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ESTABLISHING STEADY-STATE THERMAL CONDITIONS IN FLAT SLAB SPECIMENS

by

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L'ETABLISSEMENT DE CONDITIONS THERMIQUES EN REGIME PERMANENT DANS DES ECHANTILLONS DE DALLE SANS NERVURE

L'auteur examine la réponse thermique d'échantillons de dalle sans nervure soumis à quatre séries de conditions thermiques aux limites. Les conditions sont caractéristiques des dispositifs d'essai thermophysiques et les quatre séries de conditions sont semblables. Les températures initiales sont uniformes et la température requise de la surface froide est atteinte par degrés. Les conditions de la surface chaude sont au nombre de quatre: (1) température constante, (2) flux thermique constant, (3) flux thermique nul suivi d'une température constante et (4) flux thermique nul suivi de flux thermique constant. L'auteur donne des solutions des problèmes de transmission de chaleur; les équations sont tronquées, puis inverties, pour former des expressions simples et approximatives du temps de réponse. L'auteur montre que l'exactitude des équations approximatives est convenable pour fins de prédiction; les quatre séries de conditions sont rangées par ordre ascendant de temps de réponse et l'on note l'existence de situations optimales.

Authorized Reprint from
Special Technical Publication 544

C. J. Shirtliffe[1]

Establishing Steady-State Thermal Conditions in Flat Slab Specimens

REFERENCE: Shirtliffe, C. J., "**Establishing Steady-State Thermal Conditions in Flat Slab Specimens,**" *Heat Transmission Measurements in Thermal Insulations, ASTM STP 544*, American Society for Testing and Materials, 1974, pp. 13-33.

ABSTRACT: The thermal response of flat slab specimens subjected to four sets of thermal boundary conditions is examined. The conditions are typical of thermophysical test apparatus, and the four sets of conditions are similar. Initial temperatures are uniform and the cold face temperature is stepped to the required value. There are four different hot face conditions: (1) constant temperature, (2) constant heat flow, (3) zero heat flow followed by constant temperature, and (4) zero heat flow followed by constant heat flow.

Solutions are given for the heat transfer problems, and the equations are truncated, then inverted, to yield simple, approximate expressions for response time. The precision of the approximate equations is shown to be adequate for prediction purposes; the four sets of conditions are ranked in order of increasing response time; and the occurrence of optimum situations is noted.

KEY WORDS: thermal insulation, heat transmission, heat transfer, heat flow meters, flat slab, temperature, analytic functions, one-dimensional flow, reaction time, thermodynamic properties, thermophysical properties, time dependence, transient heat flow

The objectives of this present study are twofold: to determine theoretically the fastest way to achieve steady-state heat flow conditions in a flat specimen, using simple, realistic operating conditions; and to determine the theoretical settling time for specimens subjected to conditions similar to those in the four most common operating modes of guarded hot plate and heat meter apparatus.

The initial temperature is always assumed to be uniform, and only the temperature level is allowed to vary. Although the ultimate in initial condition-

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ing is to establish the correct linear temperature gradient, it is seldom practicable to precondition samples so precisely. The surface at the lower temperature was always assumed to be at constant temperature since this is typical of most apparatus where one surface is a temperature controlled heat sink.

The temperature of a body is known to approach steady-state conditions asymptotically. Thus, the time to reach steady state depends on what criteria are used to determine when steady state is reached. Temperature and heat flow may approach steady conditions at different rates. The most useful criterion is the error in the parameter to be determined, in this case, thermal conductivity. This was the approach used by Shirtliffe and Orr[2] in an earlier paper, when they considered two types of boundary conditions. Results were given for a limited range of initial temperatures and are now extended and presented in a more convenient form. The solutions for two other sets of boundary conditions are included.

The following approach was used to obtain simplified equations for settling time: (1) the series form of the solution to the heat conduction problem was derived, (2) the solution was substituted into the equation for thermal conductivity yielding an expression for the deviation from the correct value, and (3) the series in the expression were truncated and then inverted. This yielded relatively simple expressions for settling time in terms of the parameters of the problem and the error in measuring thermal conductivity.

Optimum starting conditions were identified for three cases. As the simple expressions for settling time did not hold where these optimum conditions existed, special expressions were derived for these cases. Results were compared and the boundary conditions ranked according to the speed of reaching steady state.

Simplified equations derived for the four idealized cases are useful in predicting lower limits for settling time in actual apparatus. A solution of a model for the full apparatus is necessary for an upper limit. Lower limits can be used in establishing guidelines in standard test methods. It should be noted that no existing thermal property test method attempts to establish test duration on a theoretical basis.

The equations may also be used in establishing the design of an apparatus. The advantages and disadvantages of each case are made apparent by this means. In addition, the equations can serve as a guide in experimental determination of settling time for different specimens in a particular apparatus. They are useful in establishing optimum conditioning temperatures and optimum turn-on temperature, which can be particularly useful in quality control applications where rapid measurements are required. The equations can be used, as well, in determining which of the many configurations of heat meter apparatus is fastest.

[2] Shirtliffe, C. J. and Orr, H. W., *Proceedings,* Seventh Conference on Thermal Conductivity, National Bureau of Standards, Special Publication 302, 1967, pp. 229-240.

Basic Problem

Four heat transfer problems have been solved and the solutions rearranged to yield an estimate of thermal settling time. Each problem represents a model of either a typical thermophysical test apparatus or a different mode of operation for the same apparatus.

The four cases are described in mathematical terms in the Appendix, where the solutions are given. The models are of a one-dimensional, flat slab specimen. All parameters are nondimensionalized; the space variable is divided by the specimen thickness, and time is multiplied by the thermal diffusivity and divided by the thickness squared. The temperature scale is referenced to the cold surface temperature and scaled to produce a unit temperature difference across the specimen. Heat flow is divided by the steady-state heat flow.

The range of nondimensional variables is as follows:

space: $X = 0$ to 1

time: $\tau = 0$ to ∞

temperature: $\theta = -50$ to $+50$

$= 0$ at cold surface for steady state

$= 1$ at hot surface for steady state

heat flow: $\partial \theta / \partial X = -\infty$ to $+\infty$

$= 1$ at steady state, for $X = 0$ to 1

Common Factors in Models

There are a number of factors common for all cases; specimens are always preconditioned to a constant temperature either above or below the cold plate temperature. The nondimensional initial temperature, W, can vary widely, but a range of -50 to $+50$ covers most cases of practical interest.

Step changes in surface temperature are assumed in the analysis. In a real apparatus, the cold face of the specimen is often placed against a liquid heat exchanger that can either extract or supply heat. It cannot, however, supply the infinite heat flows required to produce step changes in the surface temperature. The heat capacity of the cooling system helps to provide high heat flows and give a reasonable approximation of step changes.

Heat flow is always assumed to be one dimensional in the models. Modern apparatus usually have automatic temperature control of one or more guards, so that this assumption is reasonable.

Model Descriptions and Solutions

Case (1) Constant Temperature—The first model is of an apparatus with heat exchanger plates on both the hot and cold surfaces, which are at constant temperature after time zero.

Case (2) Constant Heat Flow—The second model is of an apparatus with a liquid heat exchanger on the cold surface and an electric heater on the hot surface. The other face of the heater is perfectly insulated, preventing heat loss. This model is typical of one half of the guarded hot plate apparatus except that in the model the heater has no thermal capacitance. The power to the heater is turned on at $\tau = 0$ and held constant thereafter.

Case (3) Zero Heat Flow Followed by Constant Temperature—The third and fourth models are of the same apparatus, but control of the power to heater is different and the initial specimen temperature, W, is always above the steady-state hot surface temperature, that is, $W \geqslant 1$. In the third model the power to the heater on the hot surface is turned on only when the hot surface of the specimen has cooled to the correct value, $\theta(1, \tau^*) = 1$. From this point on, power is controlled to maintain the hot surface temperature constant. The power on, or starting time, is termed τ^* and is determined from the solution for the case where heat flux is zero at $X = 1$. The equation must be solved by trial and error.

The solution for the rest of the problem, that is, when τ is greater than τ^* and the hot surface is at a constant temperature, was found by superposition of simple solutions.

Case (4) Zero Heat Flow Followed by Constant Heat Flow—The fourth model is similar to the third, that is, heat flow across the hot surface is zero until that surface cools to a prescribed temperature, S, when constant power is supplied to the heater. Temperature S is any value between the cold surface temperature, 0, and the initial temperature, W.

Time τ^*, as for Case (3), is found by solving essentially the same equation by trial and error. The solution to the heat transfer problem for $\tau > \tau^*$ is derived using the final temperature distribution of the initial phase as the initial distribution for the second phase.

The solutions are all in the form of three terms since they are derived by superposition. The first term is the steady-state solution, normally one in a nondimensionalized problem; the second is the transient solution for the boundary value problem, that is, the effect of boundary conditions; the third is the transient solution for the initial value problem, namely, the effect of the initial temperature distribution. The last two terms are in the form of infinite series in order to satisfy the conditions at all positions and times. The two series simplify considerably at hot surface, $X = 1$. At large times many of the terms are negligible. The solutions are listed in the Appendix, and are given in the section on inversion of the solutions in simplified form.

Simplification of Solutions

The solutions given in the Appendix are not in a particularly convenient form. They express temperature or heat flow in terms of infinite series involving X, τ, W, and S. The techniques previously used by Shirtliffe and Orr[2] may be used to define a settling time, which is defined as the time required to reach conditions where measurement of the thermal conductivity of the specimens can be made to a desired degree of precision.

The error due to neglect of the remaining transient effect is termed ϵ and is expressed as a fractional error in all formulas. The results are presented in such a way that ϵ can be specified by the user according to the application, and the required settling time can then be determined. It is assumed that in most thermophysical applications an error value of less than 10 percent will be desired.

The derivations of the equations for ϵ in terms of temperature and heat flow are given in the Appendix. The equations are essentially identical to those given by Shirtliffe and Orr.[2]

At $X = 1$, for temperature specified

$$\epsilon = \frac{\partial \theta}{\partial X}\bigg|_{1,\tau} - 1 \tag{1a}$$

for heat flux specified,

$$\epsilon/(1 + \epsilon) = 1 - \theta(1, \tau) \tag{1b}$$

The solutions for each heat transfer problem may be substituted in the appropriate expression for error, and the resulting solution plotted for a range of each variable, W, τ, and S. The solution could then be "graphically inverted" to yield a solution for settling time τ. This is basically the tack taken by Shirtliffe and Orr[2] for the first and second cases. Figures 1 through 5 are such solutions for all cases, plotted in a more convenient manner.

Approximate solutions have been derived for all the models, but in equation form rather than graphical form. These will be shown to have adequate precision for the intended purpose. They will also be shown to be more useful in comparing the settling times for the four models.

Truncation of Solutions

The solutions for the heat transfer problems given in the Appendix are all in infinite series form rather than in the equally common, error-function form. The series form was selected because it is a simpler form for truncation and for estimating the truncation error.

The Appendix gives the truncated form of the solution for ϵ. The series are truncated after the second order term(s). Truncation error is small as long as ϵ is small, since a small ϵ only occurs at a relatively large time, τ. These equations can be further simplified because they still contain τ^* and C (n) which are in series form.

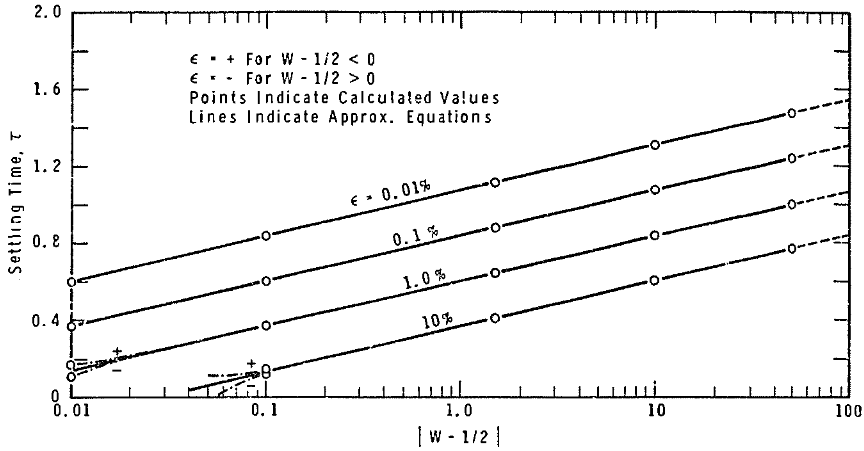

FIG. 1–*Settling time for constant temperature mode.*

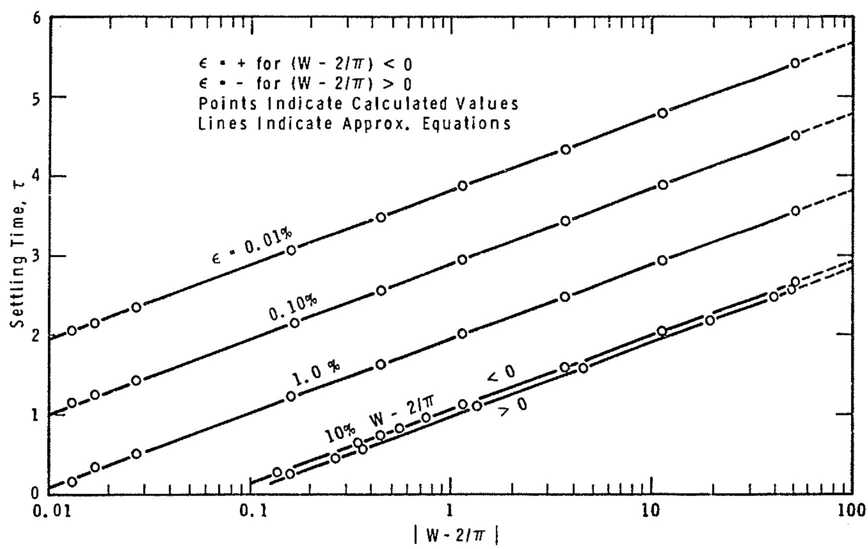

FIG. 2–*Settling time for constant heat flow mode.*

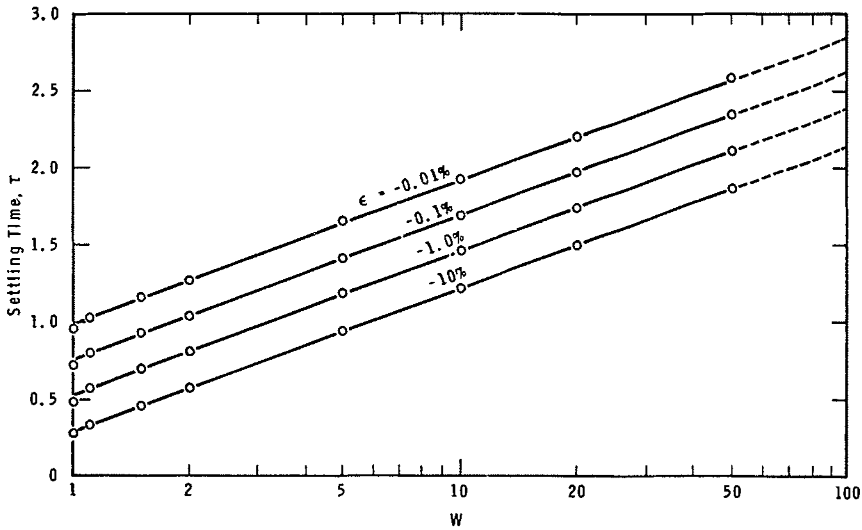

FIG. 3–*Settling time for zero heat flow followed by constant temperature at* X = *1.*

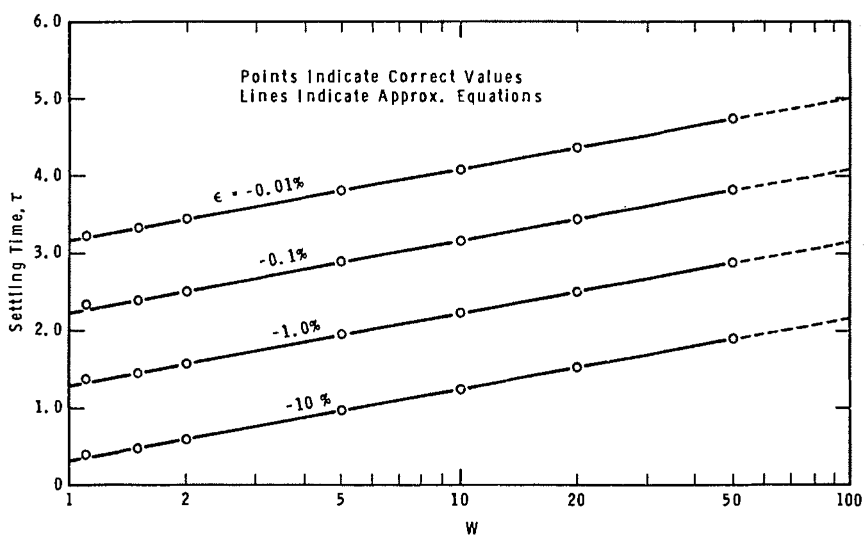

FIG. 4–*Settling time for zero heat flow followed by constant heat flow at* X = *1,* S = *1.0.*

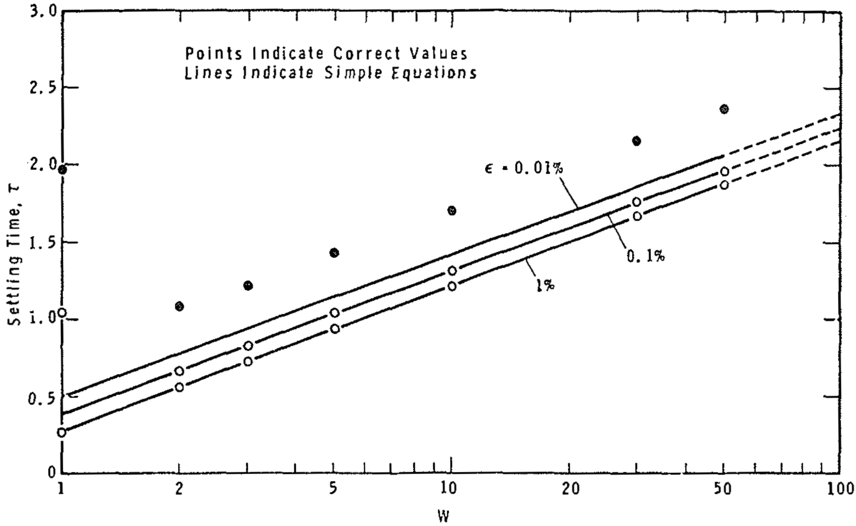

FIG. 5—*Settling time for zero heat flow followed by constant heat flow at* X = *1,* S = *0.811.*

The series solution from which τ^* was obtained had also been truncated. Figure 6 indicates that for W greater than 1.2, a single term is adequate. The simplified equation is

$$\exp\left(-\pi^2\ \tau^*/4\right) = \pi A/(4W) \tag{2}$$

The equation for the error, ϵ, in the third case contains constants C(1) and C(2) in the first and second terms, respectively. Terms that contain these constants are the effects of the initial temperature distribution on ϵ. The series for the constants C(n) have been truncated at the first term. The truncated series from which τ^* was found was then used in those truncated series to obtain an expression for C(1) and C(2) containing only W and S. These multiple truncations are shown to be justified by the results given in the section on accuracy of equations and by Figs. 3 to 5.

The equations for C(1) and C(2) are:

$$C(1) = -\pi A/(12W) \tag{3a}$$

$$C(2) = \pi A/(60W) \tag{3b}$$

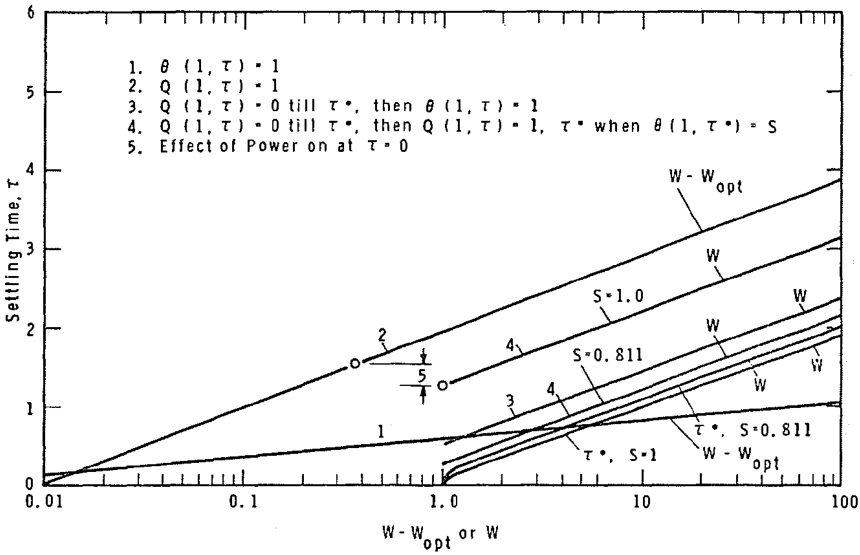

FIG. 6–*Settling time for slab–all cases, ϵ = 1 percent and τ*.*

The four equations, after truncation of the series and substitution of 1, 2, and 3, for ϵ are:

Case 1, constant temperature

$$\epsilon = (2-4W) \exp(-\pi^2 \tau) + 2 \exp(-4\pi^2 \tau) \tag{4}$$

Case 2, constant heat flow

$$\epsilon/(1+\epsilon) = (4/\pi)((2/\pi)-W) \exp(-\pi^2 \tau/4) + (4/(3\pi))((2/(3\pi)) + W) \exp(-9\pi^2 \tau/4) \tag{5}$$

Case 3, zero heat flow followed by constant temperature

$$\epsilon = -(2/3)(4W/\pi)^4 \exp(-\pi^2 \tau) + (62/15)(4W/\pi)^{16} \exp(-4\pi^2 \tau) \tag{6}$$

Case 4, zero heat flow followed by constant heat flow

$$\epsilon/(1+\epsilon) = (4/\pi)(W/S)\ [((8/\pi^2) - S) \exp(-\pi^2 \tau/4) + (1/3)((8/(3\pi^2))(4W/(\pi S))^8 + S) \exp(-9\pi^2 \tau/4)] \tag{7}$$

The first terms in Eqs 4, 5, and 7 equal zero when $W = 1/2$, $W = 2/\pi$, and $S = 8/\pi^2$, respectively. The second terms then express the relation between the variables. The second terms are negligible in comparison with the first, except very near these optimum points.

Inversion of the Solutions

Equations 4 through 7 are truncated at the first term, then inverted to produce equations for τ in terms of the other variables. These do not hold at or near the points where the first terms of the original series equal zero. When the first terms are zero, the second terms can be used to give equations for τ. In the relatively small regions where the two terms are of comparable size a graphic solution must be used.

The complete set of equations follows:

At $X = 1$,

Case 1, constant temperature

$$\tau \approx (1/\pi^2) \ln [(2-4W)/\epsilon], \; W \neq 1/2, \epsilon \neq 0 \tag{8}$$

and

$$\tau \approx (1/(4\pi^2)) \ln [2/\epsilon], \; W = 1/2, \epsilon \neq 0 \tag{8a}$$

Case 2, constant heat flow

$$\tau \approx (4/\pi^2) \ln [(4/\pi)((2/\pi) - W)(1 + \epsilon)/\epsilon] \tag{9}$$

and

$$\tau \approx (4/(9\pi^2)) \ln [2(1 + \epsilon)/\epsilon] - 0.0772, \; W \neq 2/\pi, \epsilon \neq 0 \text{ or } -1 \tag{9a}$$

Case 3, zero heat flow followed by constant temperature

$$\tau \approx (1/\pi^2) \ln (-2/\epsilon) + (4/\pi^2) \ln W - 0.050, \; W > 1.2, \epsilon \neq 0 \tag{10}$$

Case 4, zero heat flow followed by constant heat flow

$$\tau \approx (4/\pi^2) \ln [(4/\pi)(W/S)((8/\pi^2) - S)(1 + \epsilon)/\epsilon]$$
$$W > 1.2, \epsilon \neq 0 \text{ or } -1, S \neq 8/\pi^2 \tag{11}$$

and

$$\tau \approx (4/(9\pi^2)) \ln [2(1 + \epsilon)/\epsilon] + (4/\pi^2) \ln W + 0.0434$$
$$W > 1.2, \epsilon \neq 0 \text{ or } -1, S = 8/\pi^2 \tag{11a}$$

The $1 + \epsilon$ could be set equal to 1 in Eqs 9, 9*a*, 11, and 11*a* if desired, because it contributes less than 0.04 to τ for $\epsilon = 0.1$ and less for smaller errors. The ϵ in Eq 3 is normally negative so that the sign cancels the minus sign. The sign of ϵ and the sign of the other terms in the square brackets are consistent and always lead to a positive value. Equation 4 should reduce to Eq 2 when $S = W$, but this is not quite the case because of the approximation used for τ^*.

Comparison of Settling Times

The equations for settling time, though not exact, reveal the most important features of the response: the occurrence of optimum conditions, and relative values of response time away from these optimum points.

The optimum points occur very near the singularities, that is, the undefined points in the logarithmic function. For constant temperature this occurs at an initial temperature $W = 0.5$, and for constant heat flow at $W = 2/\pi \approx 0.637$. These have been confirmed by calculations using the full series.

The third and fourth cases have no optimum initial temperature, W. There is an optimum S at $8/\pi^2$ for the fourth case. Calculations with the full series indicate that it is actually between 0.811 and 0.812.

The occurrence and significance of optimums for W has already been pointed out[2] and discussed. Estimated values were 0.52 and 0.64. If specimens are conditioned to the optimum temperatures, the response time can be reduced very substantially.

The occurrence of the optimum value for S is also significant. This is the optimum value to which the hot surface should be allowed to cool before power to the heater is turned on. As with the second case, constant heat flow, the correct power must be applied. The difficulty in achieving this and the consequence of making an incorrect estimate have been discussed.[2]

Away from the optimum W's, the constant temperature case can be seen to have a settling time approximately one fourth that of the constant heat flow case. This is evident from the $1/\pi^2$ and $4/\pi^2$ constant multipliers in Eqs 8 and 9 and from Figs. 6 and 7. The settling of the third case is longer than that for constant temperature but shorter than that for constant heat flux (Figs. 6 and 7). The comparison can only be made for $W > 1.2$ because of the approximation for τ^*.

Comparison of the fourth case is more difficult. It is clearly faster than the constant heat flux case and slower than the constant temperature case. It is significantly slower than the third case when $S = 1.0$, but faster when S has a value close to 0.811 (Figs. 6 and 7).

In order of increasing settling times the cases are:

(1) constant temperature,

(3) zero heat flow followed by constant temperature,

(4) zero heat flow followed by constant heat flow, and

(2) constant heat flow.

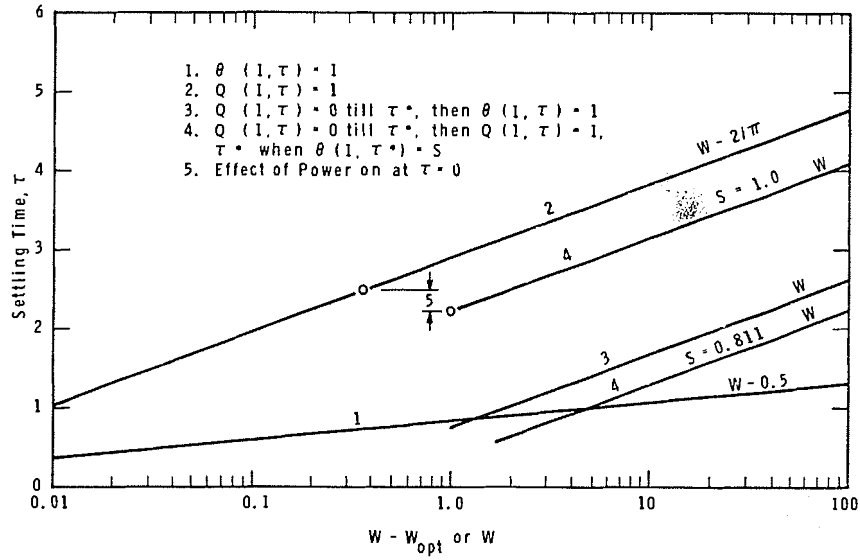

FIG. 7–*Settling time for slab–all cases, ϵ = 0.1 percent.*

In Case (4), when S is between 0.788 and 0.835 and $\epsilon = 1$ percent, the order is changed to (1), (4), (3), (2). The same order holds when $\epsilon = 0.1$ percent and S is in an even narrower range centered about 0.811.

At the optimum values of W and S the order is (1), (2), (4), (3), and this holds for a very small region on either side of the optimum values.

Accuracy of Equations

The accuracy of the approximate equations for predicting settling time is outlined in Table 1. The equations were determined by extensive calculations using the full series. Their accuracy depends on range of ϵ, error due to the transient, and nearness to optimum values. These conditions are also listed in Table 1.

The accuracy of the equations is more than adequate for estimating the settling time of the test specimens because the thermal properties are seldom known to better than 10 percent. The simplified equations are not so precise for the last two cases. The more precise equations are therefore listed. The error in the fit of either set of simplified equations to the data decreases rapidly as the distance from the optimum S increases.

Conclusions

Factors governing the settling time of flat slab specimens are thermal properties, thickness, initial temperature, maximum allowable error, and, in

TABLE 1—*Summary of errors in estimating settling time with approximate equations.*

Case	Error in τ, %	For ϵ Less Than, %	Conditions
Constant temperature	10	25	$\lvert W - 0.5\rvert < 0.2$
	1.0	15	$\lvert W - 0.5\rvert < 0.2$
	0.1	10	$\lvert W - 0.5\rvert < 0.2$
	10	10	$\lvert W - 0.5\rvert < 0.1$
	1.0	6	$\lvert W - 0.5\rvert < 0.1$
	0.1	3	$\lvert W - 0.5\rvert < 0.1$
Constant heat flow	10	6	$\lvert W - 2/\pi\rvert < 0.1$
	1	5	$\lvert W - 2/\pi\rvert < 0.1$
	0.1	4	$\lvert W - 2/\pi\rvert < 0.1$
	1	2	$\lvert W - 2/\pi\rvert < 0.04$
Zero heat flow followed by Constant Temperature (Appendix, Eq 48)	10	100	$W > 1.0$
	2	3	$W > 1.0$
	10	100	$W > 1.1$
	1	10	$W > 1.1$
	10	60	$W > 1.0$
	1	20	$W > 1.0$
	0.1	40	$W > 1.0$
	0.1	15	$W > 1.0$
Zero heat flow followed by Constant (Appendix, Eq. 48)	20	1	$S > 0.9$
(Appendix, Eq 47)	10	6	$\lvert S - 8/\pi^2\rvert < 0.1$
	1	5	$\lvert S - 8/\pi^2\rvert < 0.1$
	0.1	4	$\lvert S - 8/\pi^2\rvert < 0.1$
	10	2	$\lvert S - 8/\pi^2\rvert < 0.01$
	1	1.4	$\lvert S - 8/\pi^2\rvert < 0.01$
	0.1	1.0	$\lvert S - 8/\pi^2\rvert < 0.01$

NOTE—Error in estimating τ is less than the value given when ϵ, the error due to the transient, is less than the value given, and when the initial temperatures, W, or the turn-on temperature, S, is as indicated.

certain cases, the hot surface temperature at which the boundary conditions are applied.

Approximate equations for the thermal response of a specimen can be used to determine settling time. Accuracy is adequate for prediction purposes.

The four hot surface conditions have been ranked in order of increasing settling time. For other than near the special points they are:

(1) constant temperature,

(2) zero heat flow followed by constant temperature,

(3) zero heat flow followed by constant heat flow, and
(4) constant heat flow.

Optimum operating conditions have been determined. For constant temperature and constant heat flow conditions the optimum initial temperatures, on a scale of 0 to 1 from cold to hot surface, are 0.5 and $2/\pi$. For zero heat flow followed by constant heat flow the optimum hot surface temperature at which to supply the heat flux is 0.811 to 0.812.

The ranking of the hot surface conditions changes when the responses for optimum conditions are compared. Referring to the numbers above, these are (1), (2), (4), and (3). Figures 5, 6, and 7 give a fuller comparison.

The results are in a form that can be utilized to improve the design and operation of thermal test equipment. They are also in the correct form for incorporation in standard test methods for thermal transference properties.

Acknowledgments

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APPENDIX

Basic Equation

The nondimensionalized heat equation for an infinite slab with uniform boundary conditions is

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau} \tag{12}$$

where

$\theta = [T(x,t) - T(0,\infty)]/[T(L,\infty) - T(0,\infty)]$,
$X = x/L$, $\tau = \alpha t/L^2$,
x = space variable running from 0 to L, and
t = time, α = the thermal diffusivity.

The nondimensional heat flux is written as

$$\frac{q}{q_1} = \frac{\partial \theta}{\partial X} \tag{13}$$

where

q = heat flux per unit area,
$q_1 = -\lambda\, [T(L,\infty) - T(0,\infty)]/L$, steady-state heat flux,

λ = thermal conductivity, and
∞ = indicates the steady-state value.

Initial Conditions

The following uniform initial condition is assumed:

$$\theta(X, 0) = W \tag{14}$$

where W is a finite constant.

Boundary Conditions (BC)

Constant temperature at $X = 0$

$$\theta(0, \tau) = 0 \tag{15}$$

Four separate boundary conditions at $X = 1$ will be considered:

(1) constant temperature: $\theta(1, \tau) = 1$, (16)

(2) constant heat flux: $\left.\dfrac{\partial \theta}{\partial X}\right|_{1,\tau} = 1$, (17)

(3) with $W > 1$, zero heat flux till $\theta(1, \tau^*) = 1$, then constant temperature $\theta(1, \tau) = 1$, $\tau \geqslant \tau^*$, and (18)

(4) with $W > 1$, zero heat flux till $\theta(1, \tau^*) = S$, $0 \leqslant S \leqslant W$ then constant heat flux: $\left.\dfrac{\partial \theta}{\partial X}\right|_{1,\tau} = 1$, $\tau \geqslant \tau^*$. (19)

In every case the steady-state temperature difference and steady-state heat flux are equal to unity.

Solutions to Basic Problems

Solutions have been determined using equations derived from basic models and the "superposition principle" given by Churchill.[3] The solutions are in terms of the variable not specified at $X = 1$, that is, in terms of heat flux for the constant temperature boundary condition and in terms of temperature for the constant heat flux boundary condition. The solutions are given in general form, then for $X = 1$.

General form of solutions

(*a*) Constant temperature, BC

$$\left.\frac{\partial \theta}{\partial X}\right|_{X,\tau} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \cos(A_n X) \exp(-A_n^2 \tau) + 4W \sum_{n=1}^{\infty} \cos(B_n X) \exp(-B_n^2 \tau) \tag{20}$$

[3] Churchill, R. V., *Operational Mathematics*, 2nd ed., McGraw-Hill, New York, 1958.

where

$A_n = n\pi$, $B_n = (2n-1)\pi$, exp is the exponential function

(*b*) Constant heat flux, BC

$$\theta(X, \tau) = X - 8 \sum_{n=1}^{\infty} [(-1)^{n-1}/B_n^2] \sin(B_n X/2) \exp(-B_n^2 \tau/4)$$

$$+ 4W \sum_{n=1}^{\infty} (1/B_n) \sin(B_n X/2) \exp(-B_n^2 \tau/4) \qquad (21)$$

(*c*) Zero heat flux until $\theta(1, \tau^*) = 1$, followed by constant temperature, $\tau \geqslant \tau^*$

$$\left.\frac{\partial \theta}{\partial X}\right|_{X,\tau} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \cos(n\pi X) \exp(-A_n^2(\tau - \tau^*))$$

$$+ (32W/\pi^3) \sum_{n=1}^{\infty} C(n)(-1)^n A_n^2 \cos(n\pi X) \exp(-A_n^2(\tau - \tau^*)) \qquad (22)$$

where

$$C(n) = \sum_{m=1}^{\infty} (-1)^{m-1} (\pi/B_m)/((B_m/\pi)^2 - 4n^2) \exp(-B_m^2 \tau^*/4) \qquad (23)$$

where τ^* is found by solving the following equation

$$\sum_{n=1}^{\infty} (-1)^{n-1} (1/B_n) \exp(-B_n^2 \tau^*/4) = 1/4W \qquad (24)$$

(*d*) Zero heat flow until $\theta(1, \tau^*) = S$, followed by constant heat flow, $\tau \geqslant \tau^*$

$$\theta(X, \tau) = X - 8 \sum_{n=1}^{\infty} [(-1)^{n-1}/B_n^2] \sin(B_n X/2) \exp(-B_n^2(\tau - \tau^*)/4)$$

$$+ 4W \sum_{n=1}^{\infty} (1/B_n) \sin(B_n X/2) \exp(-B_n^2 \tau/4) \qquad (25)$$

where τ^* is found from the equation

$$\sum_{n=1}^{\infty} (-1)^{n-1} (1/B_n) \exp(-B_n^2 \tau^* 4) = S/4W \qquad (26)$$

Note—If $\tau^* = 0$, 14 reduces to 10.

Solutions for X = *1*

(*a*) Constant temperature, BC

$$\left.\frac{\partial\theta}{\partial X}\right|_{1,\tau} = 1 + 2\sum_{n=1}^{\infty} \exp(-A_n^2\tau) + 4W\sum_{n=1}^{\infty}(-1)^n \exp(-B_n^2\tau) \tag{27}$$

(*b*) Constant heat flux, BC

$$\theta(1,\tau) = 1-\sum_{n=1}^{\infty}[8(1/B_n^2) - 4W((-1)^{n-1}/B_n)]\exp(-B_n^2\tau/4) \tag{28}$$

(*c*) Zero heat flux until $\theta(1,\tau^*) = 1$ followed by constant temperature, $\tau > \tau^*$

$$\left.\frac{\partial\theta}{\partial X}\right|_{1,\tau} = 1 + \sum_{n=1}^{\infty}[2 + (32W/\pi^3)A_n^2\,\mathrm{C}(n)]\exp(-A_n^2(\tau-\tau^*)) \tag{29}$$

where

$$\mathrm{C}(n) = \sum_{m=1}^{\infty}(-1)^{m-1}((\pi/B_m)/(B_m/\pi)^2 - 4n^2)\exp(-B_m^2\tau^*/4) \tag{30}$$

and

$$B_m = (2m-1)\pi$$

τ^* is found by solving

$$\sum_{n=1}^{\infty}(-1)^{n-1}(1/B_n)\exp(-B_n^2\tau^*/4) = A/(4W) \tag{31}$$

where $B_n = (2n-1)\pi$ and $A = 1$.

(*d*) Zero heat flux until $\theta(1, \tau^*) = S$, followed by constant heat flux, $\tau > \tau^*$

$$\theta(1,\tau) = 1-8\sum_{n=1}^{\infty}(1/B_n^2)\exp(-B_n^2(\tau-\tau^*)/4)$$

$$+ 4W\sum_{n=1}^{\infty}(-1)^{n-1}(1/B_n)\exp(-B_n^2\tau/4) \tag{32}$$

where $B_n = (2n-1)\pi$.

τ^* is found by solving Eq 20 with $A = S$

Calculation of Errors

Errors in the determination of thermal conductivity, if measurements are made before steady-state conditions have been attained, are derived as follows.

Constant Temperature, BC, and Case (3)

Error at time τ and position X is expressed as

$$\text{Error} = \epsilon = \frac{q(X,\tau) - q(X,\infty)}{q(X,\infty)}$$

$$= \frac{q(X,\tau)}{q(X,\infty)} - 1$$

$$= \left.\frac{\partial\theta}{\partial X}\right|_{X,\tau} - 1 \text{ from Eq 2} \tag{33}$$

Constant Heat Flux, BC, and Case (4)

Error at time τ and position X is expressed as

$$\text{Error} = \epsilon = \frac{[\theta(X,\infty) - \theta(0,\infty)]}{[\theta(X,\tau) - \theta(0,\tau)]} - 1$$

$$= \frac{X}{\theta(X,\tau)} - 1$$

As $\theta(0,\tau) = \theta(0,\infty) = 0$ and $\theta(X,\infty) = X$

then at $X = 1$

$$\text{Error} = \epsilon = \frac{1}{\theta(1,\tau)} - 1 \tag{34}$$

and rearranging

$$-\epsilon/(1+\epsilon) = \theta(1,\tau) - 1 = H \tag{35}$$

or

$$\epsilon = -H/(1+H) \tag{36}$$

Simplified Expressions for Error Due to Transient, X = 1

Constant Temperature, BC

Truncating Eq 16 at two terms, rearranging, and substituting in Eq 22, gives

$$\epsilon = \left.\frac{\partial\theta}{\partial X}\right|_{1,\tau} - 1 \approx (2-4W)\exp(-\pi^2\tau) + 2\exp(-4\pi^2\tau) \tag{37}$$

Constant Heat Flux BC

Truncating Eq 17 at two terms, rearranging, and substituting in Eq 24, gives

$$\frac{-\epsilon}{1+\epsilon} = \theta(1,\tau) - 1 \approx -(4/\pi)\,((2/\pi)-W)\,\exp(-\pi^2\tau/4) + (4/(3\pi^2))\,((2/(3\pi)) + W)\,\exp(-9\pi^2\tau/4) \qquad (38)$$

then ϵ is found using Eq 25.

Zero Heat Flow Until $\theta(1, \tau^*) = 1$, Followed by Constant Temperature, for $\tau > \tau^*$

Truncating Eq 19 at one term, rearranging, and substituting in Eq 22 gives

$$\epsilon = \left.\frac{\partial\theta}{\partial X}\right|_{1,\tau} - 1 = ([2 + 32C(1)\,W/\pi]\,\exp(\pi^2\tau^*))\,\exp(-\pi^2\tau) \qquad (39)$$

where C(1) is defined in Eq 19.

Note—If $\tau^* = 0$ then $C(1) = -\pi/8$ and, when substituted into the first term of Eq 28, reduces to the first term of Eq 26.

Zero Heat Flow Until $\theta(1, \tau^*) = S$, *Followed by Constant Heat Flux, for* $\tau > \tau^*$

Truncating Eq 21 at two terms, rearranging, and substituting in Eq 24 gives

$$\frac{-\epsilon}{\epsilon+1} = \theta(1,\tau) - 1 = -\,[(8/\pi^2)\,\exp(\pi^2\tau^*/4) - (4W/\pi)]\,\exp(-\pi^2\tau/4) - [(8/9\pi^2)\,\exp(9\pi^2\tau^*/4) + 4W/(3\pi)]\,\exp(-9\pi^2\tau/4) \qquad (40)$$

ϵ is found using Eq 25

Simplification of τ^* and C

The Series in Eq 20 from Which τ^* is Found for Eqs 39 and 40 Can Be Truncated at One Term

$$\exp(-\pi^2\tau^*/4) \approx \pi A/(4W) \qquad (41)$$

or, solving for τ^*

$$\tau^* \approx (4/\pi^2)\ln(4W/(\pi A)) \qquad (42)$$

where ln is the natural logarithm function.

Note—The equation holds for $W \geqslant 1.2$ (see Fig. 6).

Truncating Eq 19, the Series for C(n), *at One Term*

$$C(1) \approx -(1/3)\exp(-\pi^2\tau^*/4) \qquad (43)$$

and substituting Eq 30 into Eq 32, yields

$$C(1) \approx -\pi A/(12W) \tag{43a}$$

where A is either 1 or S, as the case requires.
Similarly,

$$C(2) \approx (1/15) \exp(-\pi^2 \tau^*/4) \tag{44}$$

and substituting Eq 30 into Eq 33, yields

$$C(2) = \pi A/(60W) \tag{44a}$$

Expressions for Settling Time

Constant Temperature, BC, X = *1*

Neglecting the second term in Eq 26 and inverting

$$\tau \approx (1/\pi^2) \ln [(2-4W)/\epsilon], W \neq 1/2, \epsilon \neq 0 \tag{45}$$

When $W = 1/2$, the first term in Eq 26 is zero and the second term can be inverted to give

$$\tau \approx (1/(4\pi^2)) \ln (2/\epsilon), W = 1/2, \epsilon \neq 0 \tag{45a}$$

Note—An optimum exists at $W = 1/2$, where τ is reduced by a factor of approximately 4.

Constant Heat Flow, BC, X = 1

Neglecting the second term in Eq 27, inverting gives

$$\tau \approx (4/\pi^2) \ln [(4/\pi)((2/\pi)-W)(1+\epsilon)/\epsilon], W \neq 2/\pi, \epsilon \neq 0 \text{ or } -1 \tag{46}$$

When $W = 2/\pi$, the first term in Eq 27 is zero and the second term can be inverted to give

$$\begin{aligned}\tau &\approx (4/(9\pi^2)) \ln [32(1+\epsilon)/(9\pi^2 \epsilon)] \\ &\approx (4/(9\pi^2)) \ln [2(1+\epsilon)] - 0.0772, W = 2/\pi, \epsilon \neq 0 \text{ or } 1\end{aligned} \tag{46a}$$

Note—An optimum exists at $W = 2/\pi$, where τ is reduced by a factor of approximately 9.

Zero Heat Flux Until $\theta(1, \tau^*) = 1$, Followed by Constant Temperature, BC, X = *1, for $\tau > \tau^*$*

Equation 28 can be inverted to give

$$\tau = \tau^* + (1/\pi^2) \ln [((1/2) + (8CW/\pi))(4/\epsilon)] \tag{47}$$

Substituting for τ^* and C from Eqs 31 and 33 with $A = 1$ gives a simpler expression which holds for $W \geqslant 1.2$

$$\tau \approx (1/\pi^2) \ln [(4W/\pi)^4 (-2/(3\epsilon))], W \neq 0, \epsilon \neq 0 \tag{48}$$

Note—No optimum W exists.

Zero Heat Flow Until $\theta(1, \tau^*) = S$, Followed by Constant Heat Flux, $X = 1$, for $\tau \geqslant \tau^*$

Neglecting the second term in Eq 26 and inverting

$$\tau \approx (4/\pi^2) \ln [(4/\pi) ((1 + \epsilon)/\epsilon) ([(2/\pi) \exp (\pi^2 \tau^*/4] - W)] \tag{49}$$

Substituting for τ^* and C from Eqs 31 and 33 with $A = S$ gives a simpler expression which holds for $W \geqslant 1.2$

$$\tau \approx (4/\pi^2) \ln [(4/\pi) (W/S) ((8/\pi^2) - S) (1 + \epsilon)/\epsilon] \tag{50}$$

Note—No optimum value for W exists, but there is an optimum S at $S = 8/\pi^2 \approx 0.811$. Calculations with the full series confirm a value between 0.811 and 0.812. (51)

When the first term in Eq 29 is zero, at $S \approx 0.811$, the second term can be inverted to give

$$\tau \approx (4/(9\pi^2)) \ln [(2W/9) (\pi/2)^7 (W^8 + 9(2/\pi)^8) (1 + \epsilon)/\epsilon]$$

$$S = 8/\pi^2, \epsilon \neq 0 \text{ or } -1$$

$$\tau \approx (4/(9\pi^2)) \ln [2W^9/9) (\pi/2)^7 (1 + \epsilon)/\epsilon]$$

$$\approx (4/(9\pi^2)) \ln [2.(1 + \epsilon)/\epsilon] + (4/\pi^2) \ln W + 0.043 \tag{51a}$$

Note—At optimum S, τ is reduced by a factor of approximately 9.

Equation 39 no longer reduced to Eq 35 when $S = W$, due to the error in finding τ^* near $W = 1.0$.

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