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Internal Report No. 568

Date of issue: May 1988

Canada

A Procedure for Deriving Thermal Transfer Functions for Walls from Hot-Box Test Results: Part I

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IRC/NRC

1988

Abstract

A guarded hot-box is used to determine the U-value of a wall specimen at two different mean temperatures. With the hot side temperature held constant, the cold side temperature is varied at a constant rate during the transition between the two steady-state conditions. The results of this test are used to determine three time constants of the wall, which are then used to calculate the coefficients of the z-transfer function for that wall.

Introduction

The transfer function approach for calculating heat flow through walls and ceilings has been endorsed by ASHRAE for the past 20 years, and is widely used throughout North America. The procedure requires, as data, the coefficients of the wall's thermal transfer functions, and hourly values of the temperature for the outer face of the wall. Transfer function coefficients for many walls have been published in the ASHRAE Handbook of Fundamentals (1).

Unfortunately, these data are all based on the assumption that the heat flow through these walls is one dimensional. There is no allowance for framing members that act as heat bridges through insulation, nor for such common building materials as hollow concrete blocks. Therefore, the transfer function data in the ASHRAE Handbook of Fundamentals may not accurately represent the thermal performance of real walls. Consequently there is a growing interest in being able to derive these data from the results of tests on full scale wall specimens. This paper presents a procedure for doing this. It involves using a guarded hot-box wall testing facility; the IRC facility is shown schematically in Figure 1. A test consists of three phases:

I) An initial steady-state U-value test with the room-side environmental temperature at T_h and the climate side temperature at T_{ci}.

II) A transition phase during which the climate chamber temperature is changed from T_{ci} to T_{cf}. The temperature is changed at a constant rate during this transition.

III) A final steady-state U-value test with the room-side temperature at T_h and the climate side temperature at T_{cf}.

Figure 2 shows the temperature vs time curve obtained for a typical test, and the corresponding values for the heat flux into the room-side face of the test wall. These results can be used to determine three time-constants for the wall, plus their associated residues. These time-constants and residues plus the U-value are used as the data to calculate the coefficients for a rational z-transfer function that relates the heat flux through the room-side face of the wall to the temperature at the outside surface. The procedure for this latter calculation is basically the same as is described in Stephenson and Mitalas (2). The difference is that the time-constants and residues are derived from test results rather than from the dimensions and thermal properties of the materials that make up the wall.

Theory

The Laplace transforms of T_h and T_c are respectively θ_1 and θ_2, and the transforms of the heat fluxes Q_h and Q_c are ϕ_1 and ϕ_2 respectively. These are related by

$$\begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} \qquad (1)$$

where the elements A, B, C and D of the transmission matrix are functions of the thermal properties and dimensions of the materials in the wall and the heat transfer resistance at the surfaces. This formulation, which is taken from Carslaw and Jaeger (3), assumes that the thermal properties of the wall are constant. Equation 1 can be recast as

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} D/B & -1/B \\ 1/B & -A/B \end{bmatrix} \cdot \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad (2)$$

The element C has been eliminated by using the fact that $A \cdot D - B \cdot C = 1$. The functions D/B, 1/B and A/B are referred to as the Laplace transfer functions of the wall.

The transfer function 1/B(s) can be represented as

$$\frac{1}{B(s)} = U \sum_{n=1}^{\infty} \frac{\alpha_n}{1+\tau_n s} \qquad (3)$$

where τ_n are the time constants of the wall (i.e., the poles of 1/B(s) are at $s = -1/\tau_n$)

α_n are the residues at the poles

All the transfer functions of Equation (2) have their poles at $s = -1/\tau_n$, since B(s) is the common denominator.

When calculating the heat flux through a wall or roof it is more convenient to use z-transforms of the temperatures and fluxes rather than the Laplace transforms. (The z-transforms are sometimes referred to as the time-series representations of these quantities). An equation similar to equation 2 relates the z-transforms of heat flux to the z-transforms of temperatures: viz,

$$Z\{Q_h\} = \frac{D(z)}{B(z)} \cdot Z\{T_h\} - \frac{1}{B(z)} \cdot Z\{T_c\} \qquad (4)$$

where

$$Z\{Q_h\} = Q_{h,t} + Q_{h,t-\delta}\, z^{-1} + Q_{h,t-2\delta}\, z^{-2} + \ldots\ldots\ldots$$

$$Z\{T_h\} = T_{h,t} + T_{h,t-\delta}\, z^{-1} + T_{h,t-2\delta}\, z^{-2} + \ldots\ldots\ldots$$

and

$$Z\{T_c\} = T_{c,t} + T_{c,t-\delta}\, z^{-1} + T_{c,t-2\delta}\, z^{-2} + \ldots\ldots\ldots$$

The z-transfer functions can be approximated by

$$\frac{D(z)}{B(z)} = \frac{U \sum_{0}^{N} a_n \cdot z^{-n}}{\sum_{0}^{N} d_n \cdot z^{-n}} \qquad (5)$$

and

$$\frac{1}{B(z)} = \frac{U \sum_{0}^{N} b_n \cdot z^{-n}}{\sum_{0}^{N} d_n \cdot z^{-n}} \qquad (6)$$

where U = the overall conductance of the wall

$z = e^{s\delta}$,

δ = the time interval between successive terms in the time series for the temperatures and heat fluxes,

s = the parameter in Laplace transforms,
N = the number of terms in the various sums (N may be different for each summation)
a_n, b_n, d_n = *z-transfer function coefficients.*

Equation 4 is usually written in the time domain as

$$d_o \cdot Q_{h,t} = U \sum_{n=0}^{N} a_n \cdot T_{h,t-n\delta} - U \sum_{n=0}^{N} b_n \cdot T_{c,t-n\delta} - \sum_{n=1}^{N} d_n \cdot Q_{h,t-n\delta} \quad (7)$$

where $T_{h,t}$ is the environmental temperature in the building at time t,
$T_{c,t}$ is the environmental temperature outside at time t.

The approach used in this paper is to approximate 1/B(s) by the sum of only three (or in some cases four) terms, and then to use these values of α_n and τ_n to calculate b_n and d_n. When values of a_n are required (i.e., when inside temperature is changing), they are obtained from experimentally determined values of the wall's frequency response and the time constants, as described in Part II of this paper.

Step #1 Determination of the α_n and τ_n of a wall

If the start of phase II (see Figure 2) is taken as t = 0, Q_i as the value of heat flux during the steady-state at the end of phase I, and Q_f at the steady-state in phase III, then the heat flux, Q_t, at any time t during phase II can be represented (3) by:

$$\int \left\{ \left(\frac{Q_t - Q_i}{Q_f - Q_i} \right) t^* \right\} dt = \frac{1}{s^2} \cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{1 + \tau_n s} \qquad t < t^* \quad (8)$$

where t* is the duration of phase II. This can be inverted to give

$$Q_t = Q_i + \left(\frac{Q_f - Q_i}{t^*} \right) (t - \Gamma) + \varepsilon_t \qquad 0 < t < t^* \quad (9)$$

where

$$\Gamma = \sum_{n=1}^{\infty} \alpha_n \tau_n \quad (10)$$

and

$$\varepsilon_t = \frac{Q_f - Q_i}{t^*} \sum_{n=1}^{\infty} \alpha_n \tau_n e^{-t/\tau_n} \quad (11)$$

As $t \to \infty$, $\varepsilon_t \to 0$

and $$Q_t \to Q_i + \frac{Q_f - Q_i}{t^*}(t - \Gamma)$$

This asymptote is a straight line with a slope of $\frac{Q_f - Q_i}{t^*}$, and it equals Q_i when $t = \Gamma$. Thus the value of Γ can be obtained from the intersection of this asymptote and the value of Q_i.

In phase III,

$$Q_f - Q_t = \varepsilon_t = \frac{Q_f - Q_i}{t^*} \sum_{n=1}^{\infty} \alpha_n \tau_n e^{-t'/\tau_n} \qquad t^* < t \qquad (12)$$

where $t' = t - t^*$

Since $\varepsilon_t (0 < t < t^*)$ is equal to $\varepsilon_{t'}$ $(t^* < t)$, Equation 12 can be combined with Equation 9 to give the following expression for Γ

$$\Gamma = t - \frac{t^*}{Q_f - Q_i}\left(Q_t + Q_{t^*+t} - Q_i - Q_f\right) \qquad 0 < t < t^* \qquad (13)$$

A value of Γ can be obtained in this way from each value of Q_t, and the corresponding Q_{t^*+t}, for $0 < t < t^*$.

Equation 12 indicates that as t' becomes large

$$\frac{Q_f - Q_t}{Q_f - Q_i} \to \frac{\alpha_1 \tau_1}{t^*} e^{-t'/\tau_1} \qquad t^* < t \qquad (14)$$

where τ_1 is the largest time-constant.

Thus $$\ln\left(\frac{Q_f - Q_t}{Q_f - Q_i}\right) \to \ln\left(\frac{\alpha_1 \tau_1}{t^*}\right) - \frac{t'}{\tau_1} \qquad t^* < t \tag{15}$$

This asymptote is a straight line with a slope of $-\frac{1}{\tau_1}$, and its value at $t=0$ (i.e., $t = t^*$) is $\ln\left(\frac{\alpha_1 \tau_1}{t^*}\right)$. Thus the asymptote gives values for τ_1 and α_1.

If equation 12 is integrated from t^* to ∞ it gives

$$\int_{t^*}^{\infty} (Q_f - Q_t)dt = \left(\frac{Q_f - Q_i}{t^*}\right) \sum_{n=1}^{\infty} \alpha_n \tau_n^2$$

Thus $$\sum_{n=1}^{\infty} \alpha_n \tau_n^2 = \Delta \tag{16}$$

where $$\Delta = \frac{t^* \int_{t^*}^{\infty} (Q_f - Q_t)dt}{Q_f - Q_i} \tag{17}$$

One more relationship can be derived from the fact that

$$\frac{d}{dt}(Q_t)\bigg|_{t=0} = 0$$

When equations 9 and 11 are differentiated they give
$$\frac{d}{dt}(Q_t) = \left(\frac{Q_f - Q_i}{t^*}\right)\left(1 - \sum_{n=1}^{\infty} \alpha_n e^{-t/\tau_n}\right)$$

Therefore
$$\sum_{n=1}^{\infty} \alpha_n = 1 \tag{18}$$

Equations 10, 16 and 18 are used to determine the time constants and residues. However, if the transfer function $\frac{1}{B(s)}$ is to be approximated by a finite number of terms rather than an infinite number, the sums of the various finite series must be the same as the sums of the corresponding infinite series, i.e.

$$\sum_{n=1}^{N} \alpha_n = 1$$

$$\sum_{n=1}^{N} \alpha_n \tau_n = \Gamma \qquad (19)$$

$$\sum_{n=1}^{N} \alpha_n \tau_n^2 = \Delta$$

Values of α_1 and τ_1 can be derived directly from the test results using equation 15. Thus, if N=2 there would be only two undetermined constants, α_2 and τ_2, so it would not be possible to satisfy all three relations. On the other hand, if N=3 there would be four undetermined constants, and the three relations are not sufficient to determine all four constants. One of the constants must be used as an undetermined constant to be determined by some other constraint. It is convenient to let α_3 be the undetermined constant.

Let $\alpha_2 + \alpha_3 = F = 1-\alpha_1$

$$\alpha_2\tau_2 + \alpha_3\tau_3 = G = \Gamma-\alpha_1\tau_1 \qquad (20)$$

$$\alpha_2\tau_2^2 + \alpha_3\tau_3^2 = H = \Delta-\alpha_1\tau_1^2$$

where Γ and Δ are determined from Equation 13 and 17, respectively. These can be solved to give

$$\alpha_2 = F-\alpha_3$$

$$\tau_3 = \frac{G + \left[(G^2 - FH)\left(-\frac{\alpha_2}{\alpha_3}\right)\right]^{\frac{1}{2}}}{F} \qquad (21)$$

$$\tau_2 = \frac{G - \alpha_3\tau_3}{\alpha_2}$$

There is the additional constraint that τ_2 and τ_3 must be real and positive. This limits the possible range for α_3 as follows:

a) if $G^2 > FH$
α_3 must have the opposite sign to F

b) if $G^2 = FH$
α_3 must be zero and $\tau_2 = \frac{G}{F}$

c) if $G^2 < FH$
α_3 must be between zero and F.

The best value for α_3 is determined by trying several values within the allowable range, and for each of these determining a consistent set of values for α_2, τ_2 and τ_3. Then using these various sets of values a variance η_t is calculated as:

$$\eta_t = \frac{Q_f - Q_i}{t^*} \sum_{n=1}^{3} \{\alpha_n \tau_n e^{-t/\tau_n}\} - \varepsilon_t \qquad 0<t<t^* \qquad (22)$$

and ε_t is determined from experiment ($\varepsilon_t = Q_f - Q_t$ for $t^*<t$)

The best value for α_3 is the one that gives the smallest value for $\int_0^\infty \eta_t^2 \cdot dt$. This is a "least squares" curve fitting procedure that satisfies the constraints on Q_t and its first derivative.

<u>Example 1</u>

This procedure is illustrated in example 1. In this example, however, the values of Q_t have been calculated from the exact solution for a homogeneous slab rather than being experimental results. Hence, the values of η_t are due to the approximation of $\frac{1}{B(s)}$. In a real test there would also be some experimental error in the measured values that would also contribute to η_t.

For a homogeneous slab of the following properties

$R = 2.0\ m^2 \cdot K/W$
$L\rho c = 1.728 \times 10^5\ J/(m^2 \cdot K)$

under the following test conditions,

$T_h = 20.0°C$
$T_{ci} = 10.0°C$
$T_{cf} = 30.0°C$
$t^* = 50.0\ h$

the following values are calculated using the exact transfer function:

$RL\rho c = 3.456 \times 10^5\ s = 96\ h$
$\Gamma = \frac{96}{6} = 16\ h$
$\tau_n = \frac{96}{n^2 \pi^2}$
$\alpha_n = 2(-1)^{n+1}$

$\Delta = 179.19 \text{ h}^2$

The resulting heat flux, Q_t, and ε_t are given in Table I.

Table I

t	Q_t	ε_t	t	Q_t	ε_t
0	15.0000	3.2000			
1	15.0000	3.0000	13	15.4178	1.0178
2	15.0000	2.8000	14	15.5194	.9194
3	15.0002	2.6002	15	15.6302	.8302
4	15.0014	2.4014	16	15.7496	.7496
5	15.0066	2.2066	17	15.8767	.6767
6	15.0184	2.0184	18	16.0108	.6108
7	15.0404	1.8404	19	16.1513	.5513
8	15.0734	1.6734	20	16.2975	.4975
9	15.1184	1.5184	21	16.4490	.4490
10	15.1758	1.3758	22	16.6052	.4052
11	15.2452	1.2452	23	16.7656	.3656
12	15.3260	1.1260	24	16.9298	.3298

For the remainder of phase II, Q_t is given by

$$Q_t = 11.800 + 0.200 \left\{t + 19.4536 \, e^{-t/9.7268}\right\}$$

where $\alpha_1 = 2.000$ and $\tau_1 = \frac{96}{\pi^2} = 9.7268$ h

(These are the values that would be indicated by the results of a perfect ramp test).

From equation 20

$$F = 1.000 - 2.000 = -1.000$$
$$G = 16.000 - 19.4536 = -3.4536 \text{ [h]}$$
$$H = 179.19 - 189.22 = -10.03 \text{ [h}^2\text{]}$$
$$G^2 - FH = 1.8974 \text{ [h}^2\text{]}$$

Thus α_3 must have the opposite sign of F, i.e. positive.

Table II gives the values of α_2, τ_2 and τ_3 that are consistent with various values of α_3. The lower part of the table gives the values of η_t for three of the cases. These values show that the approximation gets better as α_3 increases, but beyond $\alpha_3 = 3$ the improvement is very slight.

Table II

Case #	1	2	3	4	5
α_1	2.00	2.00	2.00	2.00	2.00
α_2	-2.00	-3.00	-4.00	-5.00	-6.00
α_3	1.00	2.00	3.00	4.00	5.00
τ_1	9.7268	9.7268	9.7268	9.7268	9.7268
τ_2	2.4799	2.3293	2.2611	2.2220	2.1966
τ_3	1.5062	1.7671	1.8636	1.9141	1.9452
	↓		↓		↓

t	η_t (Case 1)	η_t (Case 3)	η_t (Case 5)
0	.0000	.0000	.0000
	.0029	.0021	.0020
2	.0046	.0030	.0028
	.0032	.0015	.0013
4	.0010	-.0002	-.0003
	-.0009	-.0014	-.0015
6	-.0015	-.0015	-.0015
	.0020	-.0016	-.0016
8	-.0019	-.0013	-.0013
	-.0016	-.0009	-.0008
0	-.0013	-.0006	-.0005
	-.0010	-.0004	-.0003
2	-.0007	-.0002	-.0001
	-.0005	-.0001	-.0001
4	-.0004	.0000	.0000
	-.0002	.0001	.0001
6	-.0002	.0001	.0001
	-.0001	.0001	.0001
8	-.0001	.0001	.0001
	-.0001	-	-
20	-	-	-
	-	-	-
21	-	-	-
	-	-	-
22	-	-	-
	-	-	-
24	-	-	-
	-	-	-
	$\Sigma\eta_t^2 = 58\text{x}10^{-6}$	$\Sigma\eta_t^2 = 26\text{x}10^{-6}$	$\Sigma\eta_t^2 = 23\text{x}10^{-6}$

Step #2 Determining the z-transfer function coefficients

The problem at this step is to determine the coefficients b_n and d_n that make

$$\frac{R}{B(z)} = \frac{\sum_{n=0}^{N} b_n z^{-n}}{\sum_{n=0}^{N} d_n z^{-n}}$$

equivalent to

$$\frac{R}{B(s)} = \sum_{n=1}^{3} \frac{\alpha_n}{1+\tau_n s}$$

Values of α_n and τ_n were determined in step #1. Equivalence of the two transfer functions requires that the poles of the two correspond, and that they have the same frequency response at frequencies that are of particular importance in the intended application.

The poles of the Laplace transfer function are at $s = -1/\tau_n$. Hence the poles of the z-transfer function must be at $z = e^{-\delta/\tau_n}$.

Therefore $$\sum_{n=0}^{3} d_n z^{-n} = \prod_{n=1}^{3} \left(1-e^{-\delta/\tau_n} \cdot z^{-1}\right) \tag{23}$$

which gives the coefficients d_n as:

$$\begin{aligned} d_0 &= 1.0000 \\ d_1 &= -\sum_{n=1}^{3} e^{-\delta/\tau_n} \\ d_2 &= \left(\prod_{n=1}^{3} e^{-\delta/\tau_n}\right) \sum_{n=1}^{3} e^{\delta/\tau_n} \\ d_3 &= -\prod_{n=1}^{3} e^{-\delta/\tau_n} \end{aligned} \tag{24}$$

The frequency response of $R/B\big|_{s=i(2\pi f)}$ can be calculated from

$$\frac{R}{B(s)}\bigg|_{s=i(2\pi f)} = \sum_{n=1}^{3} \frac{\alpha_n}{1+i(2\pi f\tau_n)} = X_f + i\,Y_f \tag{25}$$

where

$$X_f = \sum_{n=1}^{3} \frac{\alpha_n}{1+(2\pi f \tau_n)^2} \tag{26}$$

$$Y_f = - \sum_{n=1}^{3} \frac{\alpha_n (2\pi f \tau_n)}{1+(2\pi f \tau_n)^2} \tag{27}$$

Let

$$\sum_{n=0}^{3} d_n \cdot z^{-n} \Big|_{z=e^{i2\pi f \delta}} = V_f + i\, W_f \tag{28}$$

where

$$V_f = \sum_{n=0}^{3} d_n \cdot \cos (2\pi n \delta f) \tag{29}$$

$$W_f = - \sum_{n=0}^{3} d_n \cdot \sin (2\pi n \delta f) \tag{30}$$

Then matching the frequency response of R/B(z) to R/B(s) for a harmonic driving function with a frequency of f requires

$$\sum_{n=0}^{N} b_n \cos (2\pi n \delta f) = X_f V_f - Y_f W_f \tag{31}$$

and

$$- \sum_{n=0}^{N} b_n \sin (2\pi n \delta f) = X_f W_f + Y_f V_f \tag{32}$$

At steady-state (i.e., f = 0)

$$X_0 = \sum_{n=1}^{3} \alpha_n = 1$$

$$Y_0 = 0$$

$$V_0 = \sum_{n=0}^{3} d_n$$

$$W_0 = 0$$

Therefore

$$\sum_{n=0}^{N} b_n = \sum_{n=0}^{3} d_n \qquad (33)$$

Equations 31, 32 and 33 can be solved to find three b_n coefficients. In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \cos(2\pi\delta f) & \cos(4\pi\delta f) \\ 0 & -\sin(2\pi\delta f) & -\sin(4\pi\delta f) \end{bmatrix} \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{3} d_n \\ X_f V_f - Y_f W_f \\ X_f W_f + Y_f V_f \end{bmatrix} \qquad (34)$$

If it is desired to match the frequency response at a second frequency, there would have to be five b_n coefficients. Two more rows and two more columns would be added to the square matrix, and two more terms in the column matrix on the right side of the equation.

If the square matrix in equation 34 is M, then b_n are given by

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = [M]^{-1} \begin{bmatrix} \sum_{n=0}^{3} d_n \\ X_f V_f - Y_f W_f \\ X_f W_f + Y_f V_f \end{bmatrix} \qquad (35)$$

Example II

Taking the output of step 1 in Example I to be

$$\alpha_1 = 2.00 \qquad \tau_1 = 9.7268 \text{ h}$$

$\alpha_2 = -4.00 \qquad \tau_2 = 2.2611$ h

$\alpha_3 = 3.00 \qquad \tau_3 = 1.8636$ h

With $\delta = 1$ h, and $f = 1$ cycle/24 hours

$$[M] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0.9659 & 0.8660 \\ 0 & -0.2588 & -0.5000 \end{bmatrix}$$

Thus

$$[M]^{-1} = \begin{bmatrix} 14.6739 & -13.6739 & 5.6639 \\ -28.3478 & 28.3478 & -7.5958 \\ 14.6739 & -14.6739 & 1.9319 \end{bmatrix}$$

And from Equation 24

$$\begin{aligned} d_0 &= 1.0000 \\ d_1 &= -2.1296 \\ d_2 &= 1.4381 \\ d_3 &= \underline{-0.3390} \\ \sum_{n=0}^{3} d_n &= 0.0145 \end{aligned}$$

$$\begin{aligned} X_{24} &= -0.27165 \\ Y_{24} &= -0.109306 \\ V_{24} &= -0.0123426 \\ W_{24} &= 0.0493402 \end{aligned}$$

$$\begin{aligned} XV - YW &= 0.008746 \\ XW + YV &= -0.012054 \end{aligned}$$

$$\begin{aligned} b_0 &= 0.02491 \\ b_1 &= -0.07155 \\ b_2 &= \underline{0.06114} \end{aligned}$$

$$\sum_{n=0}^{2} b_n = 0.01450$$

Table III compares the frequency responses of R/B(s) and R/B(z) with the correct" values obtained from an analytical solution for this homogeneous slab. These values show that R/B(z) matches the correct values well when $\frac{L^2\rho cf}{\lambda} < 4$. But when this number is larger, the difference between R/B(z) and the correct frequency response is substantial. Whether this difference will lead to a significant error in Q_t depends on the frequency content of the driving function T_c.

$\frac{L^2\rho cf}{\lambda}$ is a dimensionless parameter

where
- L = thickness of slab
- ρ = density
- c = specific heat
- λ = thermal conductivity
- f = frequency

Table III

Case 3 $\frac{L^2\rho c}{\lambda} = 96$

$\frac{L^2\rho cf}{\lambda}$	R/B(s) = X + i Y				R/B(z) = X + i Y				"Exact"			
	X	Y	Mod	Arg	X	Y	Mod	Arg	X	Y	Mod	Arg
2	-0.0833	-0.5737	0.5797	-98.3°	-0.07976	-0.5733	0.5789	-97.9°	-0.0832	-0.5733	0.5793	-98.2°
4	-0.2716	-0.1093	0.2928	-158.1°	-0.2716	-0.1093	0.2928	-158.1°	-0.2686	-0.1078	0.2894	-158.1°
8	-0.0544	0.0941	0.1087	-240.0°	-0.0556	0.1024	0.1166	-241.5°	-0.0439	0.0834	0.0942	-242.2°
16	0.0376	0.0215	0.0433	-330.3°	0.0461	0.0357	0.0583	-322.3°	0.0167	-0.0003	0.0167	-361.2°

$$\frac{R}{B(s)} = \frac{2}{1 + 9.7268s} - \frac{4}{1 + 2.2611s} + \frac{3}{1 + 1.8632s}$$

$$\frac{R}{B(z)} = \frac{0.0249 - 0.0716\ z^{-1} + 0.0612\ z^{-2}}{1.0000 - 2.1296\ z^{-1} + 1.4831\ z^{-2} - 0.3390\ z^{-3}}$$

Example III - Results of a Ramp Test on a Heavy Wall with Heat Bridges

A ramp test was carried out on a wall with a 127 mm layer of foam polystyrene sandwiched between two 89 mm wythes of concrete. The sample was brought to a steady-state with T_h = 21°C and T_{ci} = -7.1°C. Then the temperature on the cold side was reduced to T_{cf} = -30.0°C at a constant rate over a period of t* = 60 hours. Table IV gives the values of temperature and total heat flow at 3-hour intervals over the 7-day test period.

The U-value of the specimen can be determined from the initial and final steady-state results. The variation of U with mean temperature can then be calculated from

$$U = \frac{Q}{\text{Area}\ (T_h - T_c)} = U_o + \frac{dU}{dT_m} \left(\frac{T_h + T_c}{2}\right)$$

For this test Q_i = 86.6 W, Q_f = 156.6 W, and the test area A = 5.946 m^2.

These values lead to:

$$U_o = 0.517\ Wm^{-2}K^{-1}$$

and

$$\frac{dU}{dT_m} = 1.66 \times 10^{-4}\ Wm^{-2}K^{-2}$$

As $\frac{dU}{dT_m}$ is very small it is valid to analyze the results from the transient part of the test as though the U-value were a constant.

Table IV

Results of a Ramp Test

	t [h]	T_c [°C]	Q_t [W]	Γ [h]
		- 7.1	86.6	
		- 7.1	86.5	
		- 7.1	86.6	
Phase I		- 7.1	86.6	
		- 7.1	86.7	
		- 7.1	86.5	
		- 7.1	86.5	
Ramp Started at 0.0				
	0.3	- 7.3	86.6	13.4
	3.3	- 8.4	86.6	13.4
	6.3	- 9.5	87.0	13.5
	9.3	-10.7	88.2	13.5
	12.3	-11.8	89.9	13.5
	15.3	-13.0	92.1	13.5
	18.3	-14.6	94.7	13.5
	21.3	-15.3	97.6	13.5
Phase II	24.3	-16.5	100.6	13.5
	27.3	-17.6	103.8	13.5
t*=60.0h	30.3	-18.7	107.1	13.5
	33.3	-20.0	110.4	13.5
	36.3	-21.1	113.8	13.5
	39.3	-22.2	117.2	13.5
	42.3	-23.4	120.6	13.5
	45.3	-24.4	124.0	13.5
	48.3	-25.5	127.5	13.5
	51.3	-26.7	131.0	13.4
	54.3	-27.9	134.5	13.4
	57.3	-29.2	138.0	13.4
Ramp Finished at 60.0				

Table IV (cont'd)

	t [h]	T_C [°C]	Q_t [W]	$\varepsilon_{t'}$ [W]	$t'=t-t^*$ [h]
	60.3	-30.0	141.5	15.3	0.3
	63.3	-30.0	144.9	11.8	3.3
	66.3	-30.0	147.9	8.8	6.3
	69.3	-30.0	150.2	6.5	9.3
Phase III	72.3	-30.0	151.9	4.7	12.3
	75.3	-30.0	153.2	3.4	15.3
	78.3	-30.0	154.1	2.5	18.3
	81.3	-30.0	155.7	1.9	21.3
	84.3	-30.0	155.2	1.4	24.3
	87.3	-30.0	155.5	1.1	27.3
	90.3	-30.0	155.7	0.9	30.3
	93.3	-30.0	155.9	0.7	33.3
	96.3	-30.0	156.0	0.6	36.3
		-30.0	156.1	0.5	39.3
	102.3	-30.0	156.2	0.4	42.3
	105.3	-30.0	156.3	0.3	45.3
	108.3	-30.0	156.3	0.3	48.3
	111.3	-30.0	156.4	0.2	51.3
	114.3	-30.0	156.4	0.2	54.3
	117.3	-30.0	156.4	0.2	57.3
	120.3	-30.0	156.4	0.2	60.3
	123.3	-30.0	156.5	0.1	63.3
	126.3	-30.0	156.5	0.1	66.3
	129.3	-30.0	156.5	0.1	69.3
	132.3	-30.0	156.6		
	135.3	-30.0	156.6		
	138.3	-30.0	156.6		
	141.3	-30.0	156.6		

Values of Γ are given in Table IV for each value of Q_t in Phase II. These were calculated using Equation 13, i.e.

$$\Gamma = t - \frac{t^*}{Q_f - Q_i} \{Q_t + Q_{t^*+t} - Q_i - Q_f\} \qquad 0 < t < t^*$$

The values in table IV show that the best value for Γ is 13.5 hours.

Table IV shows that $\varepsilon_{t'}$ has not decayed to zero when t' reaches 60.0 h (a period that equals t*). Thus ε_t will also not be zero when t = t* (i.e. at the end of phase II). This must be taken into account when calculating $\varepsilon_{t'}$ for the early part of phase III. Subtracting ε_{t^*+t} from Q_t (for $t > t^*$) corrects for the residual effect of phase II, that is

$$\varepsilon_{t'} = Q_f - (Q_t - \varepsilon_{t^*+t}) \qquad t > t^*$$

For example:

$$\varepsilon_{0.3} = 156.6 - (141.5 - 0.2) = 15.3$$

$$\varepsilon_{3.3} = 156.6 - (144.9 - 0.1) = 11.8$$

etc.

$$\varepsilon_{60.3} = 156.6 - 156.4 = 0.2$$

$$\varepsilon_{63.3} = 156.6 - 156.5 = 0.1$$

The values of $\varepsilon_{t'}$ are then used to determine α_1 and τ_1, (or as in this example α_0, τ_0, α_1 and τ_1). Figure 3 shows values of ln $\varepsilon_{t'}$ vs t'. At large values of t' the points lie along a curve that is slightly concave upward and this curvature is due to the heat bridges, which conduct some heat through the insulation in parallel with the main heat flow path. To account for this effect, values of $\varepsilon_{t'}$ for $t' > 12$ hours can be represented by the sum of two exponential terms:

$$\varepsilon_{t'} = \frac{Q_f - Q_i}{t^*} \{\alpha_0 \tau_0 e^{-t'/\tau_0} + \alpha_1 \tau_1 e^{-t'/\tau_1}\}$$

A regression fit of the data to this equation gives:

$\alpha_0 = 0.056$ $\qquad \tau_0 = 25.0$ h

$\alpha_1 = 1.670$ $\qquad \tau_1 = 8.33$ h

Integrating $\varepsilon_{t'}$ from t' = 12 to t' = ∞ yields a value of 57.3 W·h. The integral from t' = 0 to t' = 12 can be evaluated numerically. It equals 114.3 W·h. (This value was obtained using hourly values rather than the 3-hour values in Table IV.)

Thus, from Equation 17

$$\Delta = \frac{60(114.3+57.3)}{70.0} = 147.1 \text{ h}^2$$

Also from Equation 20

$$F = 1 - 0.056 - 1.670 = -0.726$$
$$G = 13.50 - 1.40 - 13.92 = -1.82 \text{ h}$$
$$H = 147.1 - 35.0 - 116.0 = -3.90 \text{ h}^2$$

Therefore $G^2 - FH = 0.48 \text{ h}^2$

As $G^2 > FH$ and $F < 0$, α_2 must be less than zero and α_3 must be greater than zero in order for τ_2 and τ_3 to be real and positive. Various pairs of values for α_2, α_3 were tried and

$$\alpha_2 = -2.00, \quad \alpha_3 = 1.274$$

gave results in good agreement with the experimental values for the early part of the test. From Equation 21 the corresponding values for τ_2 and τ_3 are:

$$\tau_2 = 1.745 \text{ h}$$

$$\tau_3 = 1.31 \text{ h}$$

These values can be used to determine the coefficients of R/B(z) just as was done in Example II.

Table V
Summary of Results

n	α_n	τ_n [h]	$\alpha_n \tau_n$ [h]	$\alpha_n \tau_n^2$ [h^2]
0	0.056	25.0	1.40	35.0
1	1.670	8.33	13.92	116.0
2	-2.000	1.74	-3.49	-6.1
3	1.274	1.31	1.67	2.2
$\sum_{n=0}^{3}$	1.000		13.50	147.1

Discussion

Figure 3 shows the importance of obtaining values of ε_t over as long a period as possible. When there are some heat bridges through layers of insulation they give rise to a long time-constant with a small associated residue. This means that the wall may appear to have reached a steady-state, but the heat flux will continue to change very slowly for many hours. Thus it is good practice to maintain the constant temperature conditions for at least a day after it seems that the steady-state has been reached.

The concept of a transfer function that embodies the thermal characteristics of a wall is valid only if the thermal properties of the wall are independent of the temperature and time. The initial and final steady-state results can be used to check whether this assumption is valid. This check should be made before making a detailed analysis of the results from the transient part of the test.

The output from a ramp test should be the values of U_o, $\frac{dU}{dT_m}$, τ_n and α_n. These values will enable a user to calculate the values of b_n and d_n for any time-step. When values of a_n are required it is necessary to have values of the A/B and D/B transfer function for the frequencies of particular interest. This procedure is explained in Part II.

Conclusion

A ramp test starting from a steady-state and ending with another steady-state yields enough information to determine three time constants of the wall, which are then used to calculate the z-transfer junction coefficient 1/B(z) for the wall.

Acknowledgement

The authors wish to express their appreciation to Mr. J. Richardson and Mr. G. Keatley for their assistance in preparing the wall specimen for test and in carrying out the ramp test. The quality of the results is due to their skill in operating this complex facility.

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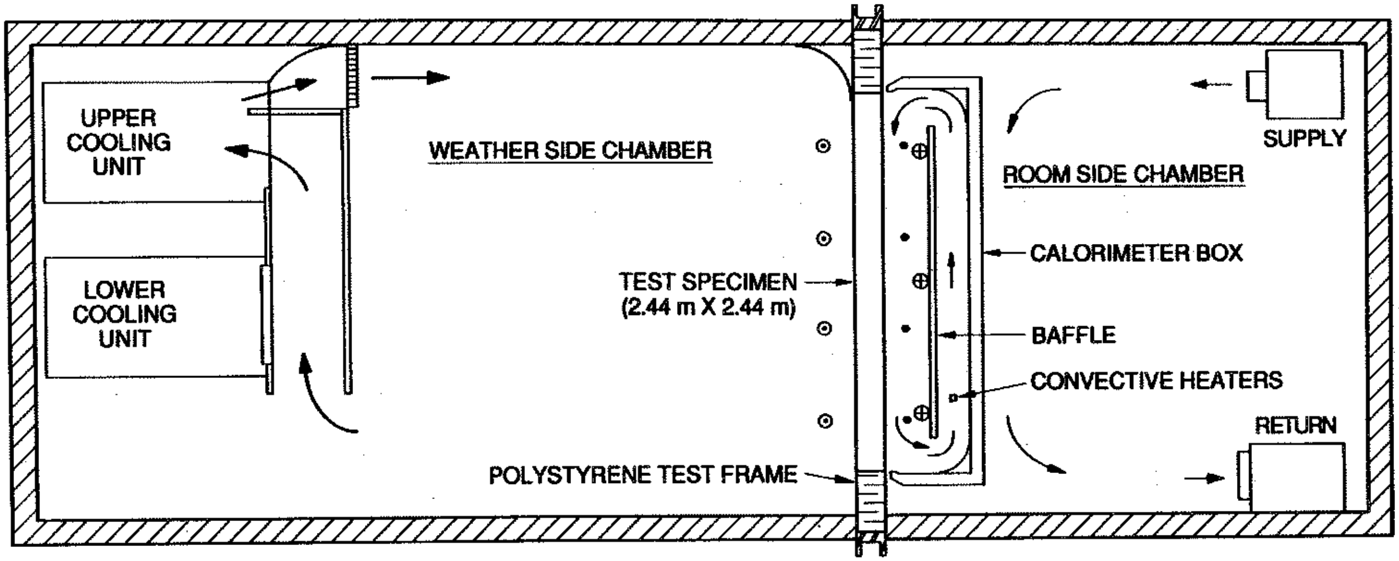

• INSIDE AIR TEMPERATURE THERMOCOUPLES (T_h)

⊕ BAFFLE TEMPERATURE THERMOCOUPLES (T_b)

⊙ OUTSIDE AIR TEMPERATURE THERMOCOUPLES (T_c)

⇉ INDICATES AIR FLOW

FIGURE 1

NRC/IRC HOT-BOX TEST FACILITY

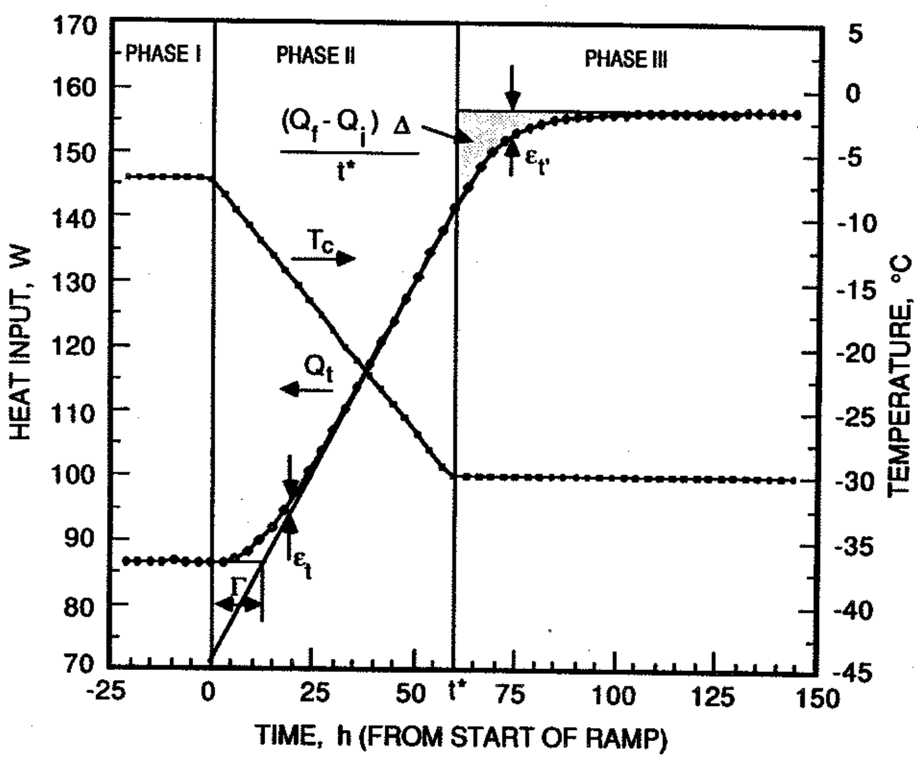

FIGURE 2

TYPICAL TEST RESULTS

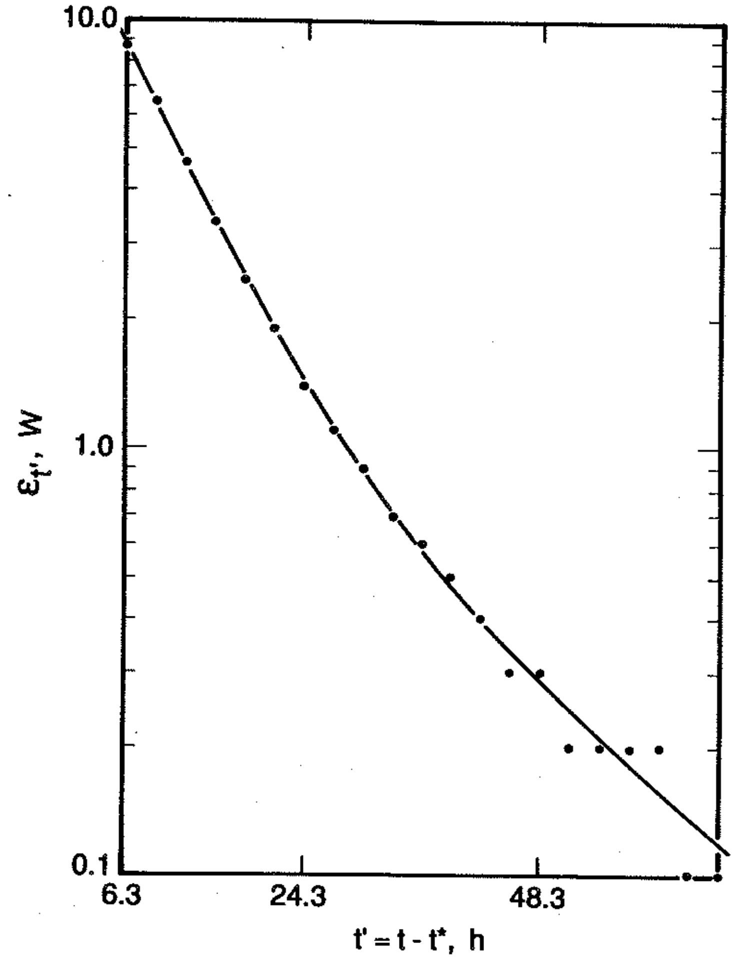

FIGURE 3

VALUE OF ε' AGAINST TIME

A Procedure for Deriving Thermal Transfer Functions for Walls from Hot-Box Test Results: Part II

By

D.G. Stephenson, K. Ouyang, W.C. Brown

Part I of this paper presents a procedure for determining the time-constants (and associated residues) for a wall from the results of a hot-box wall test. This second part presents a procedure for obtaining the frequency response of a wall from the results of a second set of hot-box tests. Both the time-constants and the frequency response are required when calculating the coefficients of the z-transfer functions.

This second part of the paper also presents a procedure for the calibration, or obtaining the transfer functions, for the hot-box.

Determining the Frequency Response of a Wall

A section of a wall is installed in a hot-box wall testing apparatus just as it would be for a U-value test. The arrangement is shown schematically in figure 1. To determine the frequency response of the $\frac{D}{B}$ transfer function the temperature in the climate chamber is kept constant while the power to the metering box is varied sinusoidally at various frequencies. The resulting variation in the temperature in the metering box is measured at regular intervals, and from these data a value of $\left.\frac{D}{B}\right|_{s=i2\pi f}$ can be derived for the particular frequency, f, used for the test.

The $\frac{1}{B}$ transfer function can also be obtained from cyclic test results. In this case the climate chamber temperature is varied sinusoidally and the power to the metering box is kept constant.

The metering box is assumed to be a linear system with three transfer functions E, I and J. The E function allows for the heat capacity and transport lag of the heating system, and I and J account for the heat capacity of the air and baffle in the metering box and for the heat that flows into (or out of) the shell of the metering box, that is

$$E\cdot\Phi = A\cdot\Phi_w + I\cdot\Theta_1 - J\cdot\Theta_g \qquad (1)$$

Φ = Laplace transform of the power, P, supplied to the metering box

A = area of the test wall

Φ_w = Laplace transform of the heat flux into the test wall from the metering box, represented by:

$$\Phi_w = \left\{\frac{D}{B}\cdot\Theta_1 - \frac{1}{B}\cdot\Theta_2\right\} \qquad (2)$$

Θ_1 = Laplace transform of T_h, the temperature in the metering box

Θ_2 = Laplace transform of T_c, the temperature in the climate chamber

Θ_g = Laplace transform of T_g, the temperature in the guard space surrounding the metering chamber

If T_g is constant it can be subtracted from T_h and T_c, and then $J\cdot\Theta_g$ can be dropped from equation 1 (i.e., T_g is the reference or zero level for the temperatures). Thus, from Equations 1 and 2

$$\frac{E}{A}\cdot\Phi = \left(\frac{D}{B} + \frac{I}{A}\right)\cdot\Theta_1 - \frac{1}{B}\cdot\Theta_2 \qquad (3)$$

This is the basic relation that can be used to determine the value of $\left.\frac{D}{B}\right|_{s=i\omega}$ and $\left.\frac{1}{B}\right|_{s=i\omega}$ when $\frac{E}{A}$ and $\frac{I}{A}$ are known for $s=i\omega$ ($\omega = 2\pi f$ where f is the frequency). Conversely when a wall is tested that has known values of $\frac{D}{B}$ and $\frac{1}{B}$ the results can be used to determine $\frac{E}{A}$ and $\frac{I}{A}$. This is the way the metering box is calibrated.

Determination of $\left.\frac{D}{B}\right|_{s=i\omega}$

The climate chamber temperature is kept at a constant value, T_c, and the power to the metering-box is varied sinusoidally at an angular velocity ω, that is

$$P = \bar{P} + \tilde{P}\, e^{i(\omega t + \beta)} \tag{4}$$

This cyclical variation of the power produces a corresponding cyclical variation of T_h

$$T_h = \bar{T}_h + \tilde{T}_h\, e^{i(\omega t + \gamma)} \tag{5}$$

Thus with T_c constant

$$\left(\frac{D}{B} + \frac{I}{A}\right)_{s=i\omega} = \left(\frac{E}{A}\right)_{s=i\omega} \cdot \left(\frac{\tilde{P}}{\tilde{T}_h} \cdot e^{i(\beta-\gamma)}\right) \tag{6}$$

The values of the real and imaginary parts in this complex expression depend on the value of ω. When $\left.\frac{E}{A}\right|_{s=i\omega}$ and $\left.\frac{I}{A}\right|_{s=i\omega}$ are known from a calibration of the hot box (as explained later), the value of $\left.\frac{D}{B}\right|_{s=i\omega}$ can be obtained from the results of a cyclical test using equation 6.

Example I

Data for the metering box obtained from calibration for f = 1 cycle/day (ω = 0.2618 rad/h)

$A = 5.95 \quad [m^2]$

$E = 1.00 \;\underline{|-0.8^\circ}$

$$I = 24.5 \;\underline{|54.1^\circ} \; [W/K]$$

Test conditions and measured quantities:

$$T_c = -20.0 \; [^\circ C]$$

$$P = 150.0 + 90.0 \;\underline{|0.0^\circ} \; [W]$$

$$T_g = 20.0 \; [^\circ C]$$

$$\omega = 0.2618 \; [rad/h]$$

$$T_h = 20.0 + 1.92 \;\underline{|-54.2^\circ} \; [^\circ C]$$

Derived quantities:

$$U = \frac{\bar{P}}{A(\bar{T}_h - \bar{T}_c)} = \frac{150.0}{5.95\ (40.0)} = 0.630 \quad [W/(m^2 \cdot K)]$$

$$R = \frac{1}{U} = 1.586 \; [m^2 \cdot K/W]$$

$$\frac{D}{B} + \frac{I}{A} = \left(\frac{1.00 \;\underline{|-0.8}}{5.95}\right) \left(\frac{90.0 \;\underline{|0.0}}{1.92 \;\underline{|-54.2}}\right)$$

$$= 7.87 \;\underline{|53.4^\circ}$$

$$= 4.69 + i\ 6.32 \quad [W/(m^2 \cdot K)]$$

But $\frac{I}{A}$ for the box is represented as

$$\frac{I}{A} = 2.41 + i\ 3.33 \quad [W/(m^2 \cdot K)]$$

Therefore

$$\frac{D}{B} = 2.28 + i\ 2.99$$

$$= 3.76 \ \underline{|52.67°} \qquad [W/(m^2 \cdot K)]$$

<u>Determination of</u> $\left.\frac{1}{B}\right|_{s=i\omega}$

For this test the climate chamber temperature, T_c, is varied sinusoidally and the power to the metering chamber is kept constant. This causes a sinusoidal variation of T_h.
In this case

$$T_c = \bar{T}_c + \tilde{T}_c\ e^{i(\omega t+\psi)}$$

and from Equation 3:

$$\left(\frac{1}{B}\right)_{s=i\omega} = \left(\frac{D}{B} + \frac{I}{A}\right)_{s=i\omega} \cdot \left(\frac{\tilde{T}_h \ \underline{|\gamma}}{\tilde{T}_c \ \underline{|\psi}}\right)_{s=i\omega} \tag{7}$$

<u>Example II</u>
Test conditions and measured quantities:

$$T_c = -20.0 + 10.0 \ \underline{|0.0} \ [°C]$$

$$P = 150.0 \ [W]$$

$$T_H = 20.0 + 0.22 \ \underline{|-211.1°} \ [°C]$$

$$T_g = 20.0 \ [°C]$$

$$\omega = 0.2618 \ [rad/h]$$

Derived quantities:

From the previous test

$$\frac{D}{B} + \frac{I}{A} = 7.87 \underline{|53.4°}$$

Thus $$\frac{1}{B} = (7.87 \underline{|53.4}) \left(\frac{0.22 \underline{|-211.1}}{10.0 \underline{|0.00}} \right)$$

$$= 0.173 \underline{|-157.7°} \ [W/(m^2 \cdot K)]$$

or $$\frac{R}{B} = 0.275 \underline{|-157.7°}$$

$$= -0.254 - i\ 0.104$$

These values for the real and imaginary parts of $\left.\frac{R}{B}\right|_{s=i\omega}$ can be compared with the values derived from the time-constants and residues. It should be noted, however, that the values derived from the results of the cyclical tests may have a relatively large uncertainty because the amplitude $\tilde{T}_h$ can be quite small. The value of $\left.\frac{D}{B}\right|_{s=i\omega}$, on the other hand, will have a much smaller probable error because of a larger magnitude of $\tilde{T}_h$.

Calibration of the Metering Box

The transfer functions $\left.\frac{E}{A}\right|_{s=i\omega}$ and $\left.\frac{I}{A}\right|_{s=i\omega}$ for the metering box can be determined for various frequencies in the same way as the frequency response of a wall is determined. But for this calibration the test wall must be one whose transfer functions are known. This is best obtained with a wall that is made of a homogeneous material whose thermal properties are known. In this case the transfer functions $\left.\frac{D}{B}\right|_{s=i\omega}$ and $\left.\frac{1}{B}\right|_{=i\omega}$ can be derived from the dimensions and thermal property values. Thus for a calibration test $\left.\frac{E}{A}\right|_{s=i\omega}$ and $\frac{I}{A}_{s=i\omega}$ are the unknowns.

In this case the first test has P held constant while T_c is varied sinusoidally, i.e. the same test conditions as for $\frac{I}{B}$.

Equation 7 can be rearranged to

$$(\frac{D}{B} + \frac{I}{A}) = (\frac{1}{B}) \left(\frac{\tilde{T}_c \angle \psi}{\tilde{T}_h \angle \gamma} \right) \tag{8}$$

As $\frac{D}{B}$ and $\frac{1}{B}$ are known for the wall that is used for the calibration, equation 8 gives the value for $\frac{I}{A}$.

For this case a 100 mm expanded polystyrene wall was used and the values of $\frac{I}{A}$ and $\frac{E}{A}$ were found to be

$$\frac{I}{A} = 4.117 \angle 54.1$$

and $$\frac{E}{A} = 0.168 \angle -0.8$$

Then having determined the value of $\frac{D}{B} + \frac{I}{A}$, it can be used in equation 6 to obtain $\frac{E}{A}$ as:

$$\frac{E}{A} = (\frac{D}{B} + \frac{I}{A}) \left(\frac{\tilde{T}_H \angle \gamma}{\tilde{P} \angle \beta} \right)$$

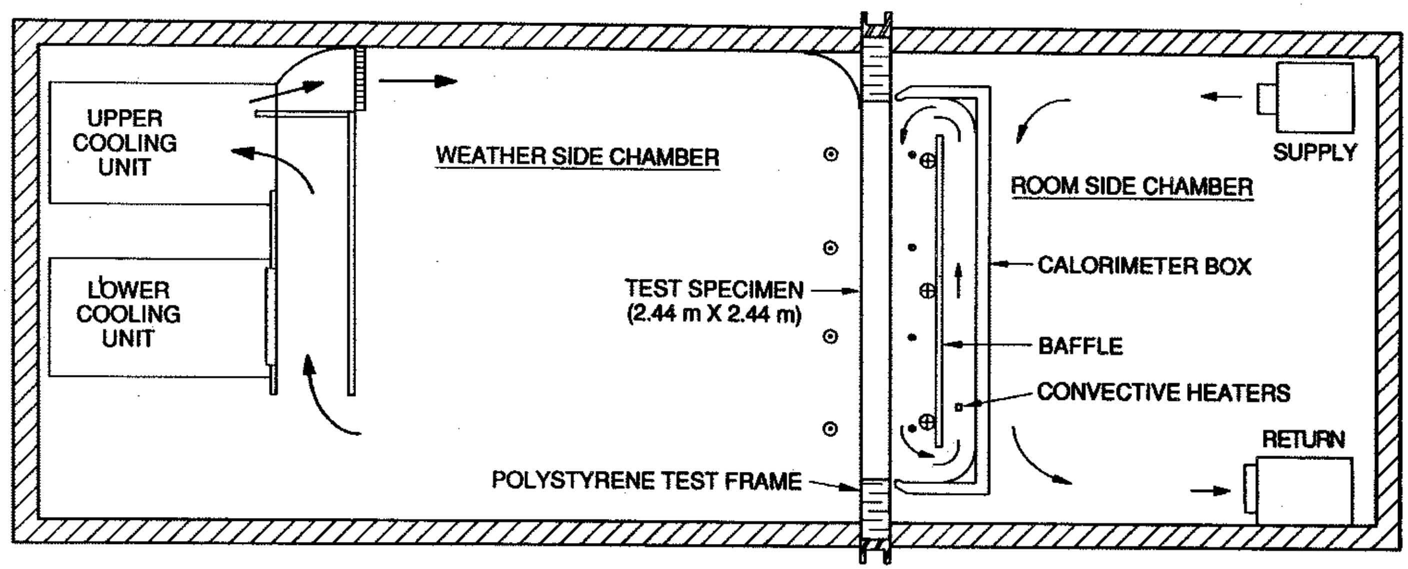

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FIGURE 1

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