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A simple criterion for structurally fixed modes

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We present a simple characterization of the maximum possible rank of the product of several real matrices, when certain entries of the matrices are constrained to be zero. Our result relates this 'generic rank' to the maximum number of independent 'information paths' through the matrices, and has as corollaries, besides several previous attempts at this problem, a new characterization of structurally fixed modes for linear time-invariant dynamical systems. Furthermore, our characterization translates immediately to an $O(n^{5/2})$ algorithm for calculating the generic rank (and therefore detecting fixed modes), by solving a network flow problem (here n is the sum of the dimensions of the matrices).

Keywords: Structurally fixed modes, Maximum flow, Generic rank, Decentralized systems.

1. Introduction

A property of linear systems is called *structural* if it can be deduced solely from the presence of zeroes at certain entries of the matrices of the system. In physical terms, the presence of zeroes in the matrices of the system indicates the absence of specified interconnections, and is therefore indicative of the structure of the system. Structural controllability has been studied for both single- and multi-input linear systems [4,10,3]. Subsequently, this was extended to the concept of *structurally fixed modes* [6,9], the structural counterpart of the notion of decentralized fixed modes [11] of systems subject to decentralization constraints. A characterization of fixed modes was obtained in [9]; however, this characterization suggests prohibitively inefficient algorithms (for example, the algorithm proposed in [6] is *exponential* in the number of control stations involved).

In this note, we obtain a new characterization of structurally fixed modes, which admits an appealing intuitive interpretation in terms of the information capacity of the system. Furthermore, our characterization reduces the problem of detecting structurally fixed modes to simple graph-theoretic problems, with extremely efficient algorithms [5]. In particular, our result suggests an $O(n^{5/2})$ algorithm for computing the generic rank or detecting fixed modes, where n is the sum of the dimensions of the matrices involved. Interestingly, this reduces the complexity of the structural problem *below that of the unstructured case*, in which the actual matrices are given, and to the same complexity level as determining the generic rank of a *single* structured matrix. All previous algorithms proposed for fixed modes [6] were exponentially slow.

From a purely algebraic point of view, we solve the more general problem of finding the maximum rank of the product $\prod_{i=1}^{k} M_i$ of matrices, subject to the constraints that certain entries of the matrices M_i be equal to zero. We present this result in the next section. In Section 3, we discuss its consequences in Control Theory, and some related problems, as well as pose an interesting open question.

2. The main result

A *structured* $m \times n$ *matrix* is a set $\mathscr{A}$ of pairs (i, j), where $1 \le i \le m$ and $1 \le j \le n$. Viewed a little differently, $\mathscr{A}$ is the set of all $m \times n$ matrices A such that $A_{ij} = 0$ whenever $(i, j) \in \mathscr{A}$. We shall use the latter viewpoint for the most part of this paper. We represent structured matrices as actual matrices, with entries 0 and $*$, where $*$ represents any real.

Let $\mathscr{M}_i$, $i = 1,\dots,k$, be a sequence of structured matrices, where M_i is $n_i \times n_{i+1}$. The *generic rank* of the product $\prod_{i=1}^{k} \mathscr{M}_i$, denoted $\rho(\prod_{i=1}^{k} \mathscr{M}_i)$, is the maximum rank of any product $\prod_{i=1}^{k} M_i$, where $M_i \in \mathscr{M}_i$, $i = 1,\dots,n$. In this section, we give a simple graph-theoretic characterization of the generic rank.

Given the sequence $\mathscr{M}_i$ of structured modes, we define a directed graph $G = (V, E)$ as follows.

The nodes in V are $\{x_i^j: i=1,\dots,k;\ j=1,\dots,n_i\}$; that is, we have a node for every row of a matrix (or column of the previous one). We have an arc from node x_i^p to node x_{i+1}^q if the (p, q)th entry of $\mathscr{M}_i$ is $*$. This completes the construction of G.

An *information path* is a path in G from a node with subscript 1 to one with subscript $k+1$. A set of information paths is *independent* if they are node-disjoint, as paths of G.

Theorem. *$\rho(\prod_{i=1}^k \mathscr{M}_i)$ is equal to p, the maximum number of independent information paths in G.*

Proof. We use E^{ij} to denote a matrix which is 1 at its (i, j)th entry, and 0 everywhere else (the dimensions will be implied by context). Suppose that there are p independent information paths in G. Then, with each such path $P_j = (x_1^{i_{1j}},\dots,x_{k+1}^{i_{k+1,j}})$ we associate the sequence of matrices $E^{i_{1j}i_{2j}},\dots,E^{i_{kj}i_{k+1,j}}$, where the ith such matrix has dimensions $n_i \times n_{i+1}$. Next, we make the following simple observations:

(i) If P_j and P_l are two of the p independent information paths, then the two products

$$E^{i_{1j}i_{2j}} \times \cdots \times E^{i_{kj}i_{k+1,j}}, \quad E^{i_{1l}i_{2l}} \times \cdots \times E^{i_{kl}i_{k+1,l}}$$

have a single non-zero (in fact, unit) entry each; these two entries are in different rows and in different columns.

(ii) Since the p paths are independent, we have

$$\sum_{j=1}^{p} \prod_{i=1}^{k} E^{i_{ij}i_{i+1,j}} = \prod_{i=1}^{k} \sum_{j=1}^{p} E^{i_{ij}i_{i+1,j}}.$$

This follows, since extra terms in the right-hand side would mean that two matrices $E^{i_{ij}i_{i+1,j}}$, $E^{i_{il}i_{i+1,l}}$ share the same row or column (that is, either $i_{ij}=i_{il}$ or $i_{i+1,j}=i_{i+1,l}$) or, equivalently, that two information paths P_j, P_l share a node, contrary to our assumption.

(iii) Let us finally observe that, for each i, the matrix

$$M_i = \sum_{j=1}^{p} E^{i_{ij}i_{i+1,j}} \in \mathscr{M}_i.$$

This is because the ith level of graph G was constructed to reflect the structure of $\mathscr{M}_i$.

From (i) we have that the right-hand side of (ii), which by (iii) equals the product $\prod_{i=1}^k M_i$, has rank at least p. From (iii) we have that each M_i is in the structured matrix $\mathscr{M}_i$, and so it follows that $\rho(\prod_{i=1}^k \mathscr{M}_i) \ge p$.

For the other direction, we need a lemma:

Lemma. *Suppose that two sequences of structured matrices $\mathscr{M}_i$, $\mathscr{M}_i'$, $i=1,\dots,k$, differ only in that, for some $i \le k$, $\mathscr{M}_i'$ is just $\mathscr{M}_i$, with one row (or column) made zero. Then,*

$$\rho\left(\prod_{i=1}^{k} \mathscr{M}_i\right) \le \rho\left(\prod_{i=1}^{k} \mathscr{M}_i'\right) + 1.$$

Proof. Let $M_1,\dots,M_k$ be the matrices that achieve $\rho(\prod_{i=1}^k \mathscr{M}_i)$. Let us write the ith matrix as $M_i = M_i^l + R_i^l$, where M_i^l is the matrix M_i with the lth row made 0, and R_i^l is the matrix that is 0 everywhere, except the lth row, which agrees with M_i (the argument with columns is identical). Then we write the product

$$\prod_{i=1}^{k} M_i = M_1 M_2 \cdots M_{i-1} M_i^l M_{i+1} \cdots M_k + M_1 M_2 \cdots M_{i-1} R_i^l M_{i+1} \cdots M_k.$$

Now, the rank of the left-hand side is $\rho(\prod_{i=1}^k \mathscr{M}_i)$, the rank of the first term of the right-hand side is at most $\rho(\prod_{i=1}^k \mathscr{M}_i')$, and that of the second term is at most one. The lemma now follows from the subadditivity of the rank. □

It follows from the lemma that $\rho(\prod_{i=1}^k \mathscr{M}_i)$ is less than or equal to the minimum number of zero lines that we have to add to the structured matrices $\mathscr{M}_i$, in order to obtain a zero product. However, adding a zero line anywhere in the given structured matrices corresponds to deleting a vertex of G; also, a product is zero iff the first and last layers of G are disconnected. Therefore, $\rho(\prod_{i=1}^k \mathscr{M}_i)$ is less than the minimum number of nodes of G that we have to remove in order to disconnect its first and last layer; by Menger's Theorem, this is equal to p, the maximum number of information paths. □

Corollary 1. *We can determine the generic rank of the product of structured matrices with sum of dimensions $n = \sum_{i=1}^{k+1} n_i$ in time $O(n^{5/2})$.*

Proof. To determine p, we construct a flow network as follows: For each node v of G we have two nodes v_1, v_2 and the arc (v_1, v_2); we also have two new nodes s, t. If v is a node of G at the first level, then we add the arc (s, v_1). Similarly, if u is a node in the last level, we have the arc (v_2, t). Finally, for each arc (u, v) of G, we add to the network the arc (u_2, v_1). All capacities are one. Notice that this network is *simple* in that the capacities are all one, and all nodes have either indegree or outdegree one or zero. The algorithm in [2] then constructs the maximum flow in time $O(n^{5/2})$ (see [5] for an exposition). The maximum flow is, by the theorem, equal to the generic rank of the product. □

It is quite interesting to note that determining the generic rank of a *single* structured matrix involves solving a bipartite matching problem, and the best algorithm known for this has the same complexity $O(n^{5/2})$. Also, to determine the rank of a single (ordinary) matrix by Gaussian elimination takes even longer: $O(n^3)$. The same holds for a simple *probabilistic* algorithm for determining the generic rank: Pick the non-zero entries of the matrices at random (with a suitably large precision), perform the matrix multiplications, and calculate the rank of the product: The result is very likely to be the generic rank of the product. If Gaussian elimination is used in the last step, then the overall complexity of the probabilistic algorithm is $O(n^3)$, worse than the deterministic one proposed here. The new asymptotically fast methods for matrix multiplication improve this performance to $O(n^{2.48\cdots})$, but are of rather theoretical value.

3. Applications, implications, and extensions

The main motivation of structured matrices comes from Control Theory. In particular, a *decentralized linear system* can be defined as a quadruple $L = (A, B, C, \mathscr{K})$, where A, B, and C are $n \times n$, $m \times n$, and $n \times p$, respectively, and $\mathscr{K}$ is an $m \times p$ structured matrix. The three matrices define the linear system

$$x = Ax + Bu, \quad y = Cx,$$

and $\mathscr{K}$ determines the set of allowed feedbacks. If a complex number λ is an eigenvalue of $A + BKC$ for all $K \in \mathscr{K}$, then we say that λ is a *fixed mode* of system L. Now, define a structured decentralized linear system to be a quadruple of structured matrices $\mathscr{L} = (\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{K})$. Finally, we say that $\mathscr{L}$ has a *structurally fixed mode* if the system $L = (A, B, C, \mathscr{K})$ has a fixed mode for all $A \in \mathscr{A}$, $B \in \mathscr{B}$, $C \in \mathscr{C}$.

In [6,9] we find some interesting results concerning structurally fixed modes. In particular, let $S = \{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, n\}$. Let B_i be the ith column of B, and Let C_j be the jth row of C. Given the set S, we define the set $\{j_1, \ldots, j_l\} = \{j$: there is an $i \notin S$ such that $\mathscr{K}_{ij} = *\}$. Finally, let

$$B^S = \left[B_{i_1}, \ldots, B_{i_k}\right], \quad C^S = \begin{bmatrix} C_{i_1} \\ \vdots \\ C_{i_k} \end{bmatrix}.$$

Then it is shown in [6,9] that the system $\mathscr{L}$ has a structurally fixed mode iff one of the following conditions hold:

(I) There is a set S and a permutation matrix P such that, for all $A \in \mathscr{A}$, $B \in \mathscr{B}$, $C \in \mathscr{C}$,

$$P'AP = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{12} & A_{22} & 0 \\ A_{13} & A_{23} & A_{33} \end{bmatrix},$$

$$P'B^S = \begin{bmatrix} 0 \\ 0 \\ B^S \end{bmatrix}, \quad C^S P = \begin{bmatrix} C_1^2 & 0 & 0 \end{bmatrix},$$

where the dimension of A_{22} must be non-zero.

(II) For all $A \in \mathscr{A}$, $B \in \mathscr{B}$, $C \in \mathscr{C}$, $K \in \mathscr{K}$, the matrix $A + BKC$ is singular. (Actually, this condition was stated slightly differently in [9].)

Note that the above two cases (to be called *structurally fixed modes of type I and II*, respectively) are fairly different qualitatively. Such a dichotomy is present even in the much simpler problem of structural controllability [4,10]. For this reason, it makes sense to apply a different algorithm to each of the two cases. Let us first deal with the simpler type I. Given a system $\mathscr{L} = (\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{K})$, we create a directed graph $G = (V, E)$. The nodes in V are

$$\{x_1, \ldots, x_n\} \cup \{u_1, \ldots, u_m\} \cup \{y_1, \ldots, y_p\}.$$

As for E, it contains an arc (x_i, x_j) iff $\mathscr{A}_{ij} = *$, an arc (x_i, y_j) iff $\mathscr{C}_{ji} = *$, an arc (y_i, u_j) iff $\mathscr{K}_{ji} = *$, and an arc (u_i, x_j) iff $\mathscr{B}_{ji} = *$. The proof of the following proposition is omitted as straight-forward:

Proposition 1. *$\mathscr{L}$ has a structurally fixed mode of type I iff there is a node x_i of G such that every directed cycle through x_i avoids all the y and u nodes.* □

Given the above proposition, an $O(n^2)$ algorithm for testing wether a system has a structurally fixed mode is suggested immediately: First find the *strongly connected components* of G, that is, the equivalence classes on the nodes defined when we think of two nodes as equivalent whenever they are traversed by the same cycle. This can be done in $O(n^2)$ time [1]. We then check whether there is an equivalence class consisting solely of x nodes.

Corollary 2. *We can test whether a system has structurally fixed modes of type I in $O(n^2)$ time.* □

For structurally fixed modes of type II, we need to compute the generic rank of the expression $\mathscr{A} + \mathscr{B}\mathscr{K}\mathscr{C}$; however, let us observe that this is the same as computing the generic rank of the product $\hat{\mathscr{B}}\hat{\mathscr{K}}\hat{\mathscr{C}}$, where

$$\hat{\mathscr{B}} = [\mathscr{I}, \mathscr{B}], \quad \hat{\mathscr{C}} = \begin{bmatrix} \mathscr{I} \\ \mathscr{C} \end{bmatrix},$$

and

$$\hat{\mathscr{K}} = \begin{bmatrix} \mathscr{A} & \mathscr{O} \\ \mathscr{O} & \mathscr{K} \end{bmatrix}.$$

Here $\mathscr{I}$ is the unit structured matrix, with $*$'s on the diagonal and 0's every where else, and $\mathscr{O}$ is the zero structured matrix, with no $*$'s. The dimensions are implicit, as usual.

Corollary 3. *We can test whether a system has a structurally fixed mode of type II in time $O(n^{5/2})$.* □

Finally, we notice that our technique can be extended to compute the generic rank of sums of monomials, whose indeterminates are *distinct* structured matrices;

$$\mathscr{A}\mathscr{B} + \mathscr{C}\mathscr{D}\mathscr{E} + \mathscr{F} + \mathscr{G}\mathscr{H}\mathscr{I}\mathscr{J}\mathscr{K}\mathscr{L}$$

is an example. We do not understand completely yet the case with repeated occurrences of the same structured matrix. As an example of a problem of this nature, what is the generic rank of $\mathscr{A}^2$, where $\mathscr{A}$ is a given structured matrix? We conjecture that it is the same as the generic rank of $\mathscr{A}\mathscr{A}'$, where $\mathscr{A}$ and $\mathscr{A}'$ are identically structured (but no longer constrained to obtain identical non-zero entries) matrices.

4. Postscriptum

After this paper was written, two other papers with related results appeared [7,8]. In the first paper, the authors of [6] describe an improved graph-theoretic necessary and sufficient condition for structurally fixed modes. Their condition is a more complicated version of the specialization of our theorem to the problem of structurally fixed modes, as outlined in the proof of Corollary 3 above. The complication stems from the fact that the authors of [7] use node-disjoint cycles, instead of our simpler information paths. Essentially the same condition as in [7] was obtained independently in [8], as an aside of its interesting treatment of the combinatorial nature of the *unstructured* case. Neither of these articles observes the connection to network flows, suggests an efficient algorithm for implementing the criterion, or generalizes the criterion to the more difficult problem of the generic rank of a product of structured matrices. These are the main contributions of the present paper.

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