

Hamiltonian Control of Quantum Dynamical Semigroups: Stabilization and Convergence Speed

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Abstract—We consider finite-dimensional Markovian open quantum systems, and characterize the extent to which time-independent Hamiltonian control may allow to stabilize a target quantum state or subspace and optimize the resulting convergence speed. For a generic Lindblad master equation, we introduce a dissipation-induced decomposition of the associated Hilbert space, and show how it serves both as a tool to analyze global stability properties for given control resources and as the starting point to synthesize controls that ensure rapid convergence. The resulting design principles are illustrated in realistic Markovian control settings motivated by quantum information processing, including quantum-optical systems and nitrogen-vacancy centers in diamond.

Index Terms—Decoherence free subspace (DFS), dissipation-induced decomposition (DID), quantum dynamical semigroup (QDS).

I. INTRODUCTION

DESIGNING effective strategies for stabilizing a desired quantum state or subsystem under general dissipative dynamics is an important problem from both a control-theoretic and quantum engineering standpoint. Significant effort has been recently devoted, in particular, to the paradigmatic class of Markovian open quantum systems, whose (continuous-time) evolution is described by a quantum dynamical semigroup [1]. Building on earlier controllability studies [2]–[4], Markovian stabilization problems have been addressed in settings ranging from the preparation of complex quantum states in multipartite systems to the synthesis of noiseless quantum information encodings by means of open-loop Hamiltonian control and reservoir engineering as well as quantum feedback [5]–[10].

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While a number of rigorous results and control protocols have emerged, the continuous progress witnessed by laboratory quantum technologies makes it imperative to develop theoretical approaches which attempt to address practical constraints and limitations.

In this work, we focus on the open-loop stability properties of quantum semigroup dynamics that is solely controlled in terms of *time-independent Hamiltonians*, with a twofold motivation in mind: i) determining under which conditions a desired target state, or more generally a subspace, may be stabilizable given limited control resources; ii) characterizing how Hamiltonian control influences the asymptotic speed of convergence to the target space. A number of analysis tools are developed to this end. We start by introducing a constructive procedure for determining whether a given invariant subspace is attractive: if successful, the algorithm identifies as a byproduct a (unique) decomposition of the Hilbert space, which we term *dissipation-induced decomposition* and will provide us with a standard form for representing and studying the underlying Markovian dynamics (Section III-A). An enhanced version of the algorithm is also presented, in order to determine which control inputs, if any, can ensure convergence in the presence of control constraints (Section III-B). Next, we illustrate two approaches for analyzing the speed of convergence of the semigroup to the target space: the first, which is system-theoretic in nature, offers in principle a quantitative way of computing the asymptotic speed of convergence (Section IV-A); the second, which builds directly on the above dissipation-induced decomposition and we term *connected basins approach*, offers instead a qualitative way of estimating the convergence speed and designing control in situations where exact analytical or numerical methods are impractical (Section IV-B). By using these tools, we show how a number of fundamental issues related to the role of the Hamiltonian in the convergence of quantum dynamical semigroups can be tackled, thus leading to further physical insight on the interplay between coherent control and dissipation [11]. A number of physically motivated examples are discussed in Section V, demonstrating how our approach can be useful in realistic quantum control scenarios.

II. QUANTUM DYNAMICAL SEMIGROUPS

A. Open-Loop Controlled QDS Dynamics

Throughout this work, we shall consider a finite-dimensional open quantum system with associated complex Hilbert space \mathcal{H} , with $\dim(\mathcal{H}) = n$. Using Dirac's notation [12], we denote vectors in \mathcal{H} by $|\psi\rangle$, and linear functionals in the dual $\mathcal{H}^\dagger \simeq \mathcal{H}$ by $\langle\phi|$. Let in addition $\mathcal{B}(\mathcal{H})$ be the set of linear operators on \mathcal{H} , with $\mathfrak{h}(\mathcal{H})$ being the Hermitian ones, and $\mathfrak{D}(\mathcal{H}) \subset \mathfrak{h}(\mathcal{H})$

the trace-one, positive semidefinite operators (or *density operators*), which represent the *states* of the system. Given a matrix representation of an operator X , we shall denote with X^* , X^T , and X^\dagger the conjugate, the transpose, and the conjugate transpose (adjoint), respectively.

The dynamics we consider are governed by a master equation [1], [13]–[15] of the Lindblad form

$$\dot{\rho} = \mathcal{L}[\rho] = -\frac{i}{\hbar}[H, \rho] + \sum_{k=1}^p \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right) \quad (1)$$

where the effective Hamiltonian $H \in \mathfrak{h}(\mathcal{H})$ and the noise operators $\{L_k\} \subset \mathcal{B}(\mathcal{H})$ describe, respectively, the coherent (unitary) and dissipative (non-unitary) contributions to the dynamics. The resulting propagator $\mathcal{T}_t[\rho] := e^{\mathcal{L}t}[\rho]$, $t \geq 0$, maps $\mathfrak{D}(\mathcal{H})$ into itself. If, as we shall assume, the generator \mathcal{L} is time-invariant, $\{\mathcal{T}_t\}$ enjoys a forward (Markov) composition law, that is $\mathcal{T}_{t+s} = \mathcal{T}_t \circ \mathcal{T}_s$, $t, s \geq 0$, and thus forms a one-parameter quantum dynamical semigroup (QDS). In what follows, we set $\hbar = 1$ unless otherwise stated.

We focus on control scenarios where the Hamiltonian H can be tuned through suitable control inputs, that is

$$H(u) = H_0 + \sum_{j=1}^{\nu} H_j u_j \quad (2)$$

where $H_0 = H_0^\dagger$ represents the free (internal) system Hamiltonian, and the controls $u_j \in \mathbb{R}$ modify the dynamics through the Hamiltonians $H_j = H_j^\dagger$. In particular, we are interested in the case of constant controls u_j taking values in some (possibly open) interval $\mathcal{C}_j \subseteq \mathbb{R}$. The set of admissible control choices is then a subset $\mathcal{C} \subseteq \mathbb{R}^\nu$.

B. Stable Subspaces for QDS Dynamics

We begin by recalling some relevant definitions and results of the linear-algebraic approach to stabilization of QDS developed in [5]–[7], [9], [16]. Consider an orthogonal decomposition of the Hilbert space $\mathcal{H} := \mathcal{H}_S \oplus \mathcal{H}_R$, with $\dim(\mathcal{H}_S) = m \leq n$. Let $\{|s_i\rangle\}$ and $\{|r_j\rangle\}$ be orthonormal sets spanning \mathcal{H}_S and \mathcal{H}_R respectively. The ordered basis $\{|s_1\rangle, \dots, |s_m\rangle, |r_1\rangle, \dots, |r_{n-m}\rangle\}$ induces the following block structure on the matrix representation of an arbitrary operator $X \in \mathcal{B}(\mathcal{H})$:

$$X = \begin{bmatrix} X_S & X_P \\ X_Q & X_R \end{bmatrix}. \quad (3)$$

We will denote the support of X by $\text{supp}(X) := \ker(X)^\perp$. It will be useful to introduce a compact notation for sets of states with support contained in a given subspace

$$\mathfrak{I}_S(\mathcal{H}) := \left\{ \rho \in \mathfrak{D}(\mathcal{H}) \mid \rho = \begin{bmatrix} \rho_S & 0 \\ 0 & 0 \end{bmatrix}, \rho_S \in \mathfrak{D}(\mathcal{H}_S) \right\}.$$

As usual in the study of dynamical systems, we say that a set of states \mathcal{W} is *invariant* for the dynamics generated by \mathcal{L} if arbitrary trajectories originating in \mathcal{W} at $t = 0$ remain confined to \mathcal{W} at all positive times. Henceforth, with a slight abuse of terminology, we will say that a *subspace* $\mathcal{H}_S \subset \mathcal{H}$ is *\mathcal{L} -invariant* (or simply invariant) when $\mathfrak{I}_S(\mathcal{H})$ is invariant for the dynamics

generated by \mathcal{L} . An algebraic characterization of “subspace-invariant” QDS generators is provided by the following Proposition (the proof is given in [5], in the more general *subsystem* case):

Proposition 1 (S-Subspace Invariance): Consider a QDS on $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, and let the generator \mathcal{L} in (1) be associated to an Hamiltonian H and a set of noise operators $\{L_k\}$. Then \mathcal{H}_S is invariant if and only if the following conditions hold:

$$\begin{cases} iH_P - \frac{1}{2} \sum_k L_{S,k}^\dagger L_{P,k} = 0, \\ L_k = \begin{bmatrix} L_{S,k} & L_{P,k} \\ 0 & L_{R,k} \end{bmatrix} \quad \forall k. \end{cases} \quad (4)$$

In order to find equivalent conditions for the invariance of \mathcal{H}_R , which will be useful in the next sections, it suffices to swap the role of the subspaces, reorder the blocks and apply Proposition 1. By recalling that $H_P = H_Q^\dagger$, this yields:

Corollary 1 (R-Subspace Invariance): Consider a QDS on $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, and let the generator \mathcal{L} in (1) be associated to an Hamiltonian H and a set of noise operators $\{L_k\}$. Then \mathcal{H}_R is invariant if and only if the following conditions hold:

$$\begin{cases} iH_P + \frac{1}{2} \sum_k L_{Q,k}^\dagger L_{R,k} = 0, \\ L_k = \begin{bmatrix} L_{S,k} & 0 \\ L_{Q,k} & L_{R,k} \end{bmatrix} \quad \forall k. \end{cases} \quad (5)$$

One of our aims in this paper is to determine a choice of controls that render an invariant subspace also Globally Asymptotically Stable (GAS). That is, we wish the target subspace \mathcal{H}_S to be both invariant and *attractive*, so that the following property is obeyed:

$$\lim_{t \rightarrow \infty} \delta(\mathcal{T}_t[\rho], \mathfrak{I}_S(\mathcal{H})) = 0, \quad \forall \rho \in \mathfrak{D}(\mathcal{H})$$

where $\delta(\sigma, \mathcal{W}) := \inf_{\tau \in \mathcal{W}} \|\sigma - \tau\|$. In [6], a number of results concerning the stabilization of pure states and subspaces by both open-loop and feedback protocols have been established. For time-independent Hamiltonian control, in particular, the following condition may be derived from (4) above:

Corollary 2 (Open-Loop Invariant Subspace): Let $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$. Assume that we can modify the system Hamiltonian as $H' = H + H_c$, with H_c being an arbitrary, time-independent control Hamiltonian. Then $\mathfrak{I}_S(\mathcal{H})$ can be made invariant under \mathcal{L} if and only if $L_{Q,k} = 0$ for every k .

In addition, the following theorems from [6] will be needed: the first provides necessary and sufficient conditions for attractivity, while the second establishes when Hamiltonian control, without control restrictions, is able to achieve stabilization:

Theorem 1 (Subspace Attractivity): Let $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, and assume that \mathcal{H}_S is an invariant subspace for the QDS dynamics in (1). Let

$$\mathcal{H}_{R'} = \bigcap_{k=1}^p \ker(L_{P,k}) \quad (6)$$

with each matrix block $L_{P,k}$ representing a linear operator from \mathcal{H}_R to \mathcal{H}_S . Then \mathcal{H}_S is GAS under \mathcal{L} if and only if $\mathcal{H}_{R'}$ does not support any invariant subsystem.

Theorem 2 (Open-Loop Subspace Attractivity): Let $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, with \mathcal{H}_S supporting an invariant subsystem. Assume that we can apply arbitrary, time-independent control Hamiltonians. Then $\mathfrak{I}_S(\mathcal{H})$ can be made GAS under \mathcal{L} if and only if $\mathfrak{I}_R(\mathcal{H})$ is *not* invariant.

From a practical standpoint, the assumption of access to an arbitrary control Hamiltonian H_c is too strong. Thus, we shall develop (Section III-B) an approach that allows us to determine whether and how a given stabilization task can be attained with available (in general restricted) time-independent Hamiltonian controls, as well as to characterize the role of the Hamiltonian component in the resulting speed of convergence.

III. ANALYSIS AND SYNTHESIS TOOLS

A. Stability and Dissipation-Induced Decomposition

Suppose that we are given a target subspace $\mathcal{H}_S \subseteq \mathcal{H}$. By using Proposition 1, it can be easily checked if \mathcal{H}_S is invariant for a given QDS. In this section, we introduce an algorithm that further determines whether \mathcal{H}_S is also GAS. The main idea is to use Theorem 1 *iteratively*, so as to restrict the subspace on which an undesired invariant set could be supported. Notice in fact that $\mathcal{H}_{R'}$ in (6) is strictly contained in \mathcal{H}_R as soon as one of the off-diagonal $L_{P,k}$ blocks is not zero. If they are all zero, either the Hamiltonian destabilizes \mathcal{H}_R , or the latter is invariant. In the first case, one can refine the decomposition as $\mathcal{H}_R = \mathcal{H}_T \oplus \mathcal{H}_{R'}$, with \mathcal{H}_T a subspace which is dynamically connected to \mathcal{H}_S . The reasoning can be iterated, by focusing on the dynamics in $\mathcal{H}_{R'}$, until either the remainder is invariant, or there is no invariant subspace. We begin by presenting the algorithm, and then prove that its successful completion ensures attractivity of the target subspace.

Algorithm for GAS Verification: Let \mathcal{H}_S be invariant. Call $\mathcal{H}_R^{(0)} := \mathcal{H}_R$, $\mathcal{H}_S^{(0)} := \mathcal{H}_S$, choose an orthonormal basis for the subspaces and write the matrices with respect to that basis. Rename the matrix blocks as follows: $H_S^{(0)} := H_S$, $H_P^{(0)} := H_P$, $H_R^{(0)} := H_R$, $L_{S,k}^{(0)} := L_{S,k}$, $L_{P,k}^{(0)} := L_{P,k}$, and $L_{R,k}^{(0)} := L_{R,k}$.

For $j \geq 0$, consider the following iterative procedure:

- 1) Compute the matrix blocks $L_{P,k}^{(j)}$ according to the decomposition $\mathcal{H}^{(j)} = \mathcal{H}_S^{(j)} \oplus \mathcal{H}_R^{(j)}$.
- 2) Define $\mathcal{H}_R^{(j+1)} := \bigcap_k \ker L_{P,k}^{(j)}$.
- 3) Consider the following three sub-cases:
 - a. If $\mathcal{H}_R^{(j+1)} = \{0\}$, define $\mathcal{H}_T^{(j+1)} := \mathcal{H}_R^{(j)}$. The iterative procedure is successfully completed.
 - b. If $\mathcal{H}_R^{(j+1)} \neq \{0\}$, but $\mathcal{H}_R^{(j+1)} \subsetneq \mathcal{H}_R^{(j)}$, define $\mathcal{H}_T^{(j+1)}$ as the orthogonal complement of $\mathcal{H}_R^{(j+1)}$ in $\mathcal{H}_R^{(j)}$, that is, $\mathcal{H}_T^{(j+1)} = \mathcal{H}_R^{(j)} \ominus \mathcal{H}_R^{(j+1)}$.
 - c. If $\mathcal{H}_R^{(j+1)} = \mathcal{H}_R^{(j)}$ (that is, $L_{P,k}^{(j)} = 0 \forall k$), define

$$\tilde{\mathcal{L}}_P^{(j)} := -iH_P^{(j)} - \frac{1}{2} \sum_k L_{Q,k}^{(j)\dagger} L_{R,k}^{(j)}.$$

— If $\tilde{\mathcal{L}}_P^{(j)} \neq 0$, re-define $\mathcal{H}_R^{(j+1)} := \ker(\tilde{\mathcal{L}}_P^{(j)})$. If $\mathcal{H}_R^{(j+1)} = \{0\}$, define $\mathcal{H}_T^{(j+1)} := \mathcal{H}_R^{(j)}$, and the iterative procedure is successfully completed. Otherwise define $\mathcal{H}_T^{(j+1)} := \mathcal{H}_R^{(j)} \ominus \mathcal{H}_R^{(j+1)}$.

— If $\tilde{\mathcal{L}}_P^{(j)} = 0$, then, by Corollary 1, $\mathcal{H}_R^{(j)}$ is invariant, and thus, by Theorem 1, \mathcal{H}_S cannot be GAS. Exit the algorithm.

- 4) Define $\mathcal{H}_S^{(j+1)} := \mathcal{H}_S^{(j)} \oplus \mathcal{H}_T^{(j+1)}$. To construct a basis for $\mathcal{H}_S^{(j+1)}$, append to the *already defined* basis for $\mathcal{H}_S^{(j)}$ an orthonormal basis for $\mathcal{H}_T^{(j+1)}$.

- 5) Increment the counter j and go back to step 1).

The algorithm ends in a finite number of steps, since at every iteration it either stops or the dimension of $\mathcal{H}_R^{(j)}$ is reduced by at least one. As anticipated, its main use is as a constructive procedure to test attractivity of a given subspace \mathcal{H}_S :

Proposition 2: The algorithm is successfully completed if and only if the target subspace \mathcal{H}_S is GAS ($\mathfrak{I}_S(\mathcal{H})$ is GAS).

Proof: If the algorithm stops because $\tilde{\mathcal{L}}_P^{(j)} = 0$, for some j , then Corollary 1 implies that \mathcal{H}_R contains an invariant subspace, hence \mathcal{H}_S cannot be GAS. On the other hand, let us assume that the algorithm runs to completion, achieved at $j \equiv q$. Then we have obtained a decomposition $\mathcal{H}_R = \mathcal{H}_T^{(1)} \oplus \mathcal{H}_T^{(2)} \oplus \dots \oplus \mathcal{H}_T^{(q)}$, and we can prove by (finite) induction that no invariant subspace is contained in \mathcal{H}_R .

Let us start from $\mathcal{H}_T^{(q)}$. By definition, since the algorithm is completed when $\mathcal{H}_R^{(j+1)} = \mathcal{H}_R^{(q+1)} = \{0\}$, either $\bigcap_k \ker(L_{P,k}^{(q)}) = \{0\}$, or $L_{P,k}^{(q)} = 0$ and $\tilde{\mathcal{L}}_P^{(q)}$ is full column-rank. In the first case, Theorem 1 guarantees that $\mathcal{H}_T^{(q)}$ does not contain any invariant set since its complement is attractive. In the second case, the P -block of the whole generator can be explicitly computed to be $\rho_R(\tilde{\mathcal{L}}_P^{(q)})^\dagger$. Because $\tilde{\mathcal{L}}_P^{(q)}$ is full column-rank, for any $\rho_R \neq 0$ the P -block is not zero. This means that the dynamics drives any state with support only in $\mathcal{H}_T^{(q)}$ out of the subspace, which cannot thus contain any invariant set.

Now assume (inductive hypothesis) that $\mathcal{H}_T^{(\ell+1)} \oplus \dots \oplus \mathcal{H}_T^{(q)}$, $\ell+1 \leq q$, does not contain invariant subspaces, and that (by contradiction) $\mathcal{H}_T^{(\ell)} \oplus \mathcal{H}_T^{(\ell+1)} \oplus \dots \oplus \mathcal{H}_T^{(q)}$ does. Then the invariant subspace should be non-orthogonal to $\mathcal{H}_T^{(\ell)}$, which is, by definition, the orthogonal complement of either $\bigcap_k \ker(L_{P,k}^{(\ell-1)})$ or $\ker(\tilde{\mathcal{L}}_P^{(\ell-1)})$. But then any state ρ with support only on $\mathcal{H}_T^{(\ell)} \oplus \mathcal{H}_T^{(\ell+1)} \oplus \dots \oplus \mathcal{H}_T^{(q)}$ and non-trivial support on $\mathcal{H}_T^{(\ell)}$ would violate the invariance conditions and, by arguments analogue to the ones above it would leave the subspace. Therefore, $\mathcal{H}_T^{(\ell)} \oplus \dots \oplus \mathcal{H}_T^{(q)}$ does not contain invariant subspaces. By iterating until $\ell = 1$, we infer that \mathcal{H}_R cannot contain invariant subspaces and, by Theorem 1, the conclusion follows. ■

Formally, the above construction motivates the following:

Definition 1: Let $\mathfrak{I}_S(\mathcal{H})$ be GAS for the QDS dynamics in (1). The Hilbert space decomposition given by

$$\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_T^{(1)} \oplus \mathcal{H}_T^{(2)} \dots \oplus \mathcal{H}_T^{(q)} \quad (7)$$

as obtained from the previous algorithm, is called the *Dissipation-Induced Decomposition* (DID). Each of the subspaces $\mathcal{H}_T^{(i)}$ in the direct sum is referred to as a *basin*.

Partitioning each matrix L_k in blocks according to the DID results in the following standard structure, where the upper block-diagonal blocks establish the *dissipation-induced connections* between the different basins $\mathcal{H}_T^{(i)}$:

$$L_k = \begin{bmatrix} L_S & \hat{L}_P^{(0)} & 0 & \cdots \\ 0 & L_T^{(1)} & \hat{L}_P^{(1)} & 0 & \cdots \\ \vdots & L_Q^{(1)} & L_T^{(2)} & \hat{L}_P^{(1)} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}_k \quad (8)$$

Since, in step 3.b of the GAS verification algorithm, the basin $\mathcal{H}_T^{(j+1)}$ is defined to be in the *complement* of $\mathcal{H}_R^{(j+1)} = \bigcap_k \ker L_{P,k}^{(j)}$, at each iteration the only non-zero parts of the $L_P^{(j)}$ blocks must be in the $(j, j+1)$ block, which we have denoted by $\hat{L}_{P,k}^{(j)}$ in (8). In the upper-triangular part of the matrix, the other blocks of any row are thus zero by construction. If some $\hat{L}_{P,k}^{(j)} = 0 \forall k$, then either the dynamical connection is established by the Hamiltonian H , through the block $H_P^{(j)}$ (as checked in step 3.c), or the target subspace is not GAS.

Corollary 3: The DID in (7) is unique, and so is the associated matrix representation, up to a choice of basis in each of the orthogonal components $\mathcal{H}_S, \mathcal{H}_T^{(i)}, i = 1, \dots, q$.

The corollary is immediately proven, by noting that the algorithm is deterministic and does not allow for any arbitrary choice other than picking a basis in each of the $\mathcal{H}_T^{(i)}$.

Remark: It is worth observing that a different decomposition of the Hilbert space into a “collective” and “decaying” subspaces has been previously introduced in [17] for studying dissipative Lindblad dynamics. The approach of [17] begins by characterizing the structure of the invariant sets (thus emphasis is on the *collecting* basin) for the full generator, and then proceeds by iterating the reasoning on reduced models for the decaying subspace, disregarding how the latter is dynamically connected to the collecting one. Our focus is rather on characterizing the structure of the *decaying* subspace, in order to determine how the noise operators and the Hamiltonian drive the evolution towards the collecting subspace, or a larger subspace that contains it. Beside being motivated by control-oriented considerations, the DID we propose is thus different from their decomposition in [17], and depends on the target invariant subspace. Its uses will be detailed in the following sections.

We conclude this section by illustrating the algorithm with an Example, which will be further considered in Section V.

Example 1: Consider a bipartite quantum system consisting of two two-level systems (qubits), and on each subsystem choose a basis $\{|0\rangle_n, |1\rangle_n\}$, with $n = 1, 2$ labeling the qubit. The standard (computational) basis for the whole system is then given by $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where $|xy\rangle := |x\rangle_1 \otimes |y\rangle_2$. As customary, let in addition $\{\sigma_a, a = x, y, z\}$ denote Pauli pseudo-spin matrices [12], with the “ladder” operator

$\sigma_+ := (\sigma_x + i\sigma_y)/2 \equiv |0\rangle\langle 1|$. Assume that the dynamics is driven by the following QDS generator:

$$\dot{\rho} = \mathcal{L}[\rho] = -i[H, \rho] + L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\} \quad (9)$$

where

$$H = \left(\frac{1}{2}\sigma_z + \sigma_x\right) \otimes I + I \otimes \left(-\frac{1}{2}\sigma_z + \sigma_x\right), \quad (10)$$

$$L = \sigma_+ \otimes I + I \otimes \sigma_+. \quad (11)$$

It is easy to verify that the (entangled) state $\rho_d = |\psi_0\rangle\langle\psi_0|$, with

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}(|00\rangle - |01\rangle + |10\rangle)$$

is invariant, that is, $\mathcal{L}[\rho_d] = 0$. We can then construct the DID and verify that such state is also GAS. By definition, $\mathcal{H}_S^{(0)} = \text{span}\{|\psi_0\rangle\}$, and one can write its orthogonal complement as $\mathcal{H}_R^{(0)} = \text{span}\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$, with an orthonormal basis being given for instance by

$$|\psi_1\rangle = |11\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad (12)$$

$$|\psi_3\rangle = -\sqrt{\frac{2}{3}}(|00\rangle + \frac{1}{2}(|01\rangle - |10\rangle)). \quad (13)$$

We begin the iteration with $j = 0$ (step 1), having $L_P^{(0)} = [0 \ \sqrt{2/3} \ 0]$. We move on (step 2), by defining $\mathcal{H}_R^{(1)} := \ker(L_P^{(0)}) = \text{span}\{|\psi_1\rangle, |\psi_3\rangle\}$. We next get (step 3.b)

$$\mathcal{H}_T^{(1)} := \mathcal{H}_R^{(0)} \ominus \mathcal{H}_R^{(1)} = \text{span}\{|\psi_2\rangle\}$$

so that (step 4)

$$\mathcal{H}_S^{(1)} = \mathcal{H}_S^{(0)} \oplus \mathcal{H}_T^{(1)} = \text{span}\{|\psi_0\rangle, |\psi_2\rangle\}.$$

We thus set $j = 1$, represent the matrices with respect to the ordered basis $\{|\psi_0\rangle, |\psi_2\rangle\} \cup \{|\psi_1\rangle, |\psi_3\rangle\}$ for $\mathcal{H}_S^{(1)} \oplus \mathcal{H}_R^{(1)}$ and iterate, obtaining

$$L_P^{(1)} = \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix},$$

$$\mathcal{H}_R^{(2)} = \ker(L_P^{(1)}) = \text{span}\{|\psi_3\rangle\}, \mathcal{H}_T^{(2)} = \text{span}\{|\psi_1\rangle\},$$

$$\mathcal{H}_S^{(2)} = \text{span}\{|\psi_0\rangle, |\psi_2\rangle, |\psi_1\rangle\}.$$

Thus in the third iteration, with $j = 2$, we do not need to change the basis, but only the partitioning: we find that $L_P^{(2)} = [0 \ 0 \ 0]^T$. Hence we would have $\mathcal{H}_R^{(3)} = \mathcal{H}_R^{(2)}$, so we move to step 3.c. Computing the required matrix blocks yields

$$\tilde{\mathcal{L}}_P^{(2)} = -iH_P^{(2)} - \frac{1}{2} \sum_k L_{Q,k}^{(2)\dagger} L_{R,k}^{(2)} = -i[0 \ -\sqrt{3} \ 0]^T.$$

Redefining $\mathcal{H}_R^{(3)} := \ker(\tilde{\mathcal{L}}_P^{(2)})$, we find that $\mathcal{H}_R^{(3)} = \{0\}$, thus $\mathcal{H}_T^{(3)} := \mathcal{H}_R^{(3)}$ and the algorithm is successfully completed.

Hence ρ_d is GAS, and in the basis $\{|\psi_0\rangle, |\psi_2\rangle, |\psi_1\rangle, |\psi_3\rangle\}$, consistent with the DID Equation $\mathcal{H}_S \oplus \mathcal{H}_T^{(1)} \oplus \mathcal{H}_T^{(2)} \oplus \mathcal{H}_T^{(2)}$, we have the following matrix representations [c.f. (8)]:

$$L = \begin{bmatrix} 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-2}{\sqrt{3}} & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{3} \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 \end{bmatrix}.$$

It is thus evident how the transitions from $\mathcal{H}_T^{(1)}$ to \mathcal{H}_S , and from $\mathcal{H}_T^{(2)}$ to $\mathcal{H}_T^{(1)}$, are enacted by the dissipative part of the generator, whereas only the Hamiltonian is connecting $\mathcal{H}_T^{(3)}$ to $\mathcal{H}_T^{(1)}$, destabilizing $|\psi_3\rangle$.

B. QDS Stabilization Under Control Constraints

The algorithm for GAS verification can be turned into a design tool to determine whether the available Hamiltonian control (2) may achieve stabilization when the range of the control parameters is limited, that is, $(u_1, \dots, u_\nu) \in \mathcal{C} \subsetneq \mathbb{R}^\nu$. Assume we are given a target \mathcal{H}_S , which need not be invariant or attractive. We can proceed in two steps.

1) *Imposing Invariance*: Partition H, L_k according to $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$. If $L_{Q,k} \neq 0$ for some k , then \mathcal{H}_S is not invariant and it cannot be made so by Hamiltonian control, hence it cannot be GAS. On the other hand, if $L_{P,k} = 0$ for all k , then \mathcal{H}_S cannot be made GAS by Hamiltonian open-loop control since $\mathcal{I}_R(\mathcal{H})$ would necessarily be invariant too (Theorem 2).

When $L_{Q,k} = 0$ for all k and there exists a \bar{k} such that $L_{P,\bar{k}} \neq 0$, we need to compute (Proposition 1)

$$\tilde{\mathcal{L}}_P(u) = iH_P(u) - \frac{1}{2} \sum_k L_{S,k}^\dagger L_{P,k}.$$

If $\tilde{\mathcal{L}}_P(u) \neq 0$ for all $u \in \mathcal{C}$, then the desired subspace cannot be stabilized. Let $\mathcal{C}^{(0)}$ be the set of controls (if any) such that if $\bar{u} \in \mathcal{C}^{(0)}$, then $\tilde{\mathcal{L}}_P(\bar{u}) = 0$.

2) *Exploring the Control Set for Global Stabilization*: Having identified a set of control choices that make \mathcal{H}_S invariant, we can then use the algorithm to check whether they can also enforce the target subspace to be GAS. By inspection of the algorithm, the only step in which a different choice of Hamiltonian may have a role in determining the attractivity is 3.c. Assume that we fixed a candidate control input u , we are at iteration j and we stop at 3.c. Assume, in addition, that the last constrained set of controls we have defined is $\mathcal{C}^{(\ell)}$, $0 \leq \ell < j$ (in case the algorithm has not stopped yet, this is $\mathcal{C}^{(0)}$). Two possibilities arise:

- If $\tilde{\mathcal{L}}_P^{(j)} \neq 0$, define $\mathcal{C}^{(j)}$ as the subset of $\mathcal{C}^{(\ell)}$ such that if $\bar{u} \in \mathcal{C}^{(j)}$, then it is still true that $\tilde{\mathcal{L}}_P(\bar{u}) \neq 0$. Pick a choice of $u \in \mathcal{C}^{(j)}$, and proceed with the algorithm. Notice that if there exists a control choice \hat{u} such that $\tilde{\mathcal{L}}_P(\hat{u})$ has full rank, we can pick that and stop the algorithm, having attained the desired stabilization.

- If $\tilde{\mathcal{L}}_P^{(j)} = 0$, the algorithm stops since there is not a dynamical link from $\mathcal{H}_T^{(j+1)}$ towards $\mathcal{H}_T^{(j)}$, neither enacted by the noise operator nor by the Hamiltonian. Hence, we can modify the algorithm as follows. Let us define $\mathcal{C}^{(j)}$ as the subset of $\mathcal{C}^{(\ell)}$ such that if $\bar{u} \in \mathcal{C}^{(j)}$ then $\tilde{\mathcal{L}}_P(\bar{u}) \neq 0$. If $\mathcal{C}^{(j)}$ is empty, no other choice of control could destabilize $\mathcal{H}_R^{(j+1)}$, so \mathcal{H}_S cannot be rendered GAS. Otherwise, redefine $\mathcal{C}^{(\ell)} := \mathcal{C}^{(j)}$, pick a choice of controls in the new $\mathcal{C}^{(\ell)}$ (for instance at random), and proceed with the algorithm *going back to step ℓ* .

The above procedure either stops with a successful completion of the algorithm or with an empty $\mathcal{C}^{(j)}$. In the first case the stabilization task has been attained, in the second it has not, and no admissible control can avoid the existence of invariant states in \mathcal{H}_R .

Note that if each \mathcal{C}_j (thus \mathcal{C}) is finite, for instance in the presence of quantized control parameters, the algorithm will clearly stop in a finite number of steps. More generally, in the following Proposition we prove that in the common case of a cartesian product of intervals as the set of admissible controls, the design algorithm works with probability one:

Proposition 3: If $\bar{u} = (u_1, \dots, u_\nu)^T \in \mathcal{C} = \mathcal{I}_1 \times \dots \times \mathcal{I}_\nu$, where $\mathcal{I}_k = [a_k, b_k] \subset \mathbb{R}, k = 1, \dots, \nu$, the above algorithm will end in a finite number of steps with probability one.

Proof: The critical point in attaining GAS is finding a set of control values that ensures *invariance* of the desired set when the free dynamics would not. In fact, to this end we need to find a $u \in \mathcal{C}^{(0)} = \{u \in \mathcal{C} | \tilde{\mathcal{L}}_P(u) = 0\}$. Since $\mathcal{C}^{(0)}$ is the intersection between a product of intervals and a $(\nu - 1)$ -dimensional hyperplane in \mathbb{R}^ν , $\mathcal{C}^{(0)}$ belongs to a lower-dimensional manifold than \mathcal{C} . Once invariance has been guaranteed, we are left with the opposite problem: at each iteration j , we need ensure $\tilde{\mathcal{L}}_P^{(j)} \neq 0$. This is again a $(\nu - 1)$ -dimensional hyperplane in \mathbb{R}^ν . Therefore, if a certain u_0 is such that $\tilde{\mathcal{L}}_P^{(j)} = 0$ but not all of them are, this belongs to a lower-dimensional manifold with respect to $\mathcal{C}^{(0)}$. Hence, picking a random $u \in \mathcal{C}^{(0)}$ (with respect to a uniform distribution) will almost surely guarantee that the algorithm stops in a finite number of steps. ■

C. Approximate State Stabilization

A necessary and sufficient condition for a state (not necessarily pure) to be GAS is that it is the unique stationary state for the dynamics [7]: this fact can be exploited, under appropriate assumptions, to *approximately* stabilize a desired pure state ρ_d when *exact* stabilization cannot be achieved. Assume that at the first step in the previous procedure we see that ρ_d is not invariant, even if $L_{Q,k} = 0$ for all k , since $\tilde{\mathcal{L}}_P^{(0)} = iH_P^{(0)} - 1/2 \sum_k L_{S,k}^{(0)\dagger} L_{P,k}^{(0)} \neq 0$, and there exists no choice of controls that achieve stabilization. If however the (operator) norm of $\tilde{\mathcal{L}}_P^{(0)}$ is small, in a suitable sense, we can still hope that a GAS state close to ρ_d exists. This can be checked as follows:

- Define $\tilde{H}_P := i\tilde{\mathcal{L}}_P^{(0)}$. Consider a new Hamiltonian

$$\tilde{H} := H^{(0)} + \Delta H = \begin{bmatrix} H_S & H_P \\ H_P & H_R \end{bmatrix} + \begin{bmatrix} 0 & \tilde{H}_P \\ \tilde{H}_P^\dagger & 0 \end{bmatrix}.$$

By construction, ρ_d is invariant under \tilde{H} .

- Proceed with the algorithm described in the previous subsection in order to stabilize ρ_d with \tilde{H} instead of H .

- As a by-product, the subset of control values that achieve stabilization is found. Let it be denoted by $\mathcal{S} \subseteq \mathcal{C}$.
- Determine $u_* \in \mathcal{S}$ such that $\min_{u \in \mathcal{S}} \|\tilde{\mathcal{L}}_P^{(0)}(u)\|_\infty$ is attained.

After the control synthesis, the generator for the actual system is in the form $\dot{\rho} = \tilde{\mathcal{L}}[\rho] - \Delta\mathcal{L}[\rho]$, with $\Delta\mathcal{L}[\rho] = -i[\Delta H, \rho]$, and $\tilde{\mathcal{L}}[\rho]$ having ρ_d as its unique stationary state corresponding to a unique zero eigenvalue. Because the eigenvalues and eigenvectors of a matrix are a continuous function of its entries, the perturbed generator will still have a unique zero eigenvalue, corresponding to a unique stationary state close to the desired one, provided that the (operator) norm of $\Delta\mathcal{L}$ is small (with respect to the smallest in modulus of the non-zero eigenvalues). In our setting, $\|\Delta\mathcal{L}\|$ can be bounded by $2\|\tilde{H}_P\|$; however, this condition has to be verified case by case. If the zero eigenvalue is still unique, we have rendered GAS a (generally) mixed state in a neighborhood of ρ_d , in the control-theoretic jargon, we have achieved “practical stabilization” of the target state, the size of the neighborhood depending on $\|\Delta\mathcal{L}\|$.

IV. SPEED OF CONVERGENCE OF A QDS

How quickly can the system reach the GAS subspace \mathcal{H}_S from a generic initial state? We address this question in two different ways. The first approach relies on explicitly computing the asymptotic speed of convergence by considering the spectrum of \mathcal{L} as a linear superoperator. Despite its simplicity and rigor, the resulting *worst-case bound* provides no physical intuition on what effect individual control parameters have on the overall dynamics. To this end, it would be necessary to know how the spectrum of \mathcal{L} ($\text{sp}(\mathcal{L})$ henceforth) depends on the linear action induced by a given control: unfortunately, this is not a viable solution for high-dimensional systems. In order to overcome this issue, in the second approach we argue that convergence can be *estimated* by the slowest speed of transfer from a basin subspace to the preceding one in the chain. While qualitative, this approach offers a more transparent physical picture and, eventually, some useful criteria for the design of rapidly convergent QDS dynamics.

A. System-Theoretic Approach

The basic step is to employ a vectorized form of the QDS generator \mathcal{L} (also known as “Liouville space formalism” in the literature [18]), in such a way that standard results on linear time-invariant (LTI) state-space models may be invoked. Recall that the vectorization of a $n \times m$ matrix M , denoted by $\text{vec}(M)$, is obtained by stacking vertically the m columns of M , resulting in a $n \times m$ -dimensional vector [19]. Vectorization is a powerful tool when used to express matrix multiplications as linear transformations acting on vectors. The key relevant property is the following: For any matrices X , Y and Z such that their composition XYZ is well defined, it holds that

$$\text{vec}(XYZ) = (Z^T \otimes X)\text{vec}(Y) \quad (14)$$

where the symbol \otimes is to be understood here as the Kronecker product of matrices. The following Theorem provides a necessary and sufficient condition for GAS subspaces directly in

terms of spectral properties of the (vectorized) generator (compare with Theorem 1):

Theorem 3 (Subspace Attractivity): Let $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, and assume that \mathcal{H}_S is an invariant subspace for the QDS dynamics in (1). Then \mathcal{H}_S is GAS if and only if the linear operator defined by the equation

$$\begin{aligned} \hat{\mathcal{L}}_R := & -\frac{i}{\hbar} (\mathbf{1}_R \otimes H_R - H_R^T \otimes \mathbf{1}_R) + \sum_k L_{R,k}^* \otimes L_{R,k} \\ & - \frac{1}{2} \sum_k \mathbf{1}_R \otimes (L_{P,k}^\dagger L_{P,k} + L_{R,k}^\dagger L_{R,k}) \\ & - \frac{1}{2} \sum_k (L_{P,k}^T L_{P,k}^* + L_{R,k}^T L_{R,k}^*) \otimes \mathbf{1}_R \end{aligned} \quad (15)$$

does *not* have a zero eigenvalue.

Proof: Let $\bar{\Pi}_R = [0 \quad \mathbf{1}_R]$. By explicitly computing the generator’s R -block and taking into account the invariance conditions (4), we find

$$\begin{aligned} \bar{\Pi}_R \mathcal{L}[\rho] \bar{\Pi}_R^\dagger = & -\frac{i}{\hbar} [H_R, \rho_R] + \sum_k L_{R,k} \rho_R L_{R,k}^\dagger \\ & - \frac{1}{2} \sum_k \{L_{P,k}^\dagger L_{P,k} + L_{R,k}^\dagger L_{R,k}\} \rho_R. \end{aligned} \quad (16)$$

Hence the evolution of the R block is decoupled from the rest.

Now let $\hat{\rho}_R = \text{vec}(\bar{\Pi}_R \rho \bar{\Pi}_R^\dagger)$. By using (16) and (14), we have

$$\dot{\hat{\rho}}_R = \text{vec}(\bar{\Pi}_R \mathcal{L}[\rho] \bar{\Pi}_R) = \text{vec}(\bar{\Pi}_R \mathcal{L}[\rho_R] \bar{\Pi}_R) =: \hat{\mathcal{L}}_R \hat{\rho}_R \quad (17)$$

where $\hat{\mathcal{L}}_R$ is exactly the map defined in (15).

Suppose that \mathcal{H}_S is *not* attractive. By Theorem 1, the dynamics must then admit an invariant state with support on \mathcal{H}_R . In the light of (17), this implies that $\hat{\mathcal{L}}_R$ has at least one non-trivial steady state, corresponding to a zero eigenvalue. To prove the converse, suppose that $(0, \text{vec}(X))$ is an eigenpair of $\hat{\mathcal{L}}_R$. Clearly, $X \neq 0$ by definition of eigenvector. Then, any initial state $\rho \in \mathfrak{D}(\mathcal{H})$ such that its R -block, ρ_R , has non-vanishing projection along X ($\text{trace}(\rho X) \neq 0$) cannot converge to $\mathfrak{I}_S(\mathcal{H})$, and thus \mathcal{H}_S is not attractive. Since $\mathfrak{D}(\mathcal{H})$ contains a set of generators for $\mathcal{B}(\mathcal{H})$ (e.g., the pure states), there is at least one state such that $\text{trace}(\rho X) \neq 0$. ■

Building on Theorem 3, the following Corollary gives a bound on the asymptotic convergence speed to an attractive subspace, based on the modal analysis of LTI systems:

Corollary 4 (Asymptotic Convergence Speed): Consider a QDS on $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, and let \mathcal{H}_S be a GAS subspace for the given QDS generator. Then any state $\rho \in \mathfrak{D}(\mathcal{H})$ converges asymptotically to a state with support only on \mathcal{H}_S at least as fast as $ke^{\lambda_0 t}$, where k is a constant depending on the initial condition and λ_0 is given by

$$\lambda_0 = \max_\lambda \{\Re \lambda \mid \lambda \in \text{sp}(\hat{\mathcal{L}}_R)\}. \quad (18)$$

Remark: In the case of one-dimensional \mathcal{H}_S , the “slowest” eigenvalue λ_0 is also the smallest Lyapunov exponent of the dynamical system in (1) [19].

B. Connected Basins Approach

Recall that the DID derived in Section III-A is a decomposition of the systems's Hilbert space in orthogonal subspaces

$$\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_T^{(1)} \oplus \mathcal{H}_T^{(2)} \dots \oplus \mathcal{H}_T^{(q)}.$$

By looking at the block structure of the matrices H, L_k induced by the DID, we can classify each basin depending on how it is *dynamically connected* to the preceding one in the DID. Beside \mathcal{H}_S , which is assumed to be globally attractive and we term the *collector basin*, let us consider a basin $\mathcal{H}_T^{(i)}$. We can distinguish the following three possibilities:

A. *Transition basin*: This allows a one-way connection from $\mathcal{H}_T^{(i)}$ to $\mathcal{H}_T^{(i-1)}$, when the following conditions hold:

$$\hat{L}_{P,k}^{(i-1)} \neq 0 \text{ for some } k, \quad L_{Q,k}^{(i-1)} = 0 \forall k$$

in addition to the invariance condition

$$iH_P^{(i-1)} - \frac{1}{2} \sum_k L_{S,k}^{(i-1)\dagger} L_{P,k}^{(i-1)} = 0. \quad (19)$$

In other words, $\hat{L}_{P,k}^{(i-1)}$ enacts a *probability flow* towards the beginning of the DID: states with support on $\mathcal{H}_T^{(i)}$ are attracted towards $\mathcal{H}_T^{(i-1)}$.

B. *Mixing basin*: This allows for the dynamical connection between the subspaces to be bi-directional, which occurs in the following cases, or *types*:

1. As in the transition basin, but with

$$iH_P^{(i-1)} - \frac{1}{2} \sum_k L_{S,k}^{(i-1)\dagger} L_{P,k}^{(i-1)} \neq 0;$$

2. In the generic case, when both $\hat{L}_{P,k}^{(i-1)} \neq 0, L_{Q,k'}^{(i-1)} \neq 0$ for some k, k' ;

3. When $\hat{L}_{P,k}^{(i-1)} = 0 \forall k, L_{Q,k}^{(i-1)} \neq 0$, for some k .

C. *Circulation basin*: In this case, $\hat{L}_{P,k}^{(i-1)} = 0 = L_{Q,k}^{(i-1)}$ for all k , and thus the transition is enacted *solely* by the Hamiltonian block H_P . Not only is the dynamical connection bi-directional, but it is also “symmetric”: *in the absence of internal dynamics* in $\mathcal{H}_T^{(i)}, \mathcal{H}_T^{(i-1)}$ and connections to other basins, the state would keep “circulating” between the subspaces.

How is this related to the speed of convergence? Let us consider a pair of basins $\mathcal{H}_T^{(i-1)}, \mathcal{H}_T^{(i)}$, and let us try to investigate how rapidly a state with support only in $\mathcal{H}_T^{(i)}$ can “flow” towards $\mathcal{H}_T^{(i-1)}$ in a worst case scenario. The answer depends on the dynamical connections, that is, the kind of basin the state is in. A good indicator is the probability of finding the state in $\mathcal{H}_T^{(i)}$, namely

$$P_i(\rho) = \text{trace} \left(\Pi_{\mathcal{H}_T^{(i)}} \rho \right)$$

and its rate of change, which may be estimated as follows:

(i) *Transition, type-1 and type-2 mixing basins*: The first derivative of $P_i(\rho)$ for a state with support in a transition basin, has been calculated in [6] and reads

$$\lambda_i(\rho) = \text{trace} \left(\sum_k \hat{L}_{P,k}^{(i-1)\dagger} \hat{L}_{P,k}^{(i-1)} \rho \right) \quad (20)$$

which in the worst case scenario corresponds to *the minimum eigenvalue*

$$\hat{\gamma}_i^L := \min \left\{ |\lambda| \lambda \in \text{sp} \left(\sum_k \hat{L}_{P,k}^{(i-1)\dagger} \hat{L}_{P,k}^{(i-1)} \right) \right\}. \quad (21)$$

The same quantity works as an estimate for the mixing basin of type-1 and-2, since in (20) only the effect of the $L_{P,k}$ blocks is relevant.

(ii) *Mixing basin of type-3 and circulation basin*: When $\hat{L}_{P,k}^{(i-1)} = 0$ for all k , and $L_{Q,k}^{(i-1)} \neq 0$, for some k , the exit from $\mathcal{H}_T^{(i)}$ is determined by the Hamiltonian. However, in this case we we have $\dot{P}_i(\rho) = 0$, since the Hamiltonian dynamics enters only at the second (and higher) order, and thus it is not possible to estimate the “transfer speed” as we did above. Let us focus on the relevant subspace, $\mathcal{H}_{(i-1,i)} = \mathcal{H}_T^{(i-1)} \oplus \mathcal{H}_T^{(i)}$, and write the Hamiltonian, restricted to $\mathcal{H}_{(i-1,i)}$, in block-form

$$\Pi_{\mathcal{H}_{(i-1,i)}} H \Pi_{\mathcal{H}_{(i-1,i)}} = \begin{bmatrix} H_T^{(i-1)} & H_P^{(i-1)} \\ H_P^{(i-1)\dagger} & H_T^{(i)} \end{bmatrix}.$$

We can always find a unitary change of basis $U_T^{(i-1)} \oplus U_T^{(i)}$ that preserves the DID and is such that $U_T^{(i-1)} H_P^{(i-1)} U_T^{(i)\dagger} = \Sigma_P^{(i-1)} > 0$, with $\Sigma_P^{(i-1)} = \text{diag}(s_1, \dots, s_{d_i})$ being the diagonal matrix of the singular values of $H_P^{(i-1)}$ in decreasing order. Then the effect of the off-diagonal blocks is to couple pairs of the new basis vectors in $\mathcal{H}_T^{(i-1)}$, with $\mathcal{H}_T^{(i)}$ generating simple rotations of the form $e^{-is_j \sigma_x t}$. Hence, any state in $\mathcal{H}_T^{(i)}$ will “rotate towards” $\mathcal{H}_T^{(i-1)}$ as a (generally time-varying, due to the diagonal blocks of H) combination of cosines, $\sum_k \ell_k(t) \cos(s_k t)$, for appropriate coefficients. The required estimate can thus be obtained by comparing the speed of transfer induced by the Hamiltonian coupling to the exponential decay in (21). When the noise action is dominant, $\hat{\gamma}_i^L$ can be thought as $1/T_2$ with T_2 being the “decoherence time” needed for the value e^{-1} to be reached. Comparing with the action of $H_P^{(i)}$, we have

$$\hat{\gamma}_i^H \approx \frac{\min\{s_j\}}{\arccos(e^{-1})}.$$

where the approximation reflects the fact that this formula does not take into account the effect of the *diagonal blocks of the Hamiltonian*, whose influence will be studied in Section IV-C.

It is worth remarking that the “transfer” is monotone in case (i), whereas in case (ii) it is so only in an initial time interval. Once we obtain an estimate for all the transition speeds, we can

think of *the slowest speed*, call it γ_{\min} , as the “bottleneck” to attaining fast convergence. If, in particular, $\gamma_{\min} = \hat{\gamma}_i^L$ for a certain i , the latter is not affected by the Hamiltonian and hence it provides a fundamental limit to the attainable convergence speed given purely Hamiltonian time-independent control resources. Conversely, connections enacted by the Hamiltonian can in principle be optimized, following the design prescriptions we shall outline below.

In situations where the matrices H, L_k may be expressed as functions of a limited number of parameters, a useful tool for visualizing the links between different basins in the DID is what we call the *Dynamical Connection Matrix* (DCM). The latter is simply defined as

$$C := H + \sum_k L_k \quad (22)$$

with all the matrices being represented in a basis consistent with the DID. Taking into account the block form (8), the upper diagonal blocks of C will contain information on: (i) the noise-induced links; and (ii) the links in which the Hamiltonian term can play a role. An example which clearly demonstrates the usefulness of the DCM is provided in Section V.C. While in general the DCM does *not* provide sufficient information to fully characterize the invariance of the various subspaces due to the fine-tuning conditions given in (4), it can be particularly insightful when the QDS involves only decay or excitation processes. In this case, the relevant creation/annihilation operators L_k have an upper/lower-triangular block structure in the DID basis, with zero blocks on the diagonal: it is then immediate to see that a non-zero entry C_{ij} implies that the j -th state of the basis is attracted towards the i -th one. The DCM gives a compact representation of the dynamical connections between the basins, pointing to the available options for Hamiltonian tuning: in this respect, the DCM is similar in spirit to the graph-based techniques that are commonly used to study controllability of closed quantum systems [20], [21].

In spite of its qualitative nature, the advantage of the connected basins approach is twofold: (i) Estimating the transition speed between basins is, in most practical situations, more efficient than deriving closed-form expressions for the eigenvalues of the generator; (ii) Unlike the system-theoretic approach, it yields concrete insight on which control parameters have a role in influencing the speed of convergence.

C. Tuning the Convergence Speed Via Hamiltonian Control

It is well known that the interplay between dissipative and Hamiltonian dynamics is critical for controllability [3], invariance, asymptotic stability and noiselessness [5], [6], as well as for purity dynamics [11]. By recalling the definition of $\hat{\mathcal{L}}_R$ given in (15), Corollary 4 implies that not only can the Hamiltonian have a key role in determining the stability of a state, but it can also influence significantly the convergence speed. Let us consider a simple prototypical example.

Example 2: Consider a 3-level system driven by a generator of the form (1), with operators H, L that with respect to the

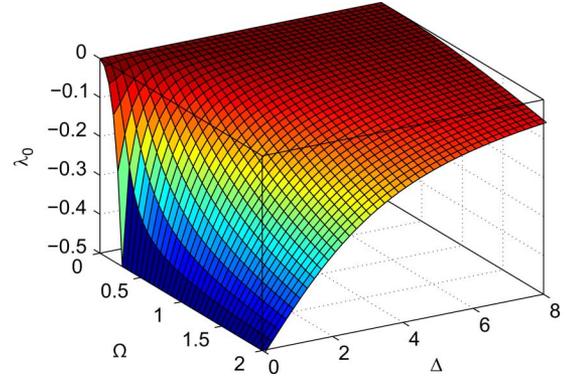


Fig. 1. Convergence speed to the GAS target state $\rho_d = |s\rangle\langle s|$ for the 3-level QDS in Example 2 as a function of the time-independent Hamiltonian control parameters Δ and Ω .

(unique, in this case) DID basis $\{|s\rangle, |r_1\rangle, |r_2\rangle\}$ have the following form:

$$H = \begin{bmatrix} \Upsilon & 0 & 0 \\ 0 & \Delta & \Omega \\ 0 & \Omega & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & \ell & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (23)$$

It is easy to show, by recalling Proposition 1, that $\rho_d = |s\rangle\langle s|$ is invariant, and that any choice of $\Omega, \ell \neq 0$ also renders ρ_d GAS. It is possible, in this case, to obtain the eigenvalues of $\hat{\mathcal{L}}_R$ and invoke Theorem 3.

Without loss of generality, let us set $\ell = 1$ so that $L = |s\rangle\langle r_1|$, and assume that Δ, Ω are positive real numbers, which makes all the relevant matrices to be real. Let $\Pi_0 := L_P^T L^P$. We can then rewrite

$$\hat{\mathcal{L}}_R = R^+ \otimes I_R + I_R \otimes R^- \quad (24)$$

where $R^\pm = \pm iH_R - \Pi_0/2$. Let $\lambda_{1,2}^\pm$ be the eigenvalues of R^+, R^- . Given the tensor structure of $\hat{\mathcal{L}}_R$, the eigenvalues of (24) are simply $\alpha_{ij} = \lambda_i^+ + \lambda_j^-$, with $i, j = 1, 2$. The real parts of the α_{ij} can be explicitly computed

$$\Re(\alpha_{ij}) = \left[\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2} \pm \frac{1}{2} \Re(\sqrt{\Gamma}) \right]$$

where $\Gamma = 1 - \Delta^2 + i\Delta - 4\Omega^2$. The behavior of $\lambda_0 = -1/2 + 1/2 \Re(\sqrt{\Gamma})$ is depicted in Fig. 1. Two features are apparent: Higher values of Ω lead to faster convergence, whereas higher values of Δ slow down convergence. The optimal scenario ($|\lambda_0| = 1/2$) is attained for $\Delta = 0$.

The above observations are instances of a general behavior of the asymptotic convergence speed that emerges when the Hamiltonian provides a “critical” dynamical connection between two subspaces. Specifically, the off-diagonal part of H_R in (25) is necessary to make ρ_d GAS, connecting the basins associated to $|r_1\rangle$ and $|r_2\rangle$. Nonetheless, the *diagonal* elements of H also have a key role: their value influences the positioning of the energy eigenvectors, which by definition are not affected by the Hamiltonian action. Intuitively, under the action of H alone, all other states “precess” unitarily around the energy eigenvectors, hence the closer the eigenvectors of H are to the

the states we aim to *destabilize*, the weaker the destabilizing action will be.

A way to make this picture more precise is to recall that $\rho_d = |s\rangle\langle s|$ is invariant, and that the basin associated to $|r_1\rangle$ is directly connected to ρ_d by dissipation. Thus, in order to make ρ_d GAS we only need to destabilize $|r_2\rangle$ using H . Consider the action of H restricted to $\mathcal{H}_R = \text{span}(|r_1\rangle, |r_2\rangle)$. The Hamiltonian's R -block spectrum is given by

$$\text{sp}(H_R) = \left\{ \frac{\Delta \pm \sqrt{\Delta^2 + 4\Omega^2}}{2} \right\} \quad (25)$$

with the correspondent eigenvectors

$$|\pm\rangle = \frac{2\Omega}{\sqrt{8\Omega^2 + 2\Delta^2 \pm 2\Delta\sqrt{\Delta^2 + 4\Omega^2}}} \begin{bmatrix} 1 \\ -\frac{\Delta \pm \sqrt{\Delta^2 + 4\Omega^2}}{2\Omega} \end{bmatrix}. \quad (26)$$

Decreasing Δ , the eigenvectors of H_R tend to $(|r_1\rangle \pm |r_2\rangle)/\sqrt{2}$, and if $\Delta = 0$, the state $|r_2\rangle$, which is unaffected by the noise action, is rotated right into $|r_1\rangle$ after half-cycle. Physically, this behavior simply follows from mapping the dynamics within the R -block to a Rabi problem (in the appropriate rotating frame), the condition $\Delta = 0$ corresponding to resonant driving [12].

Beyond the specific example, our analysis suggests two guiding principles for enhancing the speed of convergence via (time-independent) Hamiltonian design. Specifically, one can:

- Augment the dynamical connection induced by the Hamiltonian by larger off-diagonal couplings;
- Position the eigenvectors of the Hamiltonian as close as possible to balanced superpositions of the state(s) to be destabilized and the target one(s).

V. APPLICATIONS

In this section, we analyze three examples that are directly inspired by physical applications, with the goal of demonstrating how the control-theoretic tools and principles developed thus far are useful to tackle stabilization problems in realistic quantum-engineering settings.

A. Attractive Decoherence-Free Subspace in an Optical System

Consider the quantum-optical setting investigated in [22], where Lyapunov control is exploited in order to drive a dissipative four-level system into a decoherence free subspace (DFS). A schematic representation of the relevant QDS dynamics is depicted in Fig. 2. Three (degenerate) stable ground states, $|i\rangle_{i=1,2,3}$, are coupled to an unstable excited state $|e\rangle$ through three separate laser fields characterized by the coupling constants Ω_i , $i = 1, 2, 3$. In a frame that rotates with the (common) laser frequency, the Hamiltonian reads

$$H = \Delta|e\rangle\langle e| + \sum_{i=1}^3 \Omega_i(|e\rangle\langle i| + |i\rangle\langle e|) \quad (27)$$

where Δ denotes the detuning from resonance. The decay of the excited state to the stable states is a Markovian process characterized by decay rates γ_i , $i = 1, 2, 3$. The relevant Lindblad operators are thus given by the atomic lowering operators $L_i = \sqrt{\gamma_i} |i\rangle\langle e|$, $i = 1, 2, 3$.

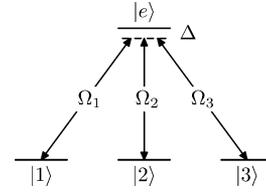


Fig. 2. Energy level configuration of the 4-level optical system discussed in Section V.A. Three degenerate stable states are coupled to an excited state through separate laser fields with a common detuning Δ and amplitude Ω_i .

Let the coupling coefficients Ω_i be parameterized as

$$\begin{cases} \Omega_1 = \Omega \sin \theta \cos \phi, \\ \Omega_2 = \Omega \sin \theta \sin \phi, \\ \Omega_3 = \Omega \cos \theta \end{cases} \quad (28)$$

where $\Omega \geq 0$ and $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$. It is known [22] that the resulting generator admits a DFS \mathcal{H}_{DFS} spanned by the following orthonormal basis:

$$\begin{cases} |d_1\rangle = -\sin \phi |1\rangle + \cos \phi |2\rangle, \\ |d_2\rangle = \cos \theta (\cos \phi |1\rangle + \sin \phi |2\rangle) - \sin \theta |3\rangle \end{cases} \quad (29)$$

provided that $\theta \neq k\pi$ and $\phi \neq \pi/2 + k\pi$. In order to formally establish that this DFS is also GAS for almost all choices of the QDS parameters $\Delta, \Omega_i, \gamma_i$, we construct the DID starting from $\mathcal{H}_S = \mathcal{H}_{\text{DFS}}$, and obtaining $\mathcal{H}_T^{(1)} = \text{span}\{|e\rangle\}$, $\mathcal{H}_T^{(2)} = \mathcal{H} \ominus (\mathcal{H}_{\text{DFS}} \oplus \mathcal{H}_T^{(1)})$. The corresponding matrix representation of the Hamiltonian and noise operators becomes

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta & \Omega' \\ 0 & 0 & \Omega' & 0 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0 & 0 & -\sqrt{\gamma_1} \sin \phi & 0 \\ 0 & 0 & \sqrt{\gamma_1} \cos \phi \cos \theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\gamma_1} |\cos \phi \sin \theta| & 0 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0 & 0 & \sqrt{\gamma_2} \cos \phi & 0 \\ 0 & 0 & \sqrt{\gamma_2} \cos \theta \sin \phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\gamma_2} \text{sign}(\cos \phi) \sin \phi |\sin \theta| & 0 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\gamma_3} \sin \theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\gamma_3} \text{sign}(\cos \phi \sin \theta) \cos \theta & 0 \end{bmatrix},$$

where $\text{sign}(x)$ is the sign function and $\Omega' := \Omega \text{sign}(\sin \theta \cos \phi)$. By Proposition 1, it follows that \mathcal{H}_{DFS} is invariant. Furthermore, the vectorized map governing the evolution of the state's R -block in (15) has the form

$$\hat{\mathcal{L}}_R = \begin{bmatrix} -\sum_i \gamma_i & -i\Omega' & i\Omega' & 0 \\ -i\Omega' & -\sum_i \frac{\gamma_i}{2} + i\Delta & 0 & i\Omega' \\ i\Omega' & 0 & -\sum_i \frac{\gamma_i}{2} - i\Delta & -i\Omega' \\ \sum_i \gamma_i \frac{\Omega_i^2}{\Omega^2} & i\Omega' & -i\Omega' & 0 \end{bmatrix}.$$

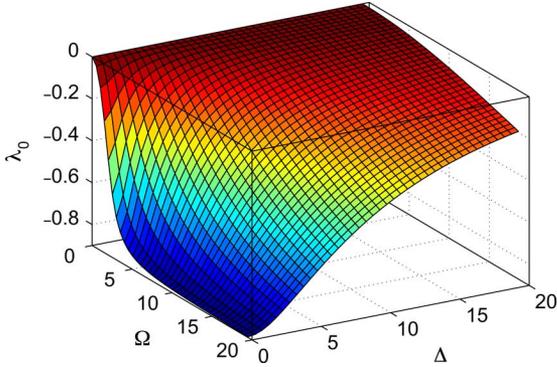


Fig. 3. Asymptotic convergence speed to the target DFS as a function of the parameters Δ and Ω . We fixed $\gamma_i = 0.9$ and $\theta = \pi/4$ and $\phi = 3/4\pi$. The value of λ_0 is computed according to (18).

Then, by Theorem 3, a sufficient and necessary condition for \mathcal{H}_{DFS} to be GAS is that the characteristic polynomial of $\hat{\mathcal{L}}_R$, $\Delta \hat{\mathcal{L}}_R(s)$, has no zero root. Explicit computation yields

$$\Delta \hat{\mathcal{L}}_R(0) = \left(\sum_i \gamma_i \right) \left(\sum_i \gamma_i (\Omega^2 - \Omega_i^2) \right) \quad (30)$$

which clearly vanishes in the trivial cases where $\gamma_i = 0 \forall i$ or $\Omega = 0$. Furthermore, there exist only isolated points in the parameter space such that $\Delta \hat{\mathcal{L}}_R$ vanishes, namely those with only one $\gamma_i \neq 0$, and the corresponding $\Omega_i = \Omega$ [recall (28)]. Otherwise, \mathcal{H}_{DFS} is attractive by Theorem 3. Notice that the Hamiltonian off-diagonal elements are strictly necessary for this DFS to be attractive, whereas the detuning parameter does not play a role in determining stability. As we anticipated in the previous section, however, the latter may significantly influence the convergence speed to the DFS.

In Fig. 3 we graph λ_0 [given in (18)] as a function of Δ and Ω , for fixed representative values of γ_i , θ , and ϕ . As in Example 2, small coupling Ω as well as high detuning Δ slow down the convergence, independently of γ_i . That a strong coupling yields faster convergence reflects the fact that the latter is fundamental to break the invariance of the subspace $\text{span}(|r_2\rangle)$. In order to elucidate the effect of the detuning, consider again the spectrum of H_R , which is given by (25)–(26). As $\Omega \rightarrow 0$, there exists an eigenvalue $\lambda \rightarrow 0$, and the same holds for $\Delta \rightarrow \infty$. Furthermore, the corresponding eigenvector tends to $|r_2\rangle$ in each of these two limits. Thus, increasing the detuning can mimic a decrease in the coupling strength, and vice-versa. Notice that, unlike in Example 2, there is a non-trivial dissipative effect linking $|e\rangle$ to $|r_2\rangle$, represented by the non-zero R -blocks of the L_i 's, however our design principles still apply. In fact, the Hamiltonian's off-diagonal terms are necessary for \mathcal{H}_{DFS} to be GAS.

B. Dissipative Entanglement Generation

The system analyzed in Example 1 is a special instance of a recently proposed scheme [10] for generating (nearly) maximal entanglement between two identical non-interacting atoms by exploiting the interplay between collective decay and Hamiltonian tuning. Assume that the two atoms are trapped in a strongly damped cavity and the detuning of the atomic transition frequencies ω_i , $i = 1, 2$, from the cavity

field frequency ω can be arranged to be symmetric, that is, $\omega_1 - \omega \equiv \Delta = -(\omega_2 - \omega)$. Under appropriate assumptions [10], the atomic dynamics is then governed by a QDS of the form (9), where (10)–(11) are generalized as follows:

$$\begin{aligned} L &= \sqrt{\gamma} (\sigma_+ \otimes I + I \otimes \sigma_+), \\ H &= \frac{\Delta}{2} (\sigma_z \otimes I - I \otimes \sigma_z) + \alpha (\sigma_x \otimes I + I \otimes \sigma_x) \end{aligned}$$

and where, without loss of generality, the parameters γ , Δ , α may be taken to be non-negative. The QDS still admits an invariant pure state $\rho_d = |\psi_0\rangle\langle\psi_0|$, which now depends on the Hamiltonian parameters Δ , α

$$|\psi_0\rangle = \frac{1}{\Omega} \left(\Delta |00\rangle - \alpha (|01\rangle - |10\rangle) \right), \quad \Omega = \sqrt{\Delta^2 + 2\alpha^2}.$$

The DID construction works as in Example 1 (where $\gamma = \Delta = \alpha = 1$), except for the fact that while $|\psi_1\rangle, |\psi_2\rangle$ are defined in the same way as in (12), the explicit form of the fourth basis state (13) is modified as follows:

$$|\psi_3\rangle = -\frac{1}{\sqrt{2}\Omega} \left(-2\alpha |00\rangle - \Delta (|10\rangle - |01\rangle) \right).$$

Therefore, in matrix representation with respect to the DID basis $\{|\psi_0\rangle, |\psi_2\rangle, |\psi_1\rangle, |\psi_3\rangle\}$, we obtain

$$\begin{aligned} L &= \begin{bmatrix} 0 & \sqrt{2}\frac{\Delta}{\Omega} & 0 & 0 \\ 0 & 0 & \sqrt{2}\gamma & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2\frac{\alpha}{\Omega} & 0 & 0 \end{bmatrix}, \\ H &= \sqrt{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & -\frac{\Omega}{\sqrt{2}} \\ 0 & \alpha & 0 & 0 \\ 0 & -\frac{\Omega}{\sqrt{2}} & 0 & 0 \end{bmatrix}. \end{aligned}$$

The entangled state $|\psi_0\rangle$ is thus GAS. Given the structure of the above matrices, the following conclusions can be drawn. First, the bottleneck to the convergence speed is determined by the element L_{12} , more precisely by the square of the ratio $\sqrt{2}\Delta/\Omega$, see (20). Assuming that $\Delta \ll \alpha$, the ratio is (approximately) linear with the detuning. This has two implications: on the one hand, the convergence speed decreases (quadratically) as $\Delta \rightarrow 0$. On the other hand, a non-zero detuning is necessary for GAS to be ensured in the first place: for $\Delta = 0$, the maximally entangled pure state $\rho_s = |\psi_s\rangle\langle\psi_s|$, with $|\psi_s\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle)$, cannot be perfectly stabilized. Likewise, although the parameter α plays no key role in determining GAS, a non-zero α is nevertheless fundamental in order for the asymptotically stable state $|\psi_0\rangle$ to be entangled.

C. State Preparation in Coupled Electron-Nuclear Systems

We consider a bipartite quantum system composed by nuclear and electronic degrees of freedom, which is motivated by the well-studied Nitrogen-Vacancy (NV) defect center in diamond [23]–[26]. While in reality both the electronic and nuclear spins (for ^{14}N isotopes) are spin-1 systems, we begin by discussing a reduced description which is common when the control field can address only selected transitions between two of the three

physical levels. The full three-level system will then be considered at the end of the section.

1) *Reduced Model*: Let both the nuclear and the electronic degrees of freedom be described as spin 1/2 particles. In addition, assume that the electronic state can transition from its energy ground state to an excited state through optical pumping while preserving its spin quantum number. The decay from the excited state can be either spin-preserving or temporarily populate a *metastable state* from which the electronic spin decays only to the spin state of lower energy [27]. We describe the optically-pumped dynamics of the NV system by constructing a QDS generator. A basis for the reduced system's state space is given by the eight states

$$|E_{el}, s_{el}\rangle \otimes |s_N\rangle \equiv |E_{el}, s_{el}, s_N\rangle$$

where the first tensor factor describes the electronic degrees of freedom, specified by the energy levels $E_{el} = g, e$, and the electron spin $s_{el} = 0, 1$ (corresponding to the spin pointing up or down, respectively), and the second factor refers to the nuclear spin, with $s_N = 0, 1$. To these states we need to add the two states belonging to the metastable energy level, denoted by $|ms\rangle \otimes |s_N\rangle$, with s_N as before. Notice that a "passage" through the metastable state erases the information on the electron spin, while it conserves the nuclear spin.

The Hamiltonian for the coupled system is of the form $H_{tot} = H_g + H_e$, where the excited-state Hamiltonian H_e and the ground-state Hamiltonian H_g share the following structure:

$$\begin{aligned} H_{g,e} &= D_{g,e} S_z^2 \otimes \mathbf{1}_N + Q \mathbf{1}_{el} \otimes S_z^2 \\ &+ B (g_{el} S_z \otimes \mathbf{1}_N + g_n \mathbf{1}_{el} \otimes S_z) \\ &+ \frac{A_{g,e}}{2} (S_x \otimes S_x + S_y \otimes S_y + 2S_z \otimes S_z). \end{aligned} \quad (31)$$

Here, $S_{x,y} = \sigma_{x,y}$, are the standard 2×2 Pauli matrices on the relevant subspace, while $S_z = 1/2(\mathbf{1} - \sigma_z)$ is a pseudo-spin¹ and $D_{g,e}, A_{g,e}, Q, g_{el}, g_n$ are fixed parameters. In particular, $A_{g,e}$ will play a key role in our analysis, determining the strength of the Hamiltonian (*hyperfine*) interaction between the electronic and the nuclear degrees of freedom. B represents the intensity of the applied static magnetic field along the z -axis, and can be thought as the available control parameter.

In order to describe the dissipative part of the evolution we employ a phenomenological model, using Lindblad terms with jump-type operators and associated pumping and decay rates. The relevant transitions are represented by the operators below: since they leave the nuclear degrees of freedom unaltered, they act as the identity operator on that tensor factor. Specifically

$$\begin{aligned} L_1 &= \sqrt{\gamma_d} |g, 0\rangle \langle e, 0| \otimes \mathbf{1}_N, \\ L_2 &= \sqrt{\gamma_d} |g, 1\rangle \langle e, 1| \otimes \mathbf{1}_N, \\ L_3 &= \sqrt{\gamma_m} |ms\rangle \langle e, 1| \otimes \mathbf{1}_N, \\ L_4 &= \sqrt{\gamma_0} |g, 0\rangle \langle ms| \otimes \mathbf{1}_N, \\ L_5 &= \sqrt{\gamma_p} |e, 0\rangle \langle g, 0| \otimes \mathbf{1}_N, \\ L_6 &= \sqrt{\gamma_p} |e, 1\rangle \langle g, 1| \otimes \mathbf{1}_N. \end{aligned} \quad (32)$$

¹This non-standard definition follows from the implemented reduction from a three- to a two-level system: specifically, we consider only $|0\rangle, |-1\rangle$ and neglect $|1\rangle$, and further map the states $0 \rightarrow 0$ and $-1 \rightarrow 1$.

The first four operators describe the decays, with associated rates $\gamma_d, \gamma_m, \gamma_0$, whereas the last two operators correspond to the optical-pumping action on the electron, with a rate γ_p . It is easy to check by inspection that the subspace

$$\mathcal{H}_S := \text{span}\{|e, 0, 0\rangle, |g, 0, 0\rangle\}$$

is invariant for the dissipative part of the dynamics: we next establish that it is also GAS, and analyze the dynamical structure associated with the DID.

Convergence Analysis: Following the procedure presented in Section III-A, we can prove that \mathcal{H}_S is attractive. This is of key interest in the study of NV-centers as a platform for solid-state quantum information processing. In fact, it corresponds to the ability to perfectly *polarize* the joint spin state of the electron-nucleus system. The proposed DID algorithm runs to completion in seven iterations, with the following basin decomposition as output:

$$\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_T^{(1)} \oplus \dots \oplus \mathcal{H}_T^{(7)},$$

where

$$\begin{aligned} \mathcal{H}_T^{(1)} &= \text{span}\{|ms, 0\rangle\}, \\ \mathcal{H}_T^{(2)} &= \text{span}\{|e, 1, 0\rangle\}, \quad \mathcal{H}_T^{(3)} = \text{span}\{|g, 1, 0\rangle\}, \\ \mathcal{H}_T^{(4)} &= \text{span}\{|e, 0, 1\rangle, |g, 0, 1\rangle\}, \\ \mathcal{H}_T^{(5)} &= \text{span}\{|ms, 1\rangle\}, \\ \mathcal{H}_T^{(6)} &= \text{span}\{|e, 1, 1\rangle\}, \quad \mathcal{H}_T^{(7)} = \text{span}\{|g, 1, 1\rangle\}. \end{aligned}$$

Given that reporting the block form of every operator would be too lengthy and not very informative, we report the relevant DCM instead, which reads:

$$C = \begin{array}{c} \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 0 & \gamma_p^{\frac{1}{2}} & & & & & & & & & \\ \gamma_d^{\frac{1}{2}} & 0 & \gamma_0^{\frac{1}{2}} & & & & & & & & \\ \hline & 0 & \gamma_m^{\frac{1}{2}} & & & & & & & & \\ \hline & & h_e & \gamma_p^{\frac{1}{2}} & A_e & 0 & & & & & \\ \hline & & \gamma_d^{\frac{1}{2}} & h_g & 0 & A_g & & & & & \\ \hline & & A_e & 0 & h_n & \gamma_p^{\frac{1}{2}} & & & & & \\ \hline & & 0 & A_g & \gamma_d^{\frac{1}{2}} & h_n & \gamma_0^{\frac{1}{2}} & & & & \\ \hline & & & & & & h_n & \gamma_m^{\frac{1}{2}} & & & \\ \hline & & & & & & & h'_e & \gamma_p^{\frac{1}{2}} & & \\ \hline & & & & & & & \gamma_d^{\frac{1}{2}} & h'_g & & \\ \hline \end{array} \end{array}$$

with $h_{e,g} := D_{e,g} - g_{el}B$, $h'_{e,g} := D_{e,g} - (g_{el} + g_n)B + Q + A_{e,g}$, and $h_n := Q - g_nB$. By definition, the block division (highlighted by the solid lines) is consistent with the DID, and all the empty blocks are zero. Since $g_n \ll g_{el}$, the diagonal entries in the Hamiltonian that are most influenced by the control parameter B are $h_{g,e}$ and $h'_{g,e}$. For typical values of the physical parameters, all the other entries of the DCM are (at most) only very weakly dependent on B .

It is immediate to see that the $\gamma_p, \gamma_m, \gamma_0$ blocks establish dynamical connections between all the neighboring basins, with the exception of $\mathcal{H}_T^{(4)}$ which is connected by the (B -independent) Hamiltonian elements A_e, A_g to $\mathcal{H}_T^{(2)}, \mathcal{H}_T^{(3)}$. The DCM also confirms the fact that \mathcal{H}_S is invariant, since its first column, except the top block, is zero. In the terminology of Section IV-B,

$\mathcal{H}_T^{(1)}$ is the only transition basin, $\mathcal{H}_T^{(4)}$ is the only circulant basin, and all the other basins are mixing basins. It is worth remarking that *any choice of the control parameter B ensures GAS of \mathcal{H}_S* . By inspection of the DCM, one finds that the bottleneck in the noise-induced connections between the basins is determined by the γ_0, γ_p parameters. Since the latter are not affected by the control parameter B , the minimum of those rates will determine the fundamental limit to the speed of convergence to \mathcal{H}_S in our setting.

Optimizing the Convergence Speed: The only transitions which are significantly influenced by B are the ones connecting $\mathcal{H}_T^{(4)}$ to $\mathcal{H}_T^{(2)}$ and $\mathcal{H}_T^{(3)}$. By appropriately choosing B one can reduce the norm of h_e or h_g to zero, mimicking the “resonance” condition of Example 2. Assume that, as in the physical system, $A_e > A_g$. Considering that $\mathcal{H}_T^{(2)}$, associated to h_e , is coupled to $\mathcal{H}_T^{(4)}$ with the largest off-diagonal Hamiltonian term (A_e) and is closer to \mathcal{H}_S in the DID, we expect that the best performance will be obtained by ensuring that $h_e = 0$, that is, by setting $B = D_e/g_{el}$.

The above qualitative analysis is confirmed by numerically computing the exact asymptotic convergence speed, (18). The behavior as a function of B is depicted in Fig. 4. It is immediate to notice that the maximum speed is indeed limited by the lowest decay rate, that is, the lifetime γ_0 of the metastable singlet state with our choice of parameters. The maximum is attained for near-resonance control values, although exact resonance, $B = D_e/g_{el}$, is actually not required. The second (lower) maximum correspond to the weaker resonance that is attained by choosing B so that $h_g = 0$. Physically, ensuring that $h_e = 0$ precisely corresponds, in our reduced model, to the excited-state “level anti-crossing” (LAC) condition that has been experimentally demonstrated in [23].

In Fig. 4, we also plot (dashed line) the speed of convergence of a simplified reduced system where the transition through the metastable state and its decay to the ground state are incorporated in a single decay operator $\tilde{L} = L_4 L_3$, with a rate γ_0 . This may seem convenient, since once the decay to the metastable level has occurred, the only possible evolution is a further decay into $|g, 1, 1\rangle$. However, by doing so the convergence speed is substantially smaller, although still in qualitative agreement with the predicted behavior (presence of the two maxima, and speed limited by the minimum decay rate). The reason lies in the fact that in this simplified model, the $\mathcal{H}_T^{(1)}, \mathcal{H}_T^{(5)}$ transition basins become (part of) mixing basins, thus the non-polarizing decay and the Hamiltonian can directly influence (slowing down) the decay dynamics associated to \tilde{L} , consistently with the general theoretical analysis. Comparison with typical experimental results indicates that the most accurate prediction is obtained by letting the two operators L_3 and L_4 act separately.

2) *Extended Model and Practical Stabilization:* A physically more realistic description of the NV-center requires representing both the electron and nucleus subsystems as three-level, spin-1 systems. In this case, a basis for the full state space is given by the 21 states

$$|E_{el}, s_{el}\rangle \otimes |s_N\rangle, |ms\rangle \otimes |s_N\rangle$$

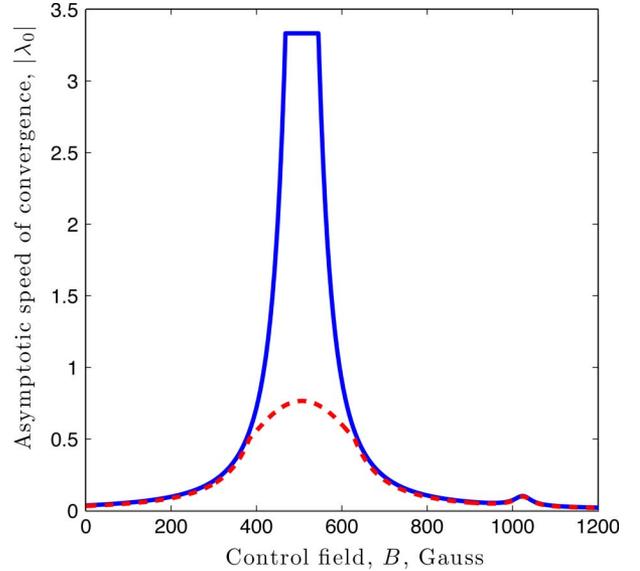


Fig. 4. Asymptotic convergence speed to \mathcal{H}_S as a function of the control parameter B for an NV-center. The blue (solid) curve is relative to the model with the metastable state, while the red (dashed) one is relative to a simplified model where the transition through the metastable state is incorporated in a single decay operator \tilde{L} with rate γ_0 (see text). Typical values for NV-centers are: $D_e = 1420$ MHz, $D_g = 2870$ MHz, $Q = 4.945$ MHz, $A_e = 40$ MHz, $A_g = 2.2$ MHz and $g_{el} = 2.8$ MHz/G, $g_n = 3.08 \times 10^{-4}$ MHz/G. We used decay rates $\gamma_d = 77$ MHz, $\gamma_m = 33$ MHz, $\gamma_0 = 3.3$ MHz, and optical-pumping rate $\gamma_p = 70$ MHz. With these values, $h'_g \approx h_g = 2870 - 2.8 B$ MHz, $h'_e \approx h_e = 1420 - 2.8 B$ MHz and $h_n \approx 4.945$ MHz.

where now $s_{el} = 1, 0, -1$ and, similarly, $s_N = 1, 0, -1$. The Hamiltonian is of the form

$$\begin{aligned} H_{g,e} = & D_{g,e} S_z^2 \otimes \mathbf{1}_N + Q \mathbf{1}_{el} \otimes S_z^2 \\ & + B (g_{el} S_z \otimes \mathbf{1}_N + g_n \mathbf{1}_{el} \otimes S_z) \\ & + A_{g,e} (S_x \otimes S_x + S_y \otimes S_y + S_z \otimes S_z) \end{aligned} \quad (33)$$

with $S_{x,y,z}$ denoting the angular momentum operators for spin-1. The dissipative part of the evolution is still formally described by the operators in (32) (where now 0,1 correspond to the spin-1 eigenstates), to which one needs to add the following Lindblad operators:

$$\begin{aligned} L_7 &= \sqrt{\gamma_d} |g, -1\rangle \langle e, -1| \otimes \mathbf{1}_N, \\ L_8 &= \sqrt{\gamma_m} |ms\rangle \langle e, -1| \otimes \mathbf{1}_N, \\ L_9 &= \sqrt{\gamma_p} |e, -1\rangle \langle g, -1| \otimes \mathbf{1}_N. \end{aligned}$$

Similar to the spin-1/2 case, the dissipative dynamics *alone* would render GAS the subspace associated to electronic spin $s_{el} = 0$, that is, $\mathcal{H}_{el,0} = \text{span}\{|e, 0, s_N\rangle, |g, 0, s_N\rangle\}$. Thus, one may hope that $\mathcal{H}_S = \text{span}\{|e, 0, 0\rangle, |g, 0, 0\rangle\}$ could still be GAS under the full dynamics. We avoid reporting the whole DCM structure, since that would be cumbersome and unnecessary to our scope: the main conclusion is that in this case *nuclear spin polarization cannot be perfectly attained*. While the hyperfine interaction components of the Hamiltonian still effectively connect the subspaces with nuclear spin 0, 1, they also have a detrimental effect: \mathcal{H}_S is no longer invariant. In fact,

represented in the DID basis, the Hamiltonian has the following form:

$$H_{tot} = \begin{bmatrix} 0 & 0 & 0 & \cdots & A_e & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & A_g & \cdots \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ A_e & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & A_g & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The presence of A_g , A_e in the H_P^\dagger block suffices to *destabilize* \mathcal{H}_S , by causing the invariance conditions in (4) to be violated. However, these terms are relatively small compared to the dominant ones, allowing for a practical stabilization attempt. Following the approach outlined in Section III-C, we neglect the H_P term and proceed with the analysis and the convergence-speed tuning. Again, the optimal speed condition is attained for B in a nearly-resonant LAC condition. By means of numerical computation, one can then show that the system still admits a *unique*, and hence attractive, steady state (which in this case is *mixed*) and that the latter is close to the desired subspace. In fact, with the same parameters we employed in the spin-1/2 example, one can ensure asymptotic preparation of a state with polarized $s_N = 1$ spin with a fidelity of about 97%.

VI. CONCLUSION

We have developed a framework for analyzing global asymptotic stabilization of a target pure state or subspace (including practical stabilization when exact stabilization cannot be attained) for finite-dimensional Markovian semigroups driven by time-independent Hamiltonian controls. A key tool for verifying stability properties is provided by a state-space decomposition into orthogonal subspaces (the DID), for which we have provided a constructive algorithm and an enhanced version that can accommodate control constraints. The DID is uniquely determined by the target subspace, the effective Hamiltonian and the Lindblad operators, and provides us with a standard form for studying convergence of the QDS. In the second part of the work, we have tackled the important practical problem of characterizing the speed of convergence to the target stable manifold and the extent to which we can manipulate it by time-independent Hamiltonian control. A quantitative system-theoretic lower bound on the attainable speed has been complemented by a connected-basin approach which builds directly on the DID and, while qualitative, offers more transparent insight on the dynamical effect of different control knobs. In particular, such an approach makes it clear that even control parameters that have no direct effect on invariance and/or attractivity properties may significantly impact the overall convergence speed.

While our results are applicable to a wide class of controlled Markovian quantum systems, a number of open problems and extensions remain for future investigation. In particular, for practical applications, an important question is whether similar analysis tools and design principles may be developed for more general classes of controls than addressed here. In this context,

the case where a set of *tunable Lindblad operators* may be applied open-loop, alone and/or in conjunction with time-independent Hamiltonian control, may be especially interesting, and potentially relevant to settings that incorporate engineered dissipation and *dissipative gadgets*, such as nuclear magnetic resonance [28] or trapped-ion and optical-lattice quantum simulators [29], [30].

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