is shown in Fig. 5. The distribution of coherence is not rotationally symmetric about the line given by Eqs. (12). Instead, it looks elliptical, similar to the diffraction pattern from volume holograms.

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