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COMMENTS ON THE Z-TRANSFER FUNCTION METHOD FOR CALCULATING HEAT TRANSFER IN BUILDINGS

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Lors de la conception des bâtiments, il faut calculer de façon détaillée la conduction de chaleur dynamique dans l'enveloppe structurale du bâtiment ainsi que dans les parois intérieures. L'une des méthodes employées se fonde sur la fonction de transfert z. Cette article traite de la méthode fondée sur la fonction de transfert z du point de vue de sa mise en application dans le calcul de la conduction de la chaleur dynamique dans les murs et les toits et de la relation qui existe entre les gains de chaleur dans les locaux, la charge de refroidissement et les pertes de chaleur. On y traite aussi des caractéristiques importantes et de l'aspect concret de la fonction de transfert z.

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COMMENTS ON THE Z-TRANSFER FUNCTION METHOD FOR CALCULATING HEAT TRANSFER IN BUILDINGS

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ABSTRACT

In the design of buildings there is a need for extensive and refined dynamic heat conduction calculations through the building envelope as well as the interior partitions. One of the methods used for these calculations is based on z-transfer functions.

This paper reviews the basic concept of the z-transfer function method in conjunction with its application for the dynamic heat conduction calculation through walls and roofs as well as for the relation between room heat gain, cooling load and heat extraction. A short discussion points out the main features as well as the physical significance of the z-transfer function.

* * * * *

Performance type energy conservation standards for buildings require an accurate method for predicting the energy consumption of a building over a full year of operation. The z-transfer function technique is particularly suitable for these calculations as it can utilize actual hour-by-hour weather data with almost no preprocessing, and allow for variations in the inside conditions that reflect the way that buildings are operated.

This paper is a condensation of the set of papers (1 to 11) prepared by D.G. Stephenson and the author on the subject of transient heat transfer calculations through the building envelope as well as the calculation of building air-conditioning loads. It also presents further discussion of the z-transfer function technique described in the ASHRAE Handbook of Fundamentals (12) for calculations of dynamic heat transfer through walls and roofs as well as the calculations of space loads and temperatures.

Z-TRANSFER FUNCTION METHOD

For a dynamic heat transfer system, dependent and independent variables can be related by differential equations. The solution of these equations may take several forms. In this case the input and output z-transforms (i.e., transforms of independent and dependent variables) are related by z-transfer function as follows:

$$O(z) = I(z) \cdot G(z) \tag{1}$$

where

$O(z)$ = z-transform* of the output
$I(z)$ = z-transform of the input
$G(z)$ = system transfer function

* z-transformation of a time function (e.g. solar radiation, outside air temperature, etc., versus time) is a sequence of values of the function at equal time intervals (usually one hour in A-C calculations) i.e., it is time series representation of the time function.

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The most important characteristic of the z-transfer function method, as compared with other calculation methods, is that the input and output are a sequence of values equally spaced in time. Thus, the weather records of outside air temperature and solar radiation, that are given on an hourly basis, can be used as an input without any or very little preprocessing.

The main limitation of the z-transfer function method is that the system under consideration must be constant with time, and linear (i.e., system where the thermal properties and heat transfer coefficients are constant and independent of temperature). It is possible to use the z-transfer function even where heat transfer coefficients are functions of temperature and/or time. The calculations in this case, however, become more involved (8).

Assuming that transfer function G(z) can be determined* and can be expressed as a ratio of two polynomials in z^{-1}.

$$\frac{O(z)}{I(z)} = G(z) = \frac{b_o + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \dots}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3} + \dots} \tag{2}$$

where

$$O(z) = O_t + O_{t-\Delta} z^{-1} + O_{t-2\Delta} z^{-2} + O_{t-3\Delta} z^{-3} + \dots \tag{3}$$

and

$$I(z) = I_t + I_{t-\Delta} z^{-1} + I_{t-2\Delta} z^{-2} + I_{t-3\Delta} z^{-3} \tag{4}$$

are output and input transforms respectively, it follows that, after inversion to real time (9)

$$\begin{aligned} O_t &= I_t b_o + I_{t-\Delta} b_1 + I_{t-2\Delta} b_2 + \dots \\ &\quad - (O_{t-\Delta} d_1 + O_{t-2\Delta} d_2 + O_{t-3\Delta} d_3 + \dots) \end{aligned} \tag{5}$$

where

t = time

Δ = time interval

$O_{t-n\Delta}$ = output value at time $(t - n\Delta)$

$I_{t-n\Delta}$ = input value at time $(t - n\Delta)$

n = integer

b's and d's = coefficients of the system transfer function

A simple and obvious, but rather important, characteristic of the z-transfer function is that the numerator, N(z), and denominator, D(z) polynomials, of the transfer function can be multiplied or divided by a factor that is a simple polynomial in z^{-1}.

The z-transfer function as given by Eq. 2, therefore, can be expressed in two other forms, namely:

$$G(z) = \frac{v_o + v_1 z^{-1} + v_2 z^{-2} + \dots + v_i z^{-i} + \dots}{1 - CR\, z^{-1}} \tag{6}$$

or

$$G(z) = r_o + r_1 z^{-1} + r_2 z^{-2} + r_3 z^{-3} + \dots + r_i z^{-i} + \dots \tag{7}$$

---

*G(z) coefficients for walls and roofs made up of homogeneous layers can be calculated by the methods described in Refs. 9, 13.

where

v_i = z-transfer function coefficients when $D(z) = 1 - CR\, z^{-1}$

r_i = z-transfer function coefficient when $D(z) = 1$

= $CR.\, r_{i-1}$ for large value of i

CR = constant, which is equal numerically to the largest negative root of the D(z) polynomial*. It is called the "Common Ratio".

The substitution of r's in place of b's and setting d's equal to zero in Eq. 5 gives:

$$0_t = r_o I_t + r_1 I_{t-\Delta} + r_2 I_{t-2\Delta} + r_3 I_{t-3\Delta} + \cdots \quad (8)$$

and substitution of v's and CR gives

$$0_t = v_o I_t + v_1 I_{t-\Delta} + v_2 I_{t-2\Delta} + \cdots - CR\, 0_{t-\Delta} \quad (9)$$

A comparison of Eqs. 5, 8 and 9 shows that Eq. 8 gives the output only in terms of the input values, Eq. 9 gives it in terms of the input values and one output value and by Eq. 5 gives the output in terms of equal numbers of both input and output values. All these expressions lead to exactly the same values for the output, but there are fewer terms in Eq. 5, which means its use requires fewer arithmetic operations than Eqs. 8 or 9. For this reason the wall and roof z-transfer functions given in the ASHRAE Handbook of Fundamentals, Chapter 22, are in a form of Eq. 2.

In some of the other publications wall and roof z-transfer functions are given in the form of Eq. 7 where the coefficients are called X, Y, Z, or W thermal response factors (3,4,5,6 and 8).

EXAMPLE PROBLEM

To illustrate some of the points in the application of the z-transfer function method in heat flux calculations for the building envelope the following sample problem is presented. Calculate the instantaneous heat flux values at the interior surface of a roof due to variable roof sol-air temperature and constant interior space temperature. The roof is made up of a 4 in. (0.1016 m) layer of heavy concrete and a 2 in. (0.0508 m) layer of insulation and it is identical to Roof No. 12 listed in Table 40 of Ref. 12. The z-transfer function coefficients for this roof are:

$b_o = 0.0000$	$d_o = 1.0000$
$b_1 = 0.0007$	$d_1 = -1.2437$
$b_2 = 0.0016$	$d_2 = 0.2877$
$b_3 = 0.0005$	$d_3 = -0.0128$

As a matter of interest, the division of the numerator by the denominator gives the response factor form:

$$\begin{aligned} G(z) = {} & 0.000000 \\ & + 0.000700z^{-1} \\ & + 0.002471z^{-2} \\ & + 0.003371z^{-3} \\ & + 0.003491z^{-4} \\ & + 0.003404z^{-5} \\ & + 0.003272z^{-6} \end{aligned}$$

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* All the roots of the D(z) polynomial are real negative numbers in the range 0.0 and -1.0 (9)

$$+ 0.003135z^{-7}$$
$$+ 0.003001z^{-8}$$
$$+ 0.002872z^{-9}$$
$$+ 0.002749z^{-10}$$
$$+ 0.002631z^{-11}$$
$$+ 0.002518z^{-12}$$
$$+ 0.002410z^{-13}$$
$$\vdots$$
$$+ r_i = r_{i-1} 0.9571$$
$$\vdots$$
$$+ r_{105} = 0.0000426$$

The hourly sol-air temperature values for the roof surface for a period of several days are plotted in Figure 1. The sol-air temperature values were calculated using measured hourly solar radiation and outside air temperature at Ottawa. The interior space temperature is constant at 75°F (23.9°C).

The hourly values of the instantaneous heat flux at the interior surface of the roof can be calculated by Eq. 58, Chapter 22, p. 425 of Ref. 12, which is similar in form to Eq. 5 of this paper; or by Eq. 2, p. 183 of Ref. 8, which is similar in form to Eq. 8 of this paper. Note that both equations give identical results. The calculated heat flux through interior roof surface is plotted in Figure 1.

To start the calculations by Eq. 5 a recent history of the surface heat flux is required. One of the simple assumptions is "zero" heat flux history, i.e., all the previous heat flux values are assumed to be zero. This necessitates starting the calculations several days prior to the time period of interest. The error due to the assumed "zero" history decays to negligible value in several days even in the case of a fairly heavy wall or roof construction.

This example shows that the non steady heat transfer through building envelope components can be calculated without much difficulty using the z-transfer function method for real inside and outside temperature variations over long periods of time, i.e., the type of calculations needed to determine annual building energy requirements.

CALCULATIONS OF ROOM COOLING LOADS, HEAT EXTRACTION AND SPACE TEMPERATURE

The detailed description and examples of room cooling load, temperature and heat extraction calculations using the z-transfer function method are given in Chapter 22, 1977 edition of ASHRAE Handbook of Fundamentals (14) and in an ASHRAE Journal article (10). The following is an attempt to extend the discussion and explain the physical significance of the room z-transfer function coefficients.

One of the simplified forms of a room z-transfer function that relates the z-transforms of room heat gain, I(z), and room cooling load, CL(z) is

$$\frac{CL(z)}{I(z)} = G(z) = \frac{V_o + V_1 z^{-1}}{1 + W_1 z^{-1}} \tag{10}$$

where V_o, V_1 and W_1 are z-transfer function coefficients. The numerical values of V_o and W_1 coefficients for various room z-transfer functions are listed in Tables 31 and 32 of Ref. 14 and the V_1 coefficient is calculated using the fact that at steady state conditions input equals output, thus

$$V_1 = 1 - V_o + W_1 \tag{11}$$

Based on the z-transfer function (Eq. 10) the room cooling load in real time is given by

$$CL_t = V_o I_t + V_1 I_{t-\Delta} - W_1 CL_{t-\Delta} \tag{12}$$

The z-transfer function, as given by Eq. 10, can be expressed as a single polynomial in z^{-1}, i.e.,

$$G(z) = \frac{V_o + V_1 z^{-1}}{1 + W_1 z^{-1}} = R_o + R_1 z^{-1} + R_2 z^{-2} + \dots + R_j z^{-j} + \dots \tag{13}$$

and cooling load, CL_t, expressed in terms of R's is

$$CL_t = R_o I_t + R_1 I_{t-\Delta} + R_2 I_{t-2\Delta} + \dots + R_j I_{t-j\Delta} + \dots \tag{14}$$

This is the expression used in Refs. 4, 5 and 6 for cooling load calculations. It gives the same results as Eq. 12, but requires many more arithmetic operations.

In the application of the z-transfer function method for calculations of room cooling loads, the selection of appropriate V_o and W_1 coefficients for a specific room construction and furnishings is the most important step. The following explanation of the physical significance of V_o and W_1 coefficients may make the selection of these coefficients for a specific room more rational.

As shown by Eq. 12 the V_o fraction of the input appears as a cooling load instantaneously while $(1-V_o)$ fraction is stored (assuming no losses) in the room envelope and furnishings. This V_o fraction is a weak function of the room total heat storage capacity since initially only the immediate subsurface mass is active as heat sink or source. It is much more dependent on the type of input, e.g., initially a much larger fraction of convective heat input appears as a load as compared with a radiation input.

The air circulation in a room is another important factor that affects V_o. The surface heat transfer coefficient increases with room air circulation and thus a larger fraction of radiant heat input to the room appears as a cooling load with increased circulation rate. In a similar way, a thick carpet increases V_o value since a larger fraction of the radiant input to the room envelope and furniture surface is dissipated to the room air rather than stored.

The change of the rate of heat flow into or out of the storage (i.e., floor slab or other heavier room envelope components as well as furnishings) is defined by the W_1 coefficient. This coefficient is a function of the total room heat storage capacity, room envelope construction, type of floor slab and its covering and room air circulation rate (see Table 31, Ref. 14).

A fraction of the heat input (heat gain) to a room is lost to surroundings and thus the cooling load is less than heat input even at steady state conditions. Depending on the construction of the room enclosure and the type of heat input this fraction can be as high as 20 per cent of the input. An estimate of this fraction, F, can be made by the procedure given in Chapter 22, Ref. 14 and it is used to modify V coefficients as follows

$$V_{o,m} = V_o (1-F) \text{ and } V_{1,m} = V_1 (1-F)$$

where subscript m indicates modified V coefficient to allow for the loss.

The z-transfer function that relates the z-transform of space temperature, O(z), and net heat input to the space, NH(z)*, is a special function of the room z-transfer function set. The simple form of this function is

* The net heat input, NH_t, is an algebraic sum of all heat flow components into the space, i,e., it is a sum of heat extraction by air conditioning, and all cooling load components.

$$\frac{NH(z)}{\Theta(z)} = G(z) = \frac{g_o z^o + g_1 z^{-1} + g_2 z^{-2}}{1 + p_1 z^{-1}} \tag{15}$$

where g's and p_1 are transfer function coefficients given in Table 34 Ref. 14. Based on Eq. 15 the net heat input and space temperature can be related by

$$NH_t = g_o \Theta_t + g_1 \Theta_{t-\Delta} + g_2 \Theta_{t-2\Delta} - p_1 NH_{t-\Delta} \tag{16}$$

or

$$\Theta_t = \frac{1}{g_o} [(NH_t + p_1 NH_{t-\Delta}) - g_1 \Theta_{t-\Delta} - g_2 \Theta_{t-2\Delta}] \tag{17}$$

The selection of g's and p_1 for a specific room construction is more involved than the selection of V_o and W_1.

The first condition that can be used in this selction is that

$$g_o + g_1 + g_2 = 0 \tag{18}$$

thus

$$g_2 = -(g_o + g_1)$$

The g_o coefficient for a specific room construction (i.e., weight of mass per unit floor area) can be obtained by interpolation based on the values given in Table 34, Ref. 14.

The p_1 value can be selected by Table 31, Ref. 14 since $p_1 = W_1$. The g_1 value is selected so that the temperature change of the room air at steady state (i.e, large t) due to unit pulse of heat input calculated by Eq. 17 is equal to

$$\Theta_{t=\infty} - \Theta_{t=o} = 1/SH \cdot M$$

where

SH = bulk specific heat of building envelope and furnishings

M = weight of building envelope and furnishings per unit floor area

As in the case of the other room z-transfer functions the g coefficients must be modified to allow for the heat loss to the surroundings as described in Chapter 22, Ref. 14.

The sample calculations of space temperature and heat extraction given in Chapter 22 show that z-transfer function method can cope with building operating conditions such as full and part load operation of A-C system, variable thermostat setting (i.e., night or weekend setback) and undersized cooling capacity to maintain prescribed inside temperature conditions. All these situations occur in real buildings during the course of a year of operation and affect building energy consumption. Thus, the z-transfer function method gives results that reflect real building operation.

CONCLUSION

The main advantage of z-transfer function method is that the input variables can be hourly values of temperature, solar radiation, infiltration, and/or heat generated inside the space This allows a designer to calculate various heat flows in a building using real weather records and makes it particularly useful in calculations of building energy requirements.

The z-transfer coefficients for walls and roofs that are made up of homogeneous layers can be determined by calculations (9,13). For a more complicated construction these coefficients may be determined by tests.

The room z-transfer function coefficients given in Tables 31, 32 and 34 of Ref. 14 were calculated by a special computer program except the coefficients for lights. These were determined by tests (7).

Additional sets of room z-transfer function coefficients can be determined by NBSLD computer program (15) using suitable inputs and output.

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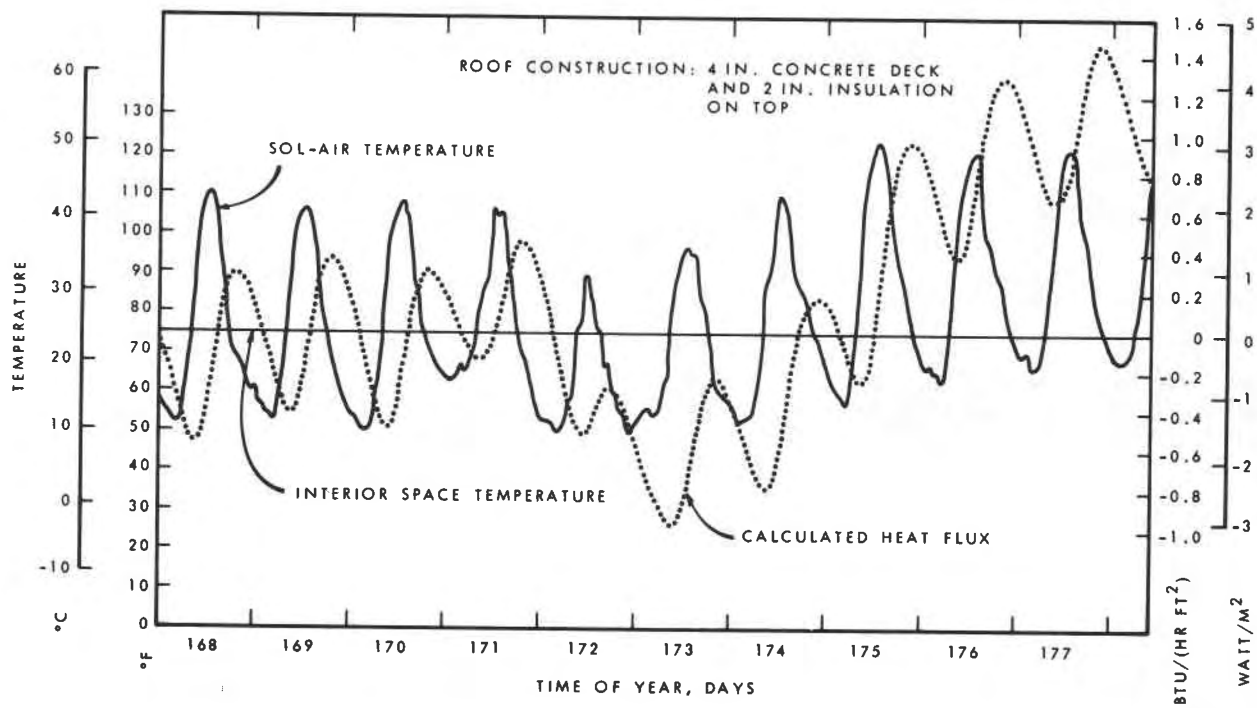

Fig. 1 Sol-air temperature and interior surface heat flux for a roof

DISCUSSION

FREDERICK WINKELMANN, Energy and Environment Division, Lawrence Berkeley Lab., Berkeley, CA: In reference to the slide showing sol-air temperature and heat flux for a several-day period for an insulated concrete roof section: for the last three days, sol-air temperature is almost periodic but the heat flux increases dramatically. Why is this?

G.P. MITALAS: It should be noted that the roof under consideration (4-in. concrete and 2-in. insulation) has relatively high thermal lag (i.e., Common Ratio = 0.957). This means that the sol-air temperature at any particular time has a non-negligible effect on heat flux values a few days later. Thus, the continuous increase in the daily average heat flux during day 175, 176 and 177 is caused by the diminishing effect of the low sol-air temperature during days 172 and 173 even though the sol-air temperature during the last three days is essentially steady periodic.

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