Chapter 3 Exercises: Solutions

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Answers in Brief (for the impatient)

3.1 **a)** $\mathbf{N}_a\mathbf{M}_\mu$ is a Kraus operator for the identity superoperator $\mathcal{I}$.

b) $\mathbf{M}_\nu^\dagger\mathbf{M}_\mu = \sum_a \mathbf{M}_\nu^\dagger\mathbf{N}_a^\dagger\mathbf{N}_a\mathbf{M}_\mu = \sum_a(\mathbf{N}_a\mathbf{M}_\nu)^\dagger\mathbf{N}_a\mathbf{M}_\mu$
$= (\sum_a \lambda_{a\nu}^*\lambda_{a\mu})\mathbf{1} = \gamma_{\nu\mu}\mathbf{1}$.

c) (b) implies $M_\nu \propto M_\mu$ for each μ,ν, so the Kraus decomposition has only one term and is hence unitary.

3.2 $N^4 - N^2$

3.3 **a)** $\frac{Q\hbar}{m\omega^2x^2} = 10^{-18}\,\mathrm{s}$ (fast)

b) $\frac{Q\hbar}{m\omega^2x^2}\frac{\hbar\omega}{k_BT} = 10^{-31}\,\mathrm{s}$ (*really* fast)

3.4 **a)** $N_0 = \sqrt{1-\frac{p}{2}}\mathbf{1}$, $N_1 = \sqrt{\frac{p}{2}}\boldsymbol{\sigma}_3$

b) $U = \begin{pmatrix} \sqrt{\frac{2-2p}{2-p}} & 0 & -e^{i\varphi}\sqrt{\frac{p}{2-p}} \\ \sqrt{\frac{p}{4-2p}} & \frac{1}{\sqrt{2}} & e^{i\varphi}\sqrt{\frac{1-p}{2-p}} \\ \sqrt{\frac{p}{4-2p}} & \frac{-1}{\sqrt{2}} & e^{i\varphi}\sqrt{\frac{1-p}{2-p}} \end{pmatrix}$, φ is free.

c) $N_0 = \sqrt{1-\frac{\varepsilon p}{2}}\mathbf{1}$, $N_1 = \sqrt{\frac{\varepsilon p}{2}}\boldsymbol{\sigma}_3$

d) $\Gamma_{decoh} = \varepsilon\Gamma_{scatt}$

3.5 **a)** $\vec{P} \mapsto [(1-p)P_1, (1-p)P_2, P_3]$

b) $\vec{P} \mapsto [\sqrt{1-p}P_1, \sqrt{1-p}P_2, P_3 + p(1-P_3)]$

c) $\vec{P} \mapsto [(1-p)P_1, (1-2p)P_2, (1-p)P_3]$

3.6 **a)** $\partial_t X = -\frac{\Gamma}{2}\vec{\boldsymbol{\lambda}}\cdot\nabla_{\vec{\lambda}} X$; $X(\vec{\boldsymbol{\lambda}},t) = X(\vec{\boldsymbol{\lambda}}e^{-\Gamma t/2},0)$; $\vec{\boldsymbol{\lambda}} \equiv (\lambda,\lambda^*)$

b) $\rho_{cat}(t) = \frac{1}{2}\begin{pmatrix} 1 & \langle\alpha_1|\alpha_2\rangle \\ \langle\alpha_2|\alpha_1\rangle & 1 \end{pmatrix}^{(1-e^{-\Gamma t})}$ in time-varying $\left(\begin{smallmatrix}|\alpha_1 e^{-\Gamma t/2}\rangle \\ |\alpha_2 e^{-\Gamma t/2}\rangle\end{smallmatrix}\right)$ basis.

Off-diagonal terms decay at a rate proportional to $e^{-|\alpha_1-\alpha_2|^2\frac{\Gamma}{2}t}$

3.1 Invertibility of Superoperators

a) Let $\mathcal{M}$ be a superoperator and suppose $\mathcal{M}$ has a superoperator left inverse such that $\mathcal{N} \circ \mathcal{M} = \mathcal{I}$. By the Kraus Representation Theorem, $\mathcal{M}$ and $\mathcal{N}$ have operator-sum representations:

$$\begin{aligned} \mathcal{M}(\rho) &= \sum_\mu \mathbf{M}_\mu \rho \mathbf{M}_\mu^\dagger \\ \mathcal{N}(\rho) &= \sum_a \mathbf{N}_a \rho \mathbf{N}_a^\dagger \end{aligned}$$

$$\sum_\mu \mathbf{M}_\mu^\dagger \mathbf{M}_\mu = \sum_a \mathbf{N}_a^\dagger \mathbf{N}_a = \mathbf{1}.$$

Moreover, the operator-sum representation for their composition is expressible in terms of the operators $\mathbf{R}_{\{a\mu\}} = \mathbf{N}_a \mathbf{M}_\mu$:

$$\begin{aligned} \sum_{a\mu} \mathbf{R}_{\{a\mu\}} \rho \mathbf{R}_{\{a\mu\}}^\dagger &= \sum_{a\mu} \mathbf{N}_a \mathbf{M}_\mu \rho \left(\mathbf{N}_a \mathbf{M}_\mu\right)^\dagger \\ &= \sum_{a\mu} \mathbf{N}_a \mathbf{M}_\mu \rho \mathbf{M}_\mu^\dagger \mathbf{N}_a^\dagger \\ &= \mathcal{N} \circ \mathcal{M}(\rho) \end{aligned}$$

$$\begin{aligned} \sum_{a\mu} \mathbf{R}_{\{a\mu\}}^\dagger \mathbf{R}_{\{a\mu\}} &= \sum_{a\mu} \left(\mathbf{N}_a \mathbf{M}_\mu\right)^\dagger \mathbf{N}_a \mathbf{M}_\mu \\ &= \sum_{a\mu} \mathbf{M}_\mu^\dagger \mathbf{N}_a^\dagger \mathbf{N}_a \mathbf{M}_\mu \\ &= \sum_\mu \mathbf{M}_\mu^\dagger \left(\sum_a \mathbf{N}_a^\dagger \mathbf{N}_a \right) \mathbf{M}_\mu \\ &= \mathbf{1} \end{aligned}$$

However, since $\mathcal{N} \circ \mathcal{M} = \mathcal{I}$, the operators $\mathbf{R}_{\{a\mu\}}$ must also be the Kraus operators for the identity superoperator, which has the trivial representation $\mathcal{I}(\rho) = \mathbf{1}\rho\mathbf{1}^\dagger$. Since the most general ambiguity of the Kraus operators is a rotation by a unitary matrix, it follows that $\mathbf{N}_a \mathbf{M}_\mu = \lambda_{a\mu}\mathbf{1}$, where $\lambda_{a\mu}$ is an element of a unitary matrix and $\sum_{a\mu} |\lambda_{a\mu}|^2 = \mathbf{1}$ by normalization of the columns of the unitary matrix.

b) By insertion of the identity $\sum_a \mathbf{N}_a^\dagger \mathbf{N}_a = \mathbf{1}$ and the relation $\mathbf{N}_a \mathbf{M}_\mu = \lambda_{a\mu}\mathbf{1}$ from part (a), we obtain the desired result:

$$\begin{aligned} \mathbf{M}_\nu^\dagger \mathbf{M}_\mu &= \sum_a \mathbf{M}_\nu^\dagger \mathbf{N}_a^\dagger \mathbf{N}_a \mathbf{M}_\mu \\ &= \sum_a (\mathbf{N}_a \mathbf{M}_\nu)^\dagger \mathbf{N}_a \mathbf{M}_\mu \end{aligned}$$

$$
\begin{aligned}
&= (\sum_a \lambda^*_{a\nu}\lambda_{a\mu})\mathbf{1} \\
&= \gamma_{\nu\mu}\mathbf{1}.
\end{aligned}
$$

c) From part (b) we know that $\mathbf{M}^\dagger_\nu \mathbf{M}_\mu = \gamma_{\nu\mu}\mathbf{1}$. It suffices to show that this implies that all the $\mathbf{M}$ operators are proportional to one another, since then the Kraus decomposition has only one term, which by normalization must be unitary.

Because we are only considering nonzero operators, we know that $\gamma_{\mu\mu} \neq 0$. Thus

$$
\begin{aligned}
\det \mathbf{M}^\dagger_\mu \mathbf{M}_\mu &= \det \gamma_{\mu\mu}\mathbf{1} \\
\det \mathbf{M}^\dagger_\mu \det \mathbf{M}_\mu &= \left(\gamma_{\mu\mu}\right)^n \\
\left(\det \mathbf{M}_\mu\right)^* \det \mathbf{M}_\mu &\neq 0 \\
\det \mathbf{M}_\mu &\neq 0
\end{aligned}
$$

Therefore each of the $\mathbf{M}_\mu$ must be invertible and in particular we find

$$
\begin{aligned}
\mathbf{M}^\dagger_\mu \mathbf{M}_\mu &= \gamma_{\mu\mu}\mathbf{1} \\
\mathbf{M}^\dagger_\mu &= \gamma_{\mu\mu}\mathbf{M}^{-1}_\mu.
\end{aligned}
$$

From this it follows that all the $\mathbf{M}$ operators are proportional to one another:

$$
\begin{aligned}
\mathbf{M}^\dagger_\nu \mathbf{M}_\mu &= \gamma_{\nu\mu}\mathbf{1} \\
\mathbf{M}_\nu \mathbf{M}^\dagger_\nu \mathbf{M}_\mu &= \gamma_{\nu\mu}\mathbf{M}_\nu \\
\mathbf{M}_\nu \left(\gamma_{\nu\nu}\mathbf{M}^{-1}_\nu\right) \mathbf{M}_\mu &= \gamma_{\nu\mu}\mathbf{M}_\nu \\
\mathbf{M}_\mu &= \frac{\gamma_{\nu\mu}}{\gamma_{\nu\nu}}\mathbf{M}_\nu.
\end{aligned}
$$

3.2 How Many Superoperators?

We saw in class that there are three equivalent ways of stating that $ is a superoperator:

1.) $ takes density matrices to density matrices.

2.) $ is a completely positive linear map which preserves the hermiticity and trace of its arguments.

3.) $ has an operator-sum representation.

One should be able to count the number of degrees of freedom for \$ using any one of these pictures. I shall only outline approaches using pictures (1) and (3) here. In each of the approaches I take ρ to be an $N \times N$ density operator which completely describes a mixed state (*i.e.* an ensemble of pure states) in a Hilbert space of dimension N.

The density matrix ρ is a unit-trace Hermitian matrix and therefore has $N^2 - 1$ free parameters. However it is wrong to think of \$ as just a "jumbling-up" of these parameters. The basis for ρ is actually N^2-dimensional, and ρ may be written as $\rho = \frac{1}{2}\left(\mathbf{1} + \vec{\alpha} \cdot \vec{\boldsymbol{\lambda}}\right)$, where the $\boldsymbol{\lambda}_i$ are $N^2 - 1$ linearly independent basis matrices. Written this way, we see that \$ is free to not only "jumble-up" the $\boldsymbol{\lambda}$ matrices by changing $\vec{\alpha}$, but also can map the identity to a linear combination of $\mathbf{1}$ and $\vec{\boldsymbol{\lambda}}$:

$$\$\left(\tfrac{1}{2}\mathbf{1}\right) = \tfrac{1}{2}\left(\mathbf{1} + \vec{\beta} \cdot \vec{\boldsymbol{\lambda}}\right) \text{ for some } \vec{\beta}.$$

The number of free parameters in this counting is then $\left(N^2 - 1\right)^2$ parameters for the mapping of $\vec{\boldsymbol{\lambda}}$ and $N^2 - 1$ parameters for the affine shift of $\mathbf{1}$, resulting in a total count of $\left(N^2 - 1\right)^2 + N^2 - 1 = N^4 - N^2$ real parameters.

If you are not convinced about the existence of the affine shift, see how the origin of the Bloch sphere shifts under the action of the amplitude-damping channel in problem 5b.

Since \$ has an operator-sum representation, we may write it as

$$\$(\rho) = \sum_{\mu} \mathbf{M}_{\mu} \rho \mathbf{M}_{\mu}^{\dagger}$$

where each $\mathbf{M}_{\mu} \in GL(N, \mathcal{C})$ has $2N^2$ real parameters. There are N^2 linearly independent matrices in $GL(N, \mathcal{C})$, which means that, *prima facie*, \$ has at most $2N^2\left(N^2\right) = N^4$ real parameters.

The $\mathbf{M}_{\mu}$ matrices must also satisfy the normalization condition

$$\sum_{\mu} \mathbf{M}_{\mu}^{\dagger} \mathbf{M}_{\mu} = \mathbf{1}.$$

This only imposes N^2 additional constraints since the Hermitian conjugate of this equation is identical. Finally, we saw in class that the most general ambiguity in the specification of the $\mathbf{M}_{\mu}$ matrices is a unitary reshuffling of the operators:

$$\mathbf{M}_{\mu} \to U_{\mu\nu} \mathbf{M}_{\nu}$$

Since there are at most N^2 $\mathbf{M}_{\mu}$ matrices, $U_{\mu\nu} \in U\left(N^2\right)$ has N^4 real parameters. Therefore we find that \$ has at most $2N^4 - N^2 - N^4 = N^4 - N^2$ real parameters.

In either counting, we find that \$ has at most $N^4 - N^2$ real parameters.

3.3 How Fast is Decoherence?

a) The equation of motion for a damped simple harmonic oscillator is

$$m\ddot{x} + b\dot{x} + \omega^2 x = 0.$$

For small damping we expect the mean energy of the oscillator to decay exponentially:

$$\langle E(t) \rangle = E_0 e^{-\frac{b}{m}t}.$$

Thus the *amplitude* of the oscillations should decay like $e^{-\frac{b}{2m}t}$. From classical mechanics or elsewhere, we recall that the quality factor for small damping is defined as

$$\begin{aligned} Q &= 2\pi \left(\frac{\text{Total Energy}}{\text{Energy loss/period}} \right) \\ &= \frac{\omega}{(b/m)}. \end{aligned}$$

In class we found that decoherence is well-modeled by the phase-damping channel. From the master equation for this channel, we discovered that the off-diagonal terms of the density matrix in the coherent-state basis decay as

$$\rho_{nm}(t) = \rho_{nm}(0) e^{-\frac{\Gamma}{2}|n-m|^2 t}$$

where Γ is the scattering rate for a single particle in the oscillator with its environment. This form for the decay suggests that we interpret Γ as the coefficient of an effective radiative damping force with a quality factor

$$Q = \frac{\omega}{\Gamma}.$$

The time it takes for a system to decohere is, to order-of-magnitude, the time it takes for its off-diagonal terms to decay to $1/e$ of their original values:

$$t_{decoh} = \frac{2}{\Gamma \left| n - m \right|^2}.$$

The cat state we are given in the problem is not expressed in the coherent-state basis. However, for a highly localized Gaussian wave packet, we expect the eigenstate of the annihilation operator to be roughly proportional to the eigenstate of the $\widehat{x}$ operator:

$$\begin{aligned} \widehat{a} &= \sqrt{\frac{m\omega}{2\hbar}} \left(\widehat{x} - \frac{i}{m\omega} \widehat{p} \right) \\ \langle \widehat{a} \rangle &= \sqrt{\frac{m\omega}{2\hbar}} \left(\langle \widehat{x} \rangle - \frac{i}{m\omega} \langle \widehat{p} \rangle \right) \\ &= \sqrt{\frac{m\omega}{2\hbar}} x \end{aligned}$$

We therefore expect the exponent of the off-diagonal density matrix elements to be on the order of:

$$\begin{aligned} |n-m|^2 &= \frac{m\omega}{2\hbar}|x-(-x)|^2 \\ &= \frac{2m\omega x^2}{\hbar} \end{aligned}$$

We now have all the ingredients to calculate the decoherence time for the pendulum:

$$\begin{aligned} t_{decoh} &= \frac{2}{\Gamma|n-m|^2} \\ &= \frac{2Q\hbar}{\omega(2m\omega x^2)} \\ &= \frac{Q\hbar}{m\omega^2 x^2} \\ &= \frac{10^9 \cdot 10^{-34}\ \mathrm{J\ s^{-1}}}{10^{-3}\ \mathrm{kg} \cdot 1\ \mathrm{s^{-2}} \cdot 10^{-4}\ \mathrm{m^2}} \\ &= 10^{-18}\ \mathrm{s} \end{aligned}$$

b) At zero temperature, all energy levels of the oscillator were coupled to the ground state of the environment. At finite temperature, $n = \frac{kT}{\hbar\omega}$ states of the environment are available for coupling. Thus, to order-of-magnitude, the decay rate is n times faster. The time for decoherence should correspondingly diminish by this factor:

$$\begin{aligned} t_{decoh}(T) &= \frac{\hbar\omega}{kT} t_{decoh}(0) \\ &= \frac{10^{-34}\ \mathrm{J\ s^{-1}} \cdot 1\ \mathrm{s^{-1}}}{10^{-23}\ \mathrm{J\ K^{-1}} \cdot 10^{2}\ \mathrm{K}} \cdot 10^{-18}\ \mathrm{s} \\ &= 10^{-31}\ \mathrm{s} \end{aligned}$$

The moral: Decoherence is fast! It is one of the fastest physical processes we know of to date.

3.4 Phase Damping

a) By inspection we see that $\mathbf{M}_0$, $\mathbf{M}_1$, and $\mathbf{M}_2$ depend only on two linearly independent matrices ($\mathbf{1}$ and $\boldsymbol{\sigma}_3$), which suggests that an operator-sum representation using only *two* Kraus operators is possible. (In fact any time a set of Kraus operators depends on n linearly independent matrices we can find a representation with n operators.)

Let's see explicitly how the $\mathbf{M}$ operators act on a general density matrix ρ:

$$\begin{aligned} \rho &\rightarrow \mathbf{M}_0\rho\mathbf{M}_0^\dagger + \mathbf{M}_1\rho\mathbf{M}_1^\dagger + \mathbf{M}_2\rho\mathbf{M}_2^\dagger \\ &= (1-p)\rho + \frac{p}{4}(\mathbf{1}+\boldsymbol{\sigma}_3)\rho(\mathbf{1}+\boldsymbol{\sigma}_3) + \frac{p}{4}(\mathbf{1}-\boldsymbol{\sigma}_3)\rho(\mathbf{1}-\boldsymbol{\sigma}_3) \\ &= \left(1-\frac{p}{2}\right)\rho + \frac{p}{2}\boldsymbol{\sigma}_3\rho\boldsymbol{\sigma}_3. \end{aligned}$$

This form suggests we take

$$\begin{aligned} \mathbf{N}_0 &= \sqrt{1-\frac{p}{2}}\mathbf{1} \\ \mathbf{N}_1 &= \sqrt{\frac{p}{2}}\boldsymbol{\sigma}_3 \end{aligned}$$

as our Kraus operators for the channel. Indeed, $\mathbf{N}_0$ and $\mathbf{N}_1$ satisfy $\mathbf{N}_0^\dagger\mathbf{N}_0 + \mathbf{N}_1^\dagger\mathbf{N}_1 = \mathbf{1}$, and hence are properly normalized as well.

b) The relation $\mathbf{M}_\mu = U_{\mu a}\mathbf{N}_a$ results in the following set of equations for the components of $U_{\mu a}$:

$$\begin{aligned} \sqrt{1-p}\mathbf{1} &= U_{00}\left(\sqrt{1-\frac{p}{2}}\mathbf{1}\right) + U_{01}\left(\sqrt{\frac{p}{2}}\boldsymbol{\sigma}_3\right) \\ \sqrt{\frac{p}{4}}(\mathbf{1}+\boldsymbol{\sigma}_3) &= U_{10}\left(\sqrt{1-\frac{p}{2}}\mathbf{1}\right) + U_{11}\left(\sqrt{\frac{p}{2}}\boldsymbol{\sigma}_3\right) \\ \sqrt{\frac{p}{4}}(\mathbf{1}-\boldsymbol{\sigma}_3) &= U_{20}\left(\sqrt{1-\frac{p}{2}}\mathbf{1}\right) + U_{21}\left(\sqrt{\frac{p}{2}}\boldsymbol{\sigma}_3\right) \end{aligned}$$

Matching terms using the linear independence of $\mathbf{1}$ and $\boldsymbol{\sigma}_3$ yields:

$$\begin{array}{cc} U_{00} = \sqrt{\frac{2-2p}{2-p}} & U_{01} = 0 \\ U_{10} = \sqrt{\frac{p}{4-2p}} & U_{11} = \sqrt{\frac{1}{2}} \\ U_{20} = \sqrt{\frac{p}{4-2p}} & U_{21} = -\sqrt{\frac{1}{2}} \end{array}$$

All we have left to do is unitarily complete the matrix U by requiring that all the rows and columns of U be orthonormal to one another:

$$\begin{aligned} |U_{00}|^2 + |U_{01}|^2 + |U_{02}|^2 &= 1 \Longrightarrow U_{02} = e^{i\theta}\sqrt{\frac{p}{2-p}} \\ |U_{10}|^2 + |U_{11}|^2 + |U_{12}|^2 &= 1 \Longrightarrow U_{12} = e^{i\varphi}\sqrt{\frac{1-p}{2-p}} \\ |U_{20}|^2 + |U_{21}|^2 + |U_{22}|^2 &= 1 \Longrightarrow U_{22} = e^{i\psi}\sqrt{\frac{1-p}{2-p}} \end{aligned}$$

$$\begin{aligned} U_{01}^*U_{02} + U_{11}^*U_{12} + U_{21}^*U_{22} &= 0 \Longrightarrow e^{i(\varphi-\psi)} = 0 \Rightarrow \varphi = \psi \\ U_{00}^*U_{02} + U_{10}^*U_{12} + U_{20}^*U_{22} &= 0 \Longrightarrow e^{i\theta} = e^{i\varphi} \end{aligned}$$

There are no remaining constraints, so there is an overall phase ambiguity ($\mathbf{N}_2 = 0$ can't have a well-defined phase!):

$$U = \begin{pmatrix} \sqrt{\frac{2-2p}{2-p}} & 0 & -e^{i\varphi}\sqrt{\frac{p}{2-p}} \\ \sqrt{\frac{p}{4-2p}} & \sqrt{\frac{1}{2}} & e^{i\varphi}\sqrt{\frac{1-p}{2-p}} \\ \sqrt{\frac{p}{4-2p}} & -\sqrt{\frac{1}{2}} & e^{i\varphi}\sqrt{\frac{1-p}{2-p}} \end{pmatrix}$$

c) The Kraus operators for a channel having a unitary representation U_{AE} are defined to be:

$$\mathbf{M}_\mu \equiv \langle \mu_E | \, U_{AE} \, |0_E\rangle$$

where the $|\mu\rangle_E$ are *orthogonal* states of the environment. We can form an orthonormal basis for the environment out of $\{|0\rangle_E, |\gamma_0\rangle_E, |\gamma_1\rangle_E\}$ in several ways. Applying the Gram-Schmidt process to this set is one method, but instead I shall choose a basis which captures the symmetry between $|\gamma_0\rangle_E$ and $|\gamma_1\rangle_E$:

$$|\pm\rangle_E = \frac{\alpha_\pm}{\sqrt{2}} \left(|\gamma_0\rangle_E \pm |\gamma_1\rangle_E \right)$$

$$\begin{aligned} \langle \pm | \pm \rangle &= 1 \\ &\Rightarrow \; |\alpha_\pm|^2 \left(1 \pm \langle \gamma_0 | \gamma_1 \rangle \right) = 1 \\ &\Rightarrow \; \alpha_\pm = \sqrt{\frac{1}{1 \pm (1-\varepsilon)}} \end{aligned}$$

Relative to this basis, the Kraus operators are

$$\begin{aligned} \mathbf{M}_0 &= \langle 0_E | \, U_{AE} \, |0_E\rangle \\ &= \sqrt{1-p}\,\mathbf{1} \\ \mathbf{M}_\pm &= \langle \pm_E | \, U_{AE} \, |0_E\rangle \\ &= \sqrt{\frac{p}{1 \pm (1-\varepsilon)}} \begin{pmatrix} 1 \pm (1-\varepsilon) & 0 \\ 0 & \pm (1 \pm (1-\varepsilon)) \end{pmatrix} \\ &= \sqrt{p\,(1 \pm (1-\varepsilon))} \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \end{aligned}$$

These do not look like phase-damping channel operators, but we could rotate them into three operators which do. Better yet, we can rotate the operators into *two* operators which look like phase damping operators. As in part (a), we examine the action of these operators on a general ρ to find these two operators:

$$\begin{aligned} \rho &\rightarrow \mathbf{M}_0 \rho \mathbf{M}_0^\dagger + \mathbf{M}_+ \rho \mathbf{M}_+^\dagger + \mathbf{M}_- \rho \mathbf{M}_-^\dagger \\ &= (1-p)\,\rho + \frac{p\,(2-\varepsilon)}{2}\rho + \frac{p\varepsilon}{2} \boldsymbol{\sigma}_3 \rho \boldsymbol{\sigma}_3 \\ &= \left(1 - \frac{p\varepsilon}{2}\right)\rho + \frac{p\varepsilon}{2} \boldsymbol{\sigma}_3 \rho \boldsymbol{\sigma}_3 \end{aligned}$$

$$\begin{aligned} \mathbf{N}_0 &= \sqrt{1 - \frac{p\varepsilon}{2}}\mathbf{1} \\ \mathbf{N}_1 &= \sqrt{\frac{p\varepsilon}{2}}\boldsymbol{\sigma}_3 \end{aligned}$$

In this form it is manifest that these Kraus operators are for a phase-damping channel having probability εp of decohering with its environment. Note that as $\varepsilon \to 1$ we recover the phase damping channel of part (a), and as $\varepsilon \to 0$ the phase damping vanishes because the state of A no longer induces conditional dynamics on how the environment scatters from it.

d) If the channel in (c) describes a single photon scattering, we have $\Gamma_{scatt} = p\Delta t$. But decoherence only occurs when the environment can distinguish the scattering outcomes, so $\Gamma_{decoh} = \varepsilon p \Delta t$. Therefore we find that:

$$\Gamma_{decoh} = \varepsilon \Gamma_{scatt}.$$

3.5 Decoherence on the Bloch Sphere

a) Under the action of the phase-damping channel, $\rho = \frac{1}{2}\left(\mathbf{1} + \vec{P}\cdot\vec{\boldsymbol{\sigma}}\right)$ evolves as: (using the $\mathbf{M}_\mu$ operators from problem 3.4)

$$\begin{aligned}
\rho &\to \mathbf{M}_0\rho\mathbf{M}_0^\dagger + \mathbf{M}_1\rho\mathbf{M}_1^\dagger + \mathbf{M}_2\rho\mathbf{M}_2^\dagger \\
&= (1-p)\,\rho + \frac{p}{4}\left(\mathbf{1}+\boldsymbol{\sigma}_3\right)\rho\left(\mathbf{1}+\boldsymbol{\sigma}_3\right) + \frac{p}{4}\left(\mathbf{1}-\boldsymbol{\sigma}_3\right)\rho\left(\mathbf{1}-\boldsymbol{\sigma}_3\right) \\
&= \left(1-\frac{p}{2}\right)\rho + \frac{p}{2}\boldsymbol{\sigma}_3\rho\boldsymbol{\sigma}_3 \\
&= \left(1-\frac{p}{2}\right)\rho + \frac{p}{2}\left[\frac{1}{2}\left(\mathbf{1}+\boldsymbol{\sigma}_3\left(\vec{P}\cdot\vec{\boldsymbol{\sigma}}\right)\boldsymbol{\sigma}_3\right)\right] \\
&= \left(1-\frac{p}{2}\right)\rho + \frac{p}{2}\left[\frac{1}{2}\left(\mathbf{1}-\vec{P}\cdot\vec{\boldsymbol{\sigma}}+2P_3\boldsymbol{\sigma}_3\right)\right] \\
&= \frac{1}{2}\left[\left(1-\frac{p}{2}+\frac{p}{2}\right)\mathbf{1} + \left(1-\frac{p}{2}-\frac{p}{2}\right)\vec{P}\cdot\vec{\boldsymbol{\sigma}}+pP_3\boldsymbol{\sigma}_3\right] \\
&= \frac{1}{2}\left[\mathbf{1}+(1-p)\,\vec{P}\cdot\vec{\boldsymbol{\sigma}}+pP_3\boldsymbol{\sigma}_3\right] \\
&= \frac{1}{2}\left[\mathbf{1}+((1-p)P_1,\ (1-p)P_2,\ P_3)\right]
\end{aligned}$$

Thus we see that the action of the phase-damping channel is to contract the Bloch sphere to a prolate spheroid about the z–axis. The preferential treatment of the z–direction indicates that the phase-damping channel acts in a preferred basis.

b) Under the action of the amplitude-damping channel, ρ evolves as:

$$\begin{aligned}
\rho &\to \mathbf{M}_0\rho\mathbf{M}_0^\dagger + \mathbf{M}_1\rho\mathbf{M}_1^\dagger \\
&= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p}\end{pmatrix}\left(\mathbf{1}+\vec{P}\cdot\vec{\boldsymbol{\sigma}}\right)\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p}\end{pmatrix}\right] \\
&\quad+\frac{1}{2}\left[\begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0\end{pmatrix}\left(\mathbf{1}+\vec{P}\cdot\vec{\boldsymbol{\sigma}}\right)\begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0\end{pmatrix}\right] \\
&= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p}\end{pmatrix}\begin{pmatrix} 1+P_3 & P_1-iP_2 \\ P_1+iP_2 & 1-P_3\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p}\end{pmatrix}\right]
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}\left[\begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+P_3 & P_1-iP_2 \\ P_1+iP_2 & 1-P_3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix}\right] \\
= & \ \frac{1}{2}\begin{pmatrix} 1+P_3+p-pP_3 & P_1-iP_2\sqrt{(1-p)} \\ \sqrt{(1-p)}\,(P_1+iP_2) & 1-P_3-p+pP_3 \end{pmatrix} \\
= & \ \frac{1}{2}\left[\mathbf{1}+\left(\sqrt{1-p}P_1,\ \sqrt{1-p}P_2,\ P_3+p(1-P_3)\right)\cdot\vec{\boldsymbol{\sigma}}\right]
\end{aligned}
$$

Thus we see that the action of the amplitude-damping channel is to contract the Bloch sphere into an oblate spheroid about the z–axis and shift it upwards.

c) Under the action of the "two-Pauli" channel, ρ evolves as:

$$
\begin{aligned}
\rho \quad \rightarrow & \quad \mathbf{M}_0\rho\mathbf{M}_0^\dagger+\mathbf{M}_1\rho\mathbf{M}_1^\dagger+\mathbf{M}_2\rho\mathbf{M}_2^\dagger \\
= & \quad (1-p)\,\rho+\frac{p}{2}\boldsymbol{\sigma}_1\rho\boldsymbol{\sigma}_1+\frac{p}{2}\boldsymbol{\sigma}_3\rho\boldsymbol{\sigma}_3 \\
= & \quad (1-p)\,\rho+\frac{p}{2}\left[\frac{1}{2}\left(\mathbf{1}+\boldsymbol{\sigma}_1\left(\vec{P}\cdot\vec{\boldsymbol{\sigma}}\right)\boldsymbol{\sigma}_1\right)+\frac{1}{2}\left(\mathbf{1}+\boldsymbol{\sigma}_3\left(\vec{P}\cdot\vec{\boldsymbol{\sigma}}\right)\boldsymbol{\sigma}_3\right)\right] \\
= & \quad (1-p)\,\rho+\frac{p}{2}\left[\mathbf{1}-\frac{1}{2}\vec{P}\cdot\vec{\boldsymbol{\sigma}}+P_1\boldsymbol{\sigma}_1-\frac{1}{2}\vec{P}\cdot\vec{\boldsymbol{\sigma}}+P_3\boldsymbol{\sigma}_3\right] \\
= & \quad \frac{1}{2}\left[(1-p+p)\,\mathbf{1}+(1-p-p)\,\vec{P}\cdot\vec{\boldsymbol{\sigma}}+p\vec{P}\cdot\vec{\boldsymbol{\sigma}}-pP_2\boldsymbol{\sigma}_2\right] \\
= & \quad \frac{1}{2}\left[\mathbf{1}+((1-p)P_1,(1-2p)P_2,(1-p)P_3)\cdot\vec{\boldsymbol{\sigma}}\right]
\end{aligned}
$$

Thus we see that the action of the two-Pauli channel is to contract the Bloch sphere into an oblate spheroid about the y–axis for $p<\frac{1}{2}$ and to an inverted prolate spheroid about the y–axis for $p>\frac{1}{2}$.

3.6 Decoherence of the damned oscillator

a) Consider the time derivative of X:

$$
\begin{aligned}
\dot{X} \quad = & \quad \mathrm{Tr}\ \left[\dot{\rho}_I(t)e^{\lambda\mathbf{a}^\dagger}e^{-\lambda^*\mathbf{a}}\right] \\
= & \quad \Gamma\ \mathrm{Tr}\ \left[\left(\mathbf{a}\rho_I\mathbf{a}^\dagger-\frac{1}{2}\mathbf{a}^\dagger\mathbf{a}\rho_I-\frac{1}{2}\rho_I\mathbf{a}^\dagger\mathbf{a}\right)e^{\lambda\mathbf{a}^\dagger}e^{-\lambda^*\mathbf{a}}\right]
\end{aligned}
$$

To simplify this expression, we wish to bring the second two terms in the trace into the same form as the first one (in an attempt to cancel them as best as possible). We may do this by using the cyclic property of the trace and the commutation relations for the annihilation and creation operators:

$$
\begin{aligned}
\left[\mathbf{a},\mathbf{a}^\dagger\right] \quad = & \quad \mathbf{1} \\
\left[\mathbf{a},e^{\lambda\mathbf{a}^\dagger}\right] \quad = & \quad \left[\mathbf{a},\mathbf{a}^\dagger\right]\frac{\partial}{\partial\mathbf{a}^\dagger}\left(e^{\lambda\mathbf{a}^\dagger}\right)=\lambda e^{\lambda\mathbf{a}^\dagger} \\
\left[e^{-\lambda^*\mathbf{a}},\mathbf{a}^\dagger\right] \quad = & \quad \frac{\partial}{\partial\mathbf{a}}\left(e^{-\lambda^*\mathbf{a}}\right)\left[\mathbf{a},\mathbf{a}^\dagger\right]=-\lambda^*e^{-\lambda^*\mathbf{a}}.
\end{aligned}
$$

Applying these manipulations to $\dot{X}$ we find:

$$\begin{aligned}\dot{X} &= \Gamma\ \mathrm{Tr}\ \left[\left(\mathbf{a}\rho_I\mathbf{a}^\dagger - \frac{1}{2}\mathbf{a}\rho_I\left(\mathbf{a}^\dagger - \lambda^*\right) - \frac{1}{2}\left(\mathbf{a}+\lambda\right)\rho_I\mathbf{a}^\dagger\right)e^{\lambda\mathbf{a}^\dagger}e^{-\lambda^*\mathbf{a}}\right] \\ &= \frac{\Gamma}{2}\ \mathrm{Tr}\ \left[\lambda^*\rho_I e^{\lambda\mathbf{a}^\dagger}\mathbf{a}e^{-\lambda^*\mathbf{a}} - \lambda\rho_I\mathbf{a}^\dagger e^{\lambda\mathbf{a}^\dagger}e^{-\lambda^*\mathbf{a}}\right]\end{aligned}$$

We may pull the extra annihilation an creation operators back into the exponentials by use of the derivatives

$$\begin{aligned}\frac{\partial}{\partial\lambda^*}e^{-\lambda^*\mathbf{a}} &= -\mathbf{a}e^{-\lambda^*\mathbf{a}} \\ \frac{\partial}{\partial\lambda}e^{\lambda\mathbf{a}^\dagger} &= \mathbf{a}^\dagger e^{\lambda\mathbf{a}^\dagger}\end{aligned}$$

so that we obtain a PDE for X:

$$\begin{aligned}\dot{X} &= -\frac{\Gamma}{2}\ \mathrm{Tr}\ \left[\lambda^*\rho_I e^{\lambda\mathbf{a}^\dagger}\frac{\partial}{\partial\lambda^*}\left(e^{-\lambda^*\mathbf{a}}\right) + \lambda\rho_I\frac{\partial}{\partial\lambda}\left(e^{\lambda\mathbf{a}^\dagger}\right)e^{-\lambda^*\mathbf{a}}\right] \\ &= -\frac{\Gamma}{2}\ \lambda^*\frac{\partial}{\partial\lambda^*}\mathrm{Tr}\ \left[\rho_I e^{\lambda\mathbf{a}^\dagger}e^{-\lambda^*\mathbf{a}}\right] - \frac{\Gamma}{2}\ \lambda\frac{\partial}{\partial\lambda}\mathrm{Tr}\ \left[\rho_I e^{\lambda\mathbf{a}^\dagger}e^{-\lambda^*\mathbf{a}}\right] \\ &= -\frac{\Gamma}{2}\left(\ \lambda^*\frac{\partial X}{\partial\lambda^*} +\ \lambda\frac{\partial X}{\partial\lambda}\right).\end{aligned}$$

I have been somewhat cavalier here in commuting the operations of differentiation and trace. For the present purposes I shall assume all relevant functions are uniformly continuous so that this commutation is allowed.

If we use the chain rule, we may rewrite this as a linear PDE:

$$\dot{X} = -\frac{\Gamma}{2}\left(\ \frac{\partial X}{\partial\ln\lambda^*} + \frac{\partial X}{\partial\ln\lambda}\right).$$

For this linear PDE, it is natural to assume that the solution depends only linearly upon its arguments:

$$X = X\left(\alpha\ln\lambda^* + \beta\ln\lambda + \gamma t\right).$$

Substituting this *ansatz* into the PDE, we find that the coefficients are related:

$$\gamma = -\frac{\Gamma}{2}\left(\alpha+\beta\right).$$

This relation gives us the scaling law we seek:

$$\begin{aligned}X\left(\vec{\lambda},t\right) &= X\left(\alpha\ln\lambda^* + \beta\ln\lambda - \frac{\Gamma}{2}\left(\alpha+\beta\right)t\right) \\ &= X\left(\alpha\ln\lambda^* e^{-\frac{\Gamma}{2}t} + \beta\ln\lambda e^{-\frac{\Gamma}{2}t}\right) \\ &= X\left(\vec{\lambda}',0\right)\end{aligned}$$

$$\vec{\lambda}'\left(\vec{\lambda},\Gamma,t\right)=\vec{\lambda}e^{-\frac{\Gamma}{2}t}$$

b) Before beginning, I should mention that this cat is not only not normal in the everyday sense, but also in the Born interpretation sense as well. To properly normalize the cat, we need "bra-cat ket-cat" to be unity:

$$\begin{aligned}|cat\rangle &= \frac{N}{\sqrt{2}}\left(|\alpha_1\rangle+|\alpha_2\rangle\right)\\ \langle cat\ |cat\rangle &= \frac{|N|^2}{2}\left(\langle\alpha_1\ |\alpha_1\rangle+\langle\alpha_1\ |\alpha_2\rangle+\langle\alpha_2\ |\alpha_1\rangle+\langle\alpha_2\ |\alpha_2\rangle\right)=1\end{aligned}$$

Rather than going further into the details of normalizing the cat at this point, which would be distracting, I shall simply note that there should be a normalization factor in front of the cat.

The results of part (a) tell us how to relate a trace of the cat and an operator at time t to the trace at time $t=0$. However, the operator in the trace is the "displacement" operator (see, *e.g.* Peres) $\mathbf{D}_\lambda=e^{\lambda\mathbf{a}^\dagger}e^{-\lambda^*\mathbf{a}}$ which takes one coherent state to another. Since any observable for the oscillator may be expanded in these shift operators, knowing how the action of the shift operators on the cat evolves with time completely specifies the time evolution of the cat.

More specifically, part (a) tells us that:

$$\mathrm{Tr}\ \left[|cat(0)\rangle\langle cat(0)|\,e^{\lambda'\mathbf{a}^\dagger}e^{-\lambda'^*\mathbf{a}}\right]=\mathrm{Tr}\ \left[\rho_{cat}(t)e^{\lambda\mathbf{a}^\dagger}e^{-\lambda^*\mathbf{a}}\right]$$

In general, $\rho_{cat}(t)$ does not have to be a pure state (in fact we will find it won't be), but for the time being let us imagine that it is. This enables us to turn the traces into expectation values. Aided by the inner product between coherent states

$$\begin{aligned}\langle\alpha\ |\beta\rangle &= e^{\alpha^*\beta-(|\alpha|^2+|\beta|^2)/2}\\ &= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\,\mathrm{Re}(\alpha^*\beta))}e^{i\,\mathrm{Im}(\alpha^*\beta)}\\ &= e^{-\frac{1}{2}|\alpha-\beta|^2}e^{i\,\mathrm{Im}(\alpha^*\beta)}\end{aligned}$$

we find that

$$\langle cat(t)|\,e^{\lambda\mathbf{a}^\dagger}e^{-\lambda^*\mathbf{a}}\,|cat(t)\rangle = \langle cat(0)|\,e^{\lambda'\mathbf{a}^\dagger}e^{-\lambda'^*\mathbf{a}}\,|cat(0)\rangle$$

$$\begin{pmatrix} e^{\lambda\alpha_1^*(t)-\lambda^*\alpha_1(t)}+\\ e^{\lambda\alpha_2^*(t)-\lambda^*\alpha_2(t)}+\\ \langle\alpha_1(t)\ |\alpha_2(t)\rangle\,e^{\lambda\alpha_1^*(t)-\lambda^*\alpha_2(t)}+\\ \langle\alpha_2(t)\ |\alpha_1(t)\rangle\,e^{\lambda\alpha_2^*(t)-\lambda^*\alpha_1(t)}\end{pmatrix} = \begin{pmatrix} e^{(\lambda\alpha_1^*-\lambda^*\alpha_1)e^{-\Gamma t/2}}+\\ e^{(\lambda\alpha_2^*-\lambda^*\alpha_2)e^{-\Gamma t/2}}+\\ \langle\alpha_1\ |\alpha_2\rangle\,e^{(\lambda\alpha_1^*-\lambda^*\alpha_2)e^{-\Gamma t/2}}+\\ \langle\alpha_2\ |\alpha_1\rangle\,e^{(\lambda\alpha_2^*-\lambda^*\alpha_1)e^{-\Gamma t/2}}\end{pmatrix}.$$

These terms could almost be brought into agreement by assuming time evolution takes pure states into pure states, but this causes the off-diagonal terms to mismatch:

$$\begin{aligned}
\left|\alpha_{1,2}\right\rangle &\mapsto \left|\alpha_{1,2}e^{-\Gamma t/2}\right\rangle \\
e^{\lambda \mathbf{a}^\dagger} e^{-\lambda^* \mathbf{a}} \left|\alpha_{1,2}e^{-\Gamma t/2}\right\rangle &= e^{(\lambda\alpha_{1,2}^* - \lambda^*\alpha_{1,2})e^{-\Gamma t/2}} \left|\alpha_{1,2}e^{-\Gamma t/2}\right\rangle \\
\left\langle \alpha_{1,2}e^{-\Gamma t/2} \middle| \alpha_{2,1}e^{-\Gamma t/2}\right\rangle &= \left\langle \alpha_2 \middle| \alpha_1\right\rangle^{e^{-\Gamma t}} \neq \left\langle \alpha_2 \middle| \alpha_1\right\rangle
\end{aligned}$$

To correct for this off-diagonal mismatch, we must demand that the off-diagonal terms of the cat decay more rapidly than the diagonal components, even as the basis for the coherent states is decaying. This leads to the full mixed-state evolution:

$$\begin{aligned}
\left|\alpha_1\right\rangle\left\langle\alpha_1\right| &\mapsto \left|\alpha_1 e^{-\Gamma t/2}\right\rangle\left\langle\alpha_1 e^{-\Gamma t/2}\right| \\
\left|\alpha_2\right\rangle\left\langle\alpha_2\right| &\mapsto \left|\alpha_2 e^{-\Gamma t/2}\right\rangle\left\langle\alpha_2 e^{-\Gamma t/2}\right| \\
\left|\alpha_1\right\rangle\left\langle\alpha_2\right| &\mapsto \left\langle\alpha_1 \middle| \alpha_2\right\rangle^{(1-e^{-\Gamma t})} \left|\alpha_1 e^{-\Gamma t/2}\right\rangle\left\langle\alpha_2 e^{-\Gamma t/2}\right| \\
\left|\alpha_2\right\rangle\left\langle\alpha_1\right| &\mapsto \left\langle\alpha_2 \middle| \alpha_1\right\rangle^{(1-e^{-\Gamma t})} \left|\alpha_2 e^{-\Gamma t/2}\right\rangle\left\langle\alpha_1 e^{-\Gamma t/2}\right|
\end{aligned}$$

Thus our cat evolves into something more diagonal:

$$\begin{aligned}
\left|cat(0)\right\rangle\left\langle cat(0)\right| &\mapsto \frac{|N|^2}{2}\begin{pmatrix} 1 & \left\langle\alpha_1 \middle| \alpha_2\right\rangle^{(1-e^{-\Gamma t})} \\ \left\langle\alpha_2 \middle| \alpha_1\right\rangle^{(1-e^{-\Gamma t})} & 1 \end{pmatrix} \\
&= \frac{|N|^2}{2}\left[\mathbf{1} + e^{-\frac{1}{2}|\alpha_1-\alpha_2|^2\left(1-e^{\Gamma t}\right)}\left(\sigma_x \cos\theta_{21}(t) + \sigma_y \sin\theta_{21}(t)\right)\right]
\end{aligned}$$

where the density matrix above is expressed in the time-varying basis $\binom{|\alpha_1 e^{-\Gamma t/2}\rangle}{|\alpha_2 e^{-\Gamma t/2}\rangle}$ and the rotation angle $\theta_{21}(t)$ is defined as $\theta_{21}(t) = \mathrm{Im}(\alpha_2^*\alpha_1)(1-e^{-\Gamma t})$.

If we are only considering the decay of the off-diagonal terms, we may ignore the phase $\theta_{21}(t)$. For times $t \ll 1/\Gamma$, the basis states stay roughly the same:

$$\begin{aligned}
\left|\alpha_1(t)\right\rangle &\cong \left|\alpha_1\left(1-\frac{\Gamma t}{2}\right)\right\rangle \\
&\cong \left|\alpha_1\right\rangle,
\end{aligned}$$

and the amplitudes of the off-diagonal terms decay exponentially with time:

$$\begin{aligned}
e^{-\frac{1}{2}|\alpha_1-\alpha_2|^2\left(1-e^{\Gamma t}\right)} &\cong e^{-\frac{1}{2}|\alpha_1-\alpha_2|^2(1-(1-\Gamma t))} \\
&\cong e^{-\frac{\Gamma t}{2}|\alpha_1-\alpha_2|^2}
\end{aligned}$$