NATIONAL RESEARCH COUNCIL OF CANADA

CONSEIL NATIONAL DE RECHERCHES DU CANADA

EVALUATION OF STEADY-PERIODIC HEAT FLOW METHOD FOR MEASURING THE THERMAL DIFFUSIVITY OF MATERIALS WITH TEMPERATURE-DEPENDENT PROPERTIES

BY

C. J. SHIRTLIFFE AND D. G. STEPHENSON

REPRINTED WITH PERMISSION FROM
PROCEEDINGS, SEVENTH CONFERENCE ON THERMAL CONDUCTIVITY
HELD IN WASHINGTON, D. C., 1967
NATIONAL BUREAU OF STANDARDS SPECIAL PUBLICATION NO. 302
P. 219 - 228

RESEARCH PAPER NO. 397
OF THE
DIVISION OF BUILDING RESEARCH

OTTAWA

APRIL 1969

PRICE 25 CENTS

NRCC 10679

EVALUATION D'UNE METHODE DE MESURE DES ECOULEMENTS DE CHALEUR PERMANENTS ET PERIODIQUES POUR LA DETERMINATION DE LA DIFFUSIVITE DES MATERIAUX DONT LES PROPRIETES DEPENDENT DE LEUR TEMPERATURE

SOMMAIRE

Les auteurs ont calculé les températures à l'intérieur d'un matériau aux propriétés dépendant de sa température par les méthodes des différences finies. Ils ont analysé les résultats des calculs pour y déterminer les composantes harmoniques et se sont servis des valeurs calculées et de la solution de la forme linéaire des équations de conduction de la chaleur pour déterminer la valeur moyenne de la diffusivité. Ils ont calculé les erreurs de détermination des propriétés moyennes et ont dressé une table des distortions produites par les phénomènes de dépendance de la température. Les distortions de la propagation de la chaleur dans les matériaux humides gêne l'utilisation de la méthode. Les auteurs ont trouvé que la composante harmonique principale n'était que faiblement affectée aussi longtemps que les harmoniques impairs étaient seuls présents aux limites. Ils ont établi la précision de la méthode pour la plupart des constitutions de matériaux de construction humides et la gamme de paramètres qui les concernent.

Evaluation of Steady-Periodic Heat Flow Method for Measuring the Thermal Diffusivity of Materials with Temperature-Dependent Properties

C. J. Shirtliffe and D. G. Stephenson

Division of Building Research
National Research Council of Canada
Ottawa

Temperatures in a material with temperature-dependent properties were calculated by finite difference methods. The calculated values were analyzed for harmonic content and used with the solution to the linear form of the heat conduction equations to determine the mean value of thermal diffusivity. Errors in determining the mean properties were calculated and the distortion produced by the temperature dependence was tabulated. Distortion of the temperature waves in the material places limitations on the use of the method for moist materials. The main harmonic content was found to have little distortion so long as only odd harmonics were present in the boundary conditions. Accuracy of the method was established for the configuration and range of parameters anticipated for moist building materials.

Key Words: Dry materials, infinite slabs, moist materials, periodic heat flow, temperature dependence, thermal conductivity, thermal diffusivity.

1. Introduction

Water migrates when a temperature gradient is applied to a moist porous material. This fact has to be borne in mind when designing experiments to measure the thermal properties of moist materials. The movement of moisture contributes to the heat transfer and it makes the apparent thermal properties quite dependent on the temperature and moisture content of the sample. It also causes the moisture distribution through the sample to change during the course of an experiment making it difficult to say what moisture content and temperature correspond to the measured conductivity or diffusivity. This problem is alleviated by using steady periodic boundary conditions because this makes the time average condition of the sample constant.

The Angstrom method has been used, therefore, to measure the thermal diffusivity of moist materials (1, 2),[1] but it has never been established that the method gives true mean values, even for "dry" materials, when the thermal properties are temperature dependent. The objective of this study was to determine the magnitude of the errors that might occur when the Angstrom method was used to measure the thermal diffusivity of materials that contain moisture.

2. Method of Determining Errors

The error inherent in the Angstrom method was checked in the following way: (i) the temperatures at different locations through a sample were calculated for steady periodic boundary conditions assuming that the thermal properties of the sample varied in a particular way with variations of temperature; (ii) the calculated temperatures were used to compute a mean value of thermal diffusivity just as if they had been obtained by experiment; and (iii) the resulting "experimental" value of diffusivity was compared with the "correct" mean value, i. e. the mean of the values used in the

[1] Figures in parentheses indicate the literature references at the end of this paper.

original calculations; the difference was the error inherent in the method of calculating a mean value from experimental results.

The initial calculations were made using the finite difference equations given in Appendix I. This numerical approach was used because no exact analytical solution was known that would permit the simulation of an experimental set-up: quasi-linear approximate methods were tried but there was no sure way of knowing when the calculated results began to have a non-negligible error. The purely numerical results could be made as accurate as required by using finer increments in space and time and the accuracy could be checked by comparing results for successively smaller increments. This study indicated that there is an optimum value of the inverse Fourier number, $1/F_o = (\Delta x)^2/a\Delta t$, at about 6 where "a" is the thermal diffusivity. Values of Δx and Δt were selected, therefore, so as to make this parameter close to six when "a" was at its mean value. This also ensured that the finite difference expressions remained stable since the stability limit is $1/F_o > 2$; in no case did "a" vary by as much as a factor of 3 from its mean value.

The results of the finite difference calculations were first analyzed to determine the mean value and the amplitude and phase of the main harmonic of the temperature at those locations where temperatures would normally be measured. The main harmonic component of each of these temperatures was then used to calculate the experimental value of "a." The formulas used for this computation are given in Appendix II. These equations are based on the premise that the thermal properties of the sample are independent of temperature; it is this simplifying assumption that causes an error in the calculated value of diffusivity.

The errors associated with all the numerical calculations were determined by comparing the "experimental" results with the "correct" values for a case where there was no variation of the properties with temperature. This showed that the error was less than 0.003 per cent.

3. Scope of Study

The initial study was limited to problems subject to the three following conditions:

(i) two identical, two-dimensional slabs
(ii) properties varying linearly with temperature

$$\lambda = \lambda_o (1 + \alpha\theta); \; dc = dc_o (1 + \beta\theta)$$

where d is density, α and β are the temperature coefficients, λ_o and dc_o are the mean values;

(iii) boundary conditions

1. $\theta_1 = E_1 \sin wt$ and $\theta_2 = E_2 \sin (wt + \Delta_o)$

where $w = 2\pi/p$, circular frequency, Δ_o is a phase shift, p is the period

or

2. Square waves; $\pm E_1$ and $\pm E_2$; $\Delta_o = 0$ or 180°.

The range of the variables investigated was:

$a = 2.6 \times 10^{-8}\, m^2/s$ and $2.6 \times 10^{-7}\, m^2/s$ (0.001 and 0.01 ft^2/hr)

E_1 and E_2 = 2.8 to 28 C (5 to 50 F)

$p = 7.2 \times 10^2$ to 3.6×10^4 sec (0.2 to 10 hr)

α and β = -0.036 to +0.036 deg C^{-1} (-0.02 to +0.02 deg F^{-1})

The boundary conditions were selected so as to have no even harmonics. A quasi-linear analysis, to be published later, showed that this type of boundary condition should produce the best results.

4. Results

4.1 Distortion of Wave Form

The temperature dependence of the properties caused a distortion of the temperature wave form in the material. Tables 1 and 2 summarize some of the results for one set of boundary conditions. Distortion occurs for both sine-wave and square-wave boundary conditions.

Table 1. Effects of Temperature Dependence of Properties on the Temperature Variation in a Slab

Tabulated results for:

- centerline of slab
- 13.9 deg C amplitude
- 1 hr period
- $1/F_o = 5.9$
- 5th cycle results
- thickness = 0.061 m
- square-waves
- in phase
- 1000 steps/cycle
- $a_o = 2.58 \times 10^{-7}\ m^2/s$
- zero mean temp. at surfaces

Temp. Coefficients		Average Temp.	Main Harmonic		Second Harmonic		Error in a_o**
$\alpha \times 10^3$	$\beta \times 10^3$	°C	Amplitude, °C	Phase	Amplitude, °C	Phase	%
0.0*	0.0*	0.000*	6.172*	-102.22*	0.000*	-	-
0.0	0.0	0.002	6.172	-102.22	0.00005	10.88	+0.002
0.0	-3.6	-0.00050	6.172	-102.22	0.077	101.58	-0.005
0.0	-7.2	-0.0011	6.172	-102.20	0.156	101.64	+0.018
0.0	7.2	0.0011	6.172	-102.20	0.156	-78.28	+0.018
3.6	0.0	0.307	6.172	-102.17	0.040	143.48	-0.020
-3.6	0.0	-0.307	6.172	-102.17	0.040	-36.38	-0.020
-7.2	0.0	-0.612	6.167	-102.07	0.080	-34.20	-0.040
3.6	-3.6	0.301	6.178	-102.12	0.091	118.60	-0.080
3.6	3.6	0.312	6.167	-102.22	0.038	122.10	+0.011
-3.6	-3.6	-0.312	6.167	-102.22	0.038	57.74	-0.011
-3.6	-7.2	-0.317	6.161	-102.26	0.089	84.18	-0.182
3.6	7.2	-0.317	6.161	-102.26	0.089	-95.73	-0.182
7.2	-3.6	0.604	6.183	-101.92	0.546	-175.47	-0.112
7.2	3.6	0.618	6.156	-102.12	0.053	170.28	-0.268
7.2	7.2	0.623	6.150	-102.12	0.076	122.15	+0.444
-7.2	-7.2	-0.623	6.144	-102.22	0.077	57.77	+0.444
36.0	-36.0	2.797	6.644	-92.07	1.006	132.24	-7.55
36.0	36.0	2.963	5.617	-102.14	0.283	-123.63	+10.2

* Solutions for constant properties
** from real part of eq (3), Appendix II

The temperature dependence of the properties gives rise to a non-zero average temperature at the internal points, even though the average boundary temperatures are zero. Typical profiles are given in figure 1. Table 1 shows that, for the conditions studied, this effect is less than 20 per cent of the amplitude of the main harmonic. The distortion in the mean temperature depends primarily on the coefficient α and varies linearly with it.

Table 2. Effects of Frequency on the Temperature Variation in a Slab with Temperature Dependent Properties (non-metal; 5 cm thickness)

α = 0.0036/°C
L = 0.061 m
13.9 deg C sine waves, zero mean
18 slices, $1/F_o$ = 6.2, 5th cycle results
Boundaries at 0 mean temperature

β = 0.0072 /deg C
a = 2.58×10^{-7} m^2/s
120° phase shift

Results for the Centerline

Period hr	Mean Temp, °C	Main Harmonic		Second Harmonic	
		Ampl, °C	Phase	Ampl, °C	Phase
1.0	0.171	2.42	-162.24	0.0218	131.84
0.5	0.179	1.12	156.43	0.0050	36.91
0.2	0.193	0.25	75.35	0.0012	-2.34

Distortion of the main harmonic is primarily affected by the coefficient α as shown by Tables 1 and 2. These are relatively small and it can be inferred that the fundamental frequency can be used to obtain good approximations to the mean thermal diffusivity. The error in the main harmonic is also affected by β, θ max, and the period.

The temperature dependence of properties produced even harmonics at internal points. The second harmonic content (Tables 1 and 2) depends on both the temperature coefficients. The amplitude of the second harmonic is more than 15 per cent of the main harmonic for the worst case shown.

Results not shown indicate that the odd harmonics of square-wave boundary conditions have distortion similar to the main harmonic, but because of the fewer number of points used in the harmonic analysis, they have more inherent errors.

The frequency has a marked effect on the distortion (Table 2). The distortion of the average temperature increases with frequency, but the relative amplitude of the second harmonic decreases. No exact explanation can be offered for this at present.

The relative phase of the boundary temperatures affects the results. The effect is second order for mean temperature as shown in figure 1. The second harmonic distortion is worse when the boundary temperatures are in phase than when they are 180° out of phase.

4.2 Errors in Determination of Thermal Diffusivity

The results of a quasi-linear analysis, to be published later, and the basic form of the finite difference equation, given in Appendix I, indicated that the errors are functions of α, β, $\alpha - \beta$ and $\alpha + \beta$ and sums and products of these parameters. These relationships were checked for the results of the finite difference calculations by plotting error in determining thermal diffusivity against the same combinations of α and β.

A summary of the most important characteristics of the error in the determination of the thermal diffusivity is given in figures 1 and 2 for a = 2.58×10^{-7} m^2/s (0.01 ft^2/ hr), and, $\pm$ 13.9 C (25 F) square-wave boundary conditions. The error for a thermal diffusivity of $2.58 \times 10^{-8} m^2/sec$ was similar.

Figure 2 gives the errors in determining the thermal diffusivity from the real part of eq (3) in Appendix II while figure 3 gives the error when they are determined from the imaginary part of the equation. The errors are smaller using the real part, except when $\alpha = \beta$, i.e. when the thermal diffusivity is a constant. The slope of the curves indicates that the error terms are essentially quadratic functions of α and β. Figures 2 and 3 are not exact, expecially at the lower values, because of the finite difference error in the calculations.

Typical values for the temperature coefficients, α and β, for dry materials are given in Table 3. The range in α is from -0.01 to +0.01 per deg K and β from -0.003 to +0.004 per deg K. Values for moist materials are unknown, but a rough estimate would be, -0.02 to +0.02 per deg K for α and -0.005 to +0.005 for β.

Table 3. Typical Values for the Temperature Coefficients of Thermal Conductivity and Thermal Diffusivity of Dry Solids at Temperatures Above 100°K

Type of Material	Temperature Coefficient α per deg. C	Temperature Coefficient β per deg. C
metals	-0.010 to +0.010 (rare 0.020)	-0.003 to +0.004
non-metals	0 to 0.004 (rare 0.020)	-0.0003 to +0.003
semi-conductors	-0.006 to +0.006	-0.0003 to +0.003

(Data from American Institute of Physics Handbook, 2nd Edition)

The errors in determining thermal diffusivity in the range considered can be estimated from figures 2 and 3, as being less than 1 per cent for dry materials and 3 per cent for moist materials using the real part of the linear solution, and less than 2 and 6 per cent, respectively, using the imaginary part of the linear solutions.

5. Conclusions

Distortion produced by the temperature dependence of thermal properties places limitations on the use of steady-periodic methods for measuring the thermal properties of moist materials. The steady mean temperature gradient can be expected to produce a nonuniform mean moisture content in the material. The method has advantages, however, over other methods since the mean gradient is only a small fraction of the maximum temperature variation in the material.

The main harmonic must be found by accurate harmonic analysis because of the harmonic distortion that is introduced by the temperature dependence of the thermal properties.

The mean thermal diffusivity of the material determined from the main harmonic has some error but the accuracy of the method is adequate for most purposes even for large temperature dependence. Steady periodic methods can, therefore, be used to determine the thermal diffusivity of normal dry materials. Errors due to the temperature dependence can be estimated from the curves in figures 2 and 3.

6. References

(1) Higashi, A. On the thermal conductivity of soil, with special reference to that of frozen soil, Trans. Amer. Geophys. Union, 34, 5, (1963).

(2) Jackson, R.D. and Kirkham, D. Method of measurement of the real diffusivity of moist soil. Proc. Soil Sci. Amer., 22, 6, (1958).

(3) Carslaw, H.S. and Jaeger, J.C. Conduction of heat in solids. 2nd Edition, Oxford Press.

(4) Shirtliffe, C.J. and Stephenson, D.G. The determination of the thermal diffusivity and conductivity of materials subjected to periodic temperature variations. NRC/DBR, Computer Program No.12 (1962).

Appendix I

1. Finite Difference Solution

1.1 Basic Equation

The basic equation approximated by the finite differences was

$$\frac{\partial \theta}{\partial t} = \frac{1}{cd} \frac{\partial}{\partial x} \left(\lambda \frac{\partial \theta}{\partial x} \right) \tag{1}$$

with: zero initial conditions, periodic boundary temperature, cd and λ functions of temperature.

1.2 Finite Differences

Central difference approximations were used to approximate the derivatives, yielding

$$\theta(n, t+\Delta t) = \theta(n,t) + \frac{\Delta t}{\Delta x^2 dc(n)} \left[\left(\frac{\lambda(n-1) + 4\lambda(n) - \lambda(n+1)}{4} \right) \cdot \theta(n-1, t) - 2\lambda(n)\,\theta(n,t) \right.$$

$$\left. + \left(\frac{\lambda(n+1) + 4\lambda(n) - \lambda(n-1)}{4} \right) \cdot \theta(n+1, t) \right]_t \tag{2}$$

where: dc and λ are arbitrary functions of temperature, n indicates the node point, Δt indicates the time step.

If $\lambda = \lambda_o (1 + \alpha\theta)$ and $cd = c_o d (1 + \beta\theta)$, eq (2) reduces to

$$\theta(n, t+\Delta t) = \theta(n,t) + a_o \frac{\Delta t}{\Delta x^2} \left[H - \left(\frac{(\beta - \alpha)\,\theta(n,t)}{1 + \beta\theta(n,t)} \right) H - \frac{\alpha\left(\theta(n+1,t) - \theta(n-1,t)\right)^2}{4\left(1 + \beta\theta(n,t)\right)} \right]$$

where $H = \left(\theta(n+1,t) - 2\theta(n,t) + \theta(n-1,t)\right)$ and is the form for constant properties and $\Delta x^2 / a_o \Delta t$ is the normal non-dimensional inverse Fourier number, $1/F_o$.

The second term is maximum when $\alpha = -\beta$.

Appendix II

2. Linear Solution for Two Slabs

2.1 General

The solution for the temperature variation in an infinite slab of a material of constant properties with periodic boundary temperatures is given in Carslaw and Jaeger (3). In terms of the temperature at three points in the material, and in matrix notation the solution is

$$\begin{bmatrix} \theta_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} A & B \\ D & A \end{bmatrix} \cdot \begin{bmatrix} \theta_2 \\ q_2 \end{bmatrix} \quad (1) \qquad \text{and} \qquad \begin{bmatrix} \theta_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ D^* & A^* \end{bmatrix} \cdot \begin{bmatrix} \theta_3 \\ q_3 \end{bmatrix} \quad (2)$$

where θ_1, θ_2, and θ_3 are periodic temperatures at points 1, 2, and 3; q_1, q_2, and q_3 are the periodic heat fluxes at points 1, 2, and 3, and

$$A = \cosh(1+i)\,\Phi \qquad B = \frac{R \sinh(1+i)\Phi}{(1+i)\Phi} \qquad D = \frac{(1+i)\,\Phi \sinh(1+i)\,\Phi}{R}$$

where $\Phi = (\pi L^2/aP)^{\frac{1}{2}}$, $R = L/\lambda$, L = distance between respective points, λ = thermal conductivity, a = thermal diffusivity $= \lambda/dc$, P = period, d = density, c = specific heat per unit mass, and $*$ refers to functions for the slab between points 2 and 3.

Equations (1) and (2) hold for all pure harmonics of the boundary conditions when the properties are constants.

2.2 Similar Slabs

For the case where the three temperatures are measured at evenly spaced points in a homogeneous material, the equations can be combined to give the following expression for the thermal diffusivity (Shirtliffe and Stephenson (4)):

$$(1+i)\,\Phi = \cosh^{-1}\left(\frac{\theta_1 + \theta_3}{2\,\theta_2}\right) \tag{3}$$

Values of Φ can be obtained for both the real and imaginary part of the equation and the two values should be equal. This configuration can be used as an absolute method for finding the thermal diffusivity of a material.

2.3 Dissimilar Slabs

For two slabs of dissimilar material in contact at plane 2, the properties of the slab of material between points 1 and 2 can be found if the properties of the material between points 2 and 3 are known. Again from Ref (4) we obtain

$$\frac{\theta_1}{\theta_2} = A + B\left(\frac{q_2}{\theta_2}\right) \tag{4}$$

where

$$\frac{q_2}{\theta_2} = \frac{A^*}{B^*} + \frac{1}{B^*}\left(\frac{\theta_3}{\theta_2}\right). \tag{5}$$

The second equation shows that the reference slab is used as a "heat meter." When eq (5) is substituted in eq (4), the complex equation for θ_1/θ_2 contains the two unknowns R and Φ, or λ and a. These can be found from the real and imaginary parts of the equation. The solution has been done on a digital computer using standard numerical techniques.

2.4 Application of Equations

Apparatus based on these equations has been built and used at intervals over the past eight years. The results from the experimental investigation including considerable error analysis will be published elsewhere.

Computer analysis was found necessary to adequately handle the harmonic analysis of the temperature waves to solve the equations, and for error analysis.

This paper is a contribution from the Division of Building Research, National Research Council of Canada, and is published with the approval of the Director of the Division.

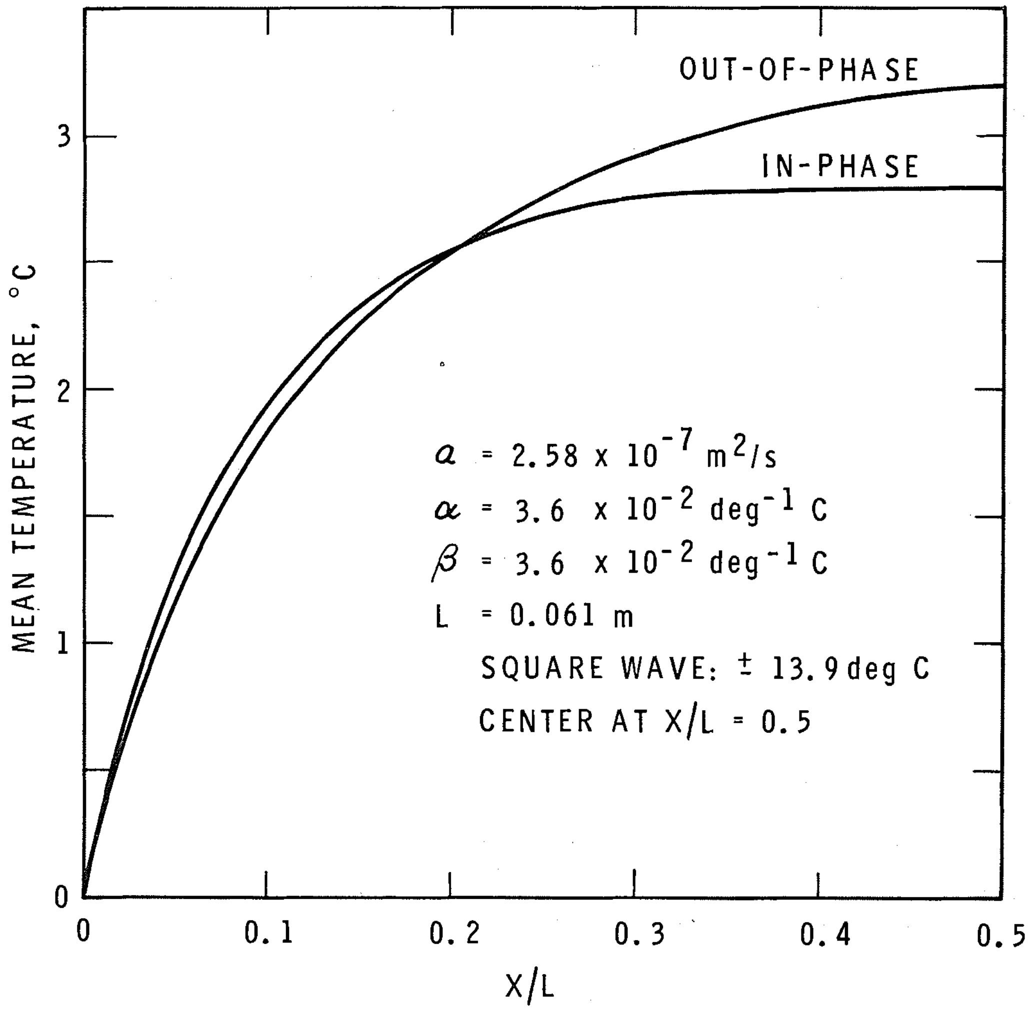

Figure 1 Mean Temperature Profile Produced by Temperature Dependence of Properties for Conditions Given.

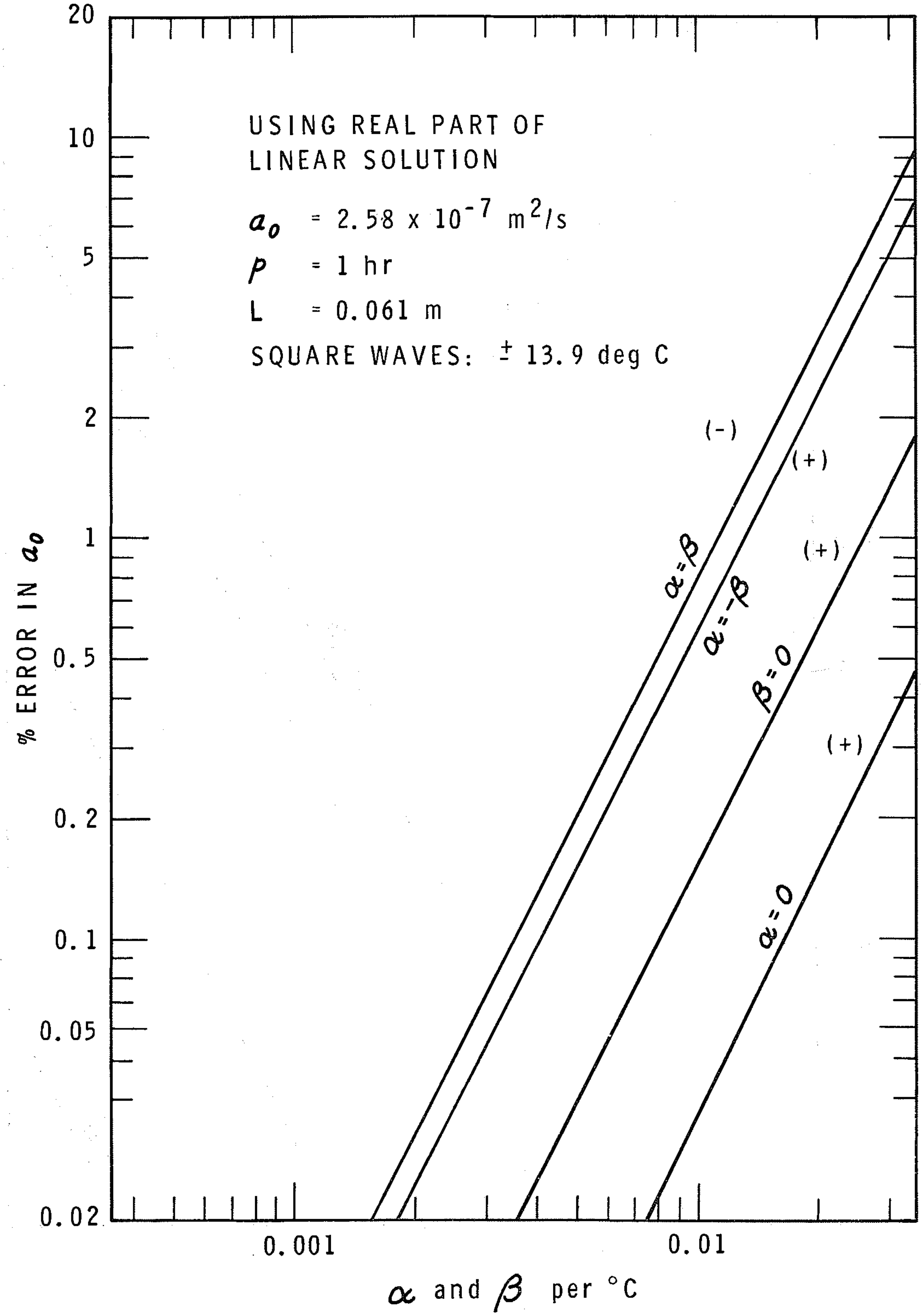

Figure 2 Error in Determining a_o Using Real Part of Linear Solution and for Particular Combinations of α and β.

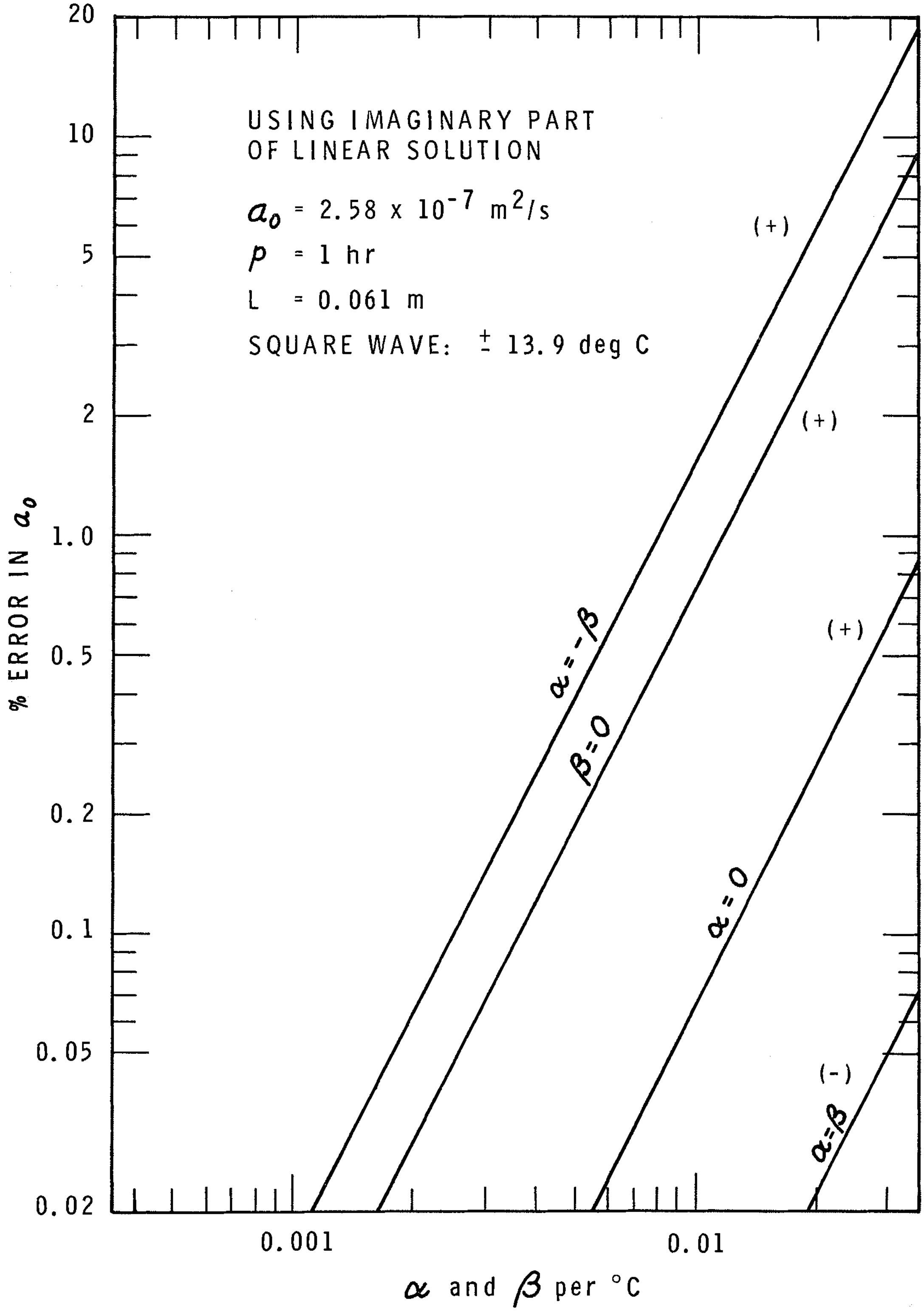

Figure 3 Error in Determining a_o Using Imaginary Part of Linear Solution and for Particular Combinations of α and β.