NATIONAL RESEARCH COUNCIL OF CANADA

DIVISION OF BUILDING RESEARCH

PERIODIC HEAT FLOW IN WALLS AND ROOFS

by

D. G. Stephenson

Report No. 132

of the

Division of Building Research

OTTAWA

September 1957

PREFACE

The Division of Building Research is currently re-examining and appraising the methods employed in predicting the thermal energy exchange between the indoor and the outdoor environment through the enclosing walls and roofs of buildings. Summer conditions which give rise to periodic flow are of special interest, and heat exchange between the ground surface and the outdoor environment is also of concern.

Although the path of the thermal energy exchange between indoors and outdoors may be broken into parts for special study, these are actually inter-related and in a rigorous treatment must be considered together. It became necessary in dealing with the exchange between exterior surfaces and the outdoor environment to consider also certain aspects of the heat flow through the building enclosure itself. One aspect of this, the case of periodic heat flow in walls and roofs, and of methods by which it may be calculated is now considered. Certain aspects of the thermal exchange between the exterior surface of a building and the outdoor environment were dealt with in DBR Report No. 121.

It is not apparent that the material contained in this report will be of sufficient interest to others to warrant publication. Opinions on this, together with any criticism and other comments are invited.

Ottawa
September 1957.

N.B. Hutcheon
Assistant Director.

PERIODIC HEAT FLOW IN WALLS AND ROOFS

by

D. G. Stephenson

ABSTRACT

Two methods of calculating the periodic heat flow through solid walls or roofs are described. The matrix method gives the accurate value of heat flow through a solid wall whether homogeneous or multilayer. One of the advantages of this method is the ease with which different surface heat transfer coefficients can be accounted for. A much simpler vector diagram method is presented for calculating the complex thermal impedance of a wall or roof. The errors which occur with this method are chiefly due to representing a continuous wall by a lumped resistance-capacitance network. The lumping error for a "T" network representation of a homogeneous wall is evaluated analytically: the same method can be applied to more complex circuits.

The following is the nomenclature used in this paper.

a, b	$a + ib = \cosh (1 + i)\varphi_n$
c, d	$c + i\,d = \dfrac{\sinh (1 + i)\varphi_n}{(1 + i)\varphi_n}$
e, f	$e + i\,f = (1 + i)\varphi_n . \sinh (1 + i)\varphi_n$
a_n	= harmonic phase angle
A_n, B_n, C_n and D_n	= elements of wall matrix for heat wave of frequency n
h_1	= outside air film coefficient of heat transfer
h_2	= inside air film coefficient of heat transfer
i	$= \sqrt{-1}$
k	= thermal conductivity of wall material
L	= thickness of homogeneous wall or of layer of multilayer wall

n	= frequency of temperature wave in cycles per day
R_1	$= 1/h_1$
R_2	$= 1/h_2$
R_w	$= L/k$
q_1	= heat flux across outside surface of wall or roof
q_2	= heat flux across inside surface of wall or roof
$q_{2,n}$	harmonic component of q_2 with frequency n
T_1	= sol-air temperature
	$= t_m + \sum_{n=1}^{\infty} t_{1,n}$
T_2	= inside air temperature
t_m	= 24 hour average of sol-air temperature
$t_{1,n}$	= harmonic component of T_1 with frequency n
	$= \lvert t_{1,n} \rvert \;\underline{\lvert 15\, n\, \theta - a_n}$
Z_n	= thermal impedance of continuous wall for a heat wave of frequency n
	$= t_{1,n}/q_{2,n}$ when T_2 = constant
α	= thermal diffusivity of wall material
φ_n	= dimensionless number characteristic of wall or wall element
	$= \sqrt{\dfrac{n.\pi L^2}{24\ .\alpha}}$
π	= 3.1416
θ	= hours after noon
ϕ_n	= harmonic lag angle
λ_n	= harmonic decrement factor

When primes are added to any of the above listed quantities, it indicates that the value is appropriate for a lumped resistance capacitance network. The number of primes equals the number of lumps.

The heat flow through the walls or roof of a building rarely reaches a steady state, so for an accurate calculation of heating or cooling load it is necessary to take account of the heat storage capacity of the wall or roof. Mackey and Wright (1, 2) have reported an accurate and approximate method for calculating the periodic heat flow through homogeneous and multilayer walls. Their work was based on the following assumptions:

(1) Periodic sol-air temperature
(2) Constant indoor air temperature
(3) The outdoor air film coefficient of heat transfer, h_1, is 4.0 B.t.u./$_{ft^2\ hr\ F°}$
(4) The indoor air film coefficient of heat transfer, h_2, is 1.65 B.t.u./$_{ft^2\ hr\ F°}$
(5) One dimensional heat flow through the wall

These give the heat flux across the inner surface as

$$q_2 = \frac{t_m - T_2}{R_1 + R_w + R_2} + h_2 \sum_{n=1}^{\infty} \lambda_n \cdot \left| t_{1,n} \right| \cdot \cos\left(15_n \theta - a_n - \phi_n\right) \qquad \ldots\ldots.(1)$$

The decrement factor λ_n and the lag angle ϕ_n are given graphically as functions of $\frac{k}{L}$ and $k.\rho.c$. The equations relating these factors to the wall parameters are included so that the values may be calculated for other values of the film coefficients. The work required to obtain λ_n and ϕ_n for other values of h_1 or h_2 is considerable even for a homogeneous wall, and much more for a composite wall. In the following sections two alternative procedures are described. The matrix method gives an exact solution with less work than the Mackey and Wright method; and the vector diagram gives a very good approximation for relatively little computational work.

Equation (1) can be written as

$$q_2 = q_{mean} + \sum_{n=1}^{\infty} q_{2,n} \qquad \ldots\ldots(2)$$

where
$$q_{mean} = \frac{t_m - T_2}{R_1 + R_w + R_2} \qquad \ldots\ldots(3)$$

$$q_{2,n} = \frac{t_{1,n}}{Z_n} \qquad \text{......(4)}$$

$$Z_n = \frac{1}{h_2 \cdot \lambda_n} \underline{|\phi_n} \qquad \text{......(5)}$$

in the terms used by Mackey and Wright.

<u>Z_n is here called the wall thermal impedance</u>

MATRIX METHOD OF OBTAINING WALL THERMAL IMPEDANCE

Van Gorcum (3) has shown that for periodic temperature oscillations the temperature and heat flux at each surface of a homogeneous slice of material are related by a linear transformation

$$\left\|\begin{matrix} t_{a,n} \\ q_{a,n} \end{matrix}\right\| = \left\|\begin{matrix} A_n & B_n \\ C_n & A_n \end{matrix}\right\| \cdot \left\|\begin{matrix} t_{b,n} \\ q_{b,n} \end{matrix}\right\| \qquad \text{......(6)}$$

where * $A_n = \cosh (1 + i) \varphi_n$ ......(7)

$$A_n \doteqdot 1 + i \varphi_n^2 - \frac{\varphi_n^4}{6} \qquad \text{......(7a)}$$

$$B_n = R_w \frac{\sinh (1 + i) \varphi_n}{(1 + i) \varphi_n} \qquad \text{......(8)}$$

$$B_n \doteqdot R_w \left[1 + \frac{i \varphi_n^2}{3} - \frac{\varphi_n^4}{30} \right] \qquad \text{......(8a)}$$

$$C_n = \frac{(1 + i)\varphi_n}{R_w} \cdot \sinh (1 + i) \varphi_n \qquad \text{......(9)}$$

$$C_n \doteqdot \frac{2}{R_w} \left[i \varphi_n^2 - \frac{\varphi_n^4}{3} \right] \qquad \text{......(9a)}$$

$$\varphi_n = \sqrt{\frac{n \cdot \pi \cdot L^2}{24 \cdot \alpha}} \qquad \text{......(10)}$$

---

* Van Gorcum expressed the matrix elements by trigonometric rather than hyperbolic functions but there is no real difference.

If this wall exchanges heat with the outside and inside air through film resistances R_1 and R_2 respectively, the relation between inside and outside heat flux and temperature is

$$\begin{Vmatrix} t_{1,n} \\ q_{1,n} \end{Vmatrix} = \begin{Vmatrix} 1 & R_1 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} A_n & B_n \\ C_n & A_n \end{Vmatrix} \cdot \begin{Vmatrix} 1 & R_2 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} t_{2,n} \\ q_{2,n} \end{Vmatrix} \qquad \ldots\ldots(11)$$

Thus when $t_{2,n} = 0$

$$\frac{t_{1,n}}{q_{2,n}} = Z_n = (R_1 + R_2)\ A_n + B_n + (R_1 \cdot R_2)\ C_n \qquad \ldots\ldots(12)$$

Substituting the approximate values of A_n, B_n and C_n given by equations 7a, 8a and 9a into equation 12 gives

$$Z_n \doteq R_1 + R_W + R_2 - \varphi_n^{\ 4} \left[\frac{R_1 + R_2}{6} + \frac{R_W}{30} + \frac{2}{3}\,\frac{R_1 \cdot R_2}{R_W} \right]$$

$$+ i\,\varphi_n^{\ 2} \left[R_1 + R_2 + \frac{R_W}{3} + \frac{2}{3}\,\frac{R_1 \cdot R_2}{R_W} \right] \qquad \ldots\ldots(13)$$

Table 1 gives the exact value of the real and imaginary parts of A_n, $\frac{B_n}{R_W}$ and $R_W \cdot C_n$ as functions of φ_n. For values of φ_n less than one the approximations given by equations 7a, 8a and 9a are in error by less than one per cent.

This matrix method can be readily adapted to composite walls since each layer of the wall can be represented by a four-term matrix. The matrix for the composite wall is just the product of the matrices for each component; viz:

$$\begin{Vmatrix} A_n & B_n \\ C_n & D_n \end{Vmatrix} = \begin{Vmatrix} A_{1,n} & B_{1,n} \\ C_{1,n} & A_{1,n} \end{Vmatrix} \cdot \begin{Vmatrix} A_{2,n} & B_{2,n} \\ C_{2,n} & A_{2,n} \end{Vmatrix} \qquad \ldots\ldots\ldots(14)$$

for a two-layer wall. The impedance formula is slightly different in this case because the wall matrix contains four different terms

$$Z_n = R_2 \cdot A_n + B_n + R_1 \cdot R_2 \cdot C_n + R_1\,D_n \qquad \ldots\ldots(15)$$

The example calculation in Appendix I illustrates the method and incidentally shows the difference in the impedance of the wall when the order of the layers is reversed. The calculation of the matrix elements for homogeneous layers of a wall is simple, but for a composite wall the matrix multiplication requires considerable computation. However, when the matrix elements for a wall have been determined it is a very simple matter to calculate the wall thermal impedance for any values of the inside and outside air film coefficients of heat transfer. The matrix elements for the more common composite walls and roofs could be tabulated in the ASHAE Guide.

A Vector Diagram Method of Obtaining Wall Thermal Impedance

The example calculation in Appendix I shows that considerable computation is involved in obtaining the exact impedance for a multilayer wall. The effort required to obtain this accurate value of the impedance is often not justifiable because of the large possible errors in the thermal constants assumed for the wall materials. For such circumstances a vector diagram solution is much easier and quicker to obtain.

The wall is first represented by a lumped resistance-capacitance network and then the temperatures and heat fluxes at the various points are determined. As an illustration, a thin homogeneous wall will be considered. The wall can be represented by a "T" network, and the inside and outside film coefficients by resistors which are in series with the wall network. The input temperature $t_{1,n}$ which will cause a unit output flux is equal to the wall thermal impedance Z_n. The output heat flux $q_{2,n}$ is represented by the unit vector along the X axis in Fig. II. The temperature at the centre of the wall is $\left(\frac{L}{2k} + \frac{1}{h_2}\right)$ and in phase with the output heat flux. The heat flux representing the heat stored in the wall is 90° ahead of the temperature at the centre of the wall and has a magnitude $(2\pi n \,.\, L \,.\, \rho \,.\, c.) \,.\, \left(\frac{L}{2k} + \frac{1}{h_2}\right)$. The heat flux across the outside surface of the wall is the vector sum of the heat output and the heat stored. The input temperature equals the vector sum of the mid wall temperature plus $\left(\frac{1}{h_1} + \frac{L}{2k}\right)$ times the heat flux into the wall. This outside temperature vector represents Z_n.

For a multilayer wall each layer can be represented by a T network and the wall by the combination of T networks in series, with the film resistors at each end of the circuit. The procedure for such a composite wall is just the same as for the simple T described above. At each network nodal point the heat flux representing the heat stored in the lump is 90° ahead of the temperature at that same point. The heat flow toward a nodal point is the vector sum of the heat flowing away and being stored. The temperature difference between nodal points is in phase with the heat flow. The total temperature drop is the vector sum of the differences between each of the nodal points. Figures III, IV, V, and VI are accurate diagrams for the composite wall considered in Appendix I.

Errors in the Vector Diagram Procedure

The impedance obtained by constructing a vector diagram differs from the correct value for two reasons, namely:

(1) Errors due to inexact construction of the diagram
(2) Errors due to representing a continuous wall by a lumped resistance-capacitance network.

The errors can be evaluated separately since the exact value of impedance can be calculated for a continuous wall and for a lumped network.

The matrix for a T network representation of a wall element is

$$\begin{Vmatrix} 1 + i\varphi_n^2 \;, & \frac{L}{K}\left(1 + \frac{i\varphi_n^2}{2}\right) \\ i\,\frac{2\,k\,\varphi_n^2}{L} \;, & 1 + i\varphi_n^2 \end{Vmatrix}$$

Thus, for a T network representation of a homogeneous wall taking account of the surface films,

$$\begin{Vmatrix} t_{1,n} \\ q'_{1,n} \end{Vmatrix} = \begin{Vmatrix} 1 & R_1 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 + i\varphi_n^2, & R_w\left(1 + \frac{i\varphi_n^2}{2}\right) \\ i\,\frac{2\varphi_n^2}{R_w} \;, & 1 + i\varphi_n^2 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & R_2 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} t_{2,n} \\ q'_{2,n} \end{Vmatrix} \qquad \ldots(16)$$

Thus, when $t_{2,n} = 0$

$$\frac{t_{1,n}}{q'_{2,n}} = Z'_n = R_1 + R_w + R_2 + i\varphi_n^2 \left\{ R_1 + R_2 + \frac{R_w}{2} + \frac{2\,R_1 \cdot R_2}{R_w} \right\} \quad (17)$$

When Z_n as given by equation (13) is subtracted from Z'_n the difference is the error due to lumping

$$Z'_n - Z_n = \wp_n^{\,4} \left[\frac{R_1 + R_2}{6} + \frac{R_W}{30} + \frac{2}{3}\,\frac{R_1 \cdot R_2}{R_W} \right] + i\,\wp_n^{\,2}\,\frac{R_W}{6} \qquad \ldots(18)$$

For a 4-in. concrete wall

$R_W = 0.320$ $\qquad \wp_1^{\,2} = 0.3975$

$R_1 = 0.167$

$R_2 = 0.606$

$Z'_1 = 1.093 + i\,(0.622) = 1.258 \;\underline{|29.6°}$

$Z'_1 - Z_1 = 0.056 + i\,(0.021)$

so that $Z_1 = 1.037 + i\,(0.601) = 1.199 \;\underline{|30.1°}$

Thus the lumping error is 5 per cent on $|Z_1|$ and - 2 per cent on the argument.

The vector diagram, Figure II, gave $Z'_1 = 1.260 \;\underline{|29.7°}$, thus the error due to construction of the diagram is less than 0.5 per cent.

Summary

The matrix method will give an accurate value of wall thermal impedance for a solid wall of any number of layers. The only assumptions necessary are: (1) sol-air temperature and inside air temperature are periodic; (2) the inside and outside air film coefficients of heat transfer are constant with respect to time; (3) the heat flow through the wall is one dimensional. Once the wall matrix has been determined the wall thermal impedance can be <u>easily</u> calculated for any values of the inside and outside air film coefficients of heat transfer. This is the chief advantage of this method compared to that of Mackey and Wright. Considerable computation is required to find the matrix for a composite wall; thus it would be desirable to have tabulated in the ASHAE Guide the matrix elements for the more common wall types.

The vector method is much simpler than the matrix method for multilayer walls if the matrix elements have to be calculated. The error in the vector diagram method is mostly due to representing a continuous wall by a lumped resistance-capacitance network. This error can be reduced by using more lumps. In the example considered in Appendix I there was an error of 10 per cent when each layer of a wall was represented by a T network, and of about 1.5 per cent when each layer was represented by a π network.

In addition to giving the over-all wall thermal impedance, the vector diagram method gives values of heat flux and temperature at several points through the wall.

References

1. Mackey, C.O. and L.T. Wright Jr., Periodic heat flow - homogeneous walls or roofs, Trans. A.S.H.V.E., Vol. 50, 1944, p. 293.

2. Mackey, C.O. and L.T. Wright Jr., Periodic heat flow - composite walls or roofs, Trans. A.S.H.V.E., Vol. 52, 1946, p. 283.

3. Van Gorcum, A.H., Theoretical considerations on the conduction of fluctuating heat flow, Applied Scientific Research, Vol. A2, 1951, p. 272.

TABLE I

Values of the Special Functions Needed for Calculating Wall Matrix Elements

φ_n	$\cosh (1+i)\varphi_n$		$\frac{\sinh (1+i)\varphi_n}{(1+i)\varphi_n}$		$(1+i)\varphi_n \sinh (1+i)\varphi_n$	
	Real	Imaginary	Real	Imaginary	Real	Imaginary
0.00	1.0000	0.0000	1.0000	0.0000	0.0000	0.0000
0.10	1.0000	0.0100	1.0000	0.0033	-0.0007	0.0200
0.20	0.9997	0.0400	0.9999	0.0133	-0.0011	0.0800
0.30	0.9986	0.0900	0.9997	0.0300	-0.0054	0.1799
0.40	0.9958	0.1600	0.9991	0.0533	-0.0170	0.3197
0.50	0.9896	0.2498	0.9979	0.0833	-0.0416	0.4990
0.60	0.9784	0.3595	0.9957	0.1200	-0.0864	0.7169
0.70	0.9600	0.4887	0.9920	0.1632	-0.1599	0.9722
0.80	0.9318	0.6371	0.9863	0.2129	-0.2725	1.2625
0.90	0.8908	0.8041	0.9782	0.2692	-0.4361	1.5846
1.00	0.8337	0.9889	0.9667	0.3318	-0.6635	1.9334

TABLE 2

Comparison of Vector Diagram and Analytical Results

4 in. concrete wall with 2 in. insulation	Insulation Inside		Insulation Outside	
	Mod Z_1	Arg Z_1	Mod Z_1	Arg Z_1
"T" Network for each layer	9.70	58°	15.1	79°
"π" Network for each layer	8.90	57°	14.0	79.4°
Accurate values	8.80	56.4°	13.8	79.2°

APPENDIX I

Problem: Find the thermal impedance of a 4-in. concrete wall with 2-in. insulation

(a) on the inside of the concrete
(b) on the outside of the concrete

Given: $R_1 = 0.167$ $\quad ft^2.hr.F°/B.t.u.$

$R_2 = 0.606$ "

$R_{concrete} = 0.320$ "

$R_{insul.} = 6.000$ "

$\varphi\, l^2$ concrete $= 0.3975$

$\varphi\, l^2$ insul. $= 0.6670$

Solution: The matrix for the concrete is

$$\begin{Vmatrix} A_c & B_c \\ C_c & A_c \end{Vmatrix} = \begin{Vmatrix} 0.974 + i\ (0.398), & 0.318 + i\ (0.0425) \\ -0.329 + i\ (2.485), & 0.974 + i\ (0.398) \end{Vmatrix}$$

and the matrix for the insulation

$$\begin{Vmatrix} A_i & B_i \\ C_i & A_i \end{Vmatrix} = \begin{Vmatrix} 0.926 + i\ (0.667), & 5.910 + i\ (1.333) \\ -0.0495 + i\ (0.222), & 0.926 + i\ (0.667) \end{Vmatrix}$$

The elements of the composite wall matrix for the case of insulation inside are

$$A_1 = A_c\ A_i + B_c\ C_i = 0.611 + i\ (1.087)$$

$$B_1 = A_c\ B_i + B_c\ A_i = 5.492 + i\ (3.902)$$

$$C_1 = C_c\ A_i + A_c\ C_i = 2.099 + i\ (2.278)$$

$$D_1 = C_c\ B_i + A_c\ A_i = -4.620 + i\ (15.266)$$

Check $A_1\ D_1 - B_1\ C_1 = 1.000 - i\ (.015)$, since this is a passive network the determinant of the matrix should be unity.

Thus

$$
\begin{aligned}
Z_1 &= .606\ A_1 & 0.370 + i\ (0.659)\\
&+ \quad B_1 & 5.492 + i\ (3.902)\\
&+ .101\ C_1 & -0.212 + i\ (0.230)\\
&+ .167\ D_1 & -0.772 + i\ (2.549)\\
&= & 4.878 + i\ (7.340)
\end{aligned}
$$

$$\text{Mod } Z_1 = 8.80$$

$$\text{Arg } Z_1 = 56.4°$$

For the case with insulation on the outside of the concrete the wall matrix is

$$\begin{Vmatrix} A_1^* & B_1^* \\ C_1^* & D_1^* \end{Vmatrix}$$

$$
\begin{aligned}
A_1^* &= A_i\ A_c + B_i\ C_c = D_1\\
B_1^* &= A_i\ B_c + B_i\ A_c = B_1\\
C_1^* &= C_i\ A_c + A_i\ C_c = C_1\\
D_1^* &= C_i\ B_c + A_i\ A_c = A_1
\end{aligned}
$$

Thus

$$
\begin{aligned}
Z_1^* &= 0.606\ D_1 & -2.800 + i\ (9.251)\\
&+ \quad B_1 & 5.492 + i\ (3.902)\\
&+ .101\ C_1 & -0.212 + i\ (0.230)\\
&+ .167\ A_1 & +0.102 + i\ (0.182)\\
&= & 2.582 + i\ (13.565)
\end{aligned}
$$

$$\text{Mod } Z_1^* = 13.8$$

$$\text{Arg } Z_1^* = 79.2°$$

This same problem is solved by the vector diagram procedure in Figures III, IV, V and VI and the results compared with those calculated above in Table 2.

Periodic Heat Flow in
Walls and Roofs

by

D.G. Stephenson

Figure Captions

Fig. I T network representation of a homogeneous wall with inside and outside film resistances.

Fig. II Vector diagram for a T network representation of a homogeneous wall with inside and outside film resistances.

Fig. III Vector diagram for a two-lump representation of a 4-in. concrete wall with 2 in. insulation on inside of concrete.

Fig. IV Vector diagram for a two-lump representation of a 4-in. concrete wall with 2 in. insulation on outside of concrete.

Fig. V Vector diagram for a four-lump representation of a 4-in. concrete wall with 2 in. insulation on inside of concrete.

Fig. VI Vector diagram for a four-lump representation of a 4-in. concrete wall with 2 in. insulation on outside of concrete.

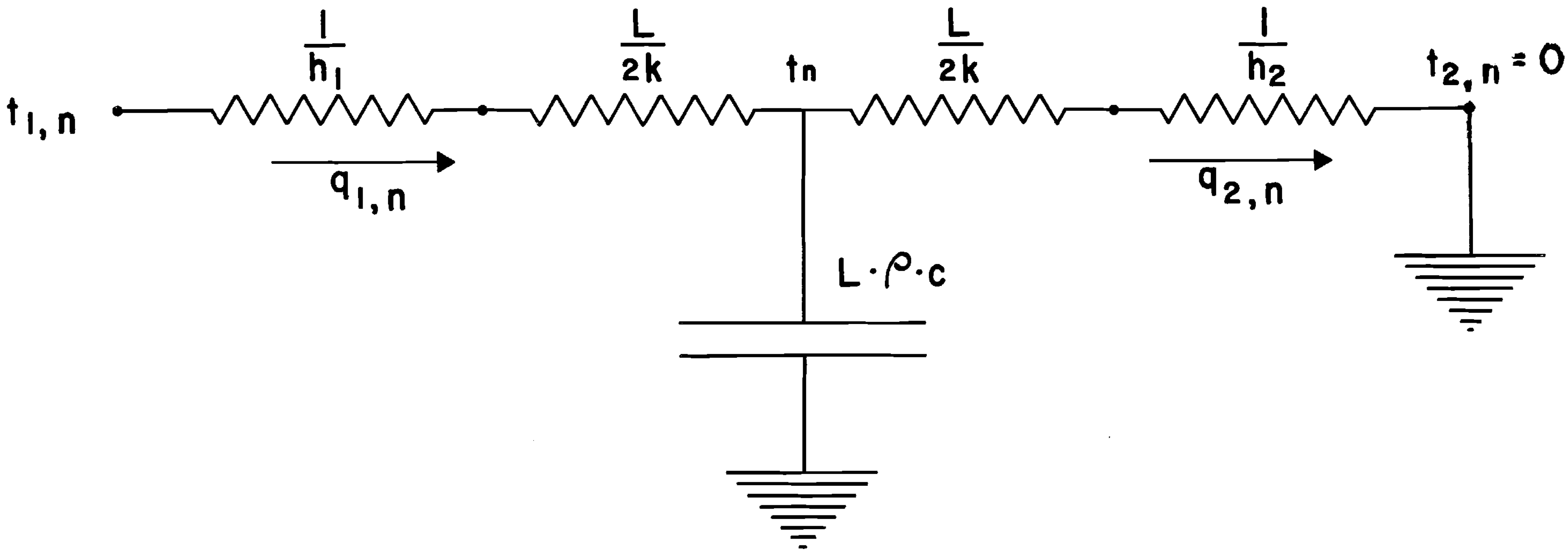

FIGURE I

(DBR REPORT 132)

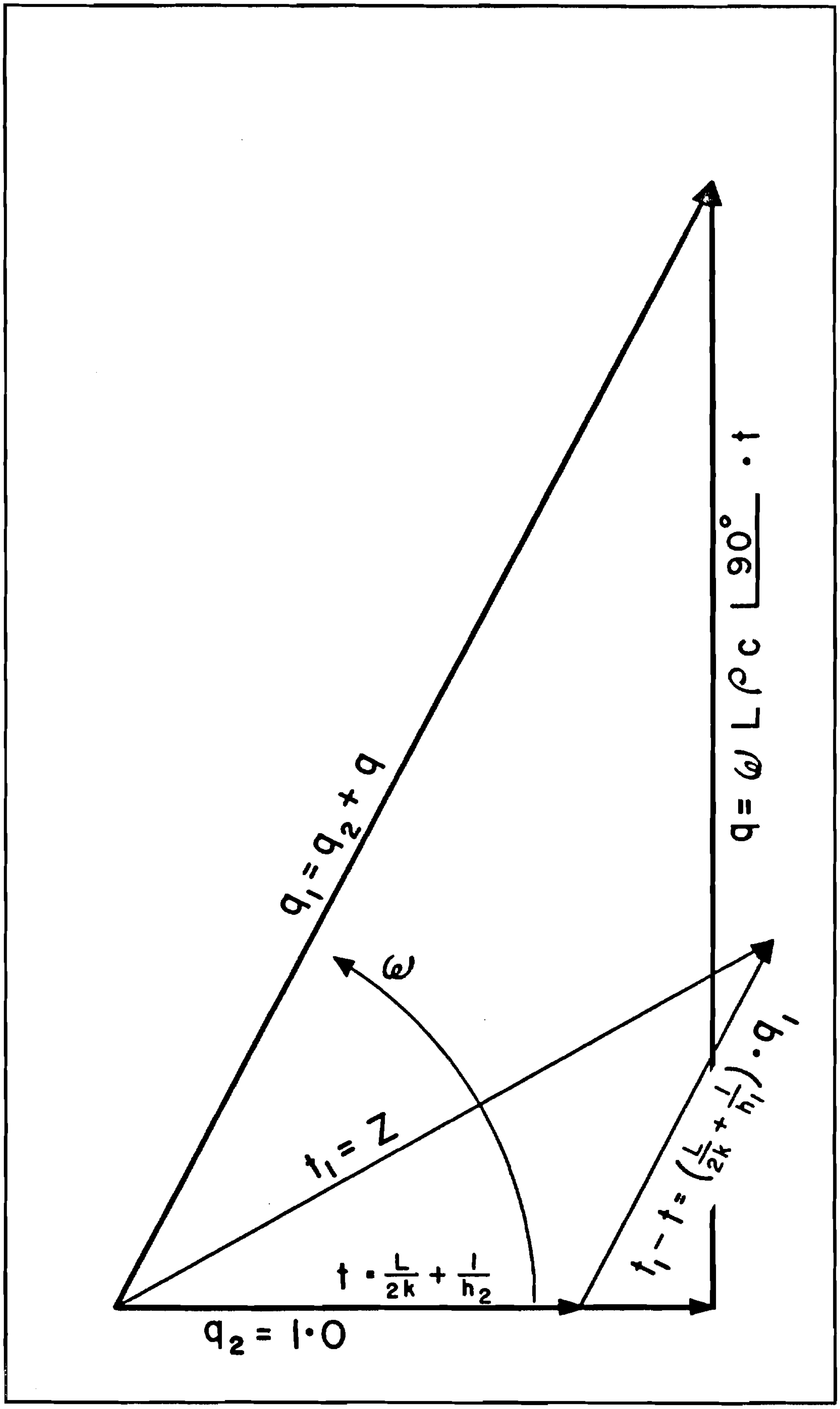

FIGURE 2

(DBR REPORT 132)

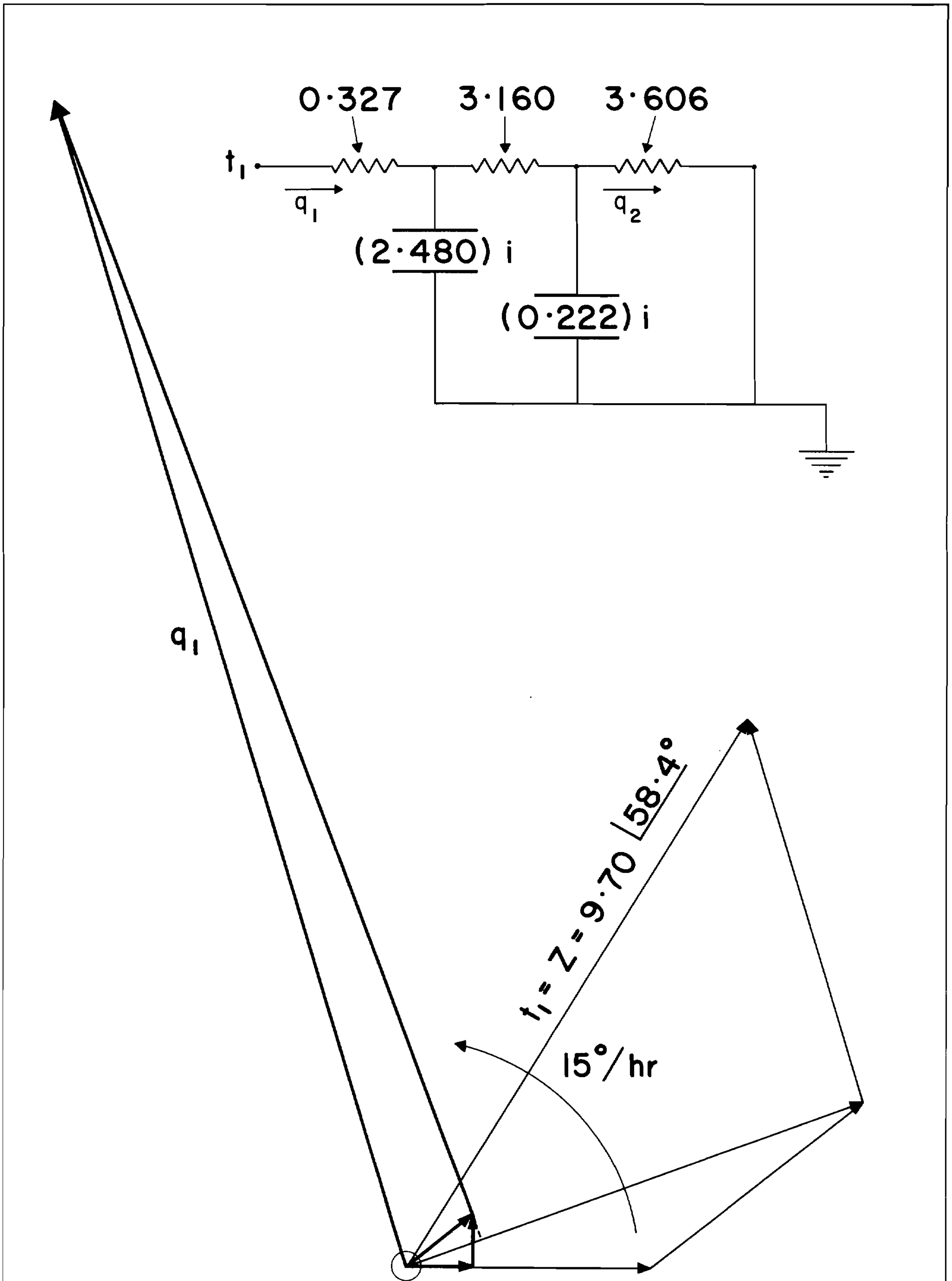

FIGURE 3

(DBR REPORT 132)

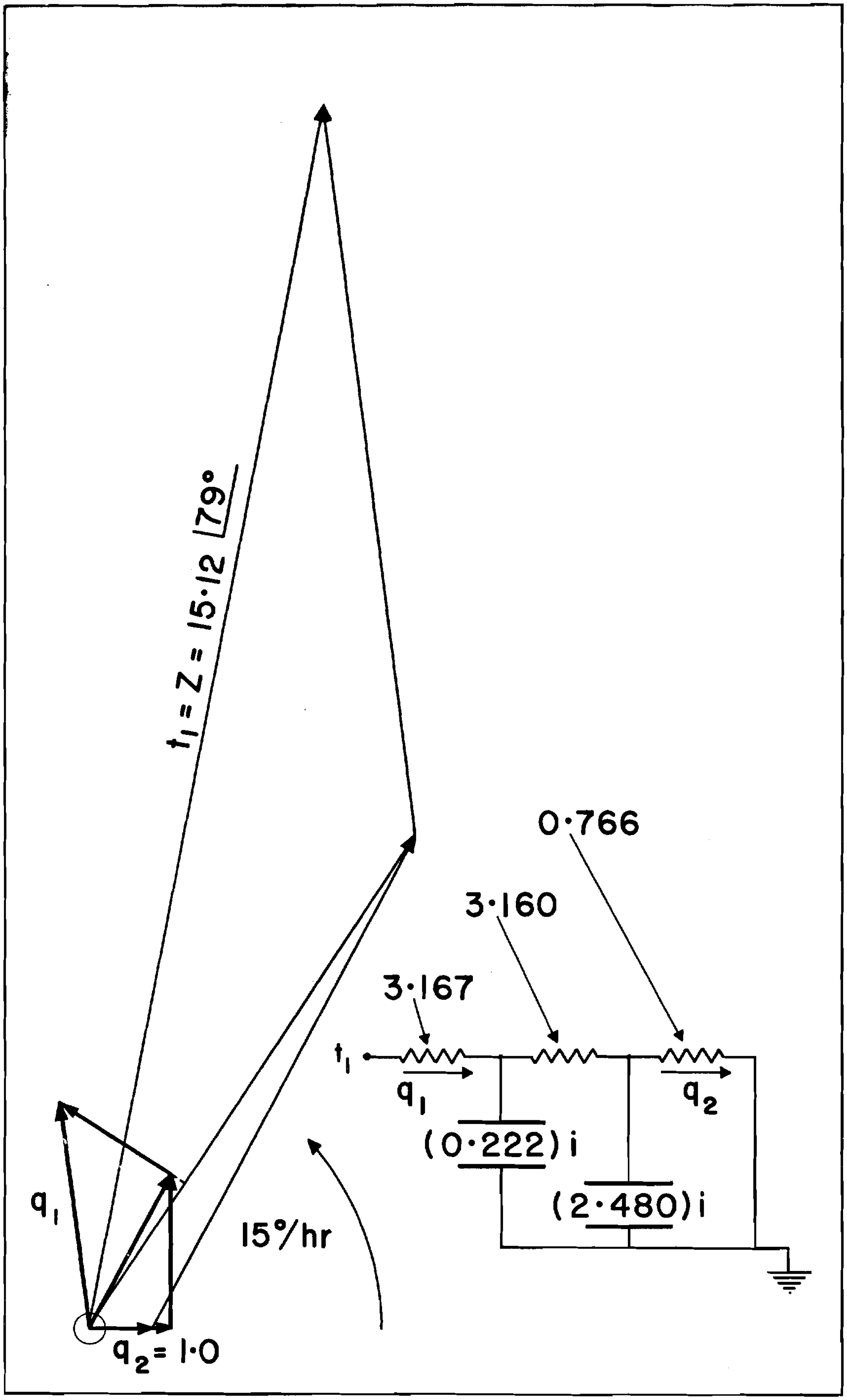

FIGURE 4

(DBR REPORT 132)

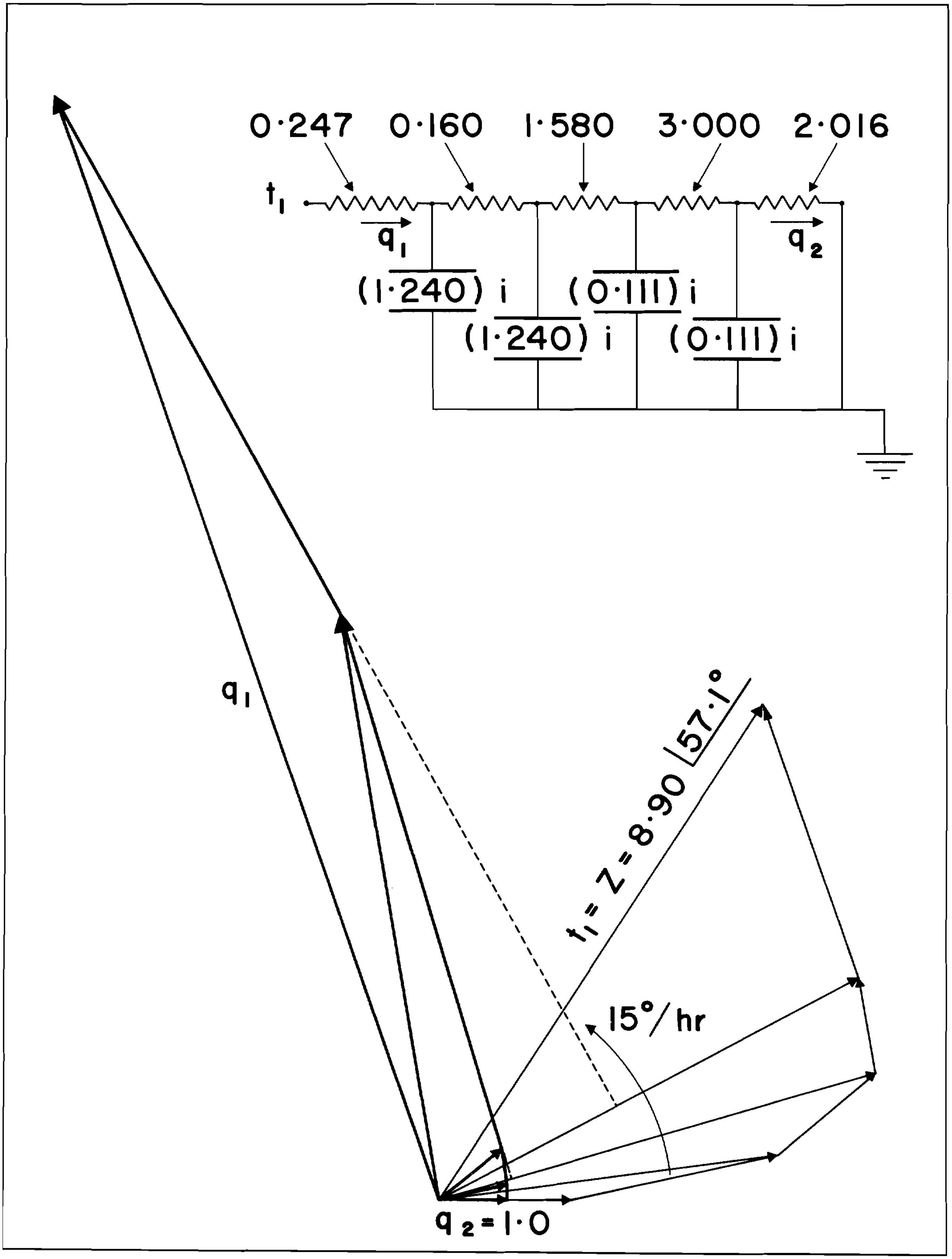
0·247
0·160
1·580
3·000
2·016
t₁
q₁
q₂
(1·240) i
(0·111) i
(1·240) i
(0·111) i
q₁
t₁ = Z = 8·90 |57·1°
15°/hr
q₂ = 1·0

FIGURE 5

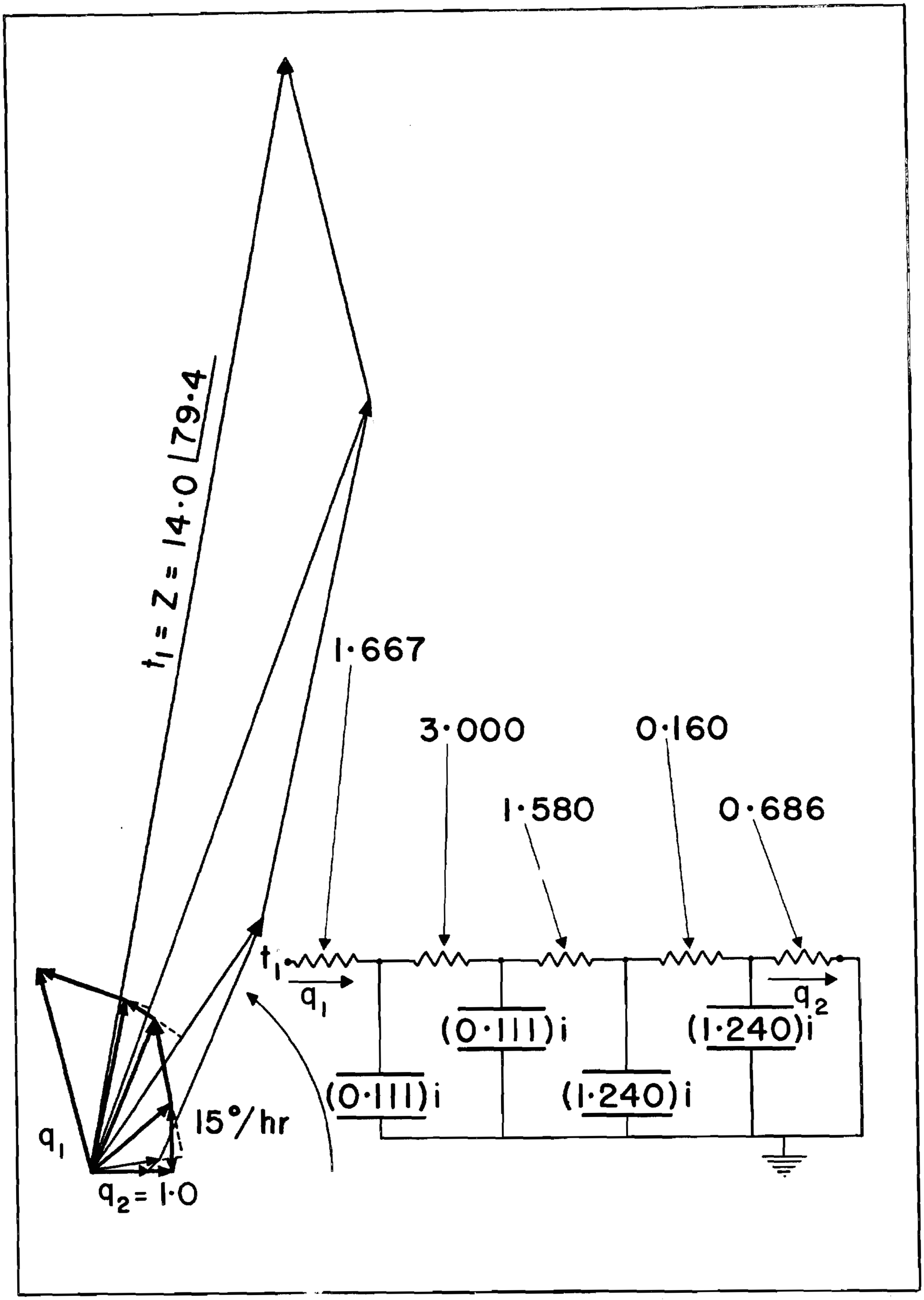

FIGURE 6

(DBR REPORT 132)