Experimental characterization of the 4D tensor monopole and topological nodal rings

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Quantum mechanics predicts the existence of the Dirac and the Yang monopoles. Although their direct experimental observation in high-energy physics is still lacking, these monopoles, together with their associated vector gauge fields, have been demonstrated in synthetic matter. On the other hand, monopoles in even-dimensional spaces have proven more elusive. A potential unifying framework—string theory—that encompasses quantum mechanics promotes the vector gauge fields to tensor gauge fields, and predicts the existence of more exotic tensor monopole in 4D space. Here we report the first experimental observation of a tensor monopole in a 4D parameter space synthesized by the spin degrees of freedom of a single solid-state defect in diamond. Using two complementary methods, we reveal the existence of the tensor monopole through measurements of its quantized topological invariant. By introducing a fictitious external field that breaks chiral symmetry, we further observe a novel phase transition to a topological nodal ring semimetal phase that is protected by mirror symmetries.

INTRODUCTION

Our current understanding of fundamental physical phenomena relies on two main pillars, general relativity, and quantum field theory. Their mutual incompatibility, however, poses critical limitations to the formulation of a unifying theory of all fundamental interactions. String theory proposes a powerful and elegant formalism to unify gravitational and quantum phenomena, providing a concrete route to quantum gravity. Within this scenario, conventional point-like particles are replaced by extended objects, such as closed and open strings, and conventional vector gauge fields are promoted to tensor (Kalb-Ramond) gauge fields. In direct analogy with the Dirac monopole, tensor gauge fields can emanate from point-like defects called tensor monopoles. In four spatial dimensions, the charge of tensor monopoles is quantized according to the topological Dixmier-Douady (DD) invariant, which generalizes the Chern number associated with the Dirac monopole.

Experimental evidence of magnetic monopoles is still lacking in high-energy-physics experiments. However, synthetic monopoles related to effective gauge fields have been recently detected in ultracold matter. Besides, momentum-space monopoles play a central role in topological matter, in particular, in the characterization of 3D Weyl semimetals. Very recently, the notions of tensor monopoles and DD invariants were shown to arise in 3D chiral topological insulators and in higher-order topological insulators.

Here we report the first experimental observation of the tensor monopole in a minimal synthetic system. We engineer a three-level model defined over a 4D parameter space using the ground state spin-1 manifold of the nitrogen-vacancy (NV) center in diamond. The model preserves the chiral symmetry and two mirror symmetries, with the tensor monopole residing at the origin of the synthesized parameter space. We present two complementary methods to experimentally measure the DD invariant that characterizes the tensor monopole. The first approach uses the quantum metric tensor to reconstruct the generalized 3-form Berry curvature and ultimately the DD invariant. The second method reveals the DD invariant through the tensor Berry connection, which plays the role of a tensor gauge potential arising from the monopole.

We further explore a novel phase transition by inducing a fictitious $S_z$ field to the system. This breaks the chiral symmetry of the system, giving rise to two doubly-degenerate nodal rings protected by mirror symmetries. Using the same experimental observables based on the metric tensor and the tensor Berry connection, we characterize this novel symmetry-protected topological phase and show the behavior of the nodal rings as we increase the external $S_z$ field strength.

Our results represent the first experimental observation of tensor monopoles in a solid-state qutrit and pave the way for simulating and implementing exotic topological phases inspired by string theory in fully controlled solid-state quantum systems. This work further lays the foundation for simulating singularity points in exotic gauge fields by exploiting the exquisite control of engineered quantum systems.
Quantum Geometric Tensor, Tensor Berry Connection and the Generalized Berry Curvature

We begin by introducing the generalized 3-form Berry curvature and the relevant Berry connection and metric tensor in 4D space. The minimal Weyl-type Hamiltonian hosting a tensor monopole is supported by a 3-band Hamiltonian\cite{14,16,19}:

$$\mathcal{H}_{4D} = \begin{pmatrix} 0 & q_x - i q_y & 0 \\ q_x + i q_y & 0 & q_z + i q_w \\ 0 & q_z - i q_w & 0 \end{pmatrix}, \quad (1)$$

defined in a 4D parameter space spanned by $q = (q_x, q_y, q_z, q_w) \in \mathbb{R}^4$. We note that the system preserves the chiral symmetry and two mirror symmetries, which protect the system keeping it gapless (Eq. S23, S26 in Supplementary Material\cite{17}). The second Chern number given by Chern-Simons theory\cite{18} is not well-defined and cannot be used to characterize the manifold topology.

The triple degenerate point of Eq. 1 located at the origin corresponds to a tensor monopole (Fig. 1 (a,b)), whose charge is determined by the DD invariant:

$$\mathcal{D} \equiv \frac{1}{2 \pi^2} \int_{S^3} \mathcal{H}_{\mu\nu\lambda} dq^\mu \wedge dq^\nu \wedge dq^\lambda, \quad (2)$$

where $\mathcal{H}_{\mu\nu\lambda}$ is the generalized 3-form (Berry) curvature associated with the ground state $|u_-\rangle$ of Eq. 1, and $(\mu, \nu, \lambda)$ are parameters of the Hamiltonian.

Both methods to measure the monopole DD require either the real or imaginary part of the quantum geometric tensor (QGT)\cite{14,16,19}. Consider the ground eigenstate $|u(q)\rangle$ of a generic quantum system $\mathcal{H}(q)$ parametrized by the generalized momentum $q$. QGT describes the geometry of this quantum state manifold and is defined as:

$$\chi_{\mu\nu} = \langle \partial_\mu u | (1 - |u\rangle \langle u|) \partial_\nu u \rangle \equiv g_{\mu\nu} + i \mathcal{F}_{\mu\nu}/2, \quad (3)$$

where $g_{\mu\nu}$ is the Fubini-Study metric tensor that defines the distance between nearby states $|u(q)\rangle, |u(q + dq)\rangle$; the imaginary part is related to the antisymmetric 2-form Berry curvature $\mathcal{F}_{\mu\nu}$ (We will refer to it as the Berry curvature, not to be confused with the 3-form curvature $\mathcal{H}_{\mu\nu\lambda}$). In 4D space, it was recently shown that the 3-form curvature is related to the metric tensor\cite{16} under chiral symmetry,

$$\mathcal{H}_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda}(\sqrt{\text{det}(g_{\mu\nu})}), \quad (4)$$

where $\mu, \nu, \lambda = \{\mu, \nu, \lambda\}$ and $\epsilon_{\mu\nu\lambda}$ is the Levi-Civita symbol. This allows measurement of the 3-form curvature through the metric tensor, which is related to experimentally observable transition rates\cite{12,20}.

Another method starts from the definition of the 3-form curvature as the external derivative of the tensor Berry connection $B_{\mu\nu}$:

$$\mathcal{H}_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}. \quad (5)$$

Here the tensor connection can be reconstructed from the Berry curvature\cite{14}:

$$B_{\mu\nu} = \Phi \mathcal{F}_{\mu\nu}, \quad \Phi = -\frac{i}{2} \log(u_1 u_2 u_3) \quad (6)$$

with $u_{1(2,3)}$ being components of the ground state vector $|u_-\rangle$. From Eq. 5, 6, it follows that in general, the 3-form curvature can be obtained through measurements of the Berry curvature combined with quantum state tomography, upon perturbations of system parameters.

RESULTS

Experiment Control

To synthesize the 4D Hamiltonian in Eq. 1, we use the ground triplet states of a single NV center in diamond (Fig. 1 (c)) at room temperature. An external magnetic field, $B = 490G$, is applied along the N-V axis to lift the degeneracy between $|m_s = \pm 1\rangle$. We apply a dual-frequency microwave pulse\cite{21}, on-resonance with the $|m_s = 0\rangle \leftrightarrow |m_s = \pm 1\rangle$ transitions. In the doubly rotating frame after the rotating wave approximation, we obtain the effective Hamiltonian (see Supplementary Material\cite{17}):

$$\mathcal{H} = \begin{pmatrix} B_{z}/\sqrt{2} & H_0 \cos(\alpha) e^{-i\beta} & 0 \\ H_0 \cos(\alpha) e^{i\beta} & 0 & H_0 \sin(\alpha) e^{i\phi} \\ 0 & -B_{z}/\sqrt{2} & 0 \end{pmatrix} \quad (7)$$

where $\alpha \in [0, \pi/2]$ and $\beta, \phi \in [0, 2\pi]$. We remark that at fixed $H_0$, our Hamiltonian can also be seen as analogous to a 3D periodic lattice model where $\alpha, \beta, \phi$ play the role of crystal momenta (see Supplementary Material\cite{17}).

The above Hamiltonian has three eigenstates $|u_-\rangle, |u_0\rangle$ and $|u_+\rangle$ with eigenvalues $-\epsilon$, $\epsilon_0$, $\epsilon_+$ in ascending order. When $B_{z} = 0$, Eq. 7 reproduces the minimal tensor monopole model in Eq. 1, with parametrization $(q_x = H_0 \cos \alpha \cos \beta, q_y = H_0 \cos \alpha \sin \beta, q_z = H_0 \sin \alpha \cos \phi, q_w = H_0 \sin \alpha \sin \phi)$. Precise modulations of the microwave frequencies, amplitudes and phases grant us full access to the 4D parameter space spanned by $(H_0, \alpha, \beta, \phi)$. We note that the above parameterization makes our system rotationally symmetric about $\beta, \phi$. Therefore the QGT and the 3-form curvatures are independent of either $\beta, \phi$.

$$\chi = \begin{pmatrix} \frac{1}{2} \sin(2\alpha) & \frac{i \sin(2\alpha)}{4} & -\frac{i \sin(2\alpha)}{4} \\ \frac{i \sin(2\alpha)}{4} & \frac{\cos^2(\alpha) - \cos(2\alpha)}{4} & -\frac{\sin^2(\alpha) - \sin^2(\alpha)}{16} \\ -\frac{i \sin(2\alpha)}{4} & -\frac{\sin^2(\alpha) - \sin^2(\alpha)}{16} & \frac{\sin^2(\alpha) - \sin^2(\alpha)}{4} \end{pmatrix}. \quad (8)$$

As a demonstration of our engineered system, we initialize the NV in the $|m_s = 0\rangle$ state, and let it evolve under the target Hamiltonian with $(H_0 = 2 \text{ MHz}, \beta = \phi = 0)$. For various $\alpha$, the resulting oscillations of all three states show excellent agreement with theory, as shown in Fig. 1(d) and Fig. S6 of the Supplementary Material\cite{17}.
When engineering our system to match the Weyl-type Hamiltonian, we choose the microwave amplitudes such that the parameters span a hypersphere with fixed radius $R_0 = H_0 = \sqrt{q_x^2 + q_y^2 + q_z^2 + q_w^2} = 2$ MHz, which encloses the tensor monopole at the origin.

Measuring the Quantum Geometric Tensor through Parametric Modulations

We now show how to measure the QGT using weak modulations of the parameters $\mu, \nu \in \{\alpha, \beta, \phi\}^{12,20}$. Setting $\mu_t = \mu_0 + m_\mu \sin(\omega t)$, $\nu_t = \nu_0 + m_\nu \sin(\omega t)$ (linear) or $\mu_t = \mu_0 + m_\mu \cos(\omega t)$, $\nu_t = \nu_0 + m_\nu \sin(\omega t)$ (elliptical), with $m_\mu, m_\nu < 1$, yields

$$\mathcal{H} \approx \mathcal{H}(\alpha_0, \beta_0, \phi_0) + m_\mu \partial_\mu \mathcal{H} \sin(\omega t)$$

+ $m_\mu \partial_\mu \mathcal{H} \cos(\omega t)$ (linear)

+ $m_\mu \partial_\mu \mathcal{H}$ (elliptical).

When the modulation frequency is resonant with the energy gap between ground and excited state, $\omega = \epsilon_0 - \epsilon_-$ ($\omega = \epsilon_+ - \epsilon_-$), the parametric modulation will coherently drive Rabi oscillations between $|u_-\rangle \leftrightarrow |u_0\rangle$ ($|u_-\rangle \leftrightarrow |u_+\rangle$). We call the $|u_-\rangle \leftrightarrow |u_0\rangle$ transition single quantum (SQ) transition and the $|u_-\rangle \leftrightarrow |u_+\rangle$ double quantum (DQ) transition, following the change in quantum number. Their Rabi frequencies are directly related to the transition matrix elements when varying one parameter $\Gamma_{\mu,-n}^\mu = |\langle u_- | \partial_\mu \mathcal{H} | n \rangle|$, or when linearly (elliptically) modulating two parameters, $\Gamma_{\mu,-n}^{\mu,\nu} = |\langle u_- | \partial_\mu \mathcal{H} \pm (i) \partial_\nu \mathcal{H} | n \rangle|$, $m_\mu = \pm m_\nu$ (see Supplementary Material\textsuperscript{17}). Here the subscript for the matrix element $-n$ stands for the transition between eigenstates $|u_-\rangle \leftrightarrow |n\rangle$. Finally, we reconstruct the diagonal and off-diagonal components of the QGT from the relations\textsuperscript{12} (see Supplementary Material\textsuperscript{17})

$$g_{\mu\mu} = \sum_{n\neq -1} \frac{(\Gamma_{-n}^{\mu})^2}{(\epsilon_+ - \epsilon_-)^2},$$

$$g_{\mu\nu} = \sum_{n\neq -1} \frac{[(\Gamma_{-n}^{\mu})^2 - (\Gamma_{-n}^{\nu})^2]}{4\epsilon_+ \epsilon_-}$$

(elliptical),

$$F_{\mu\nu} = \sum_{n\neq -1} \frac{[(\Gamma_{-n}^{\mu})^2 - (\Gamma_{-n}^{\nu})^2]}{2\epsilon_+ \epsilon_-}$$

(linear),

In each experiment needed to measure the metric tensor, we first initialize the NV in the $|m_s = 0\rangle$ state and coherently drive it to the ground state $|u_-\rangle$ of the Weyl Hamiltonian by two microwave pulses. The system is then subjected to the linear parametric modulation in Eq. \textsuperscript{9} that resonantly drives Rabi oscillations between eigenstates, and finally either the $|u_-\rangle$ or $|u_0\rangle$ state is mapped back to $|m_s = 0\rangle$ by microwave pulses, and optically read out (see Supplementary Material\textsuperscript{17}).

We start our measurements by precisely determining the resonant frequency $\omega_r = \epsilon_+ - \epsilon_- = 2(\epsilon_0 - \epsilon_-)$.

As shown in Fig. 2 (a), we fix the time and sweep the modulation frequency $\omega$ to find the resonance condition. We choose a very weak modulation amplitude to reduce power broadening and improve the precision in estimating $\omega_r$.

We then measure the coherent Rabi oscillations under linear parametric modulations at the calibrated $\omega = \omega_r/2$ ($\omega_r$) for SQ (DQ) transitions. Examples of SQ and DQ Rabi curves are shown in Fig. 2 (b) and Fig. S9-S13\textsuperscript{17}, including both single- and two-parameter modulations, associated with measuring the diagonal and off-diagonal components of the metric tensor. For every combination of modulations $\mu(\mu, \nu)$, we measure both the SQ and DQ Rabi frequencies and recover the matrix element $\Gamma_{\mu,-n}^{\mu,\nu}$ (see Supplementary Material\textsuperscript{17}). All measured matrix elements $\Gamma$ are plotted in Fig. 2 (c,d), showing good agreement with theoretical predictions.

Revealing the Tensor Monopole

As the main results of this work, we reconstruct the metric tensor and use it to determine the 3-form curvature and the $DD$ invariant.

The independent components of the metric tensor, reconstructed using Eq. 10, are shown in Fig. 3 (a). The excellent agreement between theory and our experiment demonstrates our exquisite control over the Weyl Hamiltonian, providing precise information about the geometry of the ground state manifold.

We then connect the metric tensor to the 3-form curvature using Eq. 4. The measured 3-form curvature $\mathcal{H}_{\alpha\beta\phi}$ is shown in Fig. 3 (b). We further experimentally reveal the existence of the tensor monopole characterized by a quantized unit $DD$ invariant:

$$DD_{expt} = \frac{1}{2\pi^2} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta \int_0^{2\pi} d\phi \mathcal{H}_{\alpha\beta\phi} = 0.99(3).$$

(11)

Alternatively, one can identify the tensor monopole via the Berry connection (Eq. 5, 6), where the Berry curvatures are measured through elliptical parametric modulations. This method, however, generally requires quantum state tomography that is time-consuming. Interestingly, the symmetries in our parametrization in Eq. 7 greatly simplifies the required measurements (see Supplementary Material\textsuperscript{17}) yielding:

$$\mathcal{H}_{\alpha\beta\phi} = -\frac{1}{2} (F_{\alpha\beta} + F_{\beta\alpha}).$$

(12)

We show the measured Berry curvatures in Fig. 3 (c) and the 3-form curvature in Fig. 3 (d). The associated matrix element measurements are covered in Fig. S14. This second method, supplementary to the metric tensor measurements, further confirms the existence of the tensor monopole through the measurement of its quantized
charge:
\[
\mathcal{D} \mathcal{D}_{\text{expt}} = \frac{1}{2\pi^2} \int_0^{\frac{\pi}{2}} d\alpha \int_0^{2\pi} d\beta \int_0^{2\pi} d\phi \mathcal{H}_{\alpha \beta \phi} = 1.11(3).
\] (13)

**Topological Phase Transition**

To further explore the $\mathcal{D} \mathcal{D}$ invariant as a topological characterization, we use it to study topological phase transitions. The most straightforward phase transition is induced by translating the hypersphere along one of the parameter axis, e.g., $q_x \rightarrow q_x + \delta x$. A sharp transition of $\mathcal{D} \mathcal{D} = 1 \rightarrow 0$ at $\delta z = H_0$ indicates that the tensor monopole moves out of the integration hypersphere\(^{17,22}\), resulting in a transition from a Weyl semimetal phase to a normal insulator phase inside the hypersphere. While this transition could be measured with the methods described above (see Supplementary Material\(^{17}\)), the translation breaks the symmetry about $\beta$ in our parameterization, and $\mathcal{H}_{\alpha \beta \phi}$ now depends on both $\alpha$ and $\beta$, resulting in long measurement times.

We explore instead a novel phase transition induced by adding a longitudinal field $B_z$ to the Weyl-like Hamiltonian. The field is achieved by detuning the dual-frequency microwave pulse to produce additional diagonal components $\frac{B_z}{\sqrt{2}} S_z$ in the engineered Hamiltonian (Eq. 7). The detuning acts like a $z$ field that breaks the chiral symmetry but preserves the mirror symmetries.

By introducing the external field, the system simulates a topological phase transition from the 4D Weyl semimetal phase to a new symmetry-protected topological nodal ring phase. In view of the energy spectrum, the triply degenerate Weyl node at the origin now becomes two emergent unconventional fermions, manifested by a pair of doubly degenerate nodal surfaces in $\beta - \phi$ space along $(\alpha = 0(\pi/2), B_z = H_0)$. Viewed from the $(q_x, q_y, q_z, q_\omega)$ space, this corresponds to a nodal ring in the $q_x - q_y$ ($q_x - q_\omega$) space along $q_x = q_\omega = 0$ ($q_x = q_y = 0$) with radius $B_z$, as shown in Fig. 4. Because nodal rings are more commonly encountered and studied in 3D\(^{23}\) than nodal surfaces, we will refer to the nodal structure in our model as nodal rings. These nodal rings are protected by mirror symmetries, which can be broken by introducing terms proportional to $\lambda_{4(5)}$ Gell-Mann matrices, and the degenerate rings become gapped (see Supplementary Material\(^{17}\)).

We further remark that at fixed $\beta, \phi$, with a proper unitary transformation, our Hamiltonian shares the same band structure as the three-band linearized $k \cdot p$ model for the space group (SG) SG220\(^{17,24}\), which hosts crystalline symmetry-protected free fermions beyond those existing in high-energy physics. This comparison indicates an interesting relation between the semimetal phase with unconventional fermions shown here and other models supported by different symmetry groups.

Although the classification of topological nodal line semimetals is not yet complete\(^{23}\), we glimpse signatures of the nodal rings using two observables inspired by the tensor monopole measurements, $\mathcal{G} = 8 \int \epsilon_{\mu \nu \lambda} \sqrt{\det g_{\mu \nu}} d\alpha$ and $\mathcal{B} = -\int (\mathcal{F}_{\alpha \beta} + \mathcal{F}_{\phi \alpha}) d\alpha$. They represent integration over a hyperspherical surface with radius $R_0$, and correspond to the $\mathcal{D} \mathcal{D}$ invariant when $B_z = 0$. As we fix the hypersphere radius $R_0 = H_0$, as mentioned we can interpret the periodic parameters $\alpha \in [0, \pi/2]$, $\beta, \phi \in [0, 2\pi]$ as crystal momenta. Then, we can consider the integration hypersphere as representative of the first Brillouin zone (BZ) in our model (see Supplementary Material\(^{17}\)).

As the field strength $B_z$ increases, the two nodal rings expand away from the origin. When $B_z < R_0$, they are enclosed in the integration hypersphere of fixed radius $R_0$. When $B_z = R_0$, the nodal rings cross the boundary of our enclosed manifold, triggering a sharp response of both experimental observables $\mathcal{G}, \mathcal{B}$. When $B_z > R_0$, the spectrum becomes gapped inside the integration hypersphere and the nodal rings are outside. For various $B_z$, we perform linear and elliptical parametric modulators to reconstruct the metric tensor (Fig. S15-S20) and the Berry curvature (Fig. S21-S27), from which we obtain $\mathcal{G}, \mathcal{B}$. Remarkably, both experimental observables clearly signal the nodal rings crossing the hypersphere at $B_z = R_0$, an indication of the topological nodal ring semimetal phase (Fig. 4). We also find the analytical form for $\mathcal{B}$ (see Supplementary Material\(^{17}\)) matching well with experiments:

\[
\mathcal{B} = \begin{cases} 
1, & B_z < R_0 \\
-\frac{1}{2} \left(1 - \frac{B_z}{\sqrt{B_z^2 + 8H_0}}\right) & \text{otherwise}
\end{cases}
\] (14)

These results show that, as $B_z$ increase at fixed $H_0$, our system simulates phase transitions from the Weyl semimetal phase ($B_z = 0$), to the topological nodal ring semimetal phase ($B_z < R_0$) characterized by a constant $\mathcal{B}$, and to a gapped system inside the first effective BZ when $B_z > R_0$.

**OUTLOOK**

We demonstrated precise measurements of the quantum geometric tensor, including the quantum metric tensor and the Berry curvature, that directly leads to the identification of a tensor monopole in 4D space. Our work provides a powerful tool to study both the geometric and topological properties of exotic quantum systems as well as gauge structures\(^{14,16,25}\). For example, the same approach could be applied to higher dimensions, where higher-order tensor gauge fields exist and the demonstrated tensor monopole is naturally generalized.

Furthermore, we have shown it is possible to experimentally observe novel phases of topological semimetals beyond 3D. From the viewpoint of topological protected
phases and novel quantum matters, our synthetic Hamiltonian could serve as a playground for exploring unconventional quasiparticles beyond Dirac and Weyl fermions in high dimensional space. From the viewpoint of high energy physics and exotic gauge structures, our model provides a testbed for novel non-Abelian gauge theories, especially in the case that the system has degenerate nodal rings when $B_z \neq 0$. By introducing strongly interacting spins either in diamond or using systems like atom arrays and trapped ions, one could study novel geometric features in quantum many-body systems.

Note added: During the preparation of the manuscript, we noticed another experimental work describing the observation of the tensor monopole using superconducting circuits.

AUTHOR CONTRIBUTIONS

MC, CL designed and performed the experiments, analyzed the data with assistance from PC, and input of NG on the quantum-metric measurement. MC, CL, PC discussed and interpreted the results, ran simulations and developed the analytical model describing the phase transition. MC, CL, GP analyzed the symmetries and nodal structures of the model, with inputs from all authors. All authors contributed to the writing of the manuscript. PC supervised the overall project.

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FIG. 1. Schematics of the experiments. (a) Tensor monopole in the 4D parameter space spanned by generalized momentum $q = (q_x, q_y, q_z, q_w)$. (b) Energy spectrum of the 4D Weyl-type Hamiltonian at $q_y = q_w = 0$. The triply degenerate point at the origin corresponds to the tensor monopole. (c) Model implementation with the ground state spin degrees of freedom of a single NV center (left). The spin sublevels are coupled via a dual-frequency microwave pulse. By precisely controlling the frequencies, amplitudes and phases of the microwave pulse, we can engineer arbitrary 4D-Weyl Hamiltonian of the form in Eq. 7 (right). (d) An example of $|m_s = 0\rangle$ evolving under the engineered Hamiltonian with $(q_x = q_z = 2$MHz, $q_y = q_w = 0)$. The Weyl-type Hamiltonian reduces to a spin-1 $S_z$ operator, driving Rabi oscillations between both single quantum transitions $|m_s = 0\rangle \leftrightarrow |m_s = \pm 1\rangle$ at the same rate. The dual-frequency microwave is set such that $\omega_1(\omega_2)$ is resonant with the transition $|m_s = 0\rangle \leftrightarrow |m_s = \pm 1\rangle$, $B_1 = B_2 = 2$MHz, $\phi_1 = \phi_2 = 0$. In terms of our parameterization, $(H_0 = 2$MHz, $\alpha = \pi/4, \beta = \phi = 0)$. Circles (squares, triangles) represent experiments and solid lines are sinusoidal fits.
FIG. 2. **Parametric modulations.** (a) Determining the resonance condition for parametric modulation. We fix $\tau = 7.5\mu s$, $(m_\alpha, m_\beta, m_\phi) = (0, 1/30, 1/30)$ and sweep the modulation frequency around 4MHz to find $\omega_r = 2H_0$. (b) Examples of coherent Rabi oscillations observed under parametric modulations, for the engineered Hamiltonian at $(\alpha_0 = \pi/4, \beta_0 = \phi_0 = 0)$. The Rabi frequencies are used to calculate the matrix elements $\Gamma_{\nu,-m_\nu}^{m_\nu}$ (shown in (c,d)). To extract the diagonal components of the metric tensor, we use a single-parameter modulation, as shown, e.g. by the blue curve, representing the SQ transition $(\omega = \omega_r/2)$ for $\alpha$ modulation. Due to chiral symmetry, $|\Gamma_{\nu,-m_\nu}^{m_\nu}| = |\Gamma_{\nu,m_\nu}^{m_\nu}|$. We therefore measure the population in the first excited state $|u_0\rangle$, which gives half contrast (see Supplementary Material). The other two curves represent two-parameter modulations resonant with the DQ transition $(\omega = \omega_r)$, and possess full contrast. Illustrations of the relevant single- and two-parameter modulations in the Bloch sphere representation are provided on the left. (c) Matrix elements $|\Gamma_{\nu,-m_\nu}^{m_\nu}|$ measured for SQ transitions at $\omega = \omega_r/2$. Note that many matrix elements are expected from theory to coincide and thus their measured values are superimposed at 2 MHz. (d) Matrix elements $|\Gamma_{\nu,m_\nu}^{m_\nu}|$ measured for DQ transitions at $\omega = \omega_r$. Markers are experimental data and solid lines are fits and theory.
FIG. 3. Revealing the tensor monopole. (a) shows all 6 independent components of the metric tensor as functions of $\alpha$. (b) Generalized 3-form Berry curvature $H_{\alpha\beta\phi}$ with respect to $\alpha$, calculated from the metric tensor in (a) using Eq. 4. The topological invariant $\mathcal{DD}_{expt} = 0.99(3)$ reveals the existence of a tensor monopole within the hypersphere. (c) Measurements of non-zero 2-form Berry curvature as function of $\alpha$. (d) 3-form Berry curvature $H_{\alpha\beta\phi}$, calculated from the gauge potential using Eq. 12. The result $\mathcal{DD}_{expt} = 1.11(3)$ further confirms the existence of the tensor monopole. Diamonds are experimental data and solid lines are theory. The errorbars are propagated from fitting error of resonant frequencies and Rabi oscillations.
FIG. 4. **Phase transition triggered by an external field.** The central plot shows experimental data (blue squares) and numerical simulation (green triangles) of the experimental observable $G$ based on the metric tensor, experimental results (red squares) and simulation (yellow line) of the observable $B$ based on the Berry curvature. Both methods show a sharp response at $B_z/R_0 = 1$ when the nodal rings cross the boundary of the integration hypersphere. The experimental observable $M$, $B$ correspond to the $DD$ invariant when $B_z = 0$ and chiral symmetry is preserved. On the side we show 3 representative energy spectra as the longitudinal field $B_z$ increases ($q_x = q_y = 0$). The external field splits the triply degenerate Weyl node (left) into doubly degenerate nodal rings (middle). As the field further increases, the system becomes gapped in the enclosed integration hypersphere (right).
Experimental characterization of the 4D tensor monopole and topological nodal rings: 
Supplementary Material

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I. THEORETICAL DESCRIPTION OF THE MODEL HAMILTONIAN AND THE TENSOR MONOPOLE

A. Tensor monopole

In this section we review the generalization from Chern number to DD invariant and present the relation between the 3-form curvature and the 2-form connection.

In electromagnetism, the Dirac monopole is closely related to the first Chern number. Similar to Gauss’s law for the electric charge, the Dirac monopole is revealed by integral of the well-known Berry curvature over any enclosed manifold containing the monopole:

\[ C_1 = \frac{1}{2\pi} \int_{S^2} F_{\mu\nu} dq_\mu \wedge dq_\nu. \]  

(S1)

The Berry curvature \( F_{\mu\nu} \) appears in the quantum geometric tensor (QGT):

\[ \chi_{\mu\nu} = g_{\mu\nu} + i F_{\mu\nu} / 2, \]  

(S2)

where the real part is the metric tensor. More details on the measurement of the metric tensor is covered in the next section. It has been shown that one can obtain the Berry curvature entirely from the metric tensor \( g_{\mu\nu} \):

\[ F_{\mu\nu} = 2 \epsilon_{\mu\nu} \sqrt{\det g_{\bar{\mu}\bar{\nu}}}, \]  

(S3)

where \( \epsilon_{\mu\nu} \) is the Levi-Civita symbol.

From the viewpoint of metric tensor, it is a natural generalization to have in the 4D parameter space the generalized 3-form Berry curvature:

\[ H_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda}(4 \sqrt{\det(g_{\bar{\mu}\bar{\nu}})}), \]  

(S4)

and the corresponding topological invariant (Dixmier-Douady invariant):

\[ DD = \frac{1}{2\pi^2} \int_{S^3} H_{\mu\nu\lambda} dq^\mu \wedge dq^\nu \wedge dq^\lambda. \]  

(S5)

Alternatively, the curvature \( H \) can be derived from the 2-form Berry connection \( B_{\mu\nu} \) associated with the ground state \( |u_-\rangle \):

\[ H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}, \]  

(S6)

where the 2-form tensor connection can be constructed from the state \( |u_-\rangle \):

\[ B_{\mu\nu} = \Phi F_{\mu\nu}, \Phi = \frac{-i}{2} \log(u_1 u_2 u_3) \]  

(S7)

with \( u_1(2,3) \) denoting the components of \( |u_-\rangle \), \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) being the 2-form Berry curvature. From Eq. S6 and S7, it follows that in general, the generalized curvature can be obtained by performing state tomography on the
TABLE S1. Ground state quantum geometric properties in electromagnetism in 3D and in the tensor gauge field in 4D.

eigenstate, upon external perturbations of the parameters. We will provide a simpler form of the 2-form connection and the 3-form curvature for our model, as shown in Sec. I D.

We note that the 2-form Berry connection $B_{\mu\nu}$ can be more generally constructed from a mixed set of pseudoreal and complex scalar fields $\psi_{1,2,3}$ satisfying the U(1) gauge transformation and linked to the ground state,

$$B_{\mu\nu} = \frac{i}{3} \sum_{j,k,l=1}^{3} e^{ijkl} \psi_{j} \partial_{\mu} \psi_{k} \partial_{\nu} \psi_{l} = \frac{i}{3} \psi_{j} (\partial_{\mu} \psi_{k} \partial_{\nu} \psi_{l} - \partial_{\nu} \psi_{l} \partial_{\mu} \psi_{k})$$

Choosing for example

$$\psi_{1} = -i \log(u_{1} + u_{3}), \quad \psi_{2} = u_{1}^{*} - u_{3}^{*}, \quad \psi_{3} = u_{3} - u_{1},$$

where $[u_{\pm}] = [u_{1}, u_{2}, u_{3}]^{T}$, yields a gauge-invariant $H_{\mu\nu\lambda}$ that gives the same $DD$ and $B$ as above.

We compare the quantum geometric properties in electromagnetism in 3D and in the tensor gauge field in 4D in Table S1.

B. System Hamiltonian: equivalence with SG220 linear $k \cdot p$ model

In this section we review the Hamiltonian implemented in our experiments, and point out an interesting relation between this model and a three-band linearized $k \cdot p$ Hamiltonian.$^2$

As shown in the main text, our Hamiltonian is:

$$H = H_{0} \begin{pmatrix} 0 & \cos \alpha e^{-i\beta} & 0 \\ \cos \alpha e^{i\beta} & 0 & \sin \alpha e^{i\phi} \\ 0 & \sin \alpha e^{-i\phi} & 0 \end{pmatrix} + \frac{B_{z}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $\alpha \in [0, \pi/2]$ and $\beta, \phi \in [0, 2\pi)$. The linearized $k \cdot p$ Hamiltonian for space group (SG) 220 is given by:

$$H_{220}(k) = \begin{pmatrix} 0 & k_{y} & k_{z} \\ k_{y} & 0 & -k_{z} \\ k_{x} & -k_{z} & 0 \end{pmatrix},$$

We note that when we take a slice of our Hamiltonian $k = (k_{x}, k_{y}, k_{z}) = (H_{0} \sin(\alpha - \pi/4), H_{0} \cos(\alpha - \pi/4), B_{z}/\sqrt{2})$, Eq. S10 and Eq. S11 have identical eigenvalue spectrums for any $\beta, \phi$.

For the Hamiltonian in Eq. S11, pairs of two bands are degenerate along $|k_{x}| = |k_{y}| = |k_{z}|$. This corresponds to the condition $B_{z} = H_{0}$ and $\alpha = 0, \pi/2$ in our model, matching our numerical simulation and experimental results. Due to the rotation symmetry of $\beta$ and $\phi$ in our model, we remark that a pair of doubly degenerate nodal surfaces along $(\alpha = 0, \pi/2, B_{z} = H_{0})$ emerge when $B_{z} \neq 0$ in our 4D parameter space. This corresponds to two nodal rings in the $(q_{x}, q_{y}, q_{z}, q_{w})$ coordinate. We will discuss this later in Sec. I F.

The equivalence between the SG220 model and a slice of our model presented here implies that the doubly degeneracy induced by the detuning $B_{z}$ has a correspondence to crystal symmetry-protected fermionic excitations, and the observed phase transition has a non-trivial topological meaning. It would be interesting to use our experimental system to further simulate the topological properties of the SG220 model (and other triple-degenerate point models).
C. Analytical solutions of the metric tensor and generalized Berry curvature ($B_z = 0$)

The Hamiltonian in Eq. S10 is parameterized by the three parameters $\alpha, \beta, \phi$ when $B_z = 0$. The analytical solutions for the quantum metric tensor can be easily obtained via exact diagonalization:

$$g = \begin{pmatrix}
g_{\alpha\alpha} & g_{\alpha\beta} & g_{\alpha\phi} \\
g_{\beta\alpha} & g_{\beta\beta} & g_{\beta\phi} \\
g_{\phi\alpha} & g_{\phi\beta} & g_{\phi\phi}
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{4} \cos^2 \alpha (2 - \cos^2 \alpha) & -\frac{1}{16} \sin^2 2\alpha \\
0 & -\frac{1}{16} \sin^2 2\alpha & \frac{1}{4} \sin^2 \alpha (2 - \sin^2 \alpha)
\end{pmatrix}. \tag{S12}
$$

The resulting 3-form Berry curvature is:

$$H_{\alpha\beta\phi} = \cos \alpha \sin \alpha. \tag{S13}$$

One can then easily verify that the integral of the above curvature over $\alpha, \beta$ and $\phi$ will yield $DD = 1$.

D. Analytical form of $B$ calculated from the tensor Berry connection (arbitrary $B_z$)

With the parameterization given in Eq. S10, we present the analytical calculations of the experimental observable $B$ (see main text) in the presence of external field ($B_z \neq 0$). We can easily prove that, up to a global phase, the ground state of the Hamiltonian Eq. S10 has the form

$$|u_-\rangle = [e^{-i\beta} v_1, v_2, e^{-i\phi} v_3]^T \tag{S14}$$

where $v_1, v_2, v_3$ are functions of $\alpha$ only. Then, we can calculate the vector gauge potential for this special gauge

$$A_{\alpha} = 0, \quad A_{\beta} = v_1^2, \quad A_{\phi} = v_3^2$$

and the 2-form Berry curvature $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$

$$F_{\alpha\beta} = \partial_\alpha (v_1^2), \quad F_{\phi\alpha} = -\partial_\alpha (v_3^2). \tag{S15}$$

We repeat Eq. S6 and S7 here for convenience

$$\mathcal{H} = \partial_\alpha B_{\beta\phi} + \partial_\phi B_{\alpha\beta} + \partial_\beta B_{\phi\alpha}, \tag{S16}$$

where $B_{\mu\nu} = F_{\mu\nu} \Phi$, and

$$\Phi = -\frac{i}{2} \log \prod_{i=1}^{3} u_i = -\frac{i}{2} \log (e^{-i(\phi+\beta)} v_1 v_2 v_3).$$

Combining all above, we obtain the simplified form for the curvature

$$\mathcal{H}_{\alpha\beta\phi} = \partial_\phi (\Phi F_{\alpha\beta}) + \partial_\beta (\Phi F_{\phi\alpha}) + \partial_\alpha (\Phi F_{\beta\phi})$$

$$= -\frac{1}{2} (F_{\alpha\beta} + F_{\phi\alpha})$$

$$= -\frac{1}{2} \frac{d}{d\alpha} (v_1^2 - v_3^2), \tag{S17}$$

where the second line is used in experiments to extract the curvature.

With the alternative definitions in Eq. S8 and S9 we arrive at

$$\mathcal{H}_{\alpha\beta\phi} = -\frac{2}{e^{-i\beta} v_1 + e^{-i\phi} v_3} \left[ e^{-i\phi} v_3 \frac{d}{d\alpha} (v_1^2) - e^{-i\beta} v_1 \frac{d}{d\alpha} (v_3^2) \right], \tag{S18}$$
which yields the same $B$ upon integration. Note that while the choice of the $\psi$’s fields in Eq. S9 ensures that $H_{\alpha \beta \phi}$ coincides at $B_z = 0$ with its value calculated from the QGT, there is much freedom in the choice of the field $\psi$.

As a result, there is no well-defined $B$ that can be considered a good topological number to characterize the emergent topological nodal semimetal phase protected by mirror symmetries. The classification of topological nodal line semimetals is not yet complete in 3D, and still lacking in 4D. Nevertheless, as we have shown, the observable $B$, although not topological, serves as a convenient experimental tool to glimpse signatures of the nodal ring and the associated phase transition.

We can gain further insight into the observable $B$ by explicitly evaluating the integral of Eq. S17

$$B = \frac{1}{2\pi^2} \int_{S^3} H_{\alpha \beta \phi} d\alpha d\beta d\phi = [v_1^2(0) - v_3^2(0)] - [v_1^2(\pi/2) - v_3^2(\pi/2)]$$

For $\alpha = 0, \pi/2$ the Hamiltonian eigenvectors can be calculated easily. Setting $h = B_z/R_0$, we have

$$u_1(0) = \begin{cases} \sqrt{\frac{1}{2}} \left(1 - \frac{h}{\sqrt{h^2 + 8}}\right), & h < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$u_3(0) = \begin{cases} 0, & h < 1 \\ 1, & \text{otherwise} \end{cases}$$

$$u_1(\pi/2) = 0, u_3(\pi/2) = \sqrt{\frac{1}{2}} \left(1 + \frac{h}{\sqrt{h^2 + 8}}\right)$$

Finally we obtain

$$B = \begin{cases} 1, & h < 1 \\ -\frac{1}{2} \left(1 - \frac{h}{\sqrt{h^2 + 8}}\right), & \text{otherwise} \end{cases} \quad (S19)$$

We remark that upon breaking chiral symmetry, we choose here a special gauge ($v_2 \in \mathbb{R}$) for convenience. Indeed, the 3-form curvature in Eq. S17 is not gauge-invariant and the gauge structure we defined is not universal. Nevertheless, the $B$ only depends on the eigenvector when $\alpha = 0, \pi/2$, where the two-fold degenerate points reside, and as we show analytically, numerically and experimentally, it indeed provides signatures of these non-trivial singularity points, which are related to exotic fermionic excitations.

E. Topological phase transition triggered by manifold displacement

As we mentioned in the main text, displacement of the hypersphere along one of the parameter axis, e.g., $q_x \rightarrow q_x + \delta_x$, will induce a topological phase transition when $\delta_x/H_0 = 1$. The Hamiltonian can be written as:

$$H_{\text{disp}} = \begin{pmatrix} 0 & H_0 \cos \alpha e^{-i\beta} + \delta_x & 0 \\ H_0 \cos \alpha e^{i\beta} + \delta_x & 0 & H_0 \sin \alpha e^{i\phi} \\ 0 & H_0 \sin \alpha e^{-i\phi} & 0 \end{pmatrix}.$$ \hspace{1cm} (S20)

We can analytically calculate the 3-form Berry curvature in the presence of displacement:

$$H_{\alpha \beta \phi}(H_0, \alpha, \beta, \delta_x) = \frac{H_0^2 \cos \alpha \sin \alpha (H_0 + \delta_x \cos \alpha \cos \beta)}{\left(\delta_x^2 + H_0^2 + 2H_0\delta_x \cos \alpha \cos \beta\right)^2}$$ \hspace{1cm} (S21)

and evaluate its integral to find the $\mathcal{DD}$ invariant.

The topological phase transition is characterized by the $\mathcal{DD}$ invariant, as shown in Fig. S1, where $\mathcal{DD} = 1 \rightarrow 0$ when $|\delta_x/H_0| > 1$.

However, the $\beta$ dependence of the 3-form curvature introduced by the translation poses a challenge in experiments, as it greatly prolongs the total measurement time. As we mentioned in the main text, we therefore choose to add a fictitious $z$ field to the system which preserves rotation symmetry about $\beta, \phi$, as discussed in more detail in Sec. IF next.
FIG. S1. **Phase transition triggered by a manifold displacement.** When $|\delta_x/H_0| < 1$, the manifold encloses the triply degenerate point and the system is in the Weyl semimetal phase; when $|\delta_x/H_0| > 1$, the monopole is no longer enclosed, leading to a trivial insulator phase.

**F. Topological phase transition triggered by external field**

Having been discussed in the main text and in Sec. I B, we elaborate in more detail about the topological phase transitions induced by external field $B_z$. To this purpose, it is useful to rewrite the Hamiltonian (S10) in terms of the Gell-Mann matrices that highlight its symmetries

$$H_{4D} = q_x\lambda_1 + q_y\lambda_2 + q_z\lambda_6 + q_w\lambda_7 + \frac{B_z}{2\sqrt{2}}(\lambda_3 + \sqrt{3}\lambda_8) \quad (S22)$$

Without the external field ($B_z = 0$), our Hamiltonian preserves chiral symmetry

$$\{H_{4D},U\} = 0, \quad (S23)$$

where $U = \text{diag}(1,-1,1)$.

Upon breaking the chiral symmetry ($B_z \neq 0$), the tensor monopole disappears and splits into unconventional fermions. The nodal structures are two degenerate nodal surfaces spanning the $\beta-\phi$ space along ($\alpha = 0, \pi/2, B_z = R_0$). Viewed from the ($q_x, q_y, q_z, q_w$) parameter space, for example, the eigenvalue analytical forms in the subspace of $q_z = q_w = 0$ are

$$\epsilon_- = \frac{1}{2} \left( \frac{B_z}{\sqrt{2}} - \sqrt{\frac{B_z^2}{2} + 4q_x^2 + 4q_y^2} \right), \quad \epsilon_0 = -\frac{B_z}{\sqrt{2}}, \quad \epsilon_+ = \frac{1}{2} \left( \frac{B_z}{\sqrt{2}} + \sqrt{\frac{B_z^2}{2} + 4q_x^2 + 4q_y^2} \right), \quad (S24)$$

and the lower two bands become degenerate when $q_x^2 + q_y^2 = B_z^2$, as shown in Fig. S2. Indeed, we find that there are two nodal rings, one between the middle band and the lower band (corresponding to $\alpha = 0$) and another at $\alpha = \pi/2$, between the middle and upper bands:

$$q_x^2 + q_y^2 = B_z^2, \quad q_z = q_w = 0,$$

or

$$q_x^2 + q_w^2 = B_z^2, \quad q_x = q_y = 0. \quad (S25)$$

Because nodal rings are more commonly encountered and studied in 3D than nodal surfaces, we will refer to the nodal structure in our model as nodal rings for the convenience of the reader.

As soon as the chiral symmetry is broken by $B_z \neq 0$, our system undergoes a phase transition from the Weyl semimetal phase (with tensor monopole) to the topological nodal ring semimetal phase, protected by two mirror symmetries:

$$M_1 H(q_x, q_y, q_z, q_w) M_1^{-1} = H(-q_x, -q_y, q_z, q_w)$$

$$M_2 H(q_x, q_y, q_z, q_w) M_2^{-1} = H(q_x, q_y, -q_z, -q_w), \quad (S26)$$
where $M_1 = \text{diag}(-1, 1, 1)$, $M_2 = \text{diag}(1, 1, -1)$. These mirror symmetries naturally imply inversion symmetry:

$$U_I H(q_x, q_y, q_z, q_w) U_I^{-1} = H(-q_x, -q_y, -q_z, -q_w)$$  (S27)

where $U_I = M_1 M_2$. The aforementioned nodal rings are protected by these mirror symmetries. The symmetries can be further broken by introducing terms that are proportional to $\lambda_{4(5)}$ Gell-Mann matrices,

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

and the degenerate rings will become gapped.

We further note that there exists PT symmetry for the specific 2D subsystem given by $q_y = q_w = 0$. The nodal points in the $q_x - q_z$ plane between the lower and upper two bands shown in Fig. S2 (right plot) are protected by PT symmetry. Introducing the $\lambda_4$ term breaks the mirror symmetries, but preserves the PT symmetry. Therefore it gaps the nodal rings into PT-symmetry-protected nodal points, and the system remains gapless due to the PT symmetry. An energy gap can fully open only by adding terms, such as the $\lambda_5$ Gell-Mann matrix, that break the PT symmetry. On the same note, we find that for the subsystem given by $q_x = q_z = 0$, the system Hamiltonian satisfies anti-commutation relation with the PT operator and we observe similar nodal points as mentioned above.

As examples of broken mirror symmetries and broken PT symmetry, we plot in Fig. S3 the energy spectrum projected to the $q_x$ axis, when breaking mirror symmetries by introducing $\lambda_4$ and the PT symmetry by introducing $\lambda_5$. In both cases, we break the chiral symmetry due to the $B_z$ field.

We note that at fixed $R_0$ our Hamiltonian can be seen as directly analogous to a 3D non-interacting lattice model, where $\alpha, \beta, \phi$ play the role of crystal momenta. Because the classification of topological nodal line semimetals is not yet complete in 3D, we do not have a convenient topological invariant to characterize the nodal rings. However, signatures of the nodal rings could be observed using the observable $B$ discussed in Sec. I D when the nodal rings cross the integration sphere. This is achieved by increasing $B_z$, which expands the nodal rings until their radius $B_z$ crosses the integration sphere of radius $R_0$. The analytical result of the measured quantity is already presented in Sec. I D.

Recall that the integration sphere is defined by the range $\alpha \in [0, \pi/2]$ and $\beta, \phi \in [0, 2\pi)$, and the energy spectrum is identical (up to a mirror operation on $\alpha$ and $\pi$ rotation on $\beta(\phi)$) when one considers $\alpha \in [\pi/2, \pi]$, $[\pi, 3\pi/2]$ or $[3\pi/2, 2\pi]$. The integration sphere thus behaves as the boundary of the first Brillouin zone (BZ) in our model. When $B_z = R_0$, the nodal rings cross the boundary of this enclosed manifold (BZ), triggering a sharp response of the observable $B$. When $B_z > R_0$, the spectrum becomes gapped inside the enclosed $S_3$ sphere. We discuss the experimental measurement of the observables in detail in Sec. III G.

**FIG. S2.** Nodal structures (Left) nodal ring between the lower two bands in the $q_x, q_y$ plane when $q_z = q_w = 0$ and $B_z \neq 0$. The nodal rings are protected by the mirror symmetries. (Right) nodal points between the middle band and the lower (upper) band in the $q_x, q_z$ plane when $q_y = q_w = 0$ and $B_z \neq 0$. These nodal points are protected by the PT symmetry.
FIG. S3. **Energy spectrum upon broken symmetries** We take the $q_y = q_w = 0$ slice, project the energy spectrum to the $q_x$ axis and plot the envelope of the three energy bands. The red lines correspond the envelop of the middle band. Two examples are (left) broken mirror symmetries, but preserved PT symmetry and (right) broken PT symmetry. The PT symmetry protect the nodal points and keep the system gapless. By introducing terms that break the PT symmetry (such as $\lambda_5$ matrices), we can fully open the gap.

II. QUANTUM GEOMETRIC TENSOR AND PARAMETRIC MODULATION

The quantum geometric tensor (QGT) $\chi_{\mu\nu}$ naturally appears when one defines the distance between nearby states $|n(q)\rangle$ and $|n(q + dq)\rangle$:

$$ds^2 = 1 - |\langle n(q) | n(q + dq) \rangle|^2 = dq_{\mu} \chi_{\mu\nu} dq_{\nu} + O(|dq|^3), \quad (S28)$$

where we apply the Taylor expansion about $dq = 0$ and assume a generic system parametrized by the generalized position $q$. Here $\chi_{\mu\nu}$ contains information on the geometry of the manifold for the state $|n\rangle$ and is found to be:

$$\chi_{\mu\nu}^{(n)} = \langle \partial_{\mu} n(q) | (1 - |n(q)\rangle \langle n(q)|) | \partial_{\nu} n(q) \rangle = g_{\mu\nu} + iF_{\mu\nu}/2, \quad (S29)$$

where the symmetric part is the metric tensor $g_{\mu\nu}$ and determines the distance between the states, and the anti-symmetric part is the conventional 2-form Berry curvature.

Using time-independent perturbation theory to first order we obtain

$$\left| n^{(1)} \right\rangle = |\partial_{\mu} n(q) \rangle = \sum_{k \neq n} \langle k(q) | \partial_{\mu} H | n(q) \rangle / (\epsilon_n - \epsilon_k) |k(q)\rangle \quad (S30)$$

Then we can plug in Eq. S29 and simplify to the following form

$$\chi_{\mu\nu}^{(n)} = \sum_{k \neq n} \sum_{m \neq n} \langle k | \partial_{\mu} H | n \rangle / (\epsilon_n - \epsilon_k) \langle k | (1 - |n\rangle \langle n|) \langle m | \partial_{\nu} H | n \rangle / (\epsilon_n - \epsilon_m) |m\rangle$$

$$= \sum_{m \neq n} \langle m | \partial_{\mu} H | n \rangle / (\epsilon_n - \epsilon_m) \langle m | \partial_{\nu} H | n \rangle / (\epsilon_n - \epsilon_m)$$

$$= \sum_{m \neq n} \langle n | \partial_{\mu} H | m \rangle \langle m | \partial_{\nu} H | n \rangle / (\epsilon_n - \epsilon_m)^2 \quad (S31)$$

As discussed earlier, the real part of the QGT, namely the metric tensor, contains all the information about the monopole, and it could be used to reconstruct the (generalized) Berry curvature, while the imaginary part, namely the 2-form Berry curvature, is connected with the tensor Berry connection. In the following, we show how to experimentally measure the metric tensor and 2-form Berry curvature.
A. Quantum metric tensor

The technique we use is parametric modulation of the system Hamiltonian. We apply the following linear modulations

\[
\begin{align*}
\mu_t &= \mu_0 + m_\mu \sin \omega t, \\
\nu_t &= \nu_0 + m_\nu \sin \omega t.
\end{align*}
\]

(S32)

If we modulate the parameters weakly, \(m_\mu, m_\nu \ll 1\), then

\[
\mathcal{H} \approx \mathcal{H}(q_0) + m_\mu \partial_\mu \mathcal{H} \sin \omega t + m_\nu \partial_\nu \mathcal{H} \sin \omega t.
\]

(S33)

When the modulation frequency is resonant with the energy difference between eigenstates, the parametrically modulated Hamiltonian will drive coherent Rabi oscillations between relevant eigenstates.

Here we consider the case where \(\omega = \epsilon_+ - \epsilon_-\) is resonant with the double quantum (DQ) transition between \(|u_-\rangle \leftrightarrow |u_+\rangle\). Then we have the Rabi frequency

\[
\Omega_\mu = -\frac{|\langle u_- \rangle \mathcal{H}(q_0) + m_\mu \partial_\mu \mathcal{H} + m_\nu \partial_\nu \mathcal{H} |u_+\rangle|}{|\langle u_- \rangle m_\mu \partial_\mu \mathcal{H} + m_\nu \partial_\nu \mathcal{H} |u_+\rangle|}
\]

(S34)

To measure the diagonal component \(g_{\mu\mu}\) of the metric tensor, we set \(m_\nu = 0\):

\[
\Omega_{\mu,\mu} = m_\mu \langle u_- | \partial_\mu \mathcal{H} | u_+ \rangle.
\]

(S35)

Similarly, we can obtain the contribution from SQ transition \(\Gamma_{\nu,0}\) (some complication arises for SQ, which is discussed in Sec. III C). Using the alternative form of QGT in Eq. S31, we obtain the diagonal components of the metric tensor from experimentally measurable quantities:

\[
g_{\mu\mu} = \sum_{m \in 0, +} \frac{(\Gamma_{\mu,m}^\nu)^2}{(\epsilon_m - \epsilon_+)^2}.
\]

(S36)

To measure the off-diagonal components \(g_{\mu\nu}\), we modulate both parameters such that \(m_\mu = \pm m_\nu\). Then the coherent Rabi oscillation is:

\[
\Omega_{\mu,\nu} = m_\mu \langle u_- | \partial_\mu \mathcal{H} \pm \partial_\nu \mathcal{H} | u_+ \rangle.
\]

(S37)

Setting \(\Gamma_{\mu,\nu} = \Omega_{\mu,\nu} / m_\mu\), we have

\[
(\Gamma_{\mu,\nu}^\nu)^2 - (\Gamma_{\mu,\nu}^\nu)^2 = 4 |\langle u_- | \partial_\mu \mathcal{H} | u_+ \rangle \langle u_+ | \partial_\nu \mathcal{H} | u_- \rangle|.
\]

(S38)

We thus obtain an expression for the off-diagonal components

\[
g_{\mu\nu} = \sum_{m \in 0, +} \frac{(\Gamma_{\mu,m}^\nu)^2 - (\Gamma_{\mu,m}^\nu)^2}{4(\epsilon_m - \epsilon_+)^2}.
\]

(S39)

We remark that \(g_{\mu\nu} = g_{\nu\mu}\), and there are in total 6 independent components in a 4D parameter space. The long coherence time \((T_2 > 1 \text{ ms under dynamical decoupling})\) of the NV center allows us to extract the metric tensor from measuring these Rabi oscillations.

B. 2-form Berry curvature

We next summarize the experimental details of measuring the 2-form Berry curvature, which is in turn used to get the real \(\mathcal{DD}\) invariant (see Eq.13 in main text). We use the following elliptical modulation of the system Hamiltonian:

\[
\begin{align*}
\mu_t &= \mu_0 + m_\mu \cos \omega t, \\
\nu_t &= \nu_0 + m_\nu \sin \omega t.
\end{align*}
\]

(S40)

When the modulation amplitude is small, we have

\[
\mathcal{H} \approx \mathcal{H}(q_0) + m_\mu \partial_\mu \mathcal{H} \cos \omega t + m_\nu \partial_\nu \mathcal{H} \sin \omega t,
\]

(S41)

and we could then measure \(\langle u_- | \partial_\mu \mathcal{H} \pm i \partial_\nu \mathcal{H} | m \rangle\), and obtain the imaginary part of the QGT. Next we lay out the exact realization in a 3-level system For SQ and DQ drive, where we need two different interaction pictures.
**SQ interaction picture**  In general, $\omega_1 = \epsilon_+ - \epsilon_0 \neq \omega_2 = \epsilon_0 - \epsilon_-$. We define the unitary (more details in Sec. III A)

$$V = \begin{pmatrix} e^{-i\omega t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\omega t} \end{pmatrix}$$ (S42)

For the relevant driving term, we have

$$H_x = \partial_\mu H = \begin{pmatrix} 0 & B^* & C^* \\ B & 0 & A^* \\ C & A & 0 \end{pmatrix}.$$ (S43)

After rotating wave approximation, we have

$$V^\dagger(H_x \cos(\omega t))V = \frac{1}{2} \begin{pmatrix} 0 & B^* & 0 \\ B & 0 & A^* \\ 0 & A & 0 \end{pmatrix}$$

$$V^\dagger(H_x \sin(\omega t))V = i \frac{1}{2} \begin{pmatrix} 0 & B^* & 0 \\ -B & 0 & -A^* \\ 0 & A & 0 \end{pmatrix}$$ (S44)

To measure the (3, 2) matrix element, our choice of the elliptical drive is obvious:

$$\partial_\mu H \cos(\omega t) \pm \partial_\nu H \sin(\omega t) \leftrightarrow |u_-| \partial_\mu H \pm i \partial_\nu H |u_0| \rangle$$ (S45)

However, note that here $\omega = \epsilon_- - \epsilon_0 < 0$. In experiment, we always use $\tilde{\omega} = |\omega|$, therefore here we have

$$\partial_\mu H \cos(\tilde{\omega} t) \mp \partial_\nu H \sin(\tilde{\omega} t) \leftrightarrow |u_-| \partial_\mu H \pm i \partial_\nu H |u_0| \rangle$$ (S46)

**DQ interaction picture**  For the DQ drive, the middle state is not involved. Then it is effectively a two level system

$$V = \begin{pmatrix} e^{-i\omega t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\omega t} \end{pmatrix}$$ (S47)

After rotating wave approximation,

$$V^\dagger(H_x \cos(\omega t))V = \frac{1}{2} \begin{pmatrix} 0 & 0 & C^* \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix}$$

$$V^\dagger(H_x \sin(\omega t))V = i \frac{1}{2} \begin{pmatrix} 0 & 0 & C^* \\ 0 & 0 & 0 \\ -C & 0 & 0 \end{pmatrix}$$ (S48)

To measure the (3, 1) matrix element, our choice of the elliptical drive is :

$$\partial_\mu H \cos(\omega t) \mp \partial_\nu H \sin(\omega t) \leftrightarrow |u_-| \partial_\mu H \pm i \partial_\nu H |u_+\rangle \rangle$$ (S49)

Combining the SQ and DQ analysis, we see that in both cases, we should have $\cos, -\sin$ elliptical modulations.
Degenerate SQ transitions  A special case is when $B_z = 0$, then $\omega_1 = -\omega_2$. In this case, when we modulate at the SQ transition frequency $|\omega_1| = |\omega_2|$, both SQ transitions will turn on and yield an effective DQ transition. In experiment, we start from $|u_0\rangle$. After the modulation, we perform a unitary map between $|u_-,0,+,m_s\rangle \leftrightarrow |m_s = -1,0,+,1\rangle$. The population oscillations should follow

$$
\begin{pmatrix}
  n_+ \\
  n_0 \\
  n_-
\end{pmatrix} =
\begin{pmatrix}
  \frac{B_1^2}{B_1^2 + B_2^2} \sin^2(\omega t) \\
  \frac{\cos^2(\omega t)}{B_1^2 + B_2^2} \\
  \frac{B_2^2}{B_1^2 + B_2^2} \sin^2(\omega t)
\end{pmatrix}
$$

(S50)

In terms of the parametric modulation, we have,

$$V =
\begin{pmatrix}
  e^{-i\omega t} & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & e^{i\omega t}
\end{pmatrix}
$$

(S51)

$$V^\dagger(H_x \cos(\omega t))V = \frac{1}{2}
\begin{pmatrix}
  0 & B^* & 0 \\
  B & 0 & A^* \\
  0 & A & 0
\end{pmatrix}
$$

(S52)

$$V^\dagger(H_x \sin(\omega t))V = i
\frac{1}{2}
\begin{pmatrix}
  0 & B^* & 0 \\
  -B & 0 & A^* \\
  0 & -A & 0
\end{pmatrix}
$$

Upon \(\cos/\sin\) elliptical modulation, we find the correspondence

$$
B_1 = |\langle u_+ | \partial_\mu H \pm i \partial_\nu H | u_0 \rangle| = |\langle u_- | \partial_\mu H \pm i \partial_\nu H | u_0 \rangle|
$$

$$B_2 = |\langle u_- | \partial_\mu H \mp i \partial_\nu H | u_0 \rangle|
$$

(S53)

### III. EXPERIMENTAL DETAILS

#### A. Hamiltonian under dual-frequency microwave driving

We now derive the effective Hamiltonian in the rotating frame of the dual-frequency microwave pulses following Ref.7.

The intrinsic Hamiltonian of NV is $D S_x^2 + \gamma_e B S_z$, where $\gamma_e = 2.8$ MHz/G is the gyromagnetic ratio, $D = 2.87$ GHz is the zero-field energy splitting of the NV ground state, $B = 490$ G here is the external field along N-V axis and $S_z$ is the spin-1 $z$ operator. Under the dual-frequency microwave control, the total Hamiltonian is given by:

$$H_{NV} = DS_x^2 + \gamma_e BS_z + 2\sqrt{2}[\gamma_e B_1 \cos(\omega_1 t + \phi_1) S_x + \gamma_e B_2 \cos(\omega_2 t + \phi_2)]S_z$$

(S54)

where $S_x, S_z$ are the spin-1 operators, the first line is the NV spin Hamiltonian and the second line represents the dual-frequency microwave pulse at frequencies $\omega_1, \omega_2$. In the bare NV frame with basis $| m_s = +1,0,-1 \rangle$, the above Hamiltonian can be written as:

$$H_{NV} = \begin{pmatrix}
D + \gamma_e B & 2(\gamma_e B_1 \cos(\omega_1 t + \phi_1) + \gamma_e B_2 \cos(\omega_2 t + \phi_2)) & 0 \\
2(\gamma_e B_1 \cos(\omega_1 t + \phi_1) + \gamma_e B_2 \cos(\omega_2 t + \phi_2)) & 0 & 2(\gamma_e B_1 \cos(\omega_1 t + \phi_1) + \gamma_e B_2 \cos(\omega_2 t + \phi_2)) \\
0 & 2(\gamma_e B_1 \cos(\omega_1 t + \phi_1) + \gamma_e B_2 \cos(\omega_2 t + \phi_2)) & D - \gamma_e B
\end{pmatrix}$$

(S55)

We now enter the rotating frame defined by the unitary transformation:

$$V =
\begin{pmatrix}
  e^{-i\omega_1 t} & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & e^{-i\omega_2 t}
\end{pmatrix}
$$
the Hamiltonian Eq. S55 can be rewritten as the same form as Eq.6 in main text:

\[
\mathcal{H} = \begin{pmatrix}
D + \gamma_e B - \omega_1 & B_1 e^{-i\phi_1} & 0 \\
B_1 e^{i\phi_1} & 0 & B_2 e^{i\phi_2} \\
0 & B_2 e^{-i\phi_2} & D - \gamma_e B - \omega_2
\end{pmatrix} = \begin{pmatrix}
\frac{B_z}{\sqrt{2}} & H_0 \cos \alpha e^{-i\beta} & 0 \\
H_0 \cos \alpha e^{i\beta} & 0 & H_0 \sin \alpha e^{i\phi} \\
0 & H_0 \sin \alpha e^{-i\phi} & -B_z/\sqrt{2}
\end{pmatrix}
\] (S56)

where \( B_z = D \pm \gamma_e B - \omega_{1(2)} \) corresponds to detunings in microwave frequency, \( B_1 = H_0 \cos \alpha \), \( B_2 = H_0 \sin \alpha \) and \( \phi_1 = \beta, \phi_2 = \phi \) are the amplitudes and phases of the microwave pulses, respectively. We call this Hamiltonian the double quantum (DQ) Hamiltonian in the following, for its ability to drive the \(|m_s = -1\rangle \leftrightarrow |m_s = +1\rangle\) transition, where the quantum number changes by 2.

When both microwave frequencies are on-resonance \( \omega_{1(2)} = D \pm \gamma_e B \), the eigenstates of Eq.S56 are:

\[
|u_{\pm}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{B_1 e^{-i\phi_1}}{\sqrt{B_1^2 + B_2^2}} \\
\pm 1 \\
\frac{B_2 e^{-i\phi_2}}{\sqrt{B_1^2 + B_2^2}}
\end{pmatrix},
\]

\[
|u_0\rangle = \begin{pmatrix}
\frac{B_2 e^{-i\phi_1}}{\sqrt{B_1^2 + B_2^2}} \\
0 \\
\frac{-B_1 e^{-i\phi_2}}{\sqrt{B_1^2 + B_2^2}}
\end{pmatrix},
\] (S57)

and the corresponding eigen-energies are \( \epsilon_{\pm} = \pm \gamma_e \sqrt{B_1^2 + B_2^2} \) and \( \epsilon_0 = 0 \). This Weyl-like Hamiltonian hosts a tensor monopole at the origin.

B. Generation of dual-frequency microwave pulses

To achieve precise control over the amplitude and phase of both microwave frequencies, we choose to use frequency modulation with two separate IQ mixers, shown schematically in Fig. S4. The in-phase (I) and quadrature (Q) RF signals are generated from AWG (Tektronix 5014B) using 3 separate channels. One of the outputs is further split into 0 and 90° (Mini-Circuits ZMSCQ-2-90). A two-channel microwave generator (Windfreak SynthHD) generates the Local Oscillators (LO). The IQs and LOs are combined in two IQ mixers (Texas Instrument, TRF370317; Marki Microwave, IQ-0318) that create up-converted single-sideband microwave signals. These output signals are then combined and controlled by a microwave switch (Analog Devices, ADRF5020) before amplified. For brevity, we left out pre-amplifiers in the schematics.

![FIG. S4. Setup schematics](image-url)

To characterize our engineered Weyl-type Hamiltonian in Eq. S56 under dual-frequency microwave driving, we prepare NV in the \(|m_s = 0\rangle\) state and let it evolve under the DQ Hamiltonian (Eq. S56). When both microwave
frequencies are on-resonance, we expect the following time-dependent state evolution

\[
\begin{pmatrix}
c_+(t) \\
c_0(t) \\
c_-(t)
\end{pmatrix} = \begin{pmatrix}
-iB_1 e^{-i\alpha t} \sin \omega_r t \\
\sqrt{B_1^2 + B_2^2} \cos \omega_r t \\
-iB_2 e^{-i\alpha t} \sin \omega_r t
\end{pmatrix},
\]

with the effective Rabi frequency \( \omega_r = \gamma_c \sqrt{B_1^2 + B_2^2} \). By measuring the amplitude and frequency of the Rabi oscillation, we can extract both \( B_1, B_2 \).

2D maps of the relationship between IQ voltages and the corresponding Rabi frequencies \( (\gamma_c B_1, \gamma_c B_2, \omega_r) \) are shown in Fig. S5. We choose to work in the linear regime of the microwave amplifier, where \( \omega_r = 2 \text{ MHz} \). A few examples of the state evolution under \( \alpha = 0, \pi/6, \pi/4 \) are shown in Fig. S6. Recalling that we set \( B_1 = H_0 \cos \alpha, B_2 = H_0 \sin \alpha \), we expect the amplitude of the \( |m_s = +1\rangle \) state to be \( \cos^2 \alpha \), in excellent agreement with the experiments in Fig. S6.

As a last demonstration, we show in Fig. S7 a histogram of the resonant parametric modulation frequencies \( \omega_r \) measured throughout experiments to measure the tensor monopole. See also Fig. 2(a) in the main text. The fluctuation of \( \omega_r \) is well within 2% over all the \( \alpha \in [0, \pi/2] \) range, subject to real experimental conditions including heating due to prolonged microwave driving. This result verifies the (spherical) shape of hypersphere \( R_0 = H_0 = 2 \text{ MHz} \) we choose in order to reveal the tensor monopole.

**FIG. S5. IQ voltage calibration** From left to right we show the measured \( \gamma_c B_1 \) (left), \( \gamma_c B_2 \) (middle) and \( \omega_r \) in units of \( (2\pi) \text{ MHz} \). \( x(y)\)-axis represents the IQ voltage for the AWG channels that drive the \( |m_s = 0\rangle \leftrightarrow |m_s = -1\rangle \) \( (|m_s = 0\rangle \leftrightarrow |m_s = +1\rangle) \) transition.

**FIG. S6. Double Quantum Rabi Oscillations** Evolution of the state \( |m_s = 0\rangle \) under the DQ Hamiltonian in Eq. S56. The experimental conditions are \( \alpha = 0 \) (left), \( \alpha = \pi/6 \) (middle), and \( \alpha = \pi/4 \) (right). The expected oscillation amplitude for the \( |m_s = +1\rangle \) \( (|m_s = -1\rangle) \) state is \( \cos^2 \alpha \left( \sin^2 \alpha \right) \), in good agreement with our experiments.

### C. State preparation and readout

At the beginning of every parametric modulation experiment, we polarize the NV into \( |m_s = 0\rangle \), then apply two microwave pulses to prepare NV into the ground eigenstate \( |u_-\rangle \) before subjecting the system to the engineered Weyl-type Hamiltonian.
In preparing the ground state
\[ |u_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos \alpha e^{-i\beta} \\ 1 \\ -\sin \alpha e^{-i\phi} \end{pmatrix}, \tag{S59} \]
we first apply a microwave pulse that drives the \(-1\) transition \( |m_s = 0\rangle \leftrightarrow |m_s = -1\rangle \), immediately followed by another pulse for the \(+1\) transition \( |m_s = 0\rangle \leftrightarrow |m_s = +1\rangle \). We set the Rabi frequency of both pulses to be \( \omega_{\text{init}} \). Then the durations \( t_\pm \) and phases \( \delta_\pm \) of the two initialization pulses for the \( \pm 1 \) transitions are
\[
\begin{align*}
t_- &= \sin^{-1}(\sin \alpha/\sqrt{2})/\omega_{\text{init}}, \\
\delta_- &= \phi + \pi/2, \\
t_+ &= \sin^{-1}(\cos \alpha/\sqrt{2 - \sin^2 \alpha})/\omega_{\text{init}}, \\
\delta_+ &= \beta + \pi/2.
\end{align*} \tag{S60} \]

Similarly for the first excited state
\[ |u_0\rangle = \begin{pmatrix} -\sin \alpha e^{-i\beta} \\ 0 \\ \cos \alpha e^{-i\phi} \end{pmatrix}, \tag{S61} \]
we have
\[
\begin{align*}
t_- &= \pi/\omega_{\text{init}}, \\
\delta_- &= \phi + \pi, \\
t_+ &= \pi/2\omega_{\text{init}}, \\
\delta_+ &= \beta.
\end{align*} \tag{S62} \]

When modulating at the DQ frequency \( \omega_r = 2H_0 \), Rabi oscillations occur between these two states, and we apply the inverse mapping to rotate \( |u_-\rangle \) back to \( |m_s = 0\rangle \) for fluorescent readout. When modulating at the SQ frequency \( \omega_r = H_0 \), the situation is more involved. Due to chiral symmetry of the Weyl-type Hamiltonian, both SQ transitions \( |u_0\rangle \leftrightarrow |u_\pm\rangle \) are on-resonance, and they have the same driving strength \( \Gamma = |\Gamma_{-0}| = |\Gamma_{+0}| \). This leads to an effective DQ Hamiltonian in the eigen-basis of our engineered Weyl-type Hamiltonian. After entering the DQ rotating frame and taking the rotating wave approximation similar to what we did in Eq. S55, we obtain:
\[
\mathcal{H} = \begin{pmatrix} 0 & \Gamma e^{-i\phi_1} & 0 \\ \Gamma e^{i\phi_1} & 0 & \Gamma e^{i\phi_2} \\ 0 & \Gamma e^{-i\phi_2} & 0 \end{pmatrix}, \tag{S63} \]
where $\phi_{1(2)}$ are the phases associated with $\Gamma_{0,\pm}$. We remark that although Eq. S63 is of the same form as Eq. S56, they represent two different Hamiltonians. Eq. S63 is the DQ Hamiltonian in the eigen-basis of $|m_s\rangle$, and Eq. S63 is in the eigen-basis of Eq. S56, whose eigenvectors are $|u_{0,\pm}\rangle$.

Starting from the ground state $|u_\pm\rangle$, the system under Eq. S63 evolves as:

$$\begin{pmatrix}
    c_+(t) \\
    c_0(t) \\
    c_-(t)
\end{pmatrix} = \begin{pmatrix}
    \frac{1}{\sqrt{2}} \sin(\sqrt{2} T \Gamma_0) e^{i\phi_1} \\
    \frac{1}{\sqrt{2}} \cos(\sqrt{2} T \Gamma_0) e^{i\phi_2} \\
    -\frac{1}{\sqrt{2}} \sin(\sqrt{2} T \Gamma_0) e^{i\phi_1} \\
\end{pmatrix} \left( \begin{pmatrix}
    c_+ \\
    c_0 \\
    c_-
\end{pmatrix} \right).$$

For ease of fitting, we map $|u_0\rangle$ back to $|m_s=0\rangle$ and read out the population optically. The matrix element $\Gamma$ of interest is $1/\sqrt{2}m_\mu$ of the fitted Rabi frequency.

We remark that the mapping pulses described in this section do not perform the unitary transformation between the basis $\{|m_s\rangle\}$ and $\{|u\rangle\}$. We emphasize that this unitary transformation could be achieved by three microwave pulses, and is useful in determining the relevant matrix element when the chiral symmetry of the Hamiltonian is broken by the fictitious transverse field, where $|\Gamma_{-0}| \neq |\Gamma_{+0}|$, and the modulation frequency is resonant for both SQ transitions. In this case, we prepare the initial state in $|u_0\rangle$, such that it evolves according to Eq. S58. By measuring both the amplitude and frequency of the Rabi oscillation in $|u_{\pm}\rangle$ states, we are able to reconstruct the matrix element of interest.

To determine the oscillation amplitudes accurately for all three states, we have to perform three sets of experiments. Each experiment consists of the same state preparation, parametric modulation, and the unitary map back to all three $|m_s\rangle$ states, followed by (i) no operation (ii) $\pi$ pulse between $|m_s=0\rangle \leftrightarrow |m_s=-1\rangle$ and (iii) $\pi$ pulse between $|m_s=0\rangle \leftrightarrow |m_s=+1\rangle$, and then optically read out. The fluorescence signals recorded in each experiments are labelled as $S_1$, the reference fluorescent level for each $|m_s\rangle$ states as measured in separate experiments are $r_{m_s}$, and the final population of each $|m_s\rangle$ states are $n_{m_s}$. From the three sets of experiment, we have

$$\begin{align}
r_+n_+ + r_0n_0 + r_-n_- &= S_1, \\
r_+n_+ + r_-n_0 + r_0n_- &= S_2, \\
r_0n_+ + r_+n_0 + r_-n_- &= S_3.
\end{align}$$

It is therefore straightforward to extract the populations accurately

$$\begin{pmatrix}
    n_+ \\
    n_0 \\
    n_-
\end{pmatrix} = \begin{pmatrix}
    r_+ & r_0 & r_- \\
    r_+ & r_- & r_0 \\
    r_0 & r_+ & r_- \\
\end{pmatrix}^{-1} \begin{pmatrix}
    S_1 \\
    S_2 \\
    S_3
\end{pmatrix}.$$  

In addition to the parametric modulations, this readout technique is used in e.g. Fig. S6 to reveal the accurate populations of all three states.

We remark that this method is only possible when the magnetic field is close to the excited state level anticrossing at 510 G, where the excited state electron-nuclear spin flip-flops yield distinguishable fluorescent levels for $|m_s=\pm1\rangle$.

**D. Experimental verification of rotation symmetry about $\beta, \phi$**

The Weyl-type Hamiltonian is rotationally symmetric about $\beta, \phi$ under our parametrization of $(H_0, \alpha, \beta, \phi)$. As a result, the metric tensor and generalized 3-form Berry curvature is independent of $\beta, \phi$. To show that this also occurs in our experiments, we fix $\alpha = \pi/8$ and sweep $\beta, \phi \in [0, 2\pi]$, and measure the corresponding matrix element $\Gamma$, and the metric tensor components $g_{\alpha\alpha}, g_{\beta\phi}$. The results are shown in Fig. S8, where we indeed see these measurements remain constant within experimental error for different $\beta, \phi$.

**E. Coherent Rabi oscillations under parametric modulations**

With the capability of state preparation and readout described above, we now show the coherent Rabi oscillations observed in experiments under appropriate linear parametric modulations. In Fig. S9- S13, we plot all 18 Rabi oscillations (9 for SQ transitions and 9 for DQ transitions) measured for $\alpha = 5\pi/16$, $\beta = \phi = 0$, which are in turn used to obtain the matrix elements $\Gamma$ (Fig. 2 in main text) to extract all 6 independent metric tensor components $g_{\mu\nu}$ (Fig. 3 in main text), and ultimately yields the $DD$ invariant, as described in previous sections and the main text.
FIG. S8. Rotation symmetry of $\beta, \phi$. We perform experiments at $\alpha = \pi/8$, sweeping either $\beta$ or $\phi$ and measuring $g_{\alpha\alpha}, g_{\beta\phi}$ to verify that the metric tensor is independent of $\beta, \phi$ due to the rotation symmetry of the Hamiltonian. On the left (right) we show results when fixing $\phi = 0$ ($\beta = 0$) and sweeping $\beta$ ($\phi$). The top panel shows relevant matrix element measurements $\Gamma^{\alpha}$ (green), $\Gamma^{\beta\phi}$ (blue), $\Gamma^{\beta\bar{\phi}}$ (red) in circles, and theoretical values in solid lines. The bottom panel shows extracted metric tensor components. Within experimental error they stay constant over $\beta, \phi$.

F. Matrix elements for elliptical parametric modulation

In this section we show the measured matrix elements for SQ and DQ transitions using elliptical parametric modulation under $B_z = 0$ in Fig. S14. We extract the Berry curvatures from them, which are shown in Fig. 3 (c) in the main text, and eventually reveal the tensor monopole.

G. Phase transition induced by an external field

As we discussed in the main text, detunings in the dual-frequency microwave pulse induce diagonal terms in our engineered Hamiltonian, acting as an external $z$ field (with the same form as a spin-1 $B_z S_z$ field operator). When $B_z = 0$, our observable, $\mathcal{M} = 8 \int \epsilon_{\mu\nu\lambda} \sqrt{\det \bar{g}_{\mu\nu}} \, d\alpha$, is equivalent to the $\mathcal{DD}$ invariant. The corresponding measured data for $\mathcal{G}$ and the metric tensor are shown in Fig. 2 and Fig. 3 of the main text. When $B_z \neq 0$, the diagonal terms break the chiral symmetry of our system, thus breaking the relationship between $\det \bar{g}_{\mu\nu}$ and $\mathcal{H}_{\mu\nu\lambda}$. Surprisingly, similar to the observable $\mathcal{B}$ described in Sec. 1D, we find that the chosen observable $\mathcal{M}$ can still be used to investigate the behavior of the nodal surfaces when varying $B_z$.

We have shown that it is experimentally feasible to measure the metric tensor through parametric modulation, which yields $\mathcal{G}$. Ref.1 has shown that when $B_z = 0$, $\mathcal{G}$ is equivalent to the $\mathcal{DD}$ invariant. Here we numerically simulate the case when $B_z > 0$, where the two are no longer equivalent. The simulation result is shown in green triangles in Fig. 4 in the main text. We see that $\mathcal{G}$ is a good approximation to the $|\mathcal{B}|$. When $B_z < R_0$, they quantitatively match well. At $B_z = R_0$, $\mathcal{G}$ correctly characterizes the transition when the nodal rings cross the boundary of the enclosed manifold. We therefore experimentally measure $\mathcal{G}$ to reveal the phase transition, as shown in Fig. 4 in the main text. We show additional experimental data relevant to the phase transition in Fig. S15-S20.
FIG. S9. **Coherent Rabi oscillation for** $\Omega^\alpha$ (left) and $\Omega^\beta$ (right). The experiment is performed at $(\alpha = 5\pi/16, \beta = \phi = 0)$. In each plot, blue is for SQ transition and red for DQ transition. The circles are experimental data and solid lines are sinusoidal fits.

FIG. S10. **Coherent Rabi oscillation for** $\Omega^\phi$ (left) and $\Omega^\alpha\beta$ (right). The experiment is performed at $(\alpha = 5\pi/16, \beta = \phi = 0)$. In each plot, blue is for SQ transition and red for DQ transition. The circles are experimental data and solid lines are sinusoidal fits.

While measuring the Berry connection generally involves quantum state tomography and is time-consuming, we note that thanks to the chosen parametrization for our particular Hamiltonian, measurement of $B$ does not require state tomography, as we have shown in Section 1D. To this end, we calculate and measure the observable $B$ from the Berry connection using Eq. S6, S7, S17, as shown in yellow (analytical result) and red squares (experiment) in Fig. 4 of the main text. The 2-form Berry curvature $F$ is measured by the parametric modulation, similar to the measurement of the metric tensor, as discussed in previous sections. The experimentally measured $B$ is shown in red squares in Fig. 4 of the main text, and additional experimental data relevant to the measurements are presented in Fig. S21-S27. $B$ stays constant $B = 1$ when $B_z < R_0$. At $B_z = R_0$, the two degenerate nodal rings are at the boundary of our enclosed manifold, as indicated by a sharp change in $B$ to $B(B_z/R_0 = 1) = -1/3$, as expected.

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FIG. S11. **Coherent Rabi oscillation for** $\Omega^{\alpha \beta}$ (left) and $\Omega^{\alpha \phi}$ (right). The experiment is performed at $(\alpha = 5\pi/16, \beta = \phi = 0)$. In each plot, blue is for SQ transition and red for DQ transition. The circles are experimental data and solid lines are sinusoidal fits.

FIG. S12. **Coherent Rabi oscillation for** $\Omega^{\alpha \phi}$ (left) and $\Omega^{\beta \phi}$ (right). The experiment is performed at $(\alpha = 5\pi/16, \beta = \phi = 0)$. In each plot, blue is for SQ transition and red for DQ transition. The circles are experimental data and solid lines are sinusoidal fits.


FIG. S13. Coherent Rabi oscillation for $\Omega_{\alpha,\beta}$. The experiment is performed at $(\alpha = 5\pi/16, \beta = \phi = 0)$. In each plot, blue is for SQ transition and red for DQ transition. The circles are experimental data and solid lines are sinusoidal fits.

FIG. S14. Matrix elements under elliptical parametric modulations when $B_z = 0$.

FIG. S15. Metric tensor measurements for $B_z/R_0 = 0.25\sqrt{2}$. Matrix elements $|\Gamma_{\alpha,\beta}^{(\nu)}|$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma_{\alpha,\beta}^{(\nu)}|$ measured for DQ transitions at $\omega = \omega_r$ (middle). On the right we show all 6 independent components of the metric tensor as functions of $\alpha$. Circles are experimental data and solid lines are numerical simulations.
FIG. S16. **Metric tensor measurements for** $B_z/R_0 = 0.45\sqrt{2}$. Matrix elements $|\Gamma_{\mu(\nu)}^{\mu(\nu)}|_0$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma_{\mu(\nu)}^{\mu(\nu)}|_0$ measured for DQ transitions at $\omega = \omega_r$ (middle). On the right we show all 6 independent components of the metric tensor as functions of $\alpha$. Circles are experimental data and solid lines are numerical simulations.

FIG. S17. **Metric tensor measurements for** $B_z/R_0 = 0.6\sqrt{2}$. Matrix elements $|\Gamma_{\mu(\nu)}^{\mu(\nu)}|_0$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma_{\mu(\nu)}^{\mu(\nu)}|_0$ measured for DQ transitions at $\omega = \omega_r$ (middle). On the right we show all 6 independent components of the metric tensor as functions of $\alpha$. Circles are experimental data and solid lines are numerical simulations.

FIG. S18. **Metric tensor measurements for** $B_z/R_0 = 0.825\sqrt{2}$. Matrix elements $|\Gamma_{\mu(\nu)}^{\mu(\nu)}|_0$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma_{\mu(\nu)}^{\mu(\nu)}|_0$ measured for DQ transitions at $\omega = \omega_r$ (middle). On the right we show all 6 independent components of the metric tensor as functions of $\alpha$. Circles are experimental data and solid lines are numerical simulations.
FIG. S19. Metric tensor measurements for $B_z/R_0 = \sqrt{2}$. Matrix elements $|\Gamma^\mu_{\nu,0}|$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma^\mu_{\nu,-1}|$ measured for DQ transitions at $\omega = \omega_r$ (middle). On the right we show all 6 independent components of the metric tensor as functions of $\alpha$. Circles are experimental data and solid lines are numerical simulations.

FIG. S20. Metric tensor measurements for $h = 2$. Matrix elements $|\Gamma^\mu_{\nu,0}|$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma^\mu_{\nu,-1}|$ measured for DQ transitions at $\omega = \omega_r$ (middle). On the right we show all 6 independent components of the metric tensor as functions of $\alpha$. Circles are experimental data and solid lines are numerical simulations.

FIG. S21. Berry curvature measurements for $B_z/R_0 = 0.35\sqrt{2}$. Matrix elements $|\Gamma^\mu_{\nu,0}|$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma^\mu_{\nu,-1}|$ measured for DQ transitions at $\omega = \omega_r$ (middle) using elliptical modulation. On the right we show the two relevant Berry curvatures as functions of $\alpha$. Squares are experimental data and solid lines are numerical simulations.
FIG. S22. Berry curvature measurements for $B_z/R_0 = 0.5\sqrt{2}$. Matrix elements $|\Gamma_{\mu\nu}^{\alpha}\rangle$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma_{\mu\nu}^{\alpha}\rangle$ measured for DQ transitions at $\omega = \omega_r$ (middle) using elliptical modulation. On the right we show the two relevant Berry curvatures as functions of $\alpha$. Squares are experimental data and solid lines are numerical simulations.

FIG. S23. Berry curvature measurements for $h = 0.65\sqrt{2}$. Matrix elements $|\Gamma_{\mu\nu}^{\alpha}\rangle$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma_{\mu\nu}^{\alpha}\rangle$ measured for DQ transitions at $\omega = \omega_r$ (middle) using elliptical modulation. On the right we show the two relevant Berry curvatures as functions of $\alpha$. Squares are experimental data and solid lines are numerical simulations.

FIG. S24. Berry curvature measurements for $B_z/R_0 = 0.9\sqrt{2}$. Matrix elements $|\Gamma_{\mu\nu}^{\alpha}\rangle$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma_{\mu\nu}^{\alpha}\rangle$ measured for DQ transitions at $\omega = \omega_r$ (middle) using elliptical modulation. On the right we show the two relevant Berry curvatures as functions of $\alpha$. Squares are experimental data and solid lines are numerical simulations.
FIG. S25. *Berry curvature measurements for $B_z/R_0 = 1.2\sqrt{2}$. Matrix elements $|\Gamma^{\nu}_{\mu,0}|$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma^{\nu}_{\mu,-1}|$ measured for DQ transitions at $\omega = \omega_r$ (middle) using elliptical modulation. On the right we show the two relevant Berry curvatures as functions of $\alpha$. Squares are experimental data and solid lines are numerical simulations.*

FIG. S26. *Berry curvature measurements for $B_z/R_0 = 2$. Matrix elements $|\Gamma^{\nu}_{\mu,0}|$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma^{\nu}_{\mu,-1}|$ measured for DQ transitions at $\omega = \omega_r$ (middle) using elliptical modulation. On the right we show the two relevant Berry curvatures as functions of $\alpha$. Squares are experimental data and solid lines are numerical simulations.*

FIG. S27. *Berry curvature measurements for $B_z/R_0 = 1.7\sqrt{2}$. Matrix elements $|\Gamma^{\nu}_{\mu,0}|$ measured for SQ transitions at $\omega = \omega_r/2$ (left) and Matrix elements $|\Gamma^{\nu}_{\mu,-1}|$ measured for DQ transitions at $\omega = \omega_r$ (middle) using elliptical modulation. On the right we show the two relevant Berry curvatures as functions of $\alpha$. Squares are experimental data and solid lines are numerical simulations.*