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THE CALCULATION OF SURFACE TEMPERATURE AND HEAT FLUX FROM SUBSURFACE TEMPERATURE MEASUREMENTS

BY

D.G. STEPHENSON AND G.P. MITALAS

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THE CALCULATION OF SURFACE TEMPERATURE AND HEAT FLUX FROM SUBSURFACE TEMPERATURE MEASUREMENTS

by

D. G. Stephenson and G. P. Mitalas*

SUMMARY

This paper presents an analogue and a numerical method of computing surface temperature and heat flux from subsurface temperature data. The basic equations used by both methods are:

$$T_{surface} \doteq 2T_{\Delta x} - T_{2\Delta x} + 2\left[\frac{\tau \dot{T}_{\Delta x}}{2!} + \frac{\tau^2 \ddot{T}_{\Delta x}}{4!} + \frac{\tau^3 \dddot{T}_{\Delta x}}{6!} + \ldots\right]$$

$$q_{surface} = \frac{k}{\Delta x}\left[\frac{3T_{surface} - 4T_{\Delta x} + T_{2\Delta x}}{2}\right]$$

where $\tau = \frac{(\Delta x)^2}{\alpha}$

k, α are thermal conductivity and diffusivity respectively. These calculations differ from the usual finite difference heat conduction calculations in that they require the data to be differentiated rather than integrated. Methods are presented for differentiating data that have been recorded in either analogue or numerical form. The advantages of each method of recording the data are discussed. The errors inherent in the calculation of surface conditions are discussed and illustrated by examples.

The direct measurement of the temperature of a surface is difficult in many cases for one or both of the following reasons:

1. A temperature sensing element attached to the surface may affect the heat transfer processes and consequently change the surface temperature.

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2. There may be some activity at the surface that would quickly damage any thermometer.

For example, a thermocouple attached to the outside surface of a wall may have a different absorptivity for solar radiation from that of the wall surface; it will reach a temperature different from that of the rest of the wall surface when solar radiation is a significant term in the energy balance at the surface. A sensing element on a surface can also alter the convection and long wave radiation at the surface. The direct measurement of the surface temperature of an asphalt pavement or a cultivated field would be difficult for the second reason. Surface pyrometers avoid these difficulties but they are unable to discriminate between emitted and reflected radiation.

Approximate numerical methods have been developed recently for calculating the surface conditions from measured values at points below the surface. Stolz (1) calls this the solution of an inverse problem of heat conduction. Shumakov (2) has studied the same problem and reported on it in the Russian literature. The present paper presents another numerical method that has some advantages over those previously reported, and an electronic analogue computer circuit that is particularly useful when surface conditions change rapidly.

NUMERICAL METHODS OF CALCULATING SURFACE TEMPERATURE

The methods of Stolz and Shumakov are essentially the same: they use the known relationship between a unit pulse variation of the surface temperature and the resulting change in the temperature at the depths where the temperatures are measured. The following derivation of a formula for T_o uses this approach, but is in the same terminology as the rest of this paper rather than in that used by the original authors.

Figure 1 shows the temperature response at the centre of a slab, of thickness $2\Delta x$ that results from the square pulse variation of the temperature at the surfaces. The temperature T_o can be computed from the measured temperatures at depths Δx and $2\Delta x$ by the formula.

$$T_{\Delta x,o} = r_o \left(T_{o,o} + T_{2\Delta x,o} \right) + r_1 \left(T_{o,-\delta t} + T_{2\Delta x,-\delta t} \right)$$

$$+ r_2 \left(T_{o,-2\delta t} + T_{2\Delta x,-2\delta t} \right) + \cdots$$

where the factors r_o, r_1, r_2, etc. are the ordinates of the response curve shown in Fig. 1.

By rearranging, this becomes

$$T_{o,o} + T_{2\Delta x,o} = \frac{1}{r_o}\left[T_{\Delta x,o} - r_1\left(T_{o,-\delta t} + T_{2\Delta x,-\delta t}\right) - r_2\left(T_{o,-2\delta t} + T_{2\Delta x,-2\delta t}\right) \dots\right]$$

Similarly

$$T_{o,-\delta t} + T_{2\Delta x,-\delta t} = \frac{1}{r_o}\left[T_{\Delta x,-\delta t} - r_1\left(T_{o,-2\delta t} + T_{2\Delta x,-2\delta t}\right) - \dots\right]$$

etc.

Thus

$$T_{o,o} = -T_{2\Delta x,o} + \frac{1}{r_o}T_{\Delta x,o} - \frac{r_1}{r_o^2}T_{\Delta x,-\delta t} + \frac{r_1^2 - r_2 r_o}{r_o^3}T_{\Delta x,-2\delta t} \dots$$

This expression for the surface temperature uses the data at $t \leq 0$ but does not use the values at $t > 0$ that also have a strong correlation with $T_{o,o}$.

It is shown in Appendix A that the surface temperature is related to the subsurface temperatures $T_{\Delta x}$ and $T_{2\Delta x}$ by the following exact equation:

$$T_o = -T_{2\Delta x} + 2\left[T_{\Delta x} + \frac{\tau}{2!}\dot{T}_{\Delta x} + \frac{\tau^2}{4!}\ddot{T}_{\Delta x} + \frac{\tau^3}{6!}\dddot{T}_{\Delta x} + \dots\right] \quad (A)$$

where $\tau = \frac{(\Delta x)^2}{\alpha}$,

$$\dot{T}_{\Delta x} = \frac{d}{dt}(T_{\Delta x}) \quad \text{etc.}$$

and α = thermal diffusivity.

The method of Stolz and Shumakov is equivalent to using backward difference approximations for the time derivatives of $T_{\Delta x}$ in expression A.

The following expression for T_o is based on central difference approximations for the time derivatives of $T_{\Delta x}$ and consequently is more accurate than the Stolz or Shumakov formulae.

$$T_{o,o} = a\, T_{\Delta x,-3\delta t} + b\, T_{\Delta x,-2\delta t} + c\, T_{\Delta x,-\delta t} + d\, T_{\Delta x,o} +$$

$$e\, T_{\Delta x,\delta t} + f\, T_{\Delta x,2\delta t} + g\, T_{\Delta x,3\delta t} + h\, T_{2\Delta x,o}$$

The coefficients a ... g are functions of the dimensionless number

$$M = \frac{\tau}{\delta t} .$$

The exact relationships are derived in Appendix B; and h = - 1.000. This equation takes account of all the differences of $T_{\Delta x}$ up to the sixth, i.e. it approximates the first six derivatives of $T_{\Delta x}$ and neglects all the higher differences. The first neglected term is

$$2\, \Delta_o^{v''} \left[- \frac{M}{140(2!)} + \frac{7\, M^3}{120(6!)} - \frac{M^5}{3(10!)} + \frac{M^7}{1(14!)} \right]$$

where $\Delta_o^{v''}$ is the central seventh difference of $T_{\Delta x}$ at time 0.

The error in the computed value of T_o is caused partly by neglecting the higher difference terms for $T_{\Delta x}$, and partly by errors in the measured values of $T_{\Delta x}$ and $T_{2\Delta x}$. The maximum possible error caused by errors in the data is the possible error in any measured value times the sum of the absolute value of the coefficients a ...h. It is desirable, therefore, to choose Δx and δt so that the coefficients are as small as possible, i.e. small values of Δx and large values of δt. However, as δt increases so do the difference terms and consequently the error associated with the neglected high order differences increases. Thus the optimum δt from this point of view is a compromise that makes the total error a minimum.

The choice of δt must also be based on the frequency spectrum of the data. It is well known in communications work that a sampled signal has an upper frequency limit of 1/2 δt. Thus, δt must be less than half the period of the highest frequency component that has to be considered.

If there is high frequency noise present in the data signal, the data can be improved by averaging. It is shown in Appendix C that an averaging (i.e. integration) process acts as a low pass filter with the factor

$$F = \frac{\sin \Upsilon}{\Upsilon}$$

where $\Upsilon = \dfrac{\pi \text{ (Averaging interval)}}{\text{Cycle Period}}$

Figure 2 shows log F as ordinate and log (Υ/π) as abscissa. For small values of Υ the filter factor is essentially one, but when Υ equals π or any whole number times π the filter factor is zero. The dotted line is the upper limit of the filter factor in the higher frequency region.

The following example demonstrates the magnitude of the errors that occur when $M \approx 2$ and $w_{max}\delta t \approx 1$ radian. If the temperature at the surface of a semi-infinite slab is

$$T_o = \sin(\omega t) + \sin(3\omega t) + \sin(5\omega t)$$

the temperature at depth Δx is shown in Ref. 3 to be

$$\begin{aligned} T_{\Delta x} = {} & e^{-\phi} \sin(\omega t - \phi) + \\ & e^{-3\phi} \sin 3(\omega t - \phi) + e^{-4\phi} \sin 4(\omega t - \phi) + \\ & e^{-5\phi} \sin 5(\omega t - \phi) \end{aligned}$$

Where $\phi = \left(\dfrac{w (\Delta X)^2}{2\alpha} \right)^{\frac{1}{2}}$

The temperature at depth $2\Delta x$ is

$$\begin{aligned} T_{2\Delta x} = {} & e^{-2\phi} \sin(\omega t - 2\phi) + e^{-6\phi} \sin 3(\omega t - 2\phi) + \\ & e^{-8\phi} \sin 4(\omega t - 2\phi) + e^{-10\phi} \sin 5(\omega t - 2\phi) \end{aligned}$$

Values of $T_{\Delta x}$ and $T_{2\Delta x}$ were calculated according to these formulae for $\phi = 1.0000$. This corresponds to the situation for ground temperatures measured at depths of 6 inches and 1 foot where the fundamental frequency is 1 cycle per day; or for temperatures at depths of 10 feet and 20 feet for a fundamental frequency of 1 cycle per year. These values were used to compute T_o by equation A and the resulting values are compared in Table I with the exact values obtained by evaluating the initial sine series.

Table I contains a sample of the results of the computations. The second column of values for $T_{\Delta x}$ were obtained by averaging 11 values of $T_{\Delta x}$ that had been rounded to three decimal places. The individual values were at 0.1 deg intervals over the range of ωt - 0.5 deg. to ωt + 0.5 deg. The differences between the averaged values and the exact* values show that the averaging process usually gives a result with an error of the order of one unit in the fourth decimal place. The value for ωt = 106.2 deg shows, however, that it is possible for the average to have an error of the same magnitude as the original data. In this particular case the fourth digit to the right of the decimal in each of the 11 accurate values of $T_{\Delta x}$ was a 4 - a very unusual coincidence that accounts for the unusually large error. A comparison of the exact values of T_o with those computed from the exact values of $T_{\Delta x}$ and $T_{2\Delta x}$ shows that the errors are of the order of 10 units in the third decimal place. These errors are due to the neglected high order differences. The values of T_o based on the averages of rounded values of $T_{\Delta x}$ have, in general, larger errors than in the previous case. This extra error is due to the errors in the data. Finally, the values of T_o based on individual values of $T_{\Delta x}$ and $T_{2\Delta x}$ rounded to three decimal places have the greatest error of all. The differences between the T_o values in the last two columns show the effects of reduced round-off error resulting from averaging the raw data before computing the surface temperature. The effect is small. The principal advantage of averaging is that it reduces the effects of high frequency noise. T_o was computed for 24 values of ωt at 15 deg intervals using the averaged 3-place data. These results were analysed for the harmonic components and gave the following Fourier series:

$$T_o = 0.99987 \,\mathrm{Sin}(\omega t + 0.007^\circ) + 0.99974 \,\mathrm{Sin}\,(3\omega t + 0.011^\circ) + 1.0007 \,\mathrm{Sin}(4\omega t + 0.101^\circ) + 1.0017 \,\mathrm{Sin}(5\omega t + 0.402^\circ)$$

These terms are all in excellent agreement with the initially assumed sine series for T_o.

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* 'exact' means accurate to 5 decimal places

AN ANALOGUE METHOD OF COMPUTING SURFACE TEMPERATURE

An analogue computer can be used to compute T_o if the temperature at depths $\triangle x$ and $2\triangle x$ are available as continuous electrical signals. A computer could be operated at real time with the data supplied directly from the measuring equipment. In some cases it may be more practical, however, to record $T_{\triangle x}$ $T_{2\triangle x}$ and subsequently play them back as the input to a computer. If the recorder has a range of record and play back speeds the time scale of the computer need not be the same as the time scale of the original experiment.

The basic equation for an analogue computation is the same as for the digital, i.e. equation A. The difficulty in using this equation comes from the differentiation of $T_{\triangle x}$. The circuit shown in Fig. 3 has the transfer function

$$L\left[\frac{e_{out}}{e_{in}}\right] = \frac{-s}{1 + s/G}$$

whereas an ideal differentiator would be simply s. The output of this circuit is a good approximation of the time derivative of the input for low frequencies, but the approximation gets progressively worse as the frequency increases.

The Laplace transformation of equation A is

$$L\left(\frac{T_o + T_{2\triangle x}}{2\,T_{\triangle x}}\right) = 1 + \frac{\tau s}{2!} + \frac{\tau^2 s^2}{4!} + \frac{\tau^3 s^3}{6!} + \ldots = \cosh\,(\tau s)^{\frac{1}{2}}$$

The computer circuit shown in Fig. 4 has a transfer function

$L\left(\dfrac{T_o + T_{2\Delta x}}{2\,T_{\Delta x}}\right)$ that approximates $\cosh\,(\tau s)^{\frac{1}{2}}$. The frequency response diagram for the analogue circuit is shown in Fig. 5. The transfer function is a better approximation for $\cosh\,(\tau s)^{\frac{1}{2}}$ the larger the value of $G\tau$. Unfortunately, the signal to noise ratio is reduced by the factor ω/G at each differentiation so that the choice of G must be a compromise between transfer function accuracy and noise.

Figure 6 shows the contribution to the final output of each differentiator unit for $W\tau = 8$ radians and $G\tau = 25$. The vector OP represents the ideal frequency response, $\cosh\,(i\omega\tau)^{\frac{1}{2}}$; and the vector OQ represents the frequency response of a circuit with an infinite number of approximate differentiators, i.e. $\cosh \left(\frac{i\ \omega\tau}{1 + i\omega/G} \right)^{\frac{1}{2}}$.

This shows that the fifth and all higher derivatives are negligible at this frequency. The vector OQ is a poor approximation for OP but it is possible to make the frequency response of a four differentiator circuit match the ideal response at any two values of $\omega\tau$.

$$\text{Let} \quad X = i\,\omega\tau$$

$$z = \frac{i\,\omega\tau}{1+i\omega/G}$$

$$\cosh\,(X)^{\frac{1}{2}} = 1 + \ell\frac{z}{2!} + m\frac{z^2}{4!} + n\frac{z^3}{6!} + p\frac{z^4}{8!}$$

This equation is satisfied for the values of $\omega\tau$ and $G\tau$ and the corresponding values of the coefficients ℓ, m, n, p given in Table II. The vector AB is equal to $z/2!$ whereas AB' is $\ell z/2!$; BC is equal to $z^2/4!$ and B'C' is m $z^2/4!$, etc. The primed vectors show the magnitude of the voltages from the various differentiators that makes the circuit response ideal at this frequency. Thus it is possible to obtain a very good approximation of the ideal frequency response over a specific range of $\omega\tau$ by adjusting the magnitude of the voltages from the four differentiator units.

Figure 6 also shows the frequency response of the various adjusted circuits compared with the ideal response curve. This shows that a circuit with $G\tau = 10$ closely matches the ideal curve up to $\omega\tau \approx 5$ radians. The higher values of $G\tau$ enable the circuits to handle data where $\tau\omega_{max}$ is somewhat larger than 5.

The range of $\omega\tau$ that a circuit must accommodate is determined by the position of the thermometers, the thermal diffusivity of the material and the spectrum of the temperature signals. The condition that $\omega\tau \leq 5$ requires that

$$\Delta x_{max} < \left(\frac{5\,\alpha}{\omega_{max}} \right)^{\frac{1}{2}} .$$

In fact, the smaller the value of Δx, the more accurate will be the computed surface temperature. When the data are recorded and played back to the computer at different speeds the value of τ for the computer must be adjusted inversely as the time scale has been changed. The following example illustrates the application of this type of circuit to a wall temperature computation:

Problem -

> It is required to find the surface temperature of a plaster wall (α = 0.04 ft^2/hr) allowing for frequencies up to 100 cycles per hour. What is the maximum depth at which the thermometers can be placed, and what time scale should the computer use?

Solution -

$$\omega = 2\pi \times \text{frequency}$$

$$\omega_{max} = 2\pi \times 100 = 628 \text{ radians/hour}$$

$$\Delta x_{max} < \left(\frac{5\alpha}{w_{max}} \right)^{\frac{1}{2}} = \sqrt{\frac{5 \times 0.04}{628}} = 0.0178 \text{ ft} = 0.214 \text{ in.}$$

$$\tau \omega_{max} = 5$$

$$\tau = \frac{5}{628} = 0.00796 \text{ hour} = 28.66 \text{ sec}$$

$$G\tau_{computer} = 10$$

so that

$$\tau_{computer} \approx 1 \text{ sec}$$

$$G \approx 10 \text{ sec}^{-1}$$

The computer could, therefore, operate at up to 28 times as fast as real time. The tape speed ratios of magnetic tape analogue recorders are commonly in powers of 2, i.e. ratios of 2, 4, 8, 16, 32 are possible. In this case the computer should operate 16 times as fast as real time so that

$$\tau_{computer} = \frac{28.66}{16} = 1.791 \text{ sec}$$

and

$$G = \frac{10}{\tau_{computer}} = \frac{10}{1.791} = 5.58 \text{ sec}^{-1}$$

CALCULATION OF SURFACE HEAT FLUX

The surface flux is given by

$$q_o = -k\left(\frac{\partial T}{\partial x}\right)_o$$

and

$$-\left(\frac{\partial T}{\partial x}\right)_o \doteq \frac{3T_o - 4T_{\Delta x} + T_{2\Delta x}}{2\Delta x}$$

Thus when the surface temperature T_o has been calculated it is very simple to substitute it into the above expression and obtain $q_o \left(\frac{\Delta x}{k}\right)$.

The analogue circuit in Fig. 4 includes a summing amplifier that gives an output proportional to q_o as well as a T_o signal.

For the numerical calculation

$$q_o \frac{\Delta x}{k} = \frac{3a}{2} T_{\Delta x,-3\delta t} + \frac{3b}{2} T_{\Delta x,-2\delta t} + \frac{3c}{2} T_{\Delta x,-\delta t} + \frac{3d-4}{2} T_{\Delta x} +$$

$$\frac{3e}{2} T_{\Delta x, \delta t,} + \frac{3f}{2} T_{\Delta x, 2\delta t} + \frac{3g}{2} T_{\Delta x, 3\delta t} - T_{2\Delta x}$$

where the coefficients a ... g are the functions of $\tau / \delta t$ given in Appendix B. Thus, the same computer program can be used to evaluate

$q_o \frac{\Delta x}{k}$ as for T_o; only the coefficients need to be changed.

PRACTICAL CONSIDERATIONS RELATING TO CHOICE OF METHOD

The choice between the analogue and digital methods of calculating surface conditions depends largely on the data recording apparatus. Analogue recorders generally have a larger frequency band width and larger noise to signal ratios than digital recorders do. Thus the analogue method has to be used whenever the data contain frequencies that cannot be adequately recorded by a digital recorder. The maximum frequency that a digital system can record accurately depends on the interval between readings. The ω_{max} in radians per unit time is approximately equal to the number of measurements that are made in unit time. The digital method is preferable when a digital recorder can cope with the highest frequencies in the data, because the digital recorders introduce very little noise.

CONCLUSION

The temperature and heat flux at the surface of a slab of homogeneous material can be computed from measured values of the temperature at two known depths below the surface and the values of thermal conductivity and diffusivity of the material. The accuracy of the computed surface conditions depends on the depth at which the temperatures are measured; the accuracy is best when the thermometers are close to the surface.

The computation can be made with an analogue computer or by an all-numerical technique that is suitable for programming on a digital computer. It has been shown that the numerical technique is satisfactory for values of $\delta t \approx \tau/2$ and $\omega_{max}\tau \approx 2$ radians. No other values of these parameters were tested. The analogue method has been shown to be accurate from $\omega_{max}\tau$ up to 5 radians for $G\tau = 10$ and the unusable band width can be increased by increasing the value of $G\tau$ and accepting higher noise to signal ratios.

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TABLE I

wt	$T_{\Delta x}$			$T_{2\Delta x}$	T_o			
	Rounded to 5 dec. places	Average of 11 values at 0.1^o increments each rounded to 3 dec. places	Error	Rounded to 5 dec. places	Exact rounded to 5 dec. place	Based on 5-fig. data	Based on average 3-fig. data	Based on single 3-fig. data
1.2°	-0.69927	-0.69910	0.00017	-0.09262				
16.2	- .56030	- .56023	.00007	- .15547				
31.2	- .10525	- .10530	- .00005	- .19366				
46.2	.27294	.27283	.00011	- .17058	0.52225	0.5251	0.5257	0.521
61.2	.30843	.30840	- .00003	- .10304	- .90033	- .9079	- .9072	- .902
76.2	.12986	.12993	.00007	- .04123	- .24176	- .2488	- .2494	- .256
91.2	.03843	.03843	.00000	- .01453	+1.07995	1.0828	1.0789	1.084
106.2	.13441	.13400	- .00041	- .00898	+1.36024	1.3676	1.3703	1.373
121.2	.27276	.27263	- .00013	.00384	+ .82576	.8270	.8263	.827
136.2	.32772	.32763	- .00009	.03164				
151.2	.34618	.34613	- .00005	.06276				
166.2	.40479	.40483	+ .00004	.09024				

Notes: M = 1.9098

a = 0.04702; b =-0.46046; c = 2.45647; d =-9.67903; e = 9.96234; f =-0.33273

g = 0.00640; h = -1.00000.

TABLE II

COEFFICIENTS FOR MODIFIED ANALOGUE CIRCUIT

$G\tau$	ℓ	m	n	p	For ideal response at
10	.9514	2.4034	4.6525	112.25	$\omega\tau = 2.0$ and
25	.9948	1.4849	3.4549	17.474	$\omega\tau = 4.5$
10	0.8493	2.7317	-5.3612	142.67	$\omega\tau = 2.0$ and
25	0.9849	1.4960	2.5563	18.182	$\omega\tau = 8.0$
50	0.9967	1.2417	2.0066	6.5540	
100	0.9989	1.1202	1.5258	3.1738	

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FIGURE CAPTIONS

Figure 1 Response at centre of slab to a square pulse variation of surface temperature

Figure 2 Frequency response of numerical filter that averages data over an interval of 2 γ

Figure 3 Analogue circuit for differentiation

Figure 4 Analogue circuit to compute surface temperature and heat flux

Figure 5 Frequency response of four-differentiator circuit

Figure 6 Frequency response of modified four-differentiator circuits

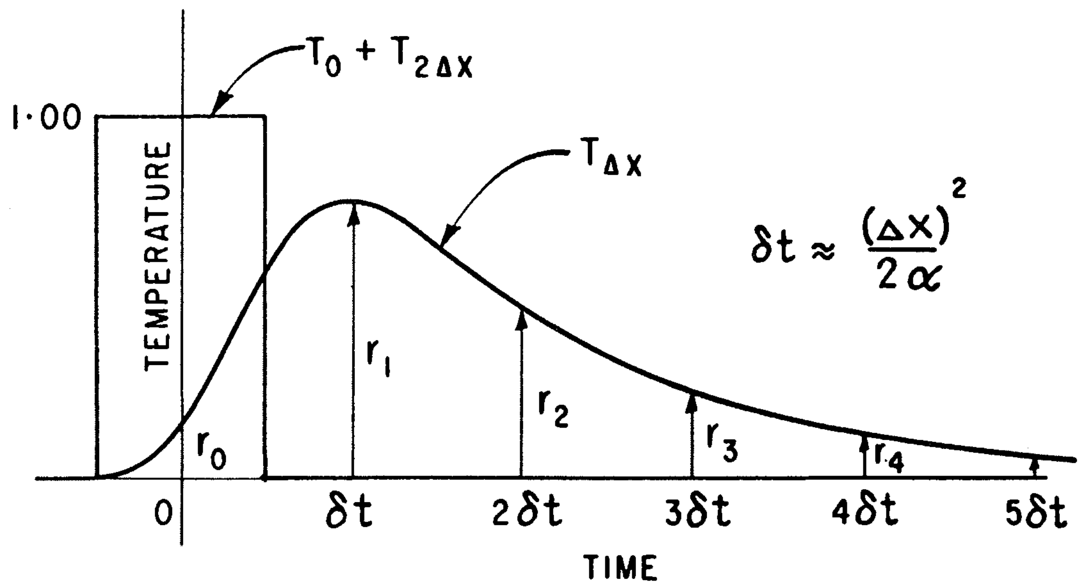


FIGURE I

RESPONSE AT CENTRE OF SLAB TO A SQUARE PULSE VARIATION OF SURFACE TEMPERATURE

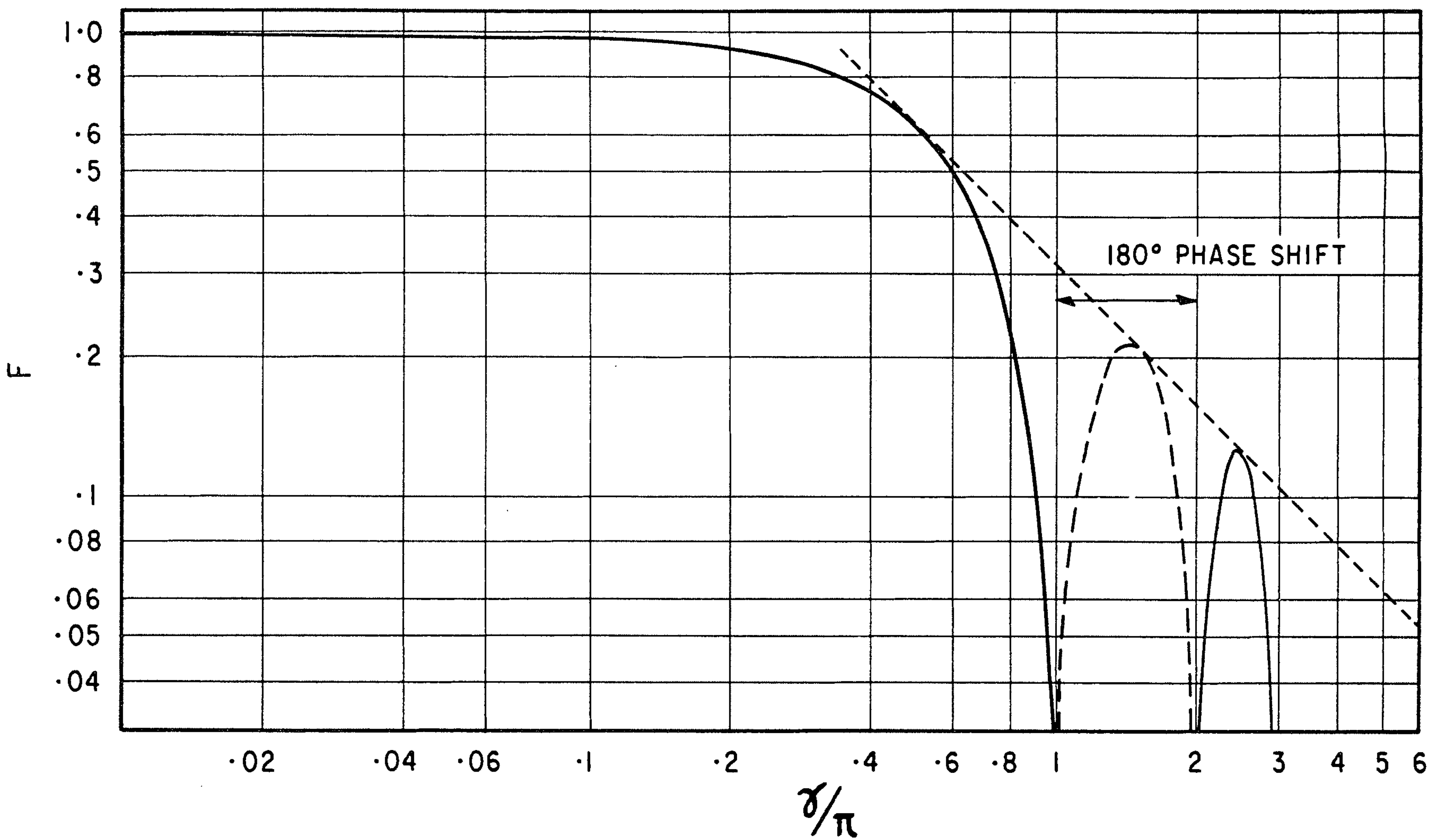
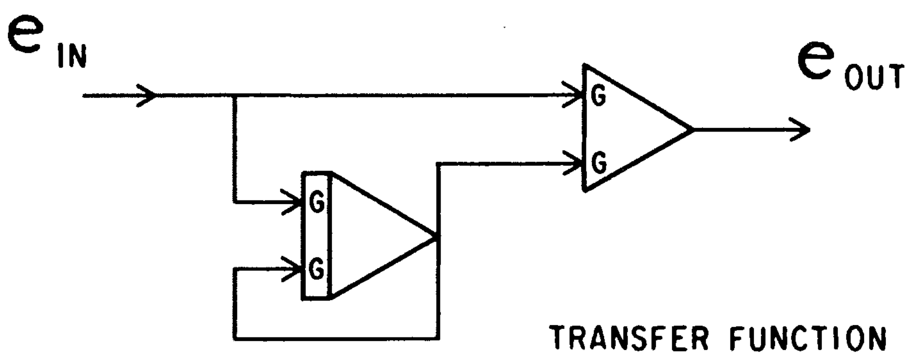


FIGURE 2

FREQUENCY RESPONSE OF NUMERICAL FILTER THAT AVERAGES DATA OVER AN INTERVAL OF 2γ



TRANSFER FUNCTION

$$L\left\{\frac{e_{OUT}}{e_{IN}}\right\} = \frac{-S}{1 + S/G}$$

FIGURE 3

ANALOG CIRCUIT FOR DIFFERENTIATION

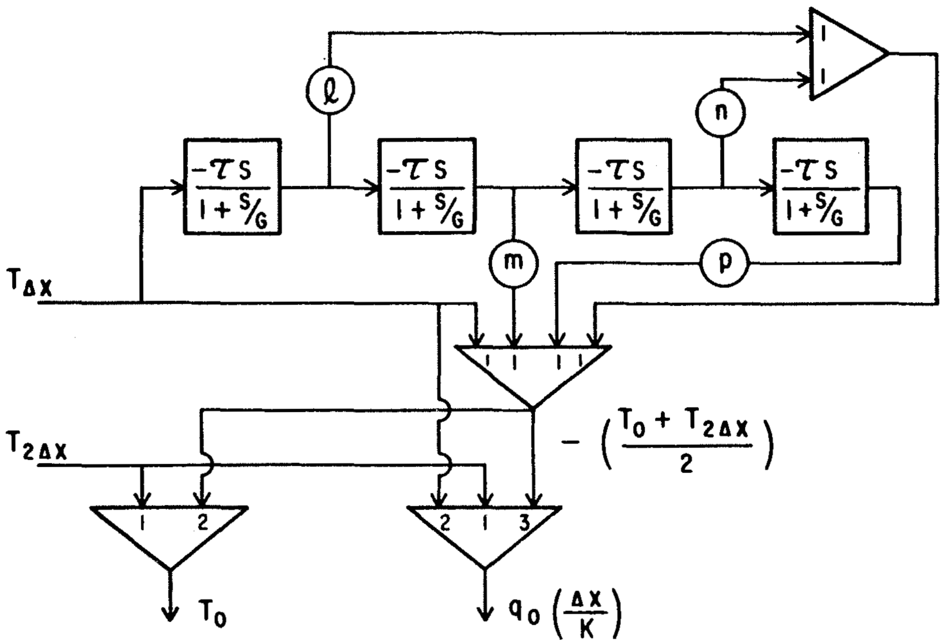


FIGURE 4

ANALOG CIRCUIT TO COMPUTE SURFACE TEMPERATURE AND HEAT FLUX

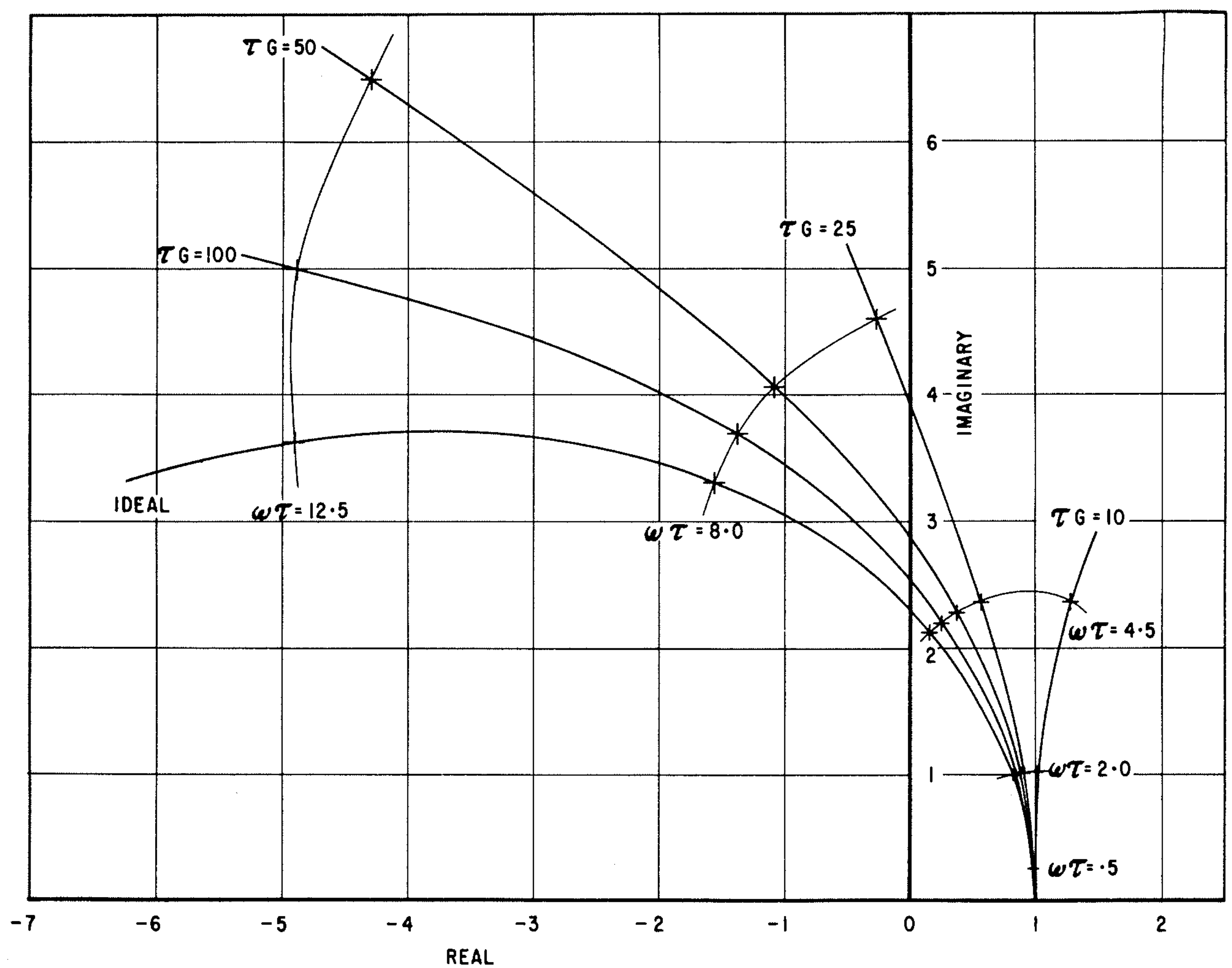


FIGURE 5 FREQUENCY RESPONSE OF FOUR DIFFERENTIATOR CIRCUIT

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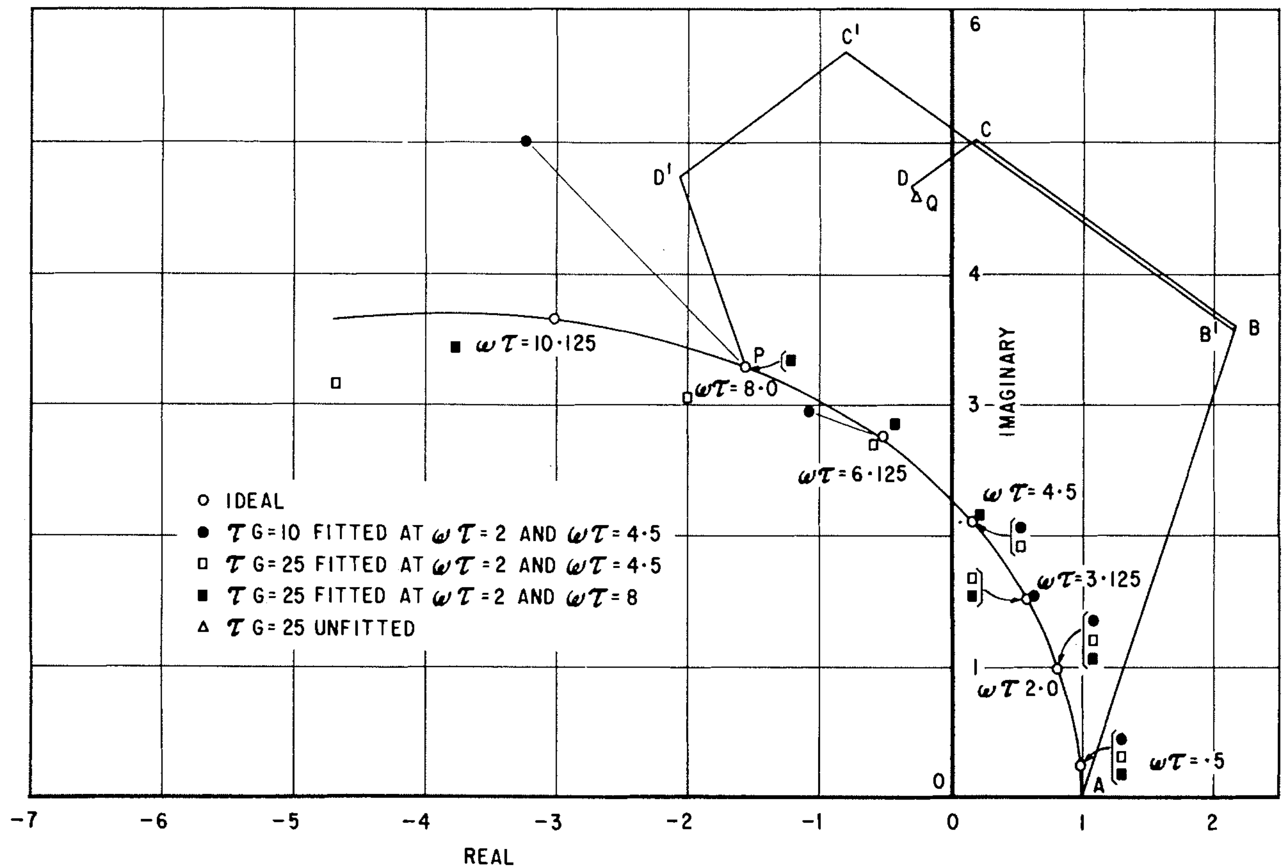

FIGURE 6 FREQUENCY RESPONSE OF MODIFIED FOUR DIFFERENTIATOR CIRCUITS

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APPENDIX A

Relationship between surface temperature and subsurface temperatures

Let the temperature at the surface of a semi-infinite slab be T_O; at a depth Δx be $T_{\Delta x}$ and at a depth $2\Delta x$ be $T_{2\Delta x}$. The temperature everywhere in the material is described by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where $\alpha =$ thermal diffusivity.

The Laplace transform of this equation gives

$$\frac{d^2\theta}{dx^2} - \frac{s}{\alpha}\theta = 0$$

where $\theta = \int_0^\infty e^{-st} \cdot T \, dt$.

The initial conditions have been assumed to be

$$T(x,o) = 0$$

$$\frac{\partial T(x,o)}{\partial t} = 0$$

This simplifies the solution without any loss of generality since the solution of interest in this case is the one that depends on the boundary condition rather than the initial condition.

The solution of this simple second order differential equation is

$$\theta_x = A\,e^{-\sqrt{\frac{sx^2}{\alpha}}} + B\,e^{\sqrt{\frac{sx^2}{\alpha}}}$$

$$\text{Let } \eta^2 = \frac{(\Delta x)^2}{\alpha} s = \tau s$$

$$\theta_o = A + B$$

$$\theta_{\Delta x} = A e^{-\eta} + B e^{\eta}$$

$$\theta_{2\Delta x} = A e^{-2\eta} + B e^{2\eta}$$

Since the heat conduction equation is linear θ_o can be thought of as the sum of two parts: the first depending on $\theta_{2\Delta x}$ for $\theta_{\Delta x} = 0$, and the second depending on $\theta_{\Delta x}$ when $\theta_{2\Delta x} = 0$. For the first case $\theta_{\Delta x} = 0$, so that

$$A = - B e^{2\eta}$$

$$\theta_o = B (1 - e^{2\eta})$$

$$\theta_{2\Delta x} = B (-1 + e^{2\eta})$$

Therefore $\qquad \theta_o = -\theta_{2\Delta x}$

For the second case $\theta_{2\Delta x} = 0$, so that

$$A = - B e^{4\eta}$$

$$\theta_o = B (1 - e^{4\eta})$$

$$\theta_{\Delta x} = B (e^{\eta} - e^{3\eta})$$

Therefore

$$\frac{\theta_o}{\theta_{\Delta x}} = \frac{(1 - e^{2\eta})(1 + e^{2\eta})}{e^{\eta}(1 - e^{2\eta})}$$

$$= e^{\eta} + e^{-\eta}$$

$$= 2 \cosh \eta$$

$$= 2 \left[1 + \frac{\eta^2}{2!} + \frac{\eta^4}{4!} + \frac{\eta^6}{6!} + \ldots\ldots \right]$$

$$= 2 \left[1 + \frac{\tau s}{2!} + \frac{(\tau s)^2}{4!} + \frac{(\tau s)^3}{6!} + \ldots\ldots \right]$$

Thus reverting to the time domain and combining the two parts gives:

$$T_o = 2\,T_{\Delta x} - T_{2\Delta x} + 2\left[\frac{\tau \dot{T}_{\Delta x}}{2!} + \frac{\tau^2 \ddot{T}_{\Delta x}}{4!} + \frac{\tau^3 \dddot{T}_{\Delta x}}{6!} + \ldots\right]$$

APPENDIX B

Finite Difference Expressions for Surface Temperature

The time derivatives of $T_{\Delta x}$ can be approximated by expressions involving the differences between successive values of $T_{\Delta x}$. This allows T_o to be expressed in terms of $T_{2\Delta x}$ and values of $T_{\Delta x}$ measured at several different times. When $T_{\Delta x}$ is measured at regular intervals of δt the various derivatives are given by:

$$\tau \quad \dot{T}_{\Delta x,o} = M\left[\Delta_o' - \frac{1}{6}\Delta_o''' + \frac{1}{30}\Delta_o^{v} - \frac{1}{140}\Delta_o^{v''} + \ldots\right]$$

$$\tau^2 \quad \ddot{T}_{\Delta x,o} = M^2\left[\Delta_o'' - \frac{1}{12}\Delta_o^{'v} + \frac{1}{90}\Delta_o^{v'} - \frac{1}{560}\Delta_o^{v'''} + \ldots\right]$$

$$\tau^3 \quad \dddot{T}_{\Delta x,o} = M^3\left[\Delta_o''' - \frac{1}{4}\Delta_o^{v} + \frac{7}{120}\Delta_o^{v''} \quad \ldots\ldots\right]$$

$$\tau^4 \quad \ddddot{T}_{\Delta x,o} = M^4\left[\Delta_o^{'v} - \frac{1}{6}\Delta_o^{v'} + \frac{7}{240}\Delta_o^{v'''} \quad \ldots\ldots\right]$$

$$\tau^5 \quad T^{\bullet\bullet\bullet\bullet\bullet}_{\Delta x,o} = M^5\left[\Delta_o^{v} - \frac{1}{3}\Delta_o^{v''} + \ldots\right]$$

$$\tau^6 \quad T^{\bullet\bullet\bullet\bullet\bullet\bullet}_{\Delta x,o} = M^6\left[\Delta_o^{v'} - \frac{1}{4}\Delta_o^{v'''} + \ldots\right]$$

where $M = \tau/\delta t$, $\tau = (\Delta x)^2/\alpha$

α = thermal diffusivity

and the differencesare given by:

$$\Delta_o' = \frac{1}{2}\left[T_{\Delta x,\delta t} - T_{\Delta x,-\delta t}\right]$$

$$\Delta_o'' = T_{\Delta x,\delta t} - 2\,T_{\Delta x,o} + T_{\Delta x,-\delta t}$$

$$\Delta_o''' = \frac{1}{2}\left[T_{\Delta x,2\delta t} - 2T_{\Delta x,\delta t} + 2T_{\Delta x,-\delta t} - T_{\Delta x,-2\delta t}\right]$$

$$\Delta_o^{'v} = T_{\Delta x,2\delta t} - 4T_{\Delta x,\delta t} + 6T_{\Delta x,o} - 4T_{\Delta x,-\delta t} + T_{\Delta x,-2\delta t}$$

$$\Delta_o^{v} = \frac{1}{2}\left[T_{\Delta x,3\delta t} - 4T_{\Delta x,2\delta t} + 5T_{\Delta x,\delta t} - 5T_{\Delta x,-\delta t} + 4T_{\Delta x,-2\delta t} - T_{\Delta x,-3\delta t}\right]$$

$$\Delta_o^{v\prime} = T_{\Delta x,3\delta t} - 6T_{\Delta x,2\delta t} + 15T_{\Delta x,\delta t} - 20T_{\Delta x,o} + 15T_{\Delta x,-\delta t} - 6T_{\Delta x,-2\delta t} + T_{\Delta x,-3\delta t} \quad .$$

When the seventh and all higher difference terms are neglected, the expression for the surface temperature derived in Appendix A becomes:

$$T_{o,o} = aT_{\Delta x,-3\delta t} + b\,T_{\Delta x,-2\delta t} + c\,T_{\Delta x,-\delta t} + d\,T_{\Delta x,o} + e\,T_{\Delta x,\delta t} + f\,T_{\Delta x,2\delta t} + g\,T_{\Delta x,3\delta t} - T_{2\Delta x,o}$$

where

$$a = -\frac{M}{15} + \frac{2M^2}{135} + \frac{M^3}{45} - \frac{2M^4}{945} - \frac{4M^5}{14,175} + \frac{8M^6}{467,775}$$

$$b = \frac{9M}{15} - \frac{M^2}{5} - \frac{8M^3}{45} + \frac{8M^4}{315} + \frac{16M^5}{14,175} - \frac{48M^6}{467,775}$$

$$c = -3M + 2M^2 + \frac{13M^3}{45} - \frac{26M^4}{315} - \frac{4M^5}{2835} + \frac{24M^6}{93,555}$$

$$d = 2 - \frac{98M^2}{27} + \frac{112M^4}{945} - \frac{32M^6}{93,555}$$

$$e = 3M + 2M^2 - \frac{13M^3}{45} - \frac{26M^4}{315} - \frac{4M^5}{2835} + \frac{24M^6}{93,555}$$

$$f = -\frac{9M}{15} - \frac{M^2}{5} + \frac{8M^3}{45} + \frac{8M^4}{315} - \frac{16M^5}{14,175} - \frac{48M^6}{467,775}$$

$$g = \frac{M}{15} + \frac{2M^2}{135} - \frac{M^3}{45} - \frac{2M^4}{945} + \frac{4M^5}{14,175} + \frac{8M^6}{467,775}$$

The first neglected term in the expression for the surface temperature is

$$2\Delta_o^{v\prime\prime}\left[-\frac{M}{140\ (2!)} + \frac{7M^3}{120\ (6!)} - \frac{M^5}{3\ (10!)} + \frac{M^7}{1\ (14!)}\right]$$

APPENDIX C

The Filtering Factor for an Averaging Process

To reduce the errors in numerical data due to 'round-off' and high frequency 'noise' it is desirable to smooth or average the data. The average value of a sine wave for angles between $\chi - \gamma$ and $\chi + \gamma$ is

$$Y = \frac{1}{2\gamma}\int_{\chi-\gamma}^{\chi+\gamma} \sin \phi \; d\phi$$

$$= \operatorname{Sin} \chi \cdot \frac{\operatorname{Sin} \gamma}{\gamma}$$

Thus the filter factor

$$F = \frac{Y}{\operatorname{Sin} \chi} = \frac{\operatorname{Sin} \gamma}{\gamma} = 1 - \frac{\gamma^2}{3!} + \frac{\gamma^4}{5!} \cdots\cdots$$

where $$\gamma = \frac{\omega \text{ (Averaging interval)}}{2} .$$