TWO CRITICAL POINTS IN THE NJL MODEL PHASE DIAGRAM FOR QCD

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Phase diagram on the $T - \mu$ plane: the standard (and simplest) view from effective models like the NJL

If $m_c = 0$

Usually obtained from $V_{\text{eff}} = U_c + \text{Loops}$

low $T$: 1st order (Stars)
Theoretical Predictions

[Reviews: Stephanov; Hatsuda & Kunihiro, etc]
Standard phase diagram is consistent with:

Effective NJL Model  [Fukushima (2008)]

Lattice

C₁ > 0

BUT what if  C₁ < 0? (As found by de Forcrand & Philipsen [JHEP 0811, 012 (2008)])

First Order region Shrinks!!

NO CP???
One possibility to recover the CP: back bending of critical surface

Our Motivation: can a simple effective model like the SU(2) standard NJL support the back bending scenario? Under which circumstances?

Previous work on the same topic:

1) Fukushima [PNJL, 3 flavors, with vector interaction, 2008]
2) Bowman & Kapusta [LSM, 2 flavors, beyond MFA, 2009]
The two-flavor NJL effective model for quarks

\[ \mathcal{L} = \bar{\psi} (i \slashed{\partial} - m_c) \psi + G \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \vec{\tau} \psi)^2 \right] \]

- Scalar & Pseudo-scalar channels only
- Non-renormalizable since \( G : 1/(\text{eV})^2 \Rightarrow \text{cut-off} \ \wedge \ \text{(new “parameter”)} \)
- \( m_c = 0 \Rightarrow \text{Chiral Symmetry dynamically broken by } G > G_c \ \text{at } T=0, \sigma=0 \)

**STANDARD PARAMETRIZATION:**

\[ \Lambda \sim 600 \text{MeV} \quad G \Lambda^2 \sim 2 \quad m_c \sim 5 \text{MeV} \]

**gives**

\[ m_\pi = 135 \text{ MeV} \quad f_\pi = 92.4 \text{ MeV} \]

\[ -\langle \bar{\psi} \psi \rangle^{1/3} = 250.8 \text{ MeV} \quad m_q^0 = 300 \text{ MeV} \]

\[ m_\pi^2 = -\frac{m_c \langle \bar{\psi} \psi \rangle}{f_\pi^2} \]
NON STANDARD Parametrization to observe NON standard behavior: SCAN mc to make contact with Lattice
Find right G with fixed cut-off
Increasing G increases size 1st order line in the MFA

So, in general by, varying parameters with MFA only changes the size of the first order line but cannot account for the shrinkage observed in lattice simulations nor for the eventual back bending in the chemical potential-quark mass plane:

WE NEED to go beyond the large-N limit!
One option: **OPTIMIZED PERTURBATION THEORY (OPT)**

[Okopinska; Duncan & Moshe; Bender et al.; Zinn-Justin; Chiku & Hatsuda; Klimenko; Yukalov + many others]

(*aka Linear Delta Expansion*)

- **Examples of OPT recent results**

1) **BEC**: \[ \Delta T_c \sim c \] where:

   - \( c = 0 \) \hspace{1cm} MFA
   - \( c = 2.33 \) \hspace{1cm} 1/N-NLO (Baym, Blaizot & Zinn-Justin)
   - \( c = 1.30 \) \hspace{1cm} Monte Carlo Simulation (Arnold & Moore)
   - \( c = 1.29 \) \hspace{1cm} OPT to 7 loops (Kneur, MPB & Neveu)
2) Phase diagram for the Gross-Neveu Model in 2+1 d

Large N misses tricritical point, Monte Carlo Simulations (Kogut et al) suspected it to be around here (for N=4) but could not locate while OPT works!
[Kneur, MBP, Ramos & Staudt]

3) $T_c$ for the Gross-Neveu Model in 1+1 d (Landau’s theorem)

Recent improvements:
1) OPT & Ren Group [Kneur & Neveu]
2) Gauge Theories: HTLpt [Andersen et al.]

- In general Large $N$ result is reproduced at first order
- Finite $N$ contributions are taken into account in a perturbative fashion.
- Non Perturbative results are generated by a variational condition.
Bosonize the 4fermion lagrangian with auxiliary fields:

\[
\mathcal{L} = \bar{\psi} (i\hat{\phi} - m_c) \psi - \bar{\psi} (\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi}) \psi - \frac{N_c}{2\lambda} (\sigma^2 + \vec{\pi}^2)
\]

Same dynamics since E-L eqs give

\[
\sigma = - (\lambda/N_c) \bar{\psi} \psi = -2G \bar{\psi} \psi \quad \text{and} \quad \vec{\pi} = - (\lambda/N_c) \bar{\psi} i\gamma_5 \vec{\tau} \psi = -2G \bar{\psi} i\gamma_5 \vec{\tau} \psi
\]

Have done \( G \rightarrow \lambda/(2N_c) \)

**OPTIMIZED PERTURBATION THEORY (OPT)**

1) **Interpolate original theory by adding/subtracting mass term**
2) **Multiply vertices by** \( \delta \)
3) **Calculate physical quantity in powers of** \( \delta \), at the end set it to unity
4) **Fix ARBITRARY mass parameter** \( \eta \) (with PMS, later)

\[
\mathcal{L} = \bar{\psi} \left[ i\hat{\phi} - m_c - \delta (\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi}) - \eta (1 - \delta) \right] \psi - \delta \frac{N_c}{2\lambda} (\sigma^2 + \vec{\pi}^2)
\]
With new (but trivial) Feynman rules EVALUATE the FREE ENERGY $\mathcal{F}$ (Effective Potential) to a given order, $k$, then do:

$$\left. \frac{d\mathcal{F}(k)}{d\eta} \right|_{\bar{\eta}, \delta = 1} = 0$$

(PMS: Principle of Minimal Sensitivity)

Here, $k=1$ so:

$$\mathcal{F} = \text{Classical} + \text{Dressed propagator:}$$

\[
\tilde{\eta} = \eta + m_c - \delta \left[ \eta - (\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi}) \right]
\]

A bit like Hartree but mass parameter does not have a fixed topology and can chop NP info from direct, exchange, vertex corrections terms etc. Depends on the information contained in the perturbative series. Easily goes beyond MFA.
FREE ENERGY TO FIRST ORDER, sigma direction, IS:

\[
\frac{\mathcal{F}}{N_c} = \frac{\sigma_c^2}{2\lambda} + 2iN_f \int \frac{d^4p}{(2\pi)^4} \ln \left[-p^2 + (\eta + m_c)^2\right] \\
- 4i\delta N_f \int \frac{d^4p}{(2\pi)^4} \frac{(\eta + m_c)(\eta - \sigma_c)}{-p^2 + (\eta + m_c)^2} \\
- 8 \frac{\delta \lambda N_f}{N_c} \left[ \int \frac{d^4p}{(2\pi)^4} \frac{p_0}{-p^2 + (\eta + m_c)^2} \right]^2 \\
+ 4 \frac{\delta \lambda N_f}{N_c} (\eta + m_c)^2 \left[ \int \frac{d^4p}{(2\pi)^4} \frac{1}{-p^2 + (\eta + m_c)^2} \right]^2
\]

NOTE:

1) Explicit 1/N corrections

2) \(p_0\) corrections only at finite chemical potential
Finite Temperature & Density (via Matsubara)

\[
\int \frac{d^4 p}{(2\pi)^4} \equiv \frac{i}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^3 p}{(2\pi)^3}, \quad p = (i\omega_n + \mu, \mathbf{p})
\]

with \( \omega_n = (2n + 1)\pi T \), \( n = 0, \pm 1, \pm 2, \ldots \)

To first order, the free energy becomes

\[
\mathcal{F} = \frac{\sigma_c^2}{4G} - 2N_f N_c I_1 + 2\delta N_f N_c (\eta + m_c) (\eta - \sigma_c) I_2
\]
\[
+ 4\delta G N_f N_c I_3^2 - 2\delta G N_f N_c (\eta + m_c)^2 I_2^2
\]

Recall the large N result: as above but with \( \eta \rightarrow \sigma_c \)
(different effective mass) and neglecting O(G) terms since they are 1/N suppressed.
Comments on the Integrals

\[ I_1(\mu, T) = \int \frac{d^3p}{(2\pi)^3} \left\{ E_p + T \ln \left[ 1 + e^{-(E_p+\mu)/T} \right] + T \ln \left[ 1 + e^{-(E_p-\mu)/T} \right] \right\} \]

\[ I_2(\mu, T) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} \left[ 1 - \frac{1}{e^{(E_p+\mu)/T} + 1} - \frac{1}{e^{(E_p-\mu)/T} + 1} \right] \]

\[ I_3(\mu, T) = \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{e^{(E_p-\mu)/T} + 1} - \frac{1}{e^{(E_p+\mu)/T} + 1} \right] \mu \]

**NOTE:** vacuum contributions of \( I_1 \) and \( I_2 \) **DIVERGE** and will be cut off. \( I_3 \) **ONLY** survives at finite chemical potential (more later). Usually, thermal contributions also integrated up to cut off but Stefan-Boltzmann not observed, we do not cut them.
PMS OPTIMIZATION to fix arbitrary mass parameter

1) PMS Eqn:

\[
\left\{ [\eta - \sigma_c - 2(\eta + m_c)G I_2] \left[ 1 + (\eta + m_c) \frac{d}{d\eta} \right] I_2 + 4G I_3 \frac{d}{d\eta} I_3 \right\}_{\eta=\bar{\eta}} = 0
\]

NOTE: when N is large the “exact” result is reproduced at any temperature and/or chemical potential.

2) Gap Eqn:

\[
\overline{\sigma}_c = 4G N_f N_c (\eta + m_c) I_2
\]

When \( \mu = 0 \),

PMS eqn is \( \bar{\eta} = \sigma_c + 2G (\bar{\eta} + m_c) I_2 \)

Then \( \bar{\eta} = \overline{\sigma}_c \mathcal{G}(N) \)

with \( \mathcal{G}(N) = \left(1 + \frac{1}{2N_f N_c}\right) \)
Ready to do the thermodynamics using the optimized:

\[ P = -\mathcal{F}(\bar{\sigma}) \]

To get \( s = (\partial P / \partial T)_\mu \) and \( \rho_q = (\partial P / \partial \mu)_T \)

\[ \epsilon = -P + Ts + \mu \rho_q \quad , \quad \Delta = \frac{\epsilon - 3P}{T^4} \quad , \quad w = \frac{P}{\epsilon} \]

\[ \chi_q = \frac{\partial \rho_q}{\partial \mu} \quad , \quad \chi_m = \frac{\partial \rho_s}{\partial m_c} \quad , \quad \text{etc...} \]

Fixing the OPT’s parameters is (much) harder than within MFA due to 2 loops

Contribute to:

\[ m_\pi \]

\[ f_\pi \]
RESULTS with STANDARD PARAMETERS (Kneur, MBP & Ramos 2010)

Large Nc reliable in the 2 flavor NJL with standard parametrization.
RESULTS with NON STANDARD PARAMETERS (Ferroni, Koch & MBP 2010)

\[ \Lambda = 590 \text{ MeV} \quad G\Lambda^2 = 3.7 \quad m_c = 0.1 \text{ MeV} \]

Give reasonable observable values: \[ f_\pi \simeq 92.2 \text{ MeV} \]
\[ m_\pi \simeq 20 \text{ MeV} \quad \left\langle \bar{\psi} \psi \right\rangle^{1/3} \simeq 263 \text{ MeV} \]

BUT: \[ m_q^0 = 781 \text{ MeV} \]
Get an unusual phase diagram with extra CP (C₁)

Exotic? Yes, but remember Weinberg with Inverse Symmetry Breaking in the scalar O(N)xO(N) and check cond matter:

Similar behavior found in metamagnetic systems on the: $B$ vs $T$ plane [Moreira, Figueiredo & Henriques (2002)]
Two CP on different planes

Critical behavior of $c_2$ is the usual and agrees with other studies (eg Costa et al.)

<table>
<thead>
<tr>
<th>CP</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$\epsilon$</th>
<th>$\epsilon_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>2</td>
<td>1/2</td>
<td>2/3</td>
</tr>
<tr>
<td>$c_2$</td>
<td>2/3</td>
<td>1/3</td>
<td>2/3</td>
<td>3</td>
<td>2/3</td>
<td>1/2</td>
</tr>
</tbody>
</table>
Other thermodynamical quantities

\[ \Delta \epsilon = 0.3 \times 10^{-2} \text{ GeV}^4 \] (C1)

\[ \Delta \epsilon = 1.72 \times 10^{-2} \text{ GeV}^4 \] (C2)

EoS parameter \( w = P/\epsilon \).
\[ \Delta = \frac{\epsilon - 3P}{T^4} \]

\[ \chi_q = \frac{\partial \rho q}{\partial \mu} \]
OPT and MFA at high $G$ and low $m_c$

MFA: NO Back Bending

OPT: Back Bending!!!
RELATION to other model theoretical approaches:

1) Fukushima (2008): PNJL 3flavors + VECTOR INTERACTION

Adding (to lagrangian):

$$-G_V (\bar{\psi} \gamma_\mu \psi)(\bar{\psi} \gamma^\mu \psi)$$

Adds (to Thermo Potential):

$$\Omega_V = G_V \rho_q^2$$

OPT: Using

$$\bar{\sigma} = \langle \sigma \rangle = -2G \langle \bar{\psi} \psi \rangle = -2G \rho_s$$

write thermodynamical potential as

$$\Omega = G \rho_s^2 + \Omega_{\text{MFAL}}(\bar{\eta}) - (\bar{\eta} + 2G \rho_s) \rho_s + \frac{G}{N_c N_f} \left( \rho_q^2 - \frac{\rho_s^2}{2} \right)$$

NOTE $N_c \rightarrow \infty$, $\bar{\eta} = -2G \rho_s$ and OPT = MFA
2) Bowman & Kapusta (2009): LSM 2flavors +thermal fluctuations

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \pi)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 - U(\sigma, \pi) + \bar{\psi} [i\gamma^\mu \partial_\mu - g (\sigma + i\gamma_5 \tau \cdot \pi)] \psi \]

\[ U(\sigma, \pi) = \frac{\lambda}{4} \left( \sigma^2 + \pi^2 - f^2 \right)^2 - H \sigma \]

CS broken at tree level
Vacuum may be neglected (?!)

**Same pattern observed in the NJL for approximately same pion masses**
CONCLUSIONS

MFA and OPT give similar results for standard parametrization

OPT predicts TWO Critical Points at very strong coupling regime and certain quark mass range

One unusual CP appears at high-T while the other is the usual, obtained with standard parameters. We have investigated their differences.

OPT results support the back bending on the 1st order surface on the chemical potential-quark mass plane conciliating lattice Simulations with model predictions. The CP should be there, at the physical values, even if the initial curvature indicates its disappearance.

NJL 2f-OPT results relate to PNJL 3f+vector interaction and to LSM 2f+thermal fluctuations
Collaborators:

**Standard NJL:** Jean-Loic Kneur (Montpellier, France) & Rudnei Ramos (Rio de Janeiro, Brazil)


**NON Standard NJL:** Volker Koch (LBNL, Berkeley) & Lorenzo Ferroni (Frankfurt, Germany)

\[ \zeta = \frac{1}{9\omega_0} \left[ T^5 \frac{\partial}{\partial T} \frac{(\epsilon - 3P)}{T^4} + 16|\epsilon_0| \right] \]