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DISCOVERY OF THERMAL CIRCUIT FOR A WALL

by
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PREFACE

The study of building heat losses and gains resulting from the Canadian climate and of the means of calculating or of predicting them is an obvious one for the Division. Attention was given first to the surface heat exchange at the exterior surfaces of buildings, and instrumentation is now being devised to measure the pertinent factors of the outdoor environment and the way in which they vary. The other two stages of the over-all study involve consideration of heat transfer under transient and cyclical conditions through the building enclosure and, finally, the energy exchanges between the inner bounding surfaces of a space, the air and any heat sources or sinks. The development of computers both digital and analog has made it feasible to consider new approaches and new techniques in research studies and perhaps, in future, in routine design calculations.

The work which is now reported describes two methods of finding equivalent analog circuits for walls. The authors, research officers in the Building Services Section of the Division, are presently engaged in the program of heat transfer studies described above.

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An accurate calculation of the heat flux and temperature of the surfaces of the walls, ceiling or floor of a room requires the simultaneous calculation of the room side surface heat flux of all elements enclosing the room. Reference 1 discusses the use of an analog computer for this type of building heat transfer problem. It is pointed out in that paper that one of the important considerations in setting up a computer for a room heat transfer calculation is, "How to simulate accurately the wall, roof, and floor sections with as few computer elements as possible." Reference 2 presents a method of designing an analog circuit to simulate an element with one-dimensional heat flow. Many of the commonly used building elements are not made up entirely of uniform layers, hence the calculation of an analog circuit is more difficult.

This report describes two methods of finding an equivalent circuit for a wall when the elements of the wall transmission matrix and steady state thermal resistance are known. These data can be obtained by experiment. A separate report on the determination of wall transmission matrix elements is being prepared.

THEORY

The sinusoidal heat flow through the surfaces and the temperature of the surfaces of a wall can be related by

$$\begin{bmatrix} \theta_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} A_w & B_w \\ D_w & E_w \end{bmatrix} \cdot \begin{bmatrix} \theta_2 \\ q_2 \end{bmatrix} \qquad (1)$$

Ref 3

where

θ = sinusoidal surface temperature with a period P

q = " " heat flux " " "

subscripts 1 and 2 refer to the two sides of the wall

A_w, B_w, D_w, E_w = Transmission matrix coefficients which are functions of wall thermal properties and cycle period P.

Relation (1) is valid only for walls where the heat flow is one-dimensional and the thermal properties independent of temperature and time. Thermal properties, however, can vary through the wall i. e. the wall can be made up of layers of different material.

A relation similar to (1) relates the voltages and currents in a four-terminal resistance-capacitance network * shown in Fig. 1

$$\begin{bmatrix} V_1 \\ \\ I_1 \end{bmatrix} = \begin{bmatrix} A_E & B_E \\ \\ D_E & E_E \end{bmatrix} \cdot \begin{bmatrix} V_2 \\ \\ I_2 \end{bmatrix} \qquad (2)$$

Ref 4

where

V = sinusoidal voltage with a period P

I = " current " " "

A_E, B_E, D_E, E_E = Transmission matrix coefficients which are functions of circuit parameters and cycle period P.

The voltages and currents in the thermal circuit will simulate the wall thermal performance if

$$A_w \approx A_E$$
$$B_w \approx B_E$$
$$D_w \approx D_E$$
$$E_w \approx E_E.$$

The difference between the respective matrix coefficients of the wall and the thermal circuit indicates the error in the simulation.

The following sections describe a numerical and an analog method for finding the most accurate two-lump thermal circuit to match any given wall transmission matrix and steady state thermal resistance.

Numerical Method

Each component of a thermal circuit (resistance or capacitance) can be represented by a simple transmission matrix.

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* subsequently called thermal circuit

Resistance

$$R \quad \begin{bmatrix} 1 \ , & R \\ 0 \ , & 1 \end{bmatrix} \tag{3}$$

Capacitance

$$C \quad \begin{bmatrix} 1 \ , & 0 \\ i\omega C, & 1 \end{bmatrix} \tag{4}$$

Ref 4

where

R = resistance

C = capacitance

ω = angular velocity

$i = \sqrt{-1}$

The transmission matrix for the network shown in Fig. 2 is the product of the matrices of the individual circuit components.

Thus

$$M_\pi = \begin{bmatrix} A_\pi \ , & B_\pi \\ D_\pi \ , & E_\pi \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \ , & aR \\ 0 \ , & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \ , & 0 \\ i\omega C_1 , & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \ , & bR \\ 0 \ , & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \ , & 0 \\ i\omega C_2 , & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \ , & dR \\ 0 \ , & 1 \end{bmatrix} \tag{4}$$

or

$$A_\pi = \{1 - 4ab\, \phi_1^2\, \phi_2^2 + i2\, (a\phi_1^2 + b\phi_2^2 + a\phi_2^2)\}$$

$$D_\pi = \frac{1}{R}\ \{- 4b\, \phi_1^2 \phi_2^2 + i2\, (\phi_1^2 + \phi_2^2)\}$$

$$B_\pi = R\ \{1 - 4abd\, \phi_1^2 \phi_2^2 + i2(ad\, \phi_1^2 + ad\, \phi_2^2 + bd\, \phi_2^2 + ab\, \phi_1^2)\}$$

$$E_\pi = \{1 - 4bd\, \phi_1^2\, \phi_2^2 + i2(d\, \phi_2^2 + d\, \phi_1^2 + b\, \phi_1^2)\} \tag{5}$$

where
$$\phi_1^2 = \omega \frac{RC_1}{2} \quad (6)$$

$$\phi_2^2 = \omega \frac{RC_2}{2} . \quad (7)$$

In general, the transmission matrix for a building component is

$$M_w = \begin{bmatrix} A_w & B_w \\ D_w & E_w \end{bmatrix} = \begin{bmatrix} m + ix, & p + iy \\ k + iv, & h + iz \end{bmatrix} \quad (8)$$

The elements of these transmission matrices are complex numbers and can be represented graphically as vectors. The differences between the vectors of the building component matrix and the corresponding vectors for a simulating network indicate the accuracy of the simulating network. A convenient over-all error criterion is

$$\Delta^2 = \left(\frac{|\Delta A|}{|A_w|}\right)^2 + \left(\frac{|\Delta B|}{|B_w|}\right)^2 + \left(\frac{|\Delta D|}{|D_w|}\right)^2 + \left(\frac{|\Delta E|}{|E_w|}\right)^2 \quad (9)$$

where

$$|\Delta A|^2 = \{m - 1 + 4ab\, \phi_1^2\, \phi_2^2\}^2 + \{x - 2\,(a\, \phi_1^2 + b\, \phi_2^2 + a\, \phi_2^2)\}^2 \quad (10)$$

$$|\Delta B|^2 = \{p - R + 4Rabd\, \phi_1^2\, \phi_2^2\}^2 + \{y - 2R\,(ad\, \phi_1^2 + ad\, \phi_2^2 + bd\, \phi_2^2 + ab\, \phi_1^2)\}^2 \quad (11)$$

$$|\Delta D|^2 = \{k + \frac{4b\, \phi_1^2\, \phi_2^2}{R}\}^2 + \{v - \frac{2\,(\phi_1^2 + \phi_2^2)}{R}\}^2 \quad (12)$$

$$|\Delta E|^2 = \{h - 1 + 4bd\, \phi_1^2\, \phi_2^2\}^2 + \{z - 2\,(d\, \phi_1^2 + d\, \phi_1^2 + b\, \phi_1^2)\}^2 \quad (13)$$

The value of R is fixed by the steady-state thermal resistance of the wall. The parameters a, b, ϕ_1 and ϕ_2 can be chosen to make Δ^2 a minimum.

Equations 10, 11, 12 and 13 show that if a, b and R values are fixed, then Δ^2 will depend on ϕ_1^2 and ϕ_2^2 .

Thus for minimum Δ^2

$$\frac{\partial (\Delta^2)}{\partial (\phi_1^2)} = 0 \tag{14}$$

$$\frac{\partial (\Delta^2)}{\partial (\phi_2^2)} = 0 \tag{15}$$

So ϕ_1^2 and ϕ_2^2 can be found by solving the two equations obtained by differentiating equation 9 with respect to ϕ_1^2 and ϕ_2^2 and by setting the derivatives equal to zero. The derivation of these equations is given in Appendix A.

Thus for each possible combination of a and b there are optimum values of ϕ_1^2 and ϕ_2^2 and a corresponding value of Δ^2. The method employed to find the optimum values of a and b is as follows:

1) Form a set of permissible combinations of a and b.

2) For each combination of a and b calculate best values of ϕ_1^2, ϕ_2^2 and the resulting Δ^2.

3) Select the a, b combination with the least value of Δ^2

4) Compare Δ^2 value for selected a, b combination with values of Δ^2 for other surrounding a, b combinations. If the difference between Δ^2 values is relatively small then this indicates optimum a, b. If the difference is still relatively large then form a new set of a, b values around the set of a, b selected in step 3 with smaller increments of a and b.

5) Repeat the calculations from step 2.

It should be pointed out that if the interval of a, b in the first set of a, b combinations (step 1) is relatively large, then the least value of Δ^2 for this set of a, b combinations does not necessarily indicate the a, b region in which minimum Δ^2 lies. In this case a few pairs of a, b values must be selected which might indicate minimum Δ^2 and for each pair calculations must be repeated from step 4.

The digital computer program for Bendix G-15 (Ref 5) was written to do these calculations automatically without any assistance from the operator, except in the first calculation where a, b intervals are relatively large. In this case the operator has to choose a few additional pairs of a, b values besides the one pair selected by the computer.

Electronic Analog Computer Method

The analog computer circuit shown in Fig. 3 can be used to find the best π network to simulate the temperatures and heat flows in a building component. The required data are the transmission matrix for the component and its steady-state thermal resistance.

In general, the relations for the sinusoidal voltages and currents in the π circuit shown in Fig. 2 are

$$\frac{dV_1}{dt} = \frac{1}{C_1}\left(\frac{V_{in}}{aR} + \frac{V_2}{bR} - \frac{V_1}{aR} - \frac{V_1}{bR}\right) \tag{16}$$

$$\frac{dV_2}{dt} = \frac{1}{C_2}\left(\frac{V_1}{bR} + \frac{V_{out}}{dR} - \frac{V_2}{bR} - \frac{V_2}{dR}\right) \tag{17}$$

$$I_{in} = \frac{V_{in} - V_1}{aR}, \quad I_{out} = \frac{V_2 - V_{out}}{aR}$$

and the relation between the input and output is

$$\begin{bmatrix} V_{in} \\ I_{in} \end{bmatrix} = \begin{bmatrix} A_\pi , & B_\pi \\ D_\pi , & E_\pi \end{bmatrix} \cdot \begin{bmatrix} V_{out} \\ I_{out} \end{bmatrix} \tag{18}$$

Ref 4

where

$$\begin{vmatrix} A_\pi , & B_\pi \\ D_\pi , & E_\pi \end{vmatrix} = 1 \tag{19}$$

therefore

$$A_\pi = \frac{V_{in}}{V_{out}} \quad \text{when } I_{out} = 0 \tag{20}$$

$$B_\pi = \frac{V_{in}}{I_{out}} \text{ when } I_{out} = 0 \tag{21}$$

$$E_\pi = \frac{V_{out}}{V_{in}} \text{ when } I_{in} = 0 \tag{22}$$

or

$$\frac{V_{in}}{A_\pi} = V_{out} \text{ when } V_2 = V_{out} \tag{23}$$

$$\frac{V_{in}}{B_\pi} = \frac{V_2}{dR} \text{ when } V_{out} = 0 \tag{24}$$

$$\frac{V_{out}}{E_\pi} = V_{in} \text{ when } V_{in} = V_1 \tag{25}$$

These signals generated by A_π, B_π and E_π circuits shown in Fig. 4 are compared with

$$\frac{V_{in}}{A_w}, \frac{V_{in}}{B_w} \text{ and } \frac{V_{out}}{E_w} \tag{26}$$

which represent the data.

$\frac{V_{in}}{A_w}$ is obtained by modifying V_{in} by a phase lag and attenuation equal respectively to $\underline{|A_w}$ and $\frac{1}{|A_w|}$. The other two reference voltages are obtained in a similar way as shown by the reference voltage circuit in Fig. 3.

The comparison circuit shown in Fig. 3 is made up of summer amplifiers and continuous recording equipment. When two signals are compared, they are fed to the summer with one signal shifted 180° in phase; thus as the signals approach equality the output of the summer approaches zero.

This comparison method of two signals is based on the absolute difference of these signals. Thus for the over-all estimate of minimum difference between $\frac{V_{in}}{A_w}$, $\frac{V_{in}}{B_w}$, $\frac{V_{out}}{E_w}$ and $\frac{V_{in}}{A_\pi}$, $\frac{V_{in}}{B_\pi}$, $\frac{V_{out}}{E_\pi}$ the difference signals for each pair are divided by the respective reference signal magnitude so that the recorded values are the relative differences with respect to the reference signals.

The operational procedure is simple; the potentiometers a, b, $(C_1 + C_2)$, and C_2 are adjusted until the recorded outputs of the comparator amplifiers are minimum. Then π circuit parameter values are calculated or read directly from a, d, b, C_1 and C_2 amplifier outputs.

Calculations of Thermal Circuit for Cooling or Heating Problems in a Building

In the previous sections methods were described for calculating a π thermal circuit parameter from a given steady-state resistance, and wall transmission matrix for sinusoidal driving functions with a particular period. For practical problems the thermal circuit must simulate the wall for driving functions that are periodic and consist of several harmonics.

In air-conditioning problems the outdoor conditions which must be considered for the heat gain or loss of a building are:

1) Outside air temperature
2) Long-wave radiation
3) Solar radiation

Cooling load calculations were based on the design outside temperature and solar irradiation of vertical surfaces facing south, as given in ASHVE Guide 1959 (6) for 18° declin., 40° lat. N. for 1 August. These were taken as typical summer outdoor conditions. They have been analysed for harmonics and the results presented in Table I. The long-wave radiation was calculated by the method given in Ref 10. These functions are shown graphically in Fig. 4.

Examination of Table I reveals that the first (24-hr period) harmonic of all three functions has the largest amplitude. Thus the thermal circuit simulating the wall should be most accurate for the 24-hr harmonic, so that the thermal circuit parameters should be calculated from a 24-hr period wall transmission matrix. The error introduced in simulation of the wall for higher harmonics by the thermal circuit constructed from the best 24-hr period parameters is small, because the best thermal circuit parameters for different frequencies are approximately the same. In Table II the best thermal circuit parameters calculated from 24-, 12- and 6-hr period wall transmission matrices are given. The thermal circuits for homogeneous walls (4-in. concrete or 2-in. insulation) have $a \approx 0.2$, $b \approx 0.6$, and variation in capacitance value less then 1 per cent for all three harmonics. No definite conclusions can be drawn for a composite wall because simulation of it by a single π thermal circuit is poor as indicated by a large value of Δ^2.

The transmission matrix coefficients of the wall and π thermal circuit are given in Table III. The 24-, 12- and 6-hr period π thermal circuit transmission matrix coefficients were calculated from the best 24-hr period circuit parameters. Comparison of respective wall and π thermal circuit matrix coefficients shows that the 24-hr π circuit parameters based on 24-hr wall matrix elements simulate the wall reasonably well for the higher harmonics also.

Extension of the Method for Thick Walls or Higher Frequencies

A thick wall, such as 4-in. concrete and 2-in. insulation, cannot be simulated very accurately by a single π circuit even for a 24-hr harmonic as indicated by the large value of Δ^2 in Table III or by the large difference between wall and π circuit matrix coefficients in Table IV. Thick walls must be simulated by two or more π circuits. In general any wall can be divided into sufficiently thin layers so that each layer can be simulated by a single π circuit. Then the over-all thermal circuit for a thick wall is a series of connected π networks which simulate the layers of the wall. The following procedure can be used to calculate the over-all thermal circuit for a thick wall:

1) Examine the construction details of the wall and calculate the transmission matrix for a homogeneous slab which approximates one outside layer of this wall. (The transmission matrix coefficients for a homogeneous slab versus ϕ are tabulated in Ref 7 or they can be calculated using the Matrix Manipulation Program Ref 8.)

2) Calculate the best π thermal circuit to simulate this slab.

3) Divide the over-all wall transmission matrix by the π thermal circuit transmission matrix calculated in step 2 to obtain the transmission matrix for the remainder of the wall. (These calculations can be done on Bendix G-15 computer using Matrix Manipulation Program Ref 8.)

4) Calculate the best π thermal circuit to simulate the remainder of the wall. Then the over-all thermal circuit for the wall is the 2 π circuits in series.

This method with small modifications can be extended for very thick walls where more than 2 π thermal circuits are required.

1) As before consider the wall construction details and divide the wall into layers.

2) Calculate transmission matrices for every layer except one.

3) Calculate the π thermal circuit to simulate each layer for which the transmission matrices were calculated in step 2.

4) Calculate the products of the π thermal circuit transmission matrices on both sides of the layer for which the transmission matrix was not calculated in step 2.

5) Premultiply and postmultiply the wall transmission matrix by the inverse of the product matrices obtained in step 4.

6) Calculate the π thermal circuit for the product obtained in step 5. This π circuit is for the layer for which the transmission matrix was not calculated in step 2.

The over-all thermal circuit is simply the π circuits for layers connected in series.

The wall transmission matrix coefficients for a composite wall and 2 π thermal circuit transmission matrix coefficients are given in Table V. The 2 π thermal circuit parameters were calculated by the procedure outlined above. The comparison of the corresponding network and wall coefficients shows that a double π thermal circuit can simulate this wall fairly well.

SUMMARY

Two methods have been presented for calculations of optimum π thermal circuit to simulate one-dimensional sinusoidal heat flow through a slab and the extension of the methods for calculation of thermal circuit for periodic, one-dimensional linear heat flow through building components.

The numerical method is only practical if a digital computer is available to do the calculations. This method of calculating optimum π circuit parameters would take too much time if the calculations were done by means of a desk calculator. (Bendix G-15 digital computer evaluates optimum π circuit parameters for one particular period to 3 significant figures in approximately $2\frac{1}{2}$ hr).

The numerical method is preferable to the electronic analog method because the accuracy of optimum π circuit parameters calculated by the latter method is limited by the accuracy of the analog computer components (servo-multipliers, potentiometers and amplifiers).

A few thermal circuits for homogeneous and composite walls were evaluated. The results of these calculations show that it is possible to simulate a building component by a simple thermal circuit with sufficient accuracy for building heat flow calculations.

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TABLE I

HARMONIC ANALYSIS OF OUTSIDE AIR TEMPERATURE, LONG-WAVE RADIATION AND SOLAR IRRADIATION OF VERTICAL SURFACE FACING SOUTH

	Air Temperature			Long-Wave Radiation			Solar Radiation		
	Ampl °F	Phase °	$\frac{\text{X har ampl}}{\text{I har ampl}}$	Ampl $\frac{\text{Btu}}{\text{hr A}^2}$	Phase °	$\frac{\text{X har ampl}}{\text{I har ampl}}$	Ampl $\frac{\text{Btu}}{\text{hr A}^2}$	Phase °	$\frac{\text{X har ampl}}{\text{I har ampl}}$
Steady State	83.1	-	-	130.45	-	-	36.8	-	-
I harm P = 24 hr	10.3	217.4	1.00	11.0	215.3	1.00	60.4	270.0	1.00
II harm P = 12 hr	1.92	23.7	.187	1.26	19.8	.110	32.6	90.0	.540
III harm P = 8 hr	.27	198.5	.026	.37	60.3	.032	10.5	270.0	.173
IV harm P = 6 hr	.18	226.4	.018	.57	269.2	.050	3.0	90.0	.049

TABLE II

BEST π CIRCUIT PARAMETERS CALCULATED BY NUMERICAL METHOD

Wall Construction	Period hr	a	b	d	C_1	C_2	Δ^2
4-in. concrete	24	.1944	.6112	.1944	4.739	4.739	.0003
	12	.1969	.6062	.1969	4.731	4.731	.0024
	6	.2016	.5968	.2016	4.713	4.713	.0200
2-in. insulation	24	.1960	.6080	.1960	.4234	.4234	.0014
	12	.2000	.6000	.2000	.4221	.4221	.0180
	6	.2061	.5878	.2061	.4208	.4208	.0923
4-in. concrete and 2-in. insulation	24	.0500	.9000	.0500	8.603	.3974	.238
	12	.0297	.9273	.0430	8.599	.4895	.741
	6	.1250	.7250	.1500	9.219	2.022	2.806

Thermal properties:

	R - $\frac{ft^2\ °F\ hr}{Btu}$	α - $\frac{ft^2}{hr}$
4-in. concrete	.3200	.03659
2-in. insulation	6.000	.005454
4-in. concrete and 2-in. insulation	6.3200	

TABLE III

π CIRCUIT TRANSMISSION MATRIX COEFFICIENTS CALCULATED FROM BEST 24-HR PERIOD π CIRCUIT PARAMETERS

4-in. concrete, $R = .3200 \frac{ft^2 \ {}^\circ F \ hr}{Btu}$, $\alpha = .0359 \frac{ft^2}{hr}$

Best 24 hr period π circuit parameters:

a = .1944 C_1 = 4.739 Btu/°F ft^2
b = .6112 C_2 = 4,739 Btu/°F ft^2
d = .1944

Period hr	|A|	$\angle A°$	|B|	$\angle B°$	|D|	$\angle D°$	|E|	$\angle E°$	
24	1.051	22.17	.3211	7.582	2.493	97.58	1.051	22.17	Wall
	1.059	22.03	.3213	7.114	2.500	96.92	1.059	22.03	π circ.
12	1.193	41.42	.3244	15.10	5.038	105.1	1.193	41.42	Wall
	1.219	40.64	.3252	14.16	5.107	103.6	1.219	40.64	π circ.
6	1.651	69.39	.3376	29.76	10.49	119.8	1.651	69.39	Wall
	1.736	66.20	.3408	27.84	11.03	115.9	1.736	66.20	π circ.

2-in. insulation, $R = 6.000 \frac{ft^2 \ {}^\circ F}{Btu}$, $\alpha = .00545 \frac{ft^2}{hr}$

Best 24 hr period π circuit parameters:

a = .1960 C_1 = .4234
b = .6080 C_2 = .4234
d = .1960

Period hr	|A|	$\angle A°$	|B|	$\angle B°$	|D|	$\angle D°$	|E|	$\angle E°$	
24	1.139	35.62	6.059	12.69	.2244	102.7	1.139	35.62	Wall
	1.157	37.07	6.070	11.96	.2262	101.4	1.157	37.07	π circ.
12	1.485	61.66	6.234	25.10	.4617	115.1	1.485	61.66	Wall
	1.547	59.32	6.278	23.62	.4783	112.0	1.547	59.32	π circ.
6	2.463	93.85	6.902	48.30	1.022	138.3	2.463	93.85	Wall
	2.665	86.63	7.099	45.14	1.141	129.0	2.665	86.63	π circ.

TABLE IV

π CIRCUIT TRANSMISSION MATRIX COEFFICIENTS CALCULATED FROM BEST 24-HR PERIOD π CIRCUIT PARAMETERS

Composite Wall: 4 in. concrete and 2 in. insulation.

Best 24 hr π circuit parameters:

a = .0500 C_1 = 8.603 Btu/F° ft
b = .9000 C_2 = .3974 Btu/F° ft
c = .0500

Period hr	|A|	∠A°	|B|	∠B°	|D|	∠D°	|E|	∠E°	
24	1.243	60.43	6.733	35.31	3.074	132.6	15.86	106.8	Wall
	1.456	66.58	7.633	35.85	2.707	119.5	13.57	87.6	π circ.
12	1.907	105.1	7.913	67.13	8.024	166.1	32.99	128.7	Wall
	2.759	104.4	10.65	57.08	7.115	138.5	27.12	91.5	π circ.
6	4.410	163.6	12.22	118.1	27.50	213.2	76.43	167.8	Wall
	7.842	137.0	18.37	76.81	23.31	154.2	54.52	96.0	π circ.

TABLE V

2 π CIRCUIT TRANSMISSION MATRIX COEFFICIENTS CALCULATED FROM BEST 24-HR PERIOD 2 π CIRCUIT PARAMETERS

Composite Wall: 4 in. concrete and 2 in. insulation.

Best 24-hr 2π circuit parameters:
(Fig. 1 notation)

$r_1 = .0622 \dfrac{\text{hr ft}^2\ °\text{F}}{\text{Btu}}$ $\qquad C_1 = 4.739$ Btu/F° ft^2

$r_2 = .1956$ " $\qquad C_2 = 4.739$ "

$r_3 = .8122$ " $\qquad C_3 = .3613$ "

$r_4 = 4.1250$ " $\qquad C_4 = .4630$ "

$r_5 = 1.1250$ "

Period hr	|A|	∠A°	|B|	∠B°	|D|	∠D°	|E|	∠E°	
24	1.243	60.4	6.733	35.31	3.074	132.6	15.86	106.8	Wall
	1.283	59.0	6.761	32.02	3.157	130.9	15.90	103.1	2π circ
12	1.907	105.1	7.913	67.1	8.029	166.1	32.99	128.7	Wall
	2.055	99.8	7.989	60.2	8.608	160.1	33.11	121.1	2π circ
6	4.410	163.6	12.22	118.1	27.50	213.2	76.43	167.8	Wall
	5.017	147.8	12.39	103.6	31.26	196.1	77.28	151.9	2π circ

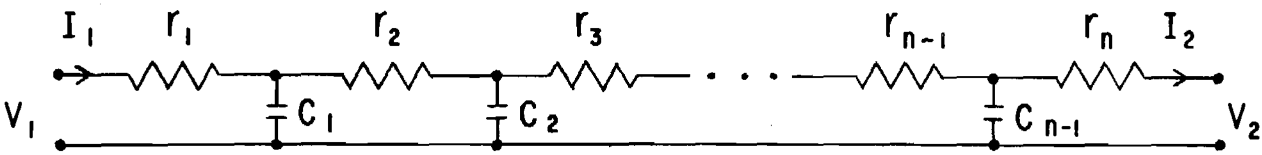

FIGURE 1

FOUR-TERMINAL RESISTANCE - CAPACITANCE NETWORK

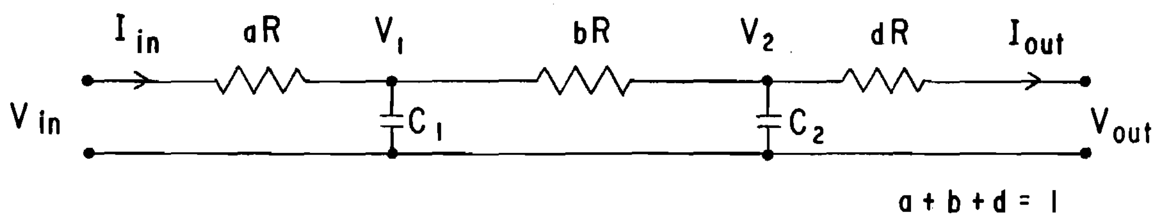

FIGURE 2 π THERMAL CIRCUIT

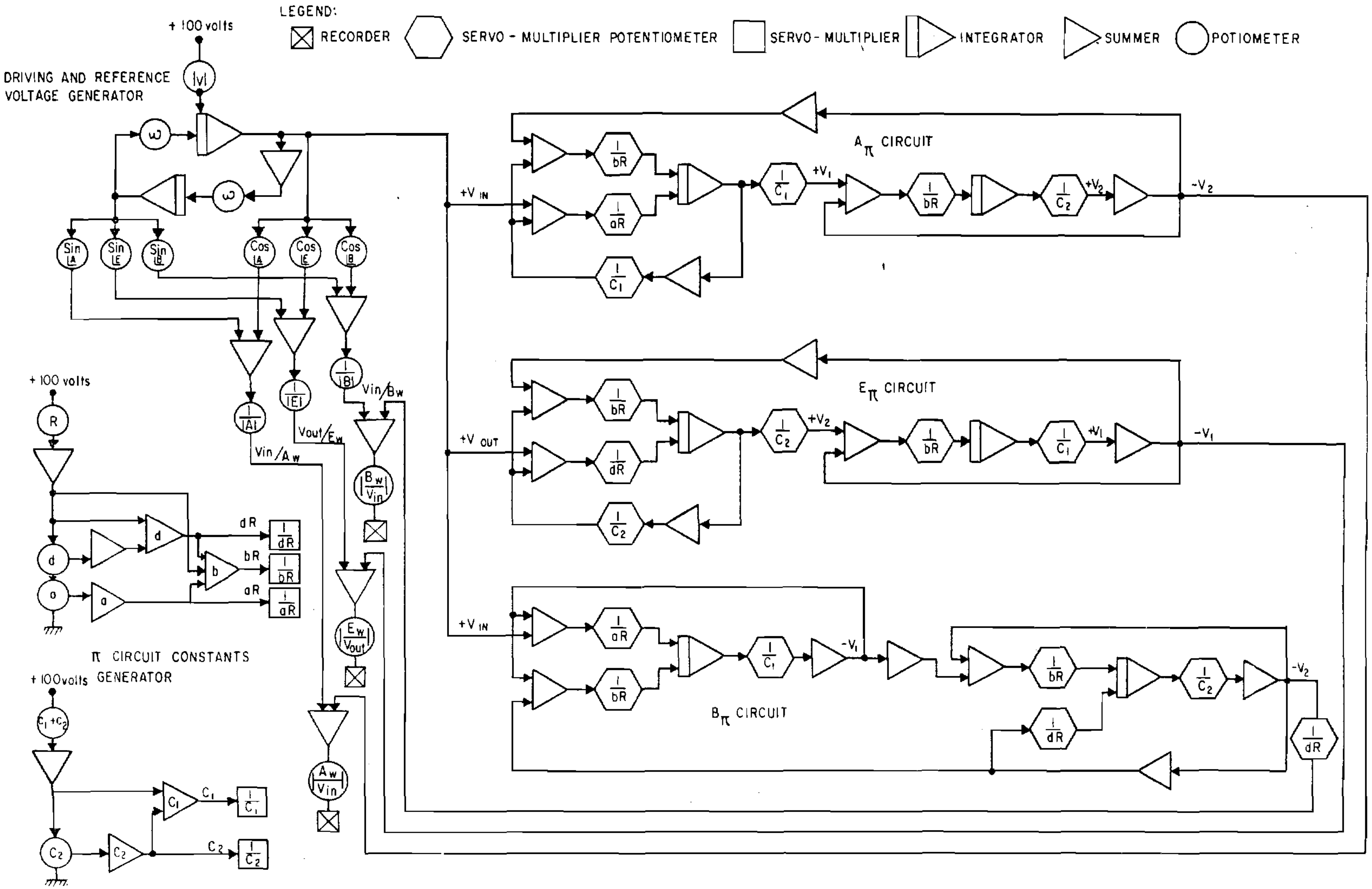

FIGURE 3

ELECTRONIC ANALOG COMPUTER CIRCUIT TO EVALUATE THE π THERMAL CIRCUIT PARAMETERS

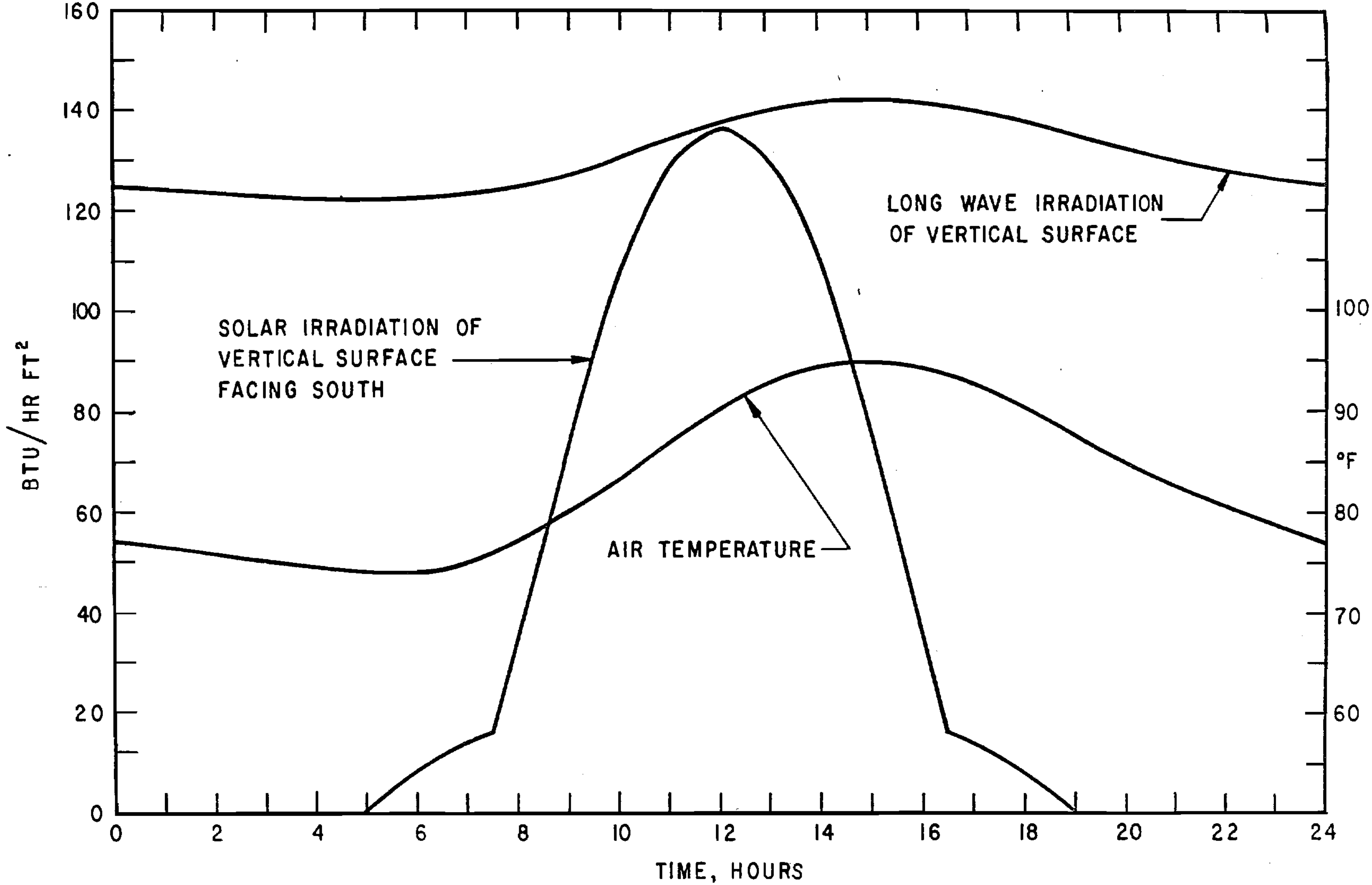

FIGURE 4

TYPICAL SUMMER AIR TEMPERATURE, LONG WAVE RADIATION AND SOLAR RADIATION FOR 18° DEC. N, 40° LAT. N, AUGUST 1

APPENDIX A

DERIVATION OF EQUATIONS FOR CALCULATIONS OF ϕ_1^2 AND ϕ_1^2 FOR MINIMUM Δ^2

By differentiating the expression for Δ^2 (Equation 9) with respect to ϕ_1^2 and ϕ_2^2 and equating to zero the following equations are obtained:

$$\frac{\partial(\Delta^2)}{\partial(\phi_1^2)} = 0 = \frac{1}{|A_w{}^2|}\left[(m-1+4ab\,\phi_1^2\,\phi_2^2)\,8ab\,\phi_2^2 + x+8a\,(a\,\phi_1^2 + b\,\phi_2^2 + a\,\phi_2^2)\right] +$$

$$+\frac{1}{|B_w{}^2|}\Big[(p-\bar{R}+4Rabd\,\phi_1^2\,\phi_2^2)8Rabd\,\phi_2^2 + y+8R^2a(b+d)(ad\,\phi_1^2 + ad\,\phi_2^2 +$$

$$+ bd\,\phi_2^2 + ab\,\phi_1^2)\Big]$$

$$+\frac{1}{|E_w{}^2|}\left[(k+\frac{4b\,\phi_1^2\,\phi_2^2}{R})\ \frac{8b\,\phi_2^2}{R} + V + \frac{8}{R^2}\ (\phi_1^2+\phi_2^2)\right] +$$

$$+\frac{1}{|D_w{}^2|}\left[(h-1+4bd\,\phi_1^2\,\phi_2^2)\ 8bd\,\phi_2^2 + z+8(b+d)\,(d\,\phi_2^2 + d\,\phi_1^2 + b\,\phi_1^2\right] \qquad (27)$$

$$\frac{\partial(\Delta^2)}{\partial(\phi_2^2)} = 0 = \frac{1}{|A_w{}^2|}\left[(m-1+4ab\,\phi_1^2\,\phi_2^2)\,8ab\,\phi_1^2 + x+8(a+b)\,(a\,\phi_1^2 + b\,\phi_1^2 + a\,\phi_2^2)\right] +$$

$$+\frac{1}{|B_w{}^2|}\left[(p-R+4Rabd\,\phi_1^2\,\phi_2^2)\,8Rabd\,\phi_1^2 + y+8R^2d(a+b)\,(ad\,\phi_1^2 + ad\,\phi_2^2 + bd\,\phi_2^2 + ab\,\phi_1^2)\right]$$

$$+\frac{1}{|E_w{}^2|}\left[(k+\frac{4b\,\phi_1^2\,\phi_2^2}{R})\ \frac{8b\,\phi_1^2}{R}\ \ V + \frac{8}{R^2}\ (\phi_1^2+\phi_2^2)\right]$$

$$+\frac{1}{|D_w{}^2|}\left[(h-1+4bd\,\phi_1^2\,\phi_2^2)\ 8bd\,\phi_1^2 + z+8\,(d\,\phi_2^2 + d\,\phi_1^2 + b\,\phi_1^2)\ \ d\right] \qquad (28)$$

or

$$K_o + K_1 \phi_1^2 + K_2 \phi_2^2 + K_3 \phi_1^2 \phi_2^4 = 0 \quad (29)$$

$$G_o + G_1 \phi_2^2 + G_2 \phi_1^2 + G_3 \phi_2^2 \phi_1^4 = 0 \quad (30)$$

where

$$K_o = -2 \left[\frac{2ax}{|A_w^2|} + \frac{2a(1-a)Ry}{|B_w^2|} + \frac{2V}{R|C_w^2|} + \frac{2(1-a)z}{|D_w^2|} \right] \quad (31)$$

$$G_o = -2 \left[\frac{2x(1-d)}{|A_w^2|} + \frac{2d(1-d)Ry}{|B_w^2|} + \frac{2V}{R|C_w^2|} + \frac{2zd}{|D_w^2|} \right] \quad (32)$$

$$K_1 = G_1 = 2 \left[\frac{4a(1-d)}{|A_w^2|} + \frac{4ad\, R^2(1-a)(1-d)}{|B_w^2|} + \frac{4}{R^2\, |C_w^2|} + \frac{4d(1-a)}{|D_w^2|} + \right.$$

$$\left. + (1-a-d)\ 8 \left\{ \frac{a(m-1)}{|A_w^2|} + \frac{ad\, R(P-R)}{|B_w^2|} + \frac{k}{R\, |C_w^2|} + \frac{d\,(h-1)}{|D_w^2|} \right\} \right] \quad (33)$$

$$K_2 = 8 \left[\frac{a^2}{|A_w^2|} + \frac{a^2\,(1-a)^2\, R^2}{|B_w^2|} + \frac{1}{R^2\, |C_w^2|} + \frac{(1-a)^2}{|D_w^2|} \right] \quad (34)$$

$$G_2 = 8 \left[\frac{(1-d)^2}{|A_w^2|} + \frac{d^2\,(1-d^2)\, R^2}{|B_w^2|} + \frac{1}{R^2\, |C_w^2|} + \frac{d^2}{|D_w^2|} \right] \quad (35)$$

$$K_3 = G_3 = 32\,(1-a-d)^2 \left[\frac{a^2}{|A_w^2|} + \frac{a^2\, d^2\, R^2}{|B_w^2|} + \frac{1}{R^2\, |C_w^2|} + \frac{d^2}{|D_w^2|} \right] \quad (36)$$

By elimination of ϕ_1^2 from the 29, 30 set of equations the following equation is obtained:

$$W_o + W_1 \phi_2^2 + W_2 \phi_2^4 + W_3 \phi_2^6 + W_4 \phi_2^8 + W_5 \phi_2^{10} = 0 \quad (37)$$

where

$$W_o = G_o K_2^2 - G_1 K_o K_2 \quad (38)$$

$$W_1 = G_2 K_2^2 + G_3 K_o^2 - G_1 K_1 K_2 \quad (39)$$

$$W_2 = 2 G_o K_2 K_3 - G_1 K_o K_3 + 2G_3 K_o K_1 \quad (40)$$

$$W_3 = 2 G_2 K_2 K_3 + G_3 K_1^2 - G_1 K_1 K_3 \quad (41)$$

$$W_4 = G_o K_3^2 \quad (42)$$

$$W_5 = G_2 K_3^2 \quad (43)$$

To evaluate the ϕ_2^2 from equation 37 the Birge-Vieta iterative method (Ref. 9) can be used. From the physical setup of the problem it is known that the ϕ_2^2 value must be positive and must be a real number. Also the value of ϕ_2^2 usually will be a small number ($0 < \phi_2^2 > 40$) therefore the initial value of ϕ_2^2 for this iteration must be a small positive real number. (In digital computer program to evaluate the best π circuit, (Ref. 5) initial value of $\phi_2^2 = 1.0$ is used. This proved to be a satisfactory choice). When the value of ϕ_2^2 is obtained, the value of ϕ_1^2 can be calculated directly by substituting ϕ_2^2 value in the equation 29.