Spin-bath narrowing with adaptive parameter estimation

Paola Cappellaro

Nuclear Science and Engineering Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

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We present a measurement scheme capable of achieving the quantum limit of parameter estimation using an adaptive strategy that minimizes the parameter’s variance at each step. The adaptive rule we propose makes the scheme robust against errors, in particular imperfect readouts, a critical requirement to extend adaptive schemes from quantum optics to solid-state sensors. Thanks to recent advances in single-shot readout capabilities for electronic spins in the solid state (such as nitrogen vacancy centers in diamond), this scheme can also be applied to estimate the polarization of a spin bath coupled to the sensor spin. In turns, the measurement process decreases the entropy of the spin bath resulting in longer coherence times of the sensor spin.

A common strategy for estimating an unknown parameter associated with a field is to prepare a probe and let it interact with the parameter-dependent field. From the probe dynamics, it is possible to derive an estimator of the parameter. The process is repeated many times to reduce the estimation uncertainty. A more efficient procedure takes advantage of the partial knowledge acquired in each successive measurement to change the probe-field interaction in order to optimize the uncertainty reduction at each step. This adaptive Bayesian scheme to the measurement of a quantum parameter: 

\[ P(b|m) \propto P^{(0)}(b)P_\theta(m|b). \]

More generally, after each measurement we can update the probability for the phase \( \varphi = b\tau \), so that after \( n \) such measurements with outcomes \( \varphi_n \), we have a p.d.f.

\[ P^{(n)}(\varphi|\varphi_n) \propto P^{(n-1)}(\varphi|\varphi_{n-1})P_\theta(\varphi_n|\varphi). \]

Thanks to the periodicity of the probability \( P(\varphi) \), we can expand it in Fourier series \[\text{[4]}, \right P^{(n)}(\varphi) = \sum_k P^{(n)}(\varphi) e^{i\varphi k}, \text{so that we can rewrite Eq. (2) as} \]

\[ P^{(n)}(\varphi|\varphi_n) \propto \frac{1}{2} P^{(n-1)}(\varphi) + \frac{1}{2} e^{-i\tau/T} \left[ e^{i(m_n+\varphi)} P^{(n-1)}(\varphi) + e^{-i(m_n+\varphi)} P^{(n-1)}(\varphi) \right]. \]

The proportionality factor is set by imposing that \( P^{(0)}(\varphi) = \frac{1}{2\pi} \) as required for a normalized p.d.f. We can further generalize this expression when the system is allowed to evolve for an integer multiple \( t_\varphi \) of the time \( \tau \), thus obtaining a general update rule for the p.d.f.:

\[ P^{(n)}(\varphi) \propto \frac{1}{2} P^{(n-1)}(\varphi) + \frac{1}{2} e^{-i\tau/T} \left[ e^{i(m_n+\varphi)} P^{(n-1)}(\varphi) + e^{-i(m_n+\varphi)} P^{(n-1)}(\varphi) \right]. \]

An adaptive strategy will then seek to choose at each step the optimal \( t_\varphi \) and \( \varphi \) that lead to the most efficient series of \( N \) measurements for a desired final uncertainty.

In order to design an adaptive strategy, we need to define a metric for the uncertainty (and accuracy) of the estimate. The Fourier transform of the p.d.f. can be used to calculate the moments of the distribution as well as other metrics and estimator. From the formula for the moments, \( \langle \varphi^2 \rangle = 2\pi^2 \int \varphi^2 P(\varphi)e^{i\varphi \varphi} d\varphi \), we can calculate the variance

\[ \langle \varphi^2 \rangle - \langle \varphi \rangle^2 = \frac{2\pi^3}{3} p_0 + 4\pi \sum_{k\neq 0} (-1)^k \frac{1}{k^2} p_k - \langle \varphi \rangle^2, \]

where the average is \( \langle \varphi \rangle = 2\pi \int \varphi P(\varphi) e^{i\varphi \varphi} d\varphi \).

The variance is often not the best estimate of the uncertainty for a periodic variable \[\text{[3]. A better metric is the Holevo variance} [9], \]

\[ V_H = (2\pi |\langle e^{i\varphi} \rangle|)^2 - 1 = (2\pi |\langle e^{-i\varphi} \rangle|)^2 - 1, \]

\[ \frac{\text{Var}(\varphi)}{\langle \varphi^2 \rangle - \langle \varphi \rangle^2} \]
where we used the fact that $\langle e^{i\psi} \rangle = p_{-1}$. We further notice that while the absolute value of $p_{-1}$ gives the phase estimate uncertainty, its argument provides an unbiased estimate of $\psi$. More generally, estimates are given by $\psi_{\text{est}} = \arg\langle e^{i\psi_n} \rangle / t = \arg(p_{-1}) / t$, giving new meaning to the Fourier coefficients of the p.d.f.

The goal of the estimation procedure is then to make $|p_{-1}|$ as large as possible. Assume for simplicity $\psi = \beta \tau = 0$ and neglect any relaxation. Then the probability of the outcome $m_x = 0$ is $P_0(0) = \frac{1}{2}(1 - \cos \theta)$. We assume that we do not have any a priori knowledge on the phase, so that $P^{(0)}(\psi) = 1/2\pi$. We fix the number of measurements $N$, each having an interrogation time $T_n = t_0 \tau = 2^{N-n} \tau$ \cite{4,10,11}. A potential strategy would be to maximize $|p_{0}^{(n)}|$ at each step $n$. However, under the assumptions made, $p_{-1}^{(n)} = 0$ until the last step $n = N$, where it is

$$ p_{-1}^{(N)} = \frac{e^{-i(m_0 \pi + \delta_x)} - e^{i\delta_x} + 1}{4\pi}. $$

Writing $p_{-1}^{(N-1)} = q e^{i\chi}$, we have

$$ 4\pi |p_{0}^{(N)}| = \sqrt{1 + 4\pi^2 q^2 + 4\pi q \cos(x + 2\theta_N)}. $$

This is maximized for $\theta_N = -x/2 = \frac{1}{2} \arg(p_{-1}^{(N-1)})$ and by maximizing $q = |p_{-1}^{(N-1)}|$. A similar argument holds for the maximization of $p_{-1}^{(N-2)}$: One has to set $\theta_{N-1} = \frac{1}{2} \arg(p_{-1}^{(N-1)})$ and maximize $|p_{-1}^{(N-2)}|$. By recursion we have that at each step we want to maximize

$$ |p_{-1}^{(n)}| = \left| \frac{e^{-i(m_0 \pi + \delta_x)} - e^{i\delta_x} + 1}{4\pi} \right| 2\pi p_{-1}^{(n-1)} e^{i\delta_x} + 1 \right|.$$

We have thus found a good adaptive rule, which fixes $t_n = 2^{N-n}$ and $\theta_n = \frac{1}{2} \arg(p_{-1}^{(n-1)})$.

With this rule we obtain the standard quantum limit (SQL) for the phase sensitivity, as we now show. Using the optimal phase, the Fourier coefficients $p_{-1}^{(n)}$ are at each step

$$ p_{-1}^{(n)} = \frac{1}{2} \left[ \frac{1}{2\pi} + p_{-1}^{(n-1)} \right] = \frac{1}{2\pi} (1 - 2^{-n}). $$

Then, for a total number of measurements $N$, the Holevo variance is $V_H = 1 - 2^{-N+1} = 2^{-N}$. The total interrogation time is $T = \tau (2^{N+1} - 1)$ yielding

$$ V_H(T) = \frac{4T \tau}{(T - \tau)^2} \approx \frac{4\pi}{T}. \quad (5) $$

We can improve the sensitivity of this scaling and reach the QML by a simple modification of this adaptive scheme. Instead of performing just one measurement of duration $t_n$ at each $n^{th}$ step, we perform two, updating the p.d.f. according to the outcomes. For $\psi = 0$, the update rule at each step is now

$$ p_{k}^{(n)} = \frac{1}{N} [6p_{k}^{(n-1)} + 4p_{k-1}^{(n-1)} + 4p_{k+1}^{(n-1)} + p_{k-2}^{(n-1)} + p_{k+2}^{(n-1)}]. $$

with the normalization factor

$$ N = 2\pi [1 + p_{-2}^{(n-1)} + p_{2}^{(n-1)}] = 2\pi [1 + p_{-2}^{(n-1)} + p_{2}^{(n-1)}]. $$

Restricting the formula above to the terms $p_{-1}^{(n)}$ gives

$$ p_{-1}^{(n)} = \frac{1}{2\pi} \left( 1 - \frac{3}{2^{n+1}} + 1 \right), $$

By recursion this yields

$$ |p_{-1}^{(n)}| = \frac{1}{2\pi} \left( 1 - \frac{3}{2^{n+1}} + 1 \right), $$

from which we obtain a Holevo variance that follows the QML,

$$ V_H = \frac{48T \tau^2 (T + 4\tau)}{(T - 4\tau)^2 (T + 6\tau)^2} \approx \frac{48\pi^2}{T^2}. \quad (7) $$

The classical and quantum scaling of the adaptive scheme with one or two measurements per step is confirmed by the p.d.f. obtained in the two cases (Fig. 1). For one measurement, the final p.d.f. Fourier coefficients are $|p_k| = \frac{1}{2\pi} \left[ 1 - 2^{-(N+1)} |k| \right]$, and the probability is well approximated by a sinc function,

$$ P^{(N)}(\psi) = \frac{2^{N+1}}{2\pi} \text{sinc}(2^{N+1} \psi)^2, $$

which gives a variance $\sigma \approx 2^{-N/2}$. For two measurements per step, the p.d.f. is well approximated by a Gaussian \cite{12} with a width $\sigma = \frac{\sqrt{T}}{T} \times 2^{-N}$.

We now consider possible sources of nonideal behavior. The first generalization is to phases $\psi \neq 0$. In this case, while the SQL is still achieved with the one-measurement scheme, two measurements per step do not always reach the QML. Indeed, at each step there is a probability $P(1 - P)$ that the two measurements will give different results; if this happens at the $n^{th}$ step, we obtain $p_{-1}^{(n)} = 0$, thus failing to properly update the p.d.f. While the probability of failure is low, a solution could be to perform three measurements and update the p.d.f. only based on the majority vote.

We can further consider the cases where the signal decays due to relaxation or there is an imperfect readout. Then the probability (1) becomes

$$ P_0(m|b) = \frac{1}{2} [1 - c(-1)^m e^{-\tau/t} \cos(b \tau + \vartheta)]. $$
In a large magnetic field along the NV axis, the hyperfine interaction between the electronic spin and the nuclear spins is truncated to its secular part, \( \mathcal{H} = S_z \sum_i A_i I_z = S_z A_z \) (where \( S \) denotes the electronic spin, \( I_z \) the nuclear spins). During a Ramsey sequence on resonance with the \( m_i = 0, 1 \) energy levels of the electronic spins, the coupled system evolves as

\[
|\psi(t)\rangle = [\sin(A_z t)|1\rangle + \cos(A_z t)|0\rangle]|\psi_C\rangle.
\]

where \( |\psi_C\rangle \) is the initial state of the nuclear spin bath. The measurement scheme (Ramsey followed by NV readout) is a quantum nondemolition measurement [23–25] for the nuclear spins, since their observable does not evolve (as long as the secular approximation holds). The adaptive process is then equivalent to determining the state-dependent (quantized) phase \( \varphi = (A_z t) \). The uncertainty on the nuclear bath state, \( \rho_C = \sum_a |\psi_a\rangle\langle\psi_a| \varphi_C \), is reflected in the p.d.f of the phase (with an injective relation if the operator \( A_z \) has nondegenerate eigenvalues). Thus updating the phase p.d.f. will update the density operator describing the state of the nuclear bath. After each readout of outcome \( m \), the system is in the state

\[
\rho^{(n)} = |m\rangle\langle m| \rho^{(n-1)}|m\rangle\langle m| = |m\rangle\langle m| \sum_a \mathcal{P}_\alpha(m|\psi_a\rangle \rho^{(n-1)}|\psi_a\rangle\langle\psi_a| \varphi_C, \tag{9}
\]

with \( \mathcal{P}_\alpha(m|\psi_a\rangle = |m\rangle\langle m| |\sin(A_z t)|1\rangle + \cos(A_z t)|0\rangle \varphi_a\rangle \). Note that in this expression the probability update rule is equivalent to Eq. (2) and thus the adaptive procedure ensures that the final state has lower entropy than the initial one.

A difference between measuring a classical field and a quantum operator is that in the latter case the resulting phase is quantized, thus it has a discrete p.d.f. An extreme case is when all the couplings to the \( N_C \) nuclear spins are equal, \( A_k = a, \forall k \). Then the eigenvalues are \( na/2 \), with \(|n| \leq N_C \) integer, each with a degeneracy \( d(n) = (\sum_{n/2}^{N_C} 1) \). The adaptive scheme needs to be modified (e.g., by considering a discrete Fourier transform), we note that since all the eigenvalues are an integer multiple of the smallest, nonzero one (\( a \) for \( N_C \) even, \( a/2 \) for \( N_C \) odd), we only need \( M \) steps, with \( 2M > \sum_{n/2}^{N_C} 1 \) [with minimum interrogation time \( \tau = 2\pi/(2aM) \)], to achieve a perfect measurement of the degenerate phase \( \varphi \) [10]. In the more common scenario where \( A_k \) varies with the nuclear spin position (and \( N_C \) is large enough), the eigenvalues give rise to an almost continuous phase [15], thus it is possible to directly use the adaptive scheme derived above.

As an example of the method, we consider one NV center surrounded by a bath of nuclear spins (\(^{13}\)C with 1.1% natural abundance). At low temperature and for NV with low strain, it is possible to perform a single-shot readout of the electronic spin state with high fidelity in tens of \( \mu s \) [6]. Optical illumination usually enhances the electronic-induced nuclear relaxation [26], due to the nonlinear part of the hyperfine interaction. This effect is however quenched in a high magnetic field (\( B > 1T \)) and the relaxation time is much longer than the measurement time (\( T_1 \geq 3ms \) [27]), a sign of a good QND measurement.

We simulated the Ramsey sequence and adaptive measurement with a bath of \( \sim2600 \) spins around the spin sensor in a large magnetic field. We considered the full anisotropic...
hyperfine interaction between the NV and the $^{13}$C spins and we took into account intrabath couplings with a disjoint cluster approximation [12,28]. Even for the longest evolution time of the Ramsey sequence required by the adaptive scheme, the fidelity $F$ of the signal with the ideal Ramsey oscillation (in the absence of couplings) is maintained. After an eight-step adaptive measurement, the nuclear spin bath is in a narrowed state. We note that in general the adaptive scheme does not polarize the spin bath (indeed a final low polarization state is more probable). However, the bath purity is increased, which is enough to ensure longer coherence times for the sensor spins, since it corresponds to a reduced variance of the phase and hence of the sensor spin dephasing. In Fig. 3, we compare the NV center spectrum for an evolution under a maximally mixed nuclear spin bath and under the narrowed spin bath. The figure shows a remarkable improvement of the NV coherence time.

In conclusion, we described an adaptive measurement scheme that has the potential to achieve the quantum metrology limit for a classical parameter estimation. We analyzed how imperfections in the measurement scheme affect the sensitivity and proposed strategies to overcome these limitations. This result could for example improve the sensitivity of spin-based magnetometers, without recurring to entangled states. In addition, we applied the scheme to the measurement of a quantum parameter, such as arising from the coupling of the sensor to a large spin bath. We showed that the adaptive scheme can be used to prepare the spin bath in a narrowed state: As the number of possible configurations for the spin bath is reduced, the coherence time of the sensor is increased. The scheme could then be a promising strategy to increase the coherence time of qubits, without the need of dynamical decoupling schemes that have large overheads and interfere with some magnetometry and quantum information tasks.

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