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ROOM DYNAMIC THERMAL RESPONSE

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RÉSUMÉ

Cette note décrit les méthodes mathématiques qu'utilise le programme informatique "Room Thermal Transfer Functions". Ce programme permet de calculer des coefficients de transmission Z pour le refroidissement, le chauffage et les prédictions des températures ambiantes d'une pièce. Le programme comprend deux parties: dans la première, le calcul des coefficients de transmission Z détaillé est basé sur une série de 29 équations du bilan thermique de diverses surfaces de la pièce.

La deuxième partie du programme utilise les coefficients calculés par la première partie du programme pour générer les fonctions de transmission de la pièce pour chaque composant intervenant (par ex. l'apport de chaleur solaire et l'apport de chaleur provenant des occupants). Il permet également de produire plusieurs approximations des fonctions qui sont représentées par un petit nombre de coefficients de fonction de transmission Z.

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ABSTRACT - The paper describes the mathematical methods used in the "Room Thermal Transfer Functions" computer program. This program calculates room Z-transfer function coefficients for cooling, heating and space-temperature predictions. The program consists of two sections: in the first, calculations of detailed Z-transfer functions are based on a set of 29 heat balance equations for the various room enclosure surfaces.

The second section of the program uses the detailed functions calculated by the first section to generate room transfer functions for each excitation component (such as solar heat gain and heat gain from occupants). It also generates several approximations of these functions that are represented by a small number of Z-transfer function coefficients.

INTRODUCTION

The increased use of the digital electronic computer in engineering offices has provided the opportunity to develop new and more powerful computational methods for use with these machines. This is especially true in the air-conditioning design field where the simple manual methods ("steady-state" load calculations) have been replaced by new calculation procedures capable of predicting the dynamic heating and cooling loads as well as the annual energy requirement. One method for calculating dynamic heating and cooling loads is the Z-transfer function method described in the ASHRAE Handbook of Fundamentals.[1]

The concept of the Z-transfer function is based on the assumption that the heat transfer processes in a room can be represented by linear algebraic and differential equations. This permits use of the superposition principle, where the driving function is represented as the sum of a series of simple pulses, and the response is the sum of the responses to each pulse considered by itself.[2,3] The relationship between a unit of input and its response in Z-transform space is defined by a Z-transfer function.

In order to correlate the room excitation components (such as the energy input components of solar radiation, lights and outdoor temperature) with the cooling, each component requires its own Z-transfer function. This is because each component has a different heat transfer mechanism or entry path into a room. A particular room, therefore, has as many Z-transfer functions as there are distinct heat gain components. In addition, a special Z-transfer function is needed to correlate a unit swing in room air temperature with the cooling that must be provided by the air conditioning unit (i.e., heat extraction).

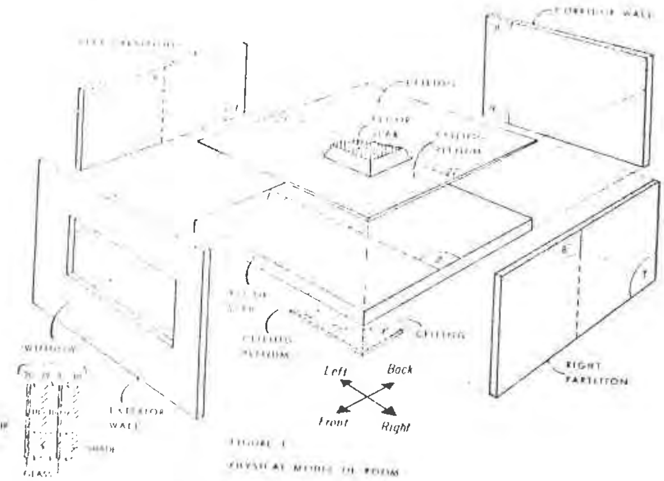
A major difficulty in applying the Z-transfer function method is the determination of the functions themselves for a specific room. A computer program entitled "Room Thermal Transfer Functions" (G.P. Mitalas; to be published as a DBR Computer Program, National Research Council Canada, Ottawa, 1983) has been written to calculate room Z-transfer functions. The present paper gives the mathematical basis for these calculations and describes a method of simplifying the functions for application to specific problems.

DETAILED ROOM Z-TRANSFER FUNCTIONS

Physical Model of Room

The geometry of the model room is shown in Fig. 1 and the numbering system for room surfaces is given in Table 1. The model incorporates most of the physical features commonly found in an office of a multistorey commercial building. Moreover, it incorporates provisions for simulating a ceiling air space between the false ceiling and the floor slab above, and for simulating a window with interior and interpane shading devices. For less elaborate rooms, redundant model features can be deleted by assigning appropriate dimensions, thermal properties or coefficients.

The model is divided into a front and a back section so that non-symmetry (such as distribution of solar radiation on interior room surfaces and heat transfer through the window) can be accounted



for. It is assumed that the room is surrounded on all sides, except the outside and corridor walls, by similar rooms so that the average heat exchange between the model room and the surrounding rooms is zero.

All furniture in the room is modeled by a single thin slab located close to and parallel to the floor surface. It is assumed that the projected area of the furniture on the floor is equal to half of the total surface of all furniture.

Thermal Model of Room

The room thermal system is described by the heat balance equations for each of the room envelope surfaces. These combine the thermal characteristics, geometry, surface excitation components, surface and air temperatures of the room, and cooling provided by the air conditioning unit (derivation of these equations will be discussed in following sections). The set of heat balance equations are represented by the matrix equation,

$$[M] \cdot [X]_t = [G]_t \qquad (1)$$

where the elements of [M] are functions of room dimensions and thermal characteristics and the elements of $[G]_t$ are functions of room excitation components, the past history of room envelope surface temperature and the thermal characteristics of the room. The subscript t indicates time and $[X]_t$ represents unknown room temperatures and the heat extraction. The solution of equation (1) is,

$$[X]_t = [M]^{-1} \cdot [G]_t \qquad (2)$$

The computation by equation (2) generates a set of room response functions of the form,

$$K^1(Z)_i = H_{i,0}z^0 + H_{i,1}z^{-1} + H_{i,2}z^{-2} + \ldots + H_{i,n}z^{-n} + \ldots \qquad (3)$$

Table 1 Numbering System of Room Envelope Surfaces, Air Spaces and Window Components

Surface Number	Back Surface Number	Description
1	25	Floor slab, upper, front surface
2	26	Floor slab, upper, back surface
3	22	Ceiling, lower, back surface
4	21	Ceiling, lower, front surface
5	7	Partition, right, back interior surface
6	8	Partition, right, front interior surface
7	5	Partition, left, back interior surface
8	6	Partition, left, front interior surface
9	*	Corridor wall interior surface
10		Window system interior surface
11		Exterior wall, below window, interior surface
12	#	Exterior wall, above window, interior surface
13		Exterior wall, right of window, interior surface
14		Exterior wall, left of window, interior surface
15	**	Furniture surface
16		Interior shade-glass pane air space
17		Window interior glass pane
18		Interpane air space
19		Interpane shade
20		Window exterior glass pane
21	4	Ceiling, upper, front surface
22	3	Ceiling, upper, back surface
23		Ceiling plenum, front, air space
24		Ceiling plenum, back, air space
25	1	Floor slab, lower front surface
26	2	Floor slab, lower back surface
27	#	Ceiling plenum exterior wall, interior surface
28	*	Ceiling plenum corridor wall, interior surface
29		Room air

Outer surface sol-air temperature $(\Theta_{o,n})$ used in place of back surface temperature.
* Corridor air temperature $(\Theta_{c,n})$ used in place of back surface temperature.
** Furniture temperature $(\Theta_{15,n})$ used as its own back surface temperature.

where,

$H_{i,n}$ = last element of $[X]_{t+n\Delta}$, the calculated heat extraction value at time $t + n\Delta$, due to a unit excitation pulse, E_i at time t. Subscript n is an integer and Δ is time interval.

A total of 34 room thermal response functions are calculated for 26 unit heat flux pulses, six surface temperature pulses and two ceiling plenum air temperature pulses. A special function that relates air temperature pulse and room heat extraction is also calculated.

A simple, but important fact should be noted at this point; calculated $\sum_{n=0}^{\infty} H_{i,n}$ is always less than one. This means that a small fraction of the energy input does not appear as a load on the air conditioning system. This fraction depends on the room thermal system resistance characteristics. For example, the sum tends to one as the room is insulated to a higher level, window area is reduced or window thermal resistance is increased. The calculated room thermal reponse functions are, therefore, normalized as follows:

$$K(Z)_i = K^1(Z)_i \Big/ \sum_{n=0}^{\infty} H_{i,n} = (H_{i,0}z^0 + H_{i,1}\, z^{-1} + ..) \Big/ \sum_{n=0}^{\infty} H_{i,n} \qquad (4)$$

For the purpose of calculating room thermal response functions, a set of 29 heat balance equations is used (i.e. [M] is a 29 by 29 matrix). One equation is used for room-air heat balance. Fourteen equations are required for interior surface heat balances since the room interior surface is subdivided into 14 segments as shown in Fig. 1. One equation is for furniture surface heat balance, five equations for window-system heat balances, and eight equations for ceiling-plenum heat balances.

The general form of the heat balance equation for surface i at time t is:

$$(U_{i,t} + R_{i,t} + Q_{i,t} + E_{i,t}) = 0 \qquad (5)$$

where,

$U_{i,t}$ = convection heat flux,
$R_{i,t}$ = net radiant heat flux due to longwave radiant heat transfer,
$Q_{i,t}$ = conduction heat flux,
$E_{i,t}$ = surface excitation heat flux; e.g. energy input to a unit surface area by solar radiation, etc.,
i = integer indicating surface number.

Heat Balance at Interior Surfaces

The convection heat transfer flux to surface i is,

$$U_{i,t} = h_i\,(\Theta_{a,t} - \Theta_{i,t}) \qquad (6)$$

where,

h_i = convection heat transfer coefficient at surface i,
$\Theta_{a,t}$ = room air temperature,
$\Theta_{i,t}$ = temperature of surface i.

The net radiant heat transfer flux to the surface i is

$$R_{i,t} = \sum_{j=1}^{15} g_{i,j}\,(\Theta_{j,t} - \Theta_{i,t}) \qquad (7)$$

where,

15 is the number of surfaces involved in the radiant heat interchange within the room,
$g_{i,j}$ = radiation heat transfer coefficient $= f_{i,j}\, 4\, \sigma\, T^3_{avg}$,
$f_{i,j}$ = radiant heat interchange factor for surfaces i and j,
σ = Stefan-Boltzmann constant,
T_{avg} = time average of all absolute surface temperatures (an assumed approximate value).[4,5]

The conduction heat transfer flux to the surface i is,

$$Q_{i,t} = a_{i,0}\Theta_{i,t} + a_{i,1}\,\Theta_{i,t-\Delta} + a_{i,2}\,\Theta_{i,t-2\Delta} + \cdots + b_{i,0}\Theta^*_{i,t} + b_{i,1}\Theta^*_{i,t-\Delta} + b_{i,2}\Theta^*_{i,t-2\Delta} \cdots + d_{i,1}\,Q_{i,t-\Delta} + d_{i,2}\,Q_{i,t-2\Delta} + \cdots \qquad (8)$$

where a, b and d are coefficients of the Z-transfer function of the room enclosure element in question, and $\Theta^*_{i,t}$ is air temperature or sol-air temperature at the opposite face of element i.[6,7]

Substitution for U_i, Q_i and R_i in equation (5) gives,

$$-\Theta_{i,t}\,(h_i + a_{i,0} + \sum_{j=1}^{15} g_{i,j}) + \sum_{j=1}^{15} g_{i,j}\,\Theta_{j,t} + b_{i,0}\,\Theta^*_{i,t} = -E_{i,t} - h_i\,\Theta_{a,t} - \sum_{n=1}^{\infty} (a_{i,n}\,\Theta_{i,(t-n\Delta)} + b_{i,n}\,\Theta^*_{i\,(t-n\Delta)} + d_{i,n}\,Q_{i,\,(t-n\Delta)}) \qquad (9)$$

The above equation is the general form of a row representing interior surface heat balance in the matrix equation (1).

Heat Balance at Furniture Surfaces

The heat balance equation for the surface of all furniture is based on equation (9) and the following assumptions:

(1) furniture surface temperature is uniform; i.e., $\Theta_{i,t} = \Theta^*_{i,t}$,
(2) one half of the furniture surface exchanges heat by radiation with the floor and the other half exchanges heat with the other enclosure surfaces. Consequently, the fraction of the floor surface covered by the furniture exchanges heat by radiation with the furniture, and the remaining fraction of floor surface exchanges heat by radiation with the other enclosure surfaces.

Window System

The heat balance equations for the window system are simplified forms of equation (9); the radiant heat flux is expressed simply as,

$$R_{i,t} = g_{i,j}\,(\Theta_{j,t} - \Theta_{i,t}) \qquad (10)$$

where $g_{i,j}$ = radiation heat transfer coefficient for an air space enclosed by two parallel surfaces i and j, and the thermal resistance values of the glass panes and the shades are negligible. Thus, the surface heat balances of a window pane or a shade are combined into a single heat balance equation.

The heat exchange between the outside air and the exterior window pane is calculated by,

$$U_{i,t} + R_{i,t} = B_i\,(\Theta_{o,t} - \Theta_{i,t}) \qquad (11)$$

where,

$\Theta_{o,t}$ = outside air temperature
B_i = combined outside surface heat transfer coefficient.

Ceiling-Floor System

The heat balance equations for the air in the ceiling plenum and the surfaces of the drop ceiling, walls and floor above, which enclose the space, are also simplified forms of equation (9). In this case, the radiant heat transfer between surfaces is expressed by,

$$R_{i,t} = g_{i,j}\,(\Theta_{j,t} - \Theta_{i,t}) \qquad (12)$$

and in the case of the air in the ceiling plenum, the heat conduction term is deleted because heat transfer by conduction is not involved in this heat balance.

Room Air

The heat balance for the room air is

$$H_t = \sum_{i=1}^{15} A_i \cdot h_i \cdot (\Theta_{i,t} - \Theta_{a,t}) \qquad (13)$$

where,

A_i = area of room interior surface i (including furniture surface).

In this case, the heat extraction (H_t) rather than temperature $(\Theta_{a,t})$ is taken as the unknown quantity.

An expanded form of equation (1) in terms of equations (6) to (13) is given in Appendix A.

COMBINED ROOM Z-TRANSFER FUNCTIONS

The detailed set of transfer functions K(Z) is cumbersome to use for room air-conditioning load calculations. For example, the cooling needed to maintain constant air temperature $(CL_{d,t})$ due to diffuse solar radiation transmitted through the window $(S_{d,t})$ can be expressed as follows using K(Z) functions:

$$CL_{d,t} = \sum_{i=1}^{15} \sum_{n=0}^{\infty} (F_{d,i}\, S_{d,(t-n)} H_{i,n}) \qquad (14)$$

where,

$F_{d,i}$ = ratio of absorbed diffuse radiation by interior surface i over the diffuse solar radiation transmitted through the window. It is assumed that $F_{d,i}$ is independent of time.

As shown by equation (14), this calculation involves many arithmetic operations and coefficients. The number of these operations can be reduced by using combined Z-transfer function C(Z). For example, $CL_{d,t}$ can be expressed as follows:

$$CL_{d,t} = \sum_{n=0}^{\infty} S_{d,(t-n)}\, Y_{d,n} \qquad (15)$$

where $Y_{d,n}$ is a coefficient of combined Z-transfer function $C(Z)_d$.

The combined transfer function can be derived by interchanging the order of summation in equation (14) and precalculating the coefficients $Y_{d,n} = \sum_{i=1}^{15} F_{d,i}\, H_{i,n}$. Thus, in general, the combined Z-transfer function for the room excitation component $(S_{e,t})$ is calculated by:

$$C(Z)_e = \sum_{i=1}^{L} F_{e,i}\, K(Z)_i = Y_{e,0}z + Y_{e,1}z^{-1} + Y_{e,2}z^{-2} + \cdot\; Y_{e,n}z^{-n} + \ldots \qquad (16)$$

where L is the total number of surface excitations $(E_{i,t})$ that make up the room excitation component $(S_{e,t})$, and $F_{e,i}$ is a fraction of $S_{e,t}$ that appears as surface i excitation heat flux, $(E_{e,i})$.

The $F_{e,i}$ values are part of the input data to the program. The selection of $F_{e,i}$ factors may be quite involved and depends on the type of room excitation component. For example, the $F_{d,i}$ value for diffuse solar radiation transmitted through a window may be taken as the geometric long-wave radiation interchange factors between the window and the other room surfaces.

On the other hand, the $F_{o,i}$ factors for outside air temperature response are:

$$F_{o,i} = \frac{A_i}{\sum_{i=1}^{L} A_i} \qquad (17)$$

where A_i = area of outside surface i, and
L = total number of outside surfaces under consideration.

SIMPLIFIED ROOM Z-TRANSFER FUNCTIONS

The detailed and combined transfer functions are unwieldy to use in the forms given by equations (3) and (16). These forms take a large number of coefficients to define K(Z) and C(Z) and, consequently, require many numerical operations. A routine has, therefore, been incorporated into the computer program (Room Thermal Transfer Functions) to express C(Z) as a ratio of two polynomials, as well as to generate a set of simplified functions of varying accuracy for each combined room transfer function, C(Z). In addition, the frequency response of each simplified function is calculated in order to check the error introduced by simplification.

The combined Z-transfer functions given by equation (16) can be expressed as a ratio of two polynomials in z^{-1}:

$$C(Z) = \frac{N(Z)}{D(Z)} = \frac{p_0 z^0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3} + \ldots + p_n z^{-n} + \ldots}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3} + \ldots + d_n z^{-n} +} \qquad (18)$$

where p and d are Z-transfer function coefficients.[8]

D(Z) and N(Z) Coefficients

The program calculates three different D(Z) polynomials and, consequently, generates three different forms of C(Z).

The fact that the room thermal response to a unit excitation is the sum of exponential (decay) functions[3,7] is used to calculate D(Z) coefficients as follows:

for large values of n,

$$\frac{Y_{n+1}}{Y_n} = CR_1 \text{ (constant)}$$

$$\frac{V_{n+1}}{V_n} = CR_2 \text{ (constant)}$$

where, $V_n = Y_n - CR_1 Y_{n-1}$

and

$$\frac{W_{n+1}}{W_n} = CR_3 \text{ (constant)}$$

where, $W_n = V_n - CR_2 V_{n-1}$.

In the case of a room of light-weight construction, the thermal decay is so rapid that CR_3 (and even CR_2) cannot be determined for a time interval of one hour. In these cases, therefore, only the available CR's are used in the following calculations.

Using the CR factors, the following three pairs of D(Z) and N(Z) polynomials are calculated:

$$D_1(Z) = 1 - CR_1 z^{-1}$$

$$D_2(Z) = (1 - CR_1 z^{-1})(1 - CR_2 z^{-1})$$

$$D_3(Z) = (1 - CR_1 z^{-1})(1 - CR_2 z^{-1})(1 - CR_3 z^{-1})$$

$$N_1(Z) = C(Z) \cdot D_1(Z)$$

$$N_2(Z) = C(Z) \cdot D_2(Z)$$

$$N_3(Z) = C(Z) \cdot D_3(Z).$$

These calculations generate a set of exact room transfer functions expressed as a ratio of two polynomials.

The transfer functions can be simplified further by noting that N(Z) coefficients diminish as the value of n increases. Thus, the N(Z) polynomial can be truncated to reduce the number of coefficients. To preserve the steady-state relation of the original function, the coefficients of the discarded portion of the N(Z) polynomial are summed up and added to the last coefficient in the truncated series.

To determine how the truncation affects the accuracy of room Z-transfer functions, the frequency responses of both the exact and truncated functions can be calculated.

The frequency response can be calculated by using the relation,[9]

$$z^{-n\Delta} = e^{-i\omega n\Delta} = \cos(\omega n\Delta) - i \sin(\omega n\Delta) \qquad (19)$$

where,

$\omega = \frac{2\pi}{P}$, the angular velocity of the sine wave,

P = period of the sine wave, and

$i = (-1)^{\frac{1}{2}}$.

The comparison of the frequency response of the exact and the truncated functions will facilitate selection of simplified room transfer functions.

The foregoing explains how the frequency response can be used to select a set of simplified room Z-transfer functions for a particular application to provide the desired accuracy with minimum computational effort. It should be noted that the transfer function in the form

$$C(Z) = \frac{p_0 z^0 + p_1 z^{-1}}{1 + d_1 z^{-1}} \quad \text{at } \Delta = 1 \text{ h} \qquad (20)$$

can adequately correlate room excitation and cooling load for most practical air-conditioning calculations. For this reason, the Z-transfer functions given in the ASHRAE Handbook of Fundmentals[1] are of this simple form.

Validity of the calculated Z-transfer functions is dependent on how well the following assumptions used to construct the model represent real conditions in a building: constant (invariable) surface convection heat transfer coefficients, one-dimensional heat conduction through the envelope (i.e., three-dimensional heat conduction effects at corners, beams and columns are neglected), uniform heat flux and temperatures at the surfaces of the various elements, simple physical and thermal model of the furnishings, room surrounded by similar rooms, uniform air temperature throughout the room.

A clear understanding of the limitations implied by these assumptions is, therefore, essential when applying the room transfer functions derived by the model.

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Appendix A

Matrix Equation $[M]\cdot[X]_n = [G]_n$

For clarification, the matrix equation is expanded using equations (6) to (13). This matrix equation is a group of four different sets of equations that correlate various temperatures and heat extraction:

Rows 1 to 15 correlate temperatures of room enclosure interior surfaces and furniture surfaces.
Rows 16 to 20 correlate temperatures of window system
Rows 21 to 28 correlate temperatures at ceiling plenum.
Row 29 correlates room air temperature, interior surface temperatures and heat extraction required to maintain constant room air temperature.

The diagonal elements of matrix $[M]$ are negative and all other elements are positive. Many elements are zeros; the zero elements indicate indirect relation between the unknowns.

For presentation the matrix $[M]$ is partitioned as follows:

$$[M] = \frac{M1 \;\text{------}\; M2}{M3 \qquad M4}$$

where the vertical and horizontal partition lines are between 15th and 16th rows and 15th and 16th columns, respectively.

a) <u>Submatrix $[M1]$</u>

Diagonal elements:

$$M1_{1,1} = -(h_1 + \Sigma^* g_{1,j} + a_{1,0})$$

$$^{**}M1_{10,10} = -(h_{10} + \Sigma\, g_{10,j} + h_{10,16} + g_{10,17})$$

$$M1_{15,15} = -(h_{15} + \Sigma\, g_{15,j} + a_{15,0} + b_{15,0})$$

---

* Σ denotes $\left(\sum_{j=1}^{15} g_{i,j} - g_{i,i}\right)$

** $M1_{2,2}$ to $M1_{14,14}$ are similar to $M1_{1,1}$, except for $M1_{10,10}$.

All other elements of $[M1]$ are radiation heat transfer coefficients, i.e., $M1_{i,j} = g_{i,j}$ except

$$M1_{5,7} = M1_{7,5} = g_{5,7} + b_{5,0}$$

$$M1_{6,8} = M1_{8,6} = g_{6,8} + b_{6,0}$$

b) <u>Submatrices $[M2]$ and $[M3]$</u>

All $[M2]$ and $[M3]$ elements are zero except:

$$M2_{1,25} = M3_{25,1} = b_{25,0}$$

$$M2_{2,26} = M3_{26,2} = b_{26,0}$$

$M2_{3,22} = M3_{22,3} = b_{22,0}$

$M2_{4,21} = M3_{21,4} = b_{21,0}$

$M2_{10,16} = M3_{16,10} = h_{10,16}$

$M2_{10,17} = M3_{17,10} = h_{10,17}$

$M3_{29,j} = A_j h_j$ for $j = 1$ to 15

c) Submatrix [M4]

$M4_{16,16} = -(h_{16,10} + h_{16,17})$

$M4_{17,17} = -(h_{17,16} + g_{17,10} + h_{17,18} + g_{17,19} + g_{17,20})$

$M4_{18,18} = -(h_{18,17} + h_{18,19} + h_{18,20})$

$M4_{19,19} = -(h_{19,18} + g_{19,17} + g_{19,20})$

$M4_{20,20} = -(h_{20,18} + g_{20,19} + g_{20,17} + B_{20})$

$M4_{21,21} = -(h_{21,23} + g_{21,25} + a_{21,0})$

$M4_{22,22} = -(h_{22,24} + g_{22,26} + a_{22,0})$

$M4_{23,23} = -(h_{23,21} + h_{23,25} + h_{23,27})$

$M4_{24,24} = -(h_{24,22} + h_{24,26} + h_{24,28})$

$M4_{25,25} = -(h_{25,23} + g_{25,21} + a_{25,0})$

$M4_{26,26} = -(h_{26,24} + g_{26,22} + a_{26,0})$

$M4_{27,27} = -(h_{27,23} + a_{27,0})$:*

$M4_{28,28} = -(h_{28,24} + a_{28,0})$: $M4_{29,29} = -1$:

$M4_{16,17} = M4_{17,16} = h_{16,17}$: $M4_{17,18} = M4_{18,17} = h_{17,18}$: $M4_{17,10} = g_{17,10}$: $M4_{17,19} = M4_{19,17} = g_{17,19}$: $M4_{18,19} = M4_{19,18} = h_{18,19}$: $M4_{18,20} = M4_{20,18} = h_{18,20}$: $M4_{19,20} = M4_{20,19} = g_{19,20}$: $M4_{20,17} = M4_{17,20} = g_{20,17}$: $M4_{21,23} = M4_{23,21} = g_{21,23}$: $M4_{21,25} = M4_{25,21} = g_{21,25}$: $M4_{22,24} = M4_{24,22} = h_{22,24}$: $M4_{22,26} = M4_{26,22} = g_{22,26}$: $M4_{23,25} = M4_{25,23} = h_{23,25}$: $M4_{23,27} = M4_{27,23} = h_{23,27}$: $M4_{24,26} = M4_{26,24} = h_{24,26}$: $M4_{24,28} = M4_{28,24} = h_{24,28}$

All other elements of [M4] are zero.

* : used for separation of matrix elements.

d) Column $[X]_t$

$X_{i,t} = \Theta_{i,t}$ for $i = 1$ to 28

$X_{29,t} = H_{i,t}$

e) Column $[G]_t$

$$G_1 = -E_1 - h_1\,\Theta_a - \sum_{n=1}^{\infty} (a_{1,n}\,\Theta_{1,t-n} + b_{25,n}\Theta_{25,t-n} + d_{1,n}\,Q_{1,t-n})$$

$${}^{*}G_{10} = -E_{10} - h_{10}\,\Theta_a$$

$$G_{11} = -E_{11} - h_{11}\,\Theta_a - \sum_{n=1}^{\infty} (a_{11,n}\,\Theta_{11,t-n} + b_{11,n}\,\Theta_{o,t-n} + d_{11,n}\,Q_{11,t-n})$$

$$^{**}G_{14} = -E_{14} - h_{14}\,\Theta_a - \sum_{n=1}^{\infty} (a_{14,n}\,\Theta_{14,t-n} + b_{14,n}\,\Theta_{o,t-n} + d_{14,n}\,Q_{14,t-n})$$

$$G_{15} = -E_{15} - h_{15}\,\Theta_a - \sum_{n=1}^{\infty} ((a_{15,n} + b_{15,n})\,\Theta_{15,t-n} + d_{15,n}\,Q_{15,t-n})$$

$$G_{16} = -E_{16}$$

$$G_{20} = -E_{20} - B_{20}\,\Theta_o$$

$$G_{21} = -E_{21} - \sum_{n=1}^{\infty} (a_{21,n}\,\Theta_{21,t-n} + b_{4,n}\,\Theta_{4,t-n} + d_{21,n}\,Q_{21,t-n})$$

$$G_{22} = -E_{22} - \sum_{n=1}^{\infty} (a_{22,n}\,\Theta_{22,t-n} + b_{3,n}\,\Theta_{3,t-n} + d_{22,n}\,Q_{22,t-n})$$

$$^{***}G_{25} = -E_{25} - \sum_{n=1}^{\infty} (a_{25,n}\,\Theta_{25,t-n} + b_{1,n}\,\Theta_{1,t-n} + d_{25,n}\,Q_{25,t-n})$$

$$G_{26} = -E_{26} - \sum_{n=1}^{\infty} (a_{26,n}\,\Theta_{26,t-n} + b_{2,n}\,\Theta_{2,t-n} + d_{26,n}\,Q_{26,t-n})$$

$$G_{27} = -E_{27} - \sum_{n=1}^{\infty} (a_{27,n}\,\Theta_{27,t-n} + b_{27,n}\,\Theta_{o,t-n} + d_{27,n}\,Q_{27,t-n})$$

$$G_{28} = -E_{28} - \sum_{n=1}^{\infty} (a_{28,n}\,\Theta_{28,t-n} + b_{28,n}\,\Theta_{c,t-n} + d_{28,n}\,Q_{28,t-n})$$

$$G_{29} = \Theta_{a},\sum_{i=1}^{15} (A_i\,h_i)$$

* G_2 to G_9 are similar to G_1.

** G_{12}, G_{13} are similar to G_{11}.

*** G_{17} to G_{19} and G_{23} and G_{24} are similar to G_{16}.

A list of all publications available from the Division may be obtained by writing to the Publications Section, Division of Building Research, National Research Council of Canada, Ottawa, Ontario, K1A 0R6.