Advanced Compressor Modeling (Compressors 102)

Calculating Gas Pulsations

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Web-site
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Outline

- Introduction-Noise Problems in Compressors
- Undamped One Dimensional Wave Equation
- Derivation of Wave Equation
- Solution of Wave Equation
- The Four Pole Concept
- Derivation of Four Poles
- Global Four Poles From Local Element
  a) Tube in series
  b) Tubes in parallel
- Calculating Gas Pulsations using Mass Flow Rate
- Implementation in MATLAB: Simulation Code.

Noise Problems in Compressors

- Compressors generate noise even if no vibrating valve reed or valve plates are present.
- Compressors generate a time varying force, which creates structural vibrations of the compressor casing. [Soedel, 2007] .
- Compressor valves flutter and add noise to the compressor. The valve flutter is caused by two mechanisms [Soedel, 2007]:
  - Sudden opening of the compressor valves
  - Negative pressure in the valve seat.
- Sound is also produced by the valve impact during opening and closing of the valve.

- Potential radiators of sound include:

- Manifolds due to transient flows and acoustic response.

- Valve due to flutter, excitation of resonances in manifold, and impact.

- Structure borne noise from casing and suction/discharge lines.

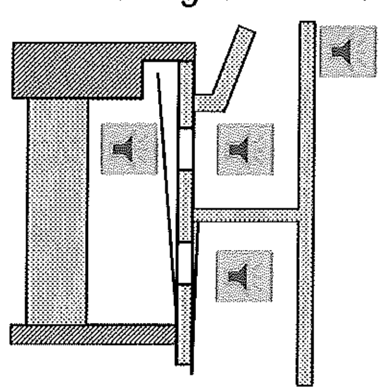


Figure 1: Compressor and likely noise sources

How Do Gas Pulsations Become Noise?

- Gas pulsations are directly radiated towards the receiver.
- The acoustic vibrations of the gas in this space excites the natural modes and frequencies of the hermetic shell or housing, which then radiates noise to the receiver [Soedel, 2007]. ?
- Discharge pipes also transmit noise because of the bends [Soedel, 2007]. change in momentum
- Discharge pipes also get excited through the vibration of the casing.

Objective

Objective:

- To calculate the pressure response in a simple tube connected to a compressor.

Response Location:

- Response calculated at the valve location at the tube inlet.
- Response is a function of both time and spatial variable.

Excitation Force:

- Mass flow rate through the tube

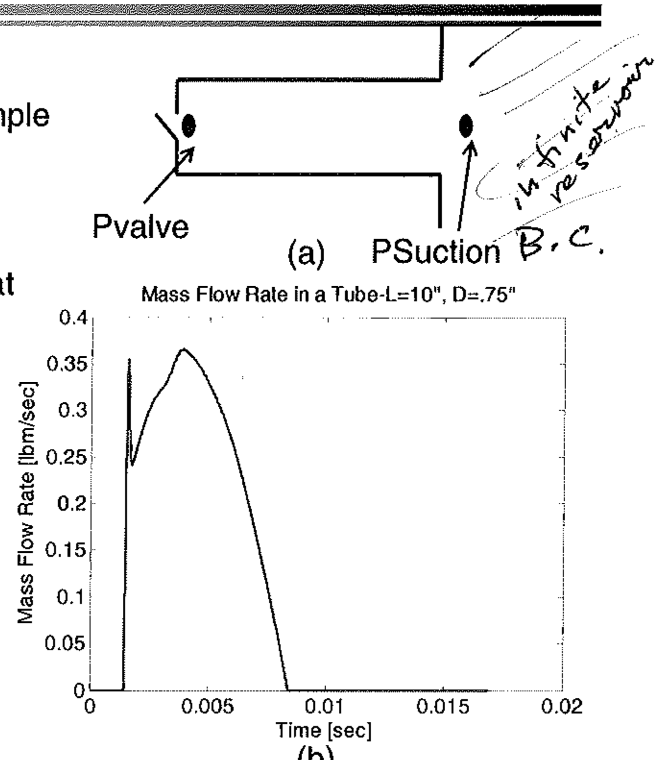
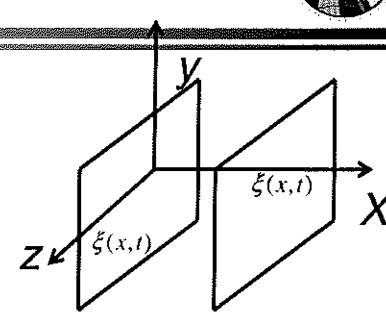


Figure 2:(a) A simple tube, (b) Mass flow rate through the tube (B.C.)

The One Dimensional Wave Equation

- Most problems related to gas pulsation can be solved using the 1D wave equation without much simplification.
- Simple tube-like compressor suction and discharge manifolds are used to apply the wave equation to calculate the gas pulsations in this kind of acoustic element.
- The same principle can be extended to more complicated shapes and multiple inputs. Composites

1D Wave Equation

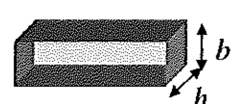


Assumptions:

- Plane wave: If all the acoustic variables are functions of only one spatial coordinate, the phase of each variable is a constant on any plane perpendicular to this coordinate:

Plane wave assumption validity

- Rectangular duct ←→ $h > b, \lambda > 2h$
- Circular duct ←→ $\lambda > (\pi/1.84)D$



λ=wavelength of interest

- Discontinuities do not introduce changes in variables if the dimensions of the tube satisfy the plane wave assumptions above.
- The fluctuations are small compared to the static pressure:

$$\frac{p(t)}{p_o} << 1, \ \frac{\rho(t)}{\rho_o} << 1$$

Derivation of 1D-Wave Equation

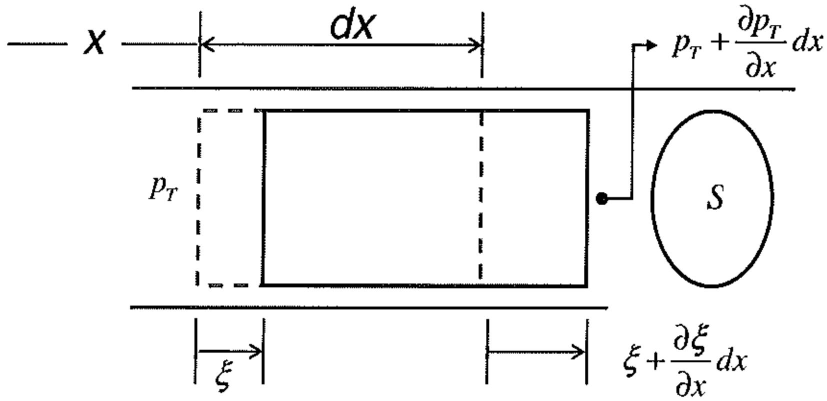


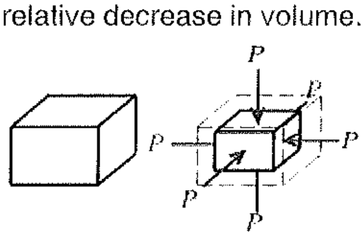
Figure 3: Element of a continuous gas column

where,
S=cross-sectional area
r_1=Damping coefficient
ρ_o=mean fluid density
ξ=particle displacement
X=direction of particle displacement
p_T=fluid pressure

Ref: Soedel (2007)

1D-Wave Equation

- Bulk Modulus (K) of a substance is the measure of the substance's resistance to uniform compression.
- It is defined as the pressure needed to cause a given relative decrease in volume.



Source: Wikipedia

Volume of the element

$$V_o = Sdx \quad (1)$$

Volume of the displaced element

$$V_o + dV = S\left(dx + \frac{\partial \xi}{\partial x} dx\right) \quad (2)$$

Volume increase is

$$dV = S\frac{\partial \xi}{\partial x} dx \quad (3)$$

Using the bulk modulus K_o formula

$$p = -K_o \frac{dV}{V_o} \quad (4)$$

$$\frac{dV}{V_0} = \frac{S\frac{\partial \xi}{\partial x} dx}{Sdx} = \frac{\partial \xi}{\partial x}$$

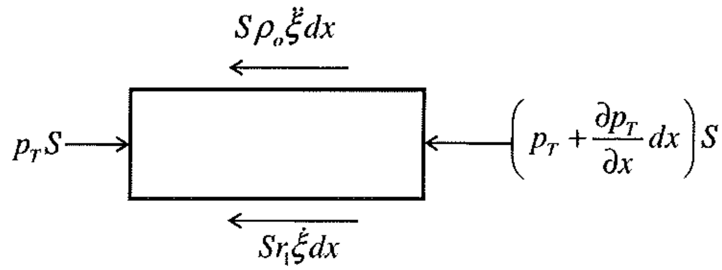
$$K_o = \rho_o c_o^2 \quad (5)$$

speed of sound

1D-Damped Wave Equation

$$p = -Ko\frac{\partial \xi}{\partial x} \quad (6)$$

From the free body diagram of the gas element



$$S\left(p_o + p\right) - S\rho_o\ddot{\xi}dx - S\left(p_o + p + \frac{\partial p}{\partial x}dx\right) - r_1 S\dot{\xi}dx = 0 \quad (7)$$

or

$$\rho_o\ddot{\xi} + r_1\dot{\xi} + \frac{\partial p}{\partial x} = 0 \quad (8)$$

1D-Damped Wave Equation

Using Eq. (6)

acceleration of Δx gas

$$\ddot{\xi} + \frac{r_1}{\rho_o}\dot{\xi} = c_o^{\,2}\frac{\partial^2 \xi}{\partial x^2} \quad (9)$$

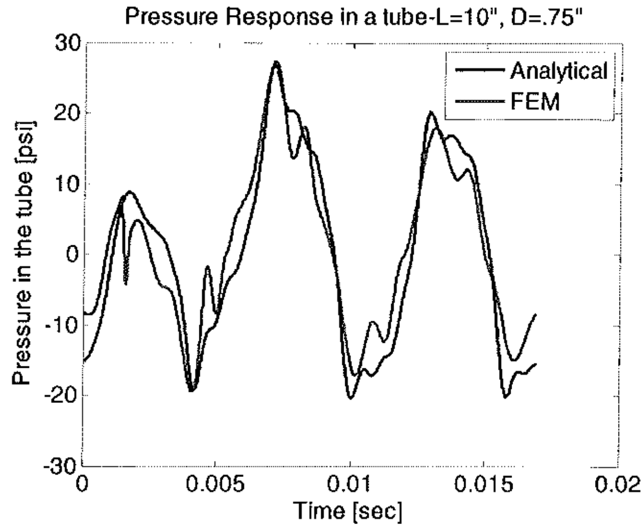
r_1= the equivalent viscous damping coefficient [N sec/m]
ρ_o=mean density [N sec^2/m^4]
c_o=mean speed of sound [m/s].

The value of r_1 can be determined by (Soedel, 2007)

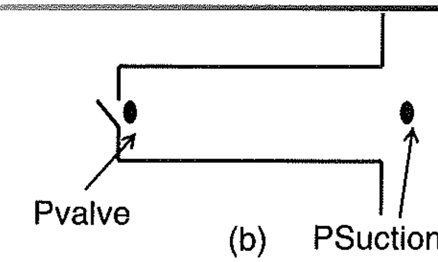
$$r_1 = \frac{2\rho_o}{d}\sqrt{2\upsilon n\Omega} \quad (10)$$

D= tube diameter [m]
υ = kinematic viscosity [m^2/sec]
Ω = rotational speed of compressor
n = harmonic number (n= 1,2,…)

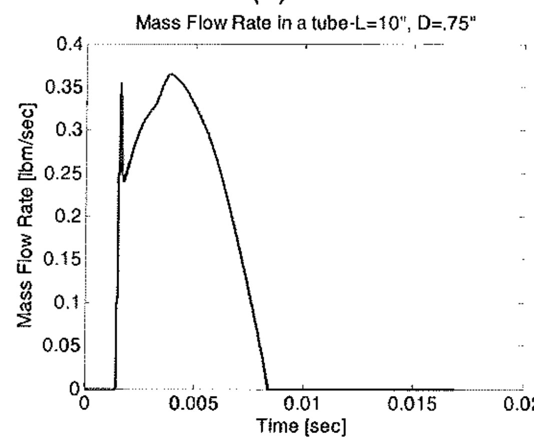
Pressure Response Calculated Using The Wave Equation



(a)



(b)



(c)

Figure 4: (a) The pressure response in the tube at valve location, (b) the simple tube connected to a compressor, (c) the mass flow rate--the excitation function.

Solution of 1D-Undamped Wave Equation

The undamped one dimensional wave equation is given by

$$\frac{\partial^2 \xi}{\partial t^2} = c_o^{\,2} \frac{\partial^2 \xi}{\partial x^2} \qquad (13)$$

no damping

D'Alembert's solution of wave equation is given by

$$\xi(x,t) = A_1 e^{j(\omega t - kx)} + B_1 e^{j(\omega t + kx)} \qquad (14)$$

right & left traveling waves

where A_1 and B_1 are arbitrary constant

$k = \frac{\omega}{c_o} = \frac{2\pi}{\lambda}$ = the wave number

$\omega = n\Omega$, where, n = the harmonic number (n=1,2,...),

Ω = rotational speed of compressor in rad/sec

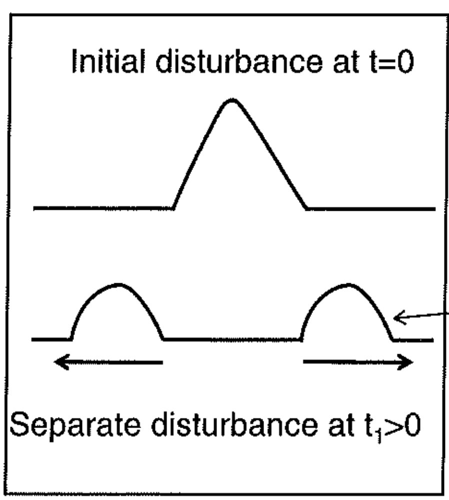


Figure 5 : Left traveling and right traveling waves

Substituting the solution in the wave equation gives

Using (6), the pressure is

$$p = jk\rho_o c_o^{\,2}[A_1 e^{j(\omega t-kx)} - B_1 e^{j(\omega t+kx)}] \qquad (15)$$

The volume velocity is

$$q(x,t) = S\dot{\xi}$$

$$q(x,t) = j\omega S[A_1 e^{j(\omega t-kx)} + B_1 e^{j(\omega t+kx)}] \qquad (16)$$

Note: The detailed derivation is given in the Appendix

The Solution of 1D-Damped Wave Equation

Motivation for Damping:

- To model the physical system more accurately.
- The actual response is matched much better by including the effects of damping in the model.
- Small changes in damping values make a significant change in the pressure response.

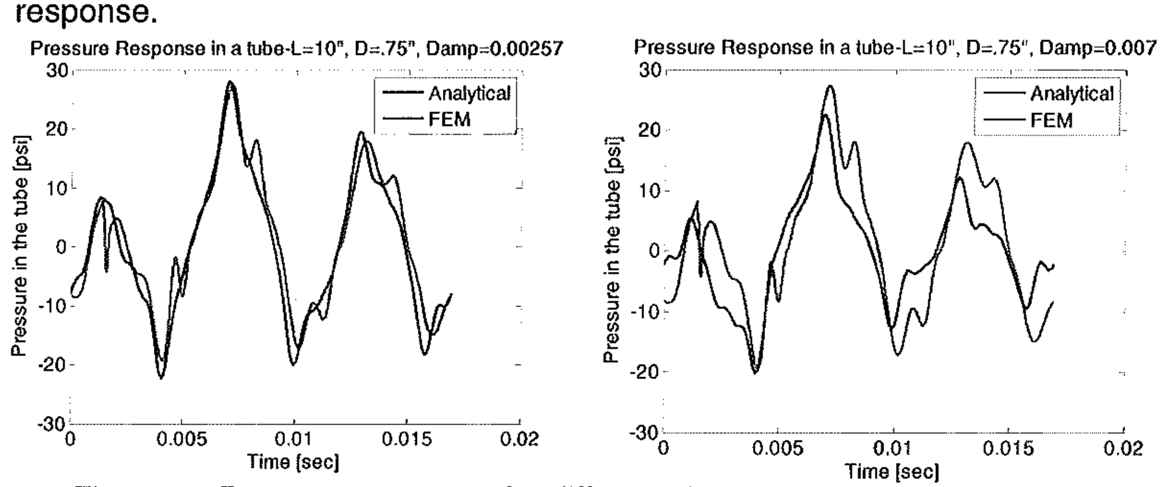


Figure 6 : Pressure response for different damping values

The Solution of 1D-Damped Wave Equation

The damped wave equation is given by

$$\frac{\partial^2 \xi}{\partial t^2} + \gamma_1 \frac{\partial \xi}{\partial t} = c_o^{\,2} \frac{\partial^2 \xi}{\partial x^2} \tag{17}$$

where $$\gamma_1 = \frac{r_1}{\rho_o} \tag{18}$$

The solution of damped wave equation for harmonic input is given by

$$\xi(x,t) = A_1 e^{j(\omega t - k_1 x)} + B_1 e^{j(\omega t + k_1 x)} \tag{19}$$

where $k_1 = \frac{\omega}{c_1} =$ modified wave number $= \frac{2\pi}{\lambda}$?

c_1 = modified speed of sound

substitute the solution in Eq. 17, and after some simplifications, we get

The Solution of 1D-Damped Wave Equation

The solution can be written as

$$p(x,t) = \rho_o c_o^{\,2} \gamma [A_1 e^{-\gamma x} - B_1 e^{+\gamma x}] e^{j\omega t} \tag{30}$$

and

$$q(x,t) = j\omega S [A_1 e^{-\gamma x} + B_1 e^{+\gamma x}] e^{j\omega t} \tag{31}$$

where $$\gamma = a + jk$$

$$k = \frac{\omega}{c_o}, \quad a = \frac{\gamma_1}{2c_o}, \text{ and } \quad \gamma_1 = \frac{r_1}{\rho_0}$$

we define $$P(x) = \rho_o c_o^{\,2} \gamma [A_1 e^{-\gamma x} - B_1 e^{+\gamma x}] \tag{33}$$

$$Q(x) = j\omega S [A_1 e^{-\gamma x} + B_1 e^{+\gamma x}] \tag{34}$$

Note: Complete derivation is given in the appendix

can be transformed to 4-pole

The Four Pole Concept

- Four pole parameters are useful for the analysis of composite acoustic systems
- Expresses flow conditions at one end of the cavity as a function of conditions at the other end of the cavity
- Q and P are complex harmonic amplitudes of the volume flow velocity, and pressure, respectively

$$\begin{Bmatrix} Q_1 \\ P_1 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} Q_2 \\ P_2 \end{Bmatrix}$$

- A, B, C, and D are called the four poles and are defined as follows

$$A = \cosh\gamma L = D$$

$$B = \frac{j\omega S}{\rho_o c_o^2}\sinh\gamma L \qquad C = \frac{\rho_0 c_0^2 \gamma}{j\omega S}\sinh\gamma L$$

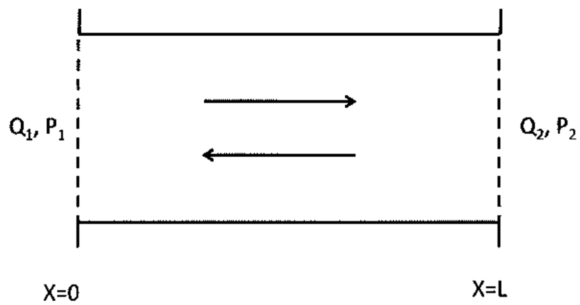


Figure 7: Four pole parameters for a simple tube

ρ=fluid density; c=speed of sound; γ=the complex wave number; L=tube length

The Global Four Poles From Local Four Pole Element-Tubes In Series

Tubes In Series:

Four pole of the first tube is

$$\begin{Bmatrix} Q_{01} \\ P_{01} \end{Bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{Bmatrix} Q_{L1} \\ P_{L1} \end{Bmatrix}$$

Four pole of the second tube is

$$\begin{Bmatrix} Q_{02} \\ P_{02} \end{Bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{Bmatrix} Q_{L2} \\ P_{L2} \end{Bmatrix}$$

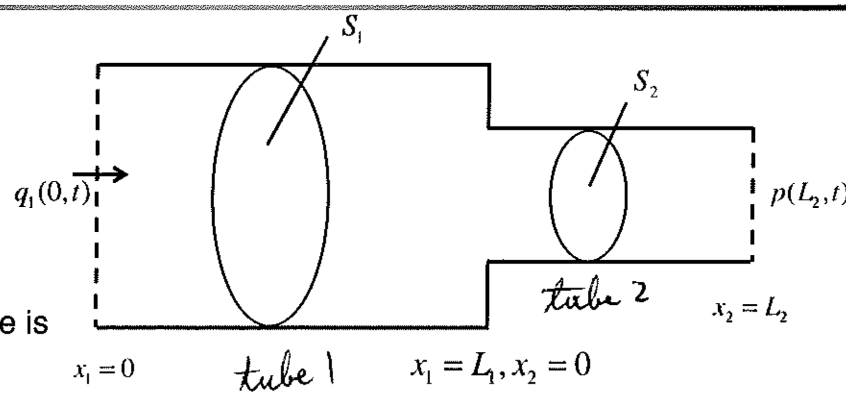


Figure 8: Combination of two tubes in series

At the boundary of the two tubes

$$\begin{Bmatrix} Q_{L1} \\ P_{L2} \end{Bmatrix} = \begin{Bmatrix} Q_{02} \\ P_{02} \end{Bmatrix}$$

The Global Four Poles From Local Four Pole Element-Tubes In Series

Thus, we get

$$\begin{Bmatrix} Q_{01} \\ P_{01} \end{Bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{Bmatrix} Q_{L2} \\ P_{L2} \end{Bmatrix} \qquad (46)$$

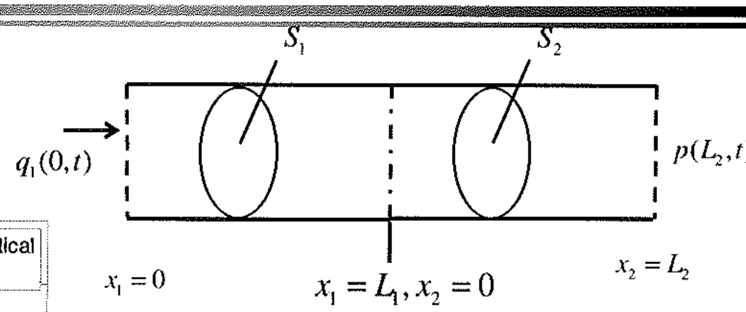
tube 1 tube 2

The same principle of connecting tubes in series can be extended to *n* tubes

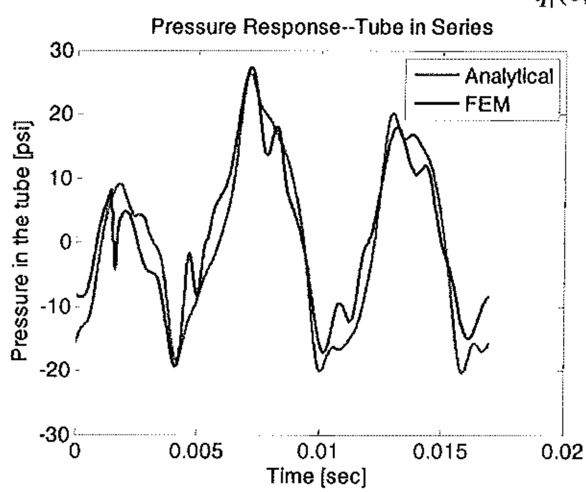
$$\begin{Bmatrix} Q_{01} \\ P_{01} \end{Bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \cdots \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \begin{Bmatrix} Q_{Ln} \\ P_{Ln} \end{Bmatrix} \qquad (47)$$

Two Tube Connected in Series

Calculating the pressure response at location x_1 in the two tubes connected in series.



(a)



(b)

$L_1 = 5''; \quad L_2 = 5''$
$d_1 = 0.75''; d_2 = 0.75''$

Figure 9: (a) Tube tubes in series, (b) pressure response at x_1=0

The Global Four Poles From Local Four Pole Element-Tubes In Parallel

Branches Tubes:

The four poles for each tube are:

transfer matrix See Appendix

$$\begin{Bmatrix} Q_{01} \\ P_{01} \end{Bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{Bmatrix} Q_{L1} \\ P_{L1} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_{02} \\ P_{02} \end{Bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{Bmatrix} Q_{L2} \\ P_{L2} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_{03} \\ P_{03} \end{Bmatrix} = \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} \begin{Bmatrix} Q_{L3} \\ P_{L3} \end{Bmatrix}$$

At branch junction, we have

$$Q_{L1} = Q_{02} + Q_{03} \quad (48)$$

$$P_{L1} = P_{02} = P_{03} \quad (49)$$

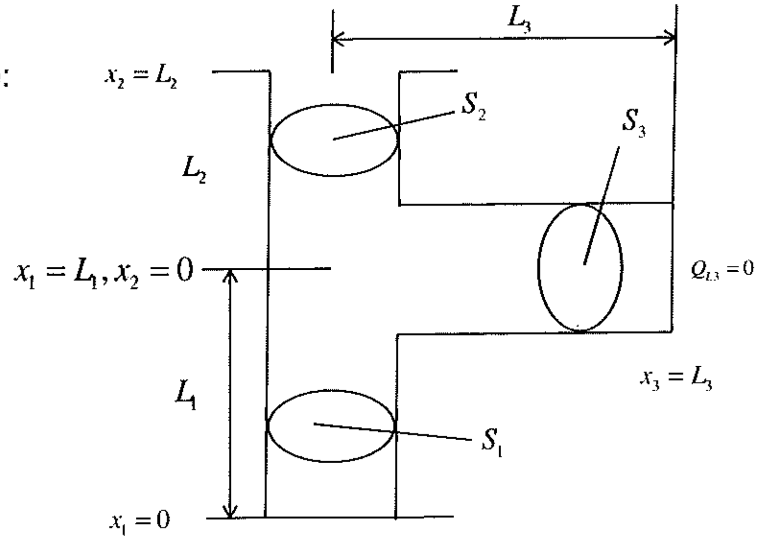


Figure 10: Branched tubes

Example of the General Formulation of Four Pole Parameters

The derivation of the general expression is shown in the Appendix -I

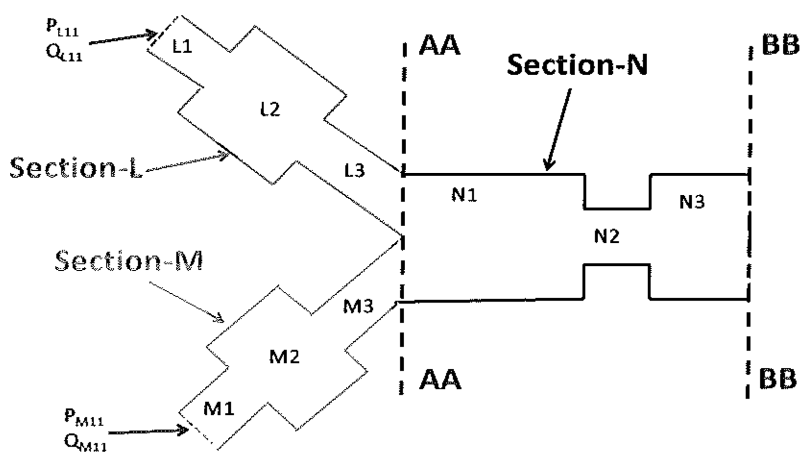


Figure 11: Schematic of a multi-cylinder compressor with each section consisting of several tubes connected in series.

Simulation Code: Tubes Connected in Series and Parallel -I

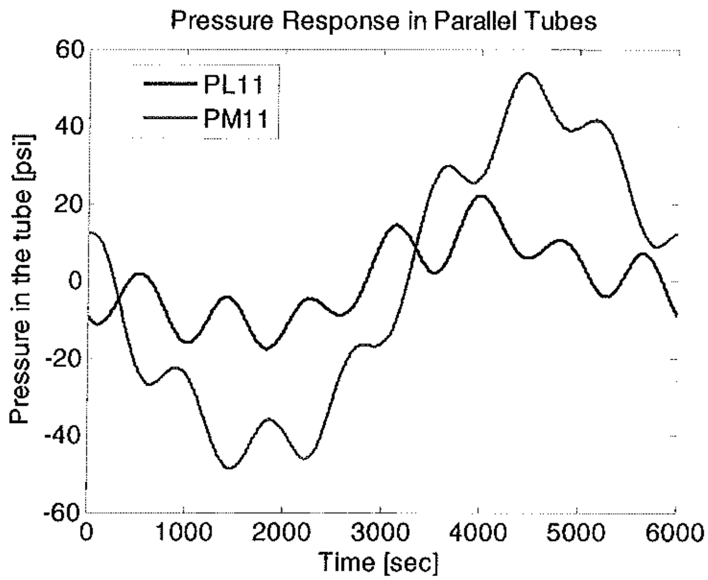


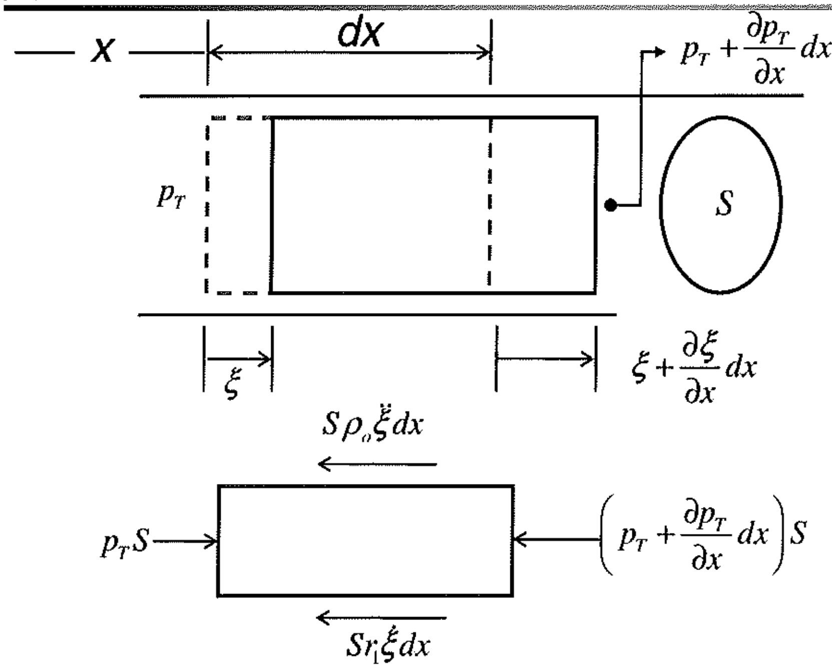
Figure 11: Pressure response in two tube connected in parallel with multiple inputs

References

1. Soedel, W., Sound and Vibrations of Positive Displacement Compressors, CRC Press, 2007.
2. Soedel, W., Introduction To Mathematical Modeling of Sound And Vibrations For Positive Displacement Compressors And Implications on Design, Short Course Notes. West Lafayette, IN, Purdue University, 1999.
3. Park, J. I., Mathematical Modeling and Simulation of a Multi-Cylinder Automotive Compressor, PhD Thesis, Purdue University, 2004.
4. O' Neil, P. V., Advanced Engineering Mathematics, 5th Edition, Thomson Brooks/cole, 2003, pp 693.

Appendix

Derivation of 1D-Wave Equation



where,
S=cross-sectional area
r_1=Damping coefficient
ρ_o=mean fluid density
ξ=particle displacement
X=direction of particle displacement
p_T=fluid pressure

Figure 1: Element of a continuous gas column

Ref: Soedel (2007)

1D-Wave Equation

Volume of the element

$$V_o = S dx \qquad (1)$$

Volume of the displaced element

$$V_o + dV = S\left(dx + \frac{\partial \xi}{\partial x} dx\right) \qquad (2)$$

Volume increase is

$$dV = S \frac{\partial \xi}{\partial x} dx \qquad (3)$$

Using the bulk modulus formula

$$p = -K_o \frac{dV}{V_o} \qquad (4)$$

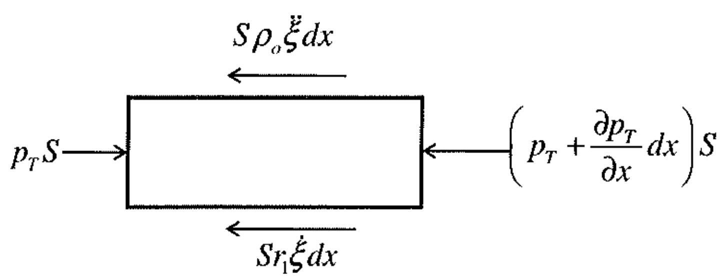
$$\frac{dV}{V_0} = \frac{S \frac{\partial \xi}{\partial x} dx}{S dx} = \frac{\partial \xi}{\partial x}$$

$$K_o = \rho_o c_o^{\,2} \qquad (5)$$

1D-Damped Wave Equation

$$p = -Ko \frac{\partial \xi}{\partial x} \qquad (6)$$

From the free body diagram, Fig. 1



$$S\left(p_o + p\right) - S\rho_o \ddot{\xi} dx - S\left(p_o + p + \frac{\partial p}{\partial x} dx\right) - r_1 S \dot{\xi} dx = 0 \qquad (7)$$

or

$$\rho_o \ddot{\xi} + r_1 \dot{\xi} + \frac{\partial p}{\partial x} = 0 \qquad (8)$$

1D-Damped Wave Equation

Using Eq. (6)

$$\ddot{\xi} + \frac{r_1}{\rho_o}\dot{\xi} = c_o^{\,2}\frac{\partial^2 \xi}{\partial x^2} \qquad (9)$$

r_1= the equivalent viscous damping coefficient [N sec/m]
ρ_o=mean density [N sec^2/m^4]
c_o=mean speed of sound [m/s].

The value of r_1 can be determined by (Soedel, 2007)

$$r_1 = \frac{2\rho_o}{d}\sqrt{2\upsilon n\Omega} \qquad (10)$$

d=effective tube diameter [m]
υ = kinematic viscosity [m^2/sec]
Ω = rotational speed of compressor
n = harmonic number (n= 1,2,…)

1D-Damped Wave Equation

It is sometime useful to write Eq. (8) in terms of volume velocities q [m^3/sec]

$$q = S\dot{\xi} \qquad (11)$$

after differentiation, it becomes

$$\ddot{q} + \frac{r_1}{\rho_o}\dot{q} = c_o^{\,2}\frac{\partial^2 q}{\partial x^2} \qquad (12)$$

Solution of 1D-Undamped Wave Equation

The undamped one dimensional wave equation is given by

$$\frac{\partial^2 \xi}{\partial t^2} = c_o{}^2 \frac{\partial^2 \xi}{\partial x^2} \quad (13)$$

D'Alembert's solution of wave equation is given by

$$\xi(x,t) = A_1 e^{j(\omega t - kx)} + B_1 e^{j(\omega t + kx)} \quad (14)$$

where A_1 and B_1 are arbitrary constant

$k = \frac{\omega}{c_o} = \frac{2\pi}{\lambda}$ = the wave number

$\omega = n\Omega$, where, n=the harmonic number (n=1,2,...),

Ω=rotational speed compressor in rad/sec

Solution of 1D-Undamped Wave Equation

Using (6), the pressure is

$$p = jk\rho_o c_o{}^2 [A_1 e^{j(\omega t - kx)} - B_1 e^{j(\omega t + kx)}] \quad (15)$$

The volume velocity is

$$q(x,t) = j\omega S [A_1 e^{j(\omega t - kx)} + B_1 e^{j(\omega t + kx)}] \quad (16)$$

The Solution of 1D-Damped Wave Equation

The damped wave equation is given by

$$\frac{\partial^2 \xi}{\partial t^2} + \gamma_1 \frac{\partial \xi}{\partial t} = c_o{}^2 \frac{\partial^2 \xi}{\partial x^2} \quad (17)$$

where $$\gamma_1 = \frac{r_1}{\rho_o} \quad (18)$$

The solution of damped wave equation for harmonic input is given by

$$\xi(x,t) = A_1 e^{j(\omega t - k_1 x)} + B_1 e^{j(\omega t + k_1 x)} \quad (19)$$

where $k_1 = \dfrac{\omega}{c_1} =$ modified wave number

c_1 = modified speed of sound

substitute the solution in Eq. 17, we get

The Solution of 1D-Damped Wave Equation

$$-\omega^2 \xi + \gamma_1 (j\omega)\xi = c_o{}^2(-k_1{}^2)\xi \quad (20)$$

or

$$k_1 = \left(\frac{\omega}{c_o}\right)\sqrt{1 - j\frac{\gamma_1}{\omega}} \quad (21)$$

where $$c_1 = \frac{c_o}{\sqrt{1 - j\frac{\gamma_1}{\omega}}} \quad (22)$$

since $\dfrac{\gamma_1}{\omega} << 1,$ the square root expansion gives the approximate solution

$$k_1 = \frac{\omega}{c_o}\left(1 - j\frac{\gamma_1}{2\omega}\right) = \frac{\omega}{c_o} - j\frac{\gamma_1}{2c_o} \quad (23)$$

The Solution of 1D-Damped Wave Equation

$$jk_1 = j\frac{\omega}{c_o} + \frac{\gamma_1}{2c_o} \quad (24)$$

let $k = \dfrac{\omega}{c_o}$ and $a = \dfrac{\gamma_1}{2c_o}$ (25 a & b)

The solution of the damped wave equation becomes

$$\xi(x,t) = A_1 e^{-ax} e^{j(\omega t - kx)} + B_1 e^{+ax} e^{j(\omega t + kx)} \quad (26)$$

The pressure is $p = -\rho_o c_o^2 \dfrac{\partial \xi}{\partial x}$ (27)

or $p(x,t) = \rho_o c_o^2 (a + jk)[A_1 e^{-ax} e^{j(\omega t - kx)} - B_1 e^{+ax} e^{j(\omega t + kx)}]$ (28)

The Solution of 1D-Damped Wave Equation

the volume velocity is

$$q(x,t) = j\omega S[A_1 e^{-ax} e^{j(\omega t - kx)} + B_1 e^{+ax} e^{j(\omega t + kx)}] \quad (29)$$

The solution can be written as

$$p(x,t) = \rho_o c_o^2 \gamma [A_1 e^{-\gamma x} - B_1 e^{+\gamma x}] e^{j\omega t} \quad (30)$$

and

$$q(x,t) = j\omega S[A_1 e^{-\gamma x} + B_1 e^{+\gamma x}] e^{j\omega t} \quad (31)$$

where $\gamma = a + jk$ (32)

we define

$$P(x) = \rho_o c_o^2 \gamma [A_1 e^{-\gamma x} - B_1 e^{+\gamma x}] \quad (33)$$

Laplace Xfer

$$Q(x) = j\omega S[A_1 e^{-\gamma x} + B_1 e^{+\gamma x}] \quad (34)$$

The Four Pole Concept

- Four pole parameters are useful for the analysis of composite acoustic systems
- Expresses flow conditions at one end of the cavity as a function of conditions at the other end of the cavity
- Q and P are complex harmonic amplitudes of the volume flow velocity, and pressure, respectively

$$\begin{Bmatrix} Q_1 \\ P_1 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} Q_2 \\ P_2 \end{Bmatrix}$$

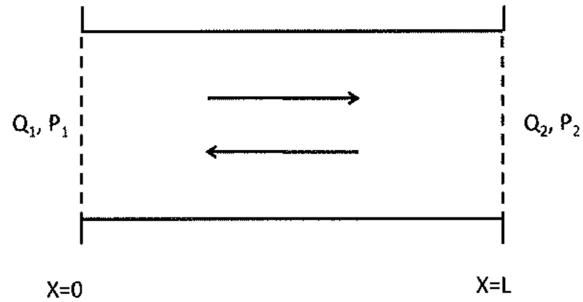


Figure 2: A constant area tube

- A, B, C, and D are called the four poles and are defined as follows

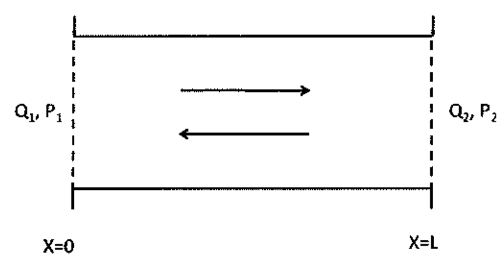
$$A = \cosh\gamma L = D$$

$$B = \frac{j\omega S}{\rho_o c_o^2} \sinh\gamma L \qquad C = \frac{\rho_0 c_0^2 \gamma}{j\omega S} \sinh\gamma L$$

ρ=fluid density; c=speed of sound; γ=the complex wave number; L=tube length

The Derivation of Four Poles

- Q_1 and Q_2 are complex harmonic amplitudes of the volume flow at inlet and outlet respectively
- The volume flow conditions at inlet and outlet are



$$Q(0) = Q_o \quad \text{and} \quad Q(L) = Q_L$$

The pressure at both ends is unknown.

Applying the boundary conditions Eq. 33 and 34

$$j\omega S\left(A_1 + B_1\right) = Q_o \tag{35}$$

$$j\omega S\left(A_1 e^{-\gamma L} + B_1 e^{+\gamma L}\right) = Q_L \tag{36}$$