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Decision Support

Pricing of fluctuations in electricity markets

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ABSTRACT

In an electric power system, demand fluctuations may result in significant ancillary cost to suppliers. Furthermore, in the near future, deep penetration of volatile renewable electricity generation is expected to exacerbate the variability of demand on conventional thermal generating units. We address this issue by explicitly modeling the ancillary cost associated with demand variability. We argue that a time-varying price equal to the suppliers' instantaneous marginal cost may not achieve social optimality, and that consumer demand fluctuations should be properly priced. We propose a dynamic pricing mechanism that explicitly encourages consumers to adapt their consumption so as to offset the variability of demand on conventional units. Through a dynamic game-theoretic formulation, we show that (under suitable convexity assumptions) the proposed pricing mechanism achieves social optimality asymptotically, as the number of consumers increases to infinity. Numerical results demonstrate that compared with marginal cost pricing, the proposed mechanism creates a stronger incentive for consumers to shift their peak load, and therefore has the potential to reduce the need for long-term investment in peaking plants.

1. Introduction

Our motivation stems from the fact that fluctuations in the demand on conventional thermal generating units typically result in significantly increased, and nontrivial, ancillary costs. Today, such demand fluctuations are mainly due to time-dependent consumer preferences. In addition, in the future, a certain percentage of electricity production is required by law in many states in the U.S. to come from renewable sources (Barbose, Wiser, Phadke, & Goldman, March 2008). The high volatility of renewable energy sources may aggravate the variability of the demand for conventional thermal generators and result in significant ancillary cost. More concretely, either a demand surge or a decrease in renewable generation may result in (i) higher energy costs due to the deployment of peaking plants with higher ramping rates but higher marginal cost, such as oil/gas combustion turbines, and (ii) the cost associated with resource redispatch[2] that the system will incur to meet reserve constraints if the demand increase (or renewable generation decrease) causes a reserve shortage.

There is general agreement that charging real-time prices (that reflect current operating conditions) to electricity consumers has the potential of reducing supplier ancillary cost, improving system efficiency, and lowering volatility in wholesale prices (Chao, 2010; Spees & Lave, 2008; US Department of Energy, 2006). Therefore, dynamic pricing, especially real-time marginal cost pricing, is often identified as a priority for the implementation of wholesale electricity markets with responsive demand (Hogan, 2010), which in turn raises many new questions. For example, should prices for a given time interval be calculated ex ante or ex post? Does real-time pricing introduce the potential for new types of market instabilities? How is supplier competition affected? In this paper, we abstract away from almost all of these questions and focus on the specific issue of whether prices should also explicitly encourage consumers to adapt their demand so as to reduce supplier ancillary cost.

To illustrate the issue that we focus on, we note that a basic model of electricity markets assumes that the cost of satisfying a given level A_t of aggregate demand during period t is of the form $C(A_t)$. It then follows that in a well-functioning wholesale market, the observed price should more or less reflect the marginal cost $C'(A_t)$. In particular, prices should be more or less determined by the aggregate demand level. Empirical data do not quite support this view. Fig. 1 plots the real-time system load and the hourly prices on February 11, 2011 and on February 16, 2011, as reported by the New England ISO

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[2] A certain level of reserve must always be maintained in an electric power system. Local reserve shortages are usually due to the quick increase of system load rather than a capacity deficiency. If the increase of system load makes the system short in reserves, the system redispatches resources to increase the amount of reserves available. Redispatch generally increases the generation cost and results in higher prices. The redispatch cost can be very high (cf. Section 2.3.2 of ISO New England Inc., 2010).

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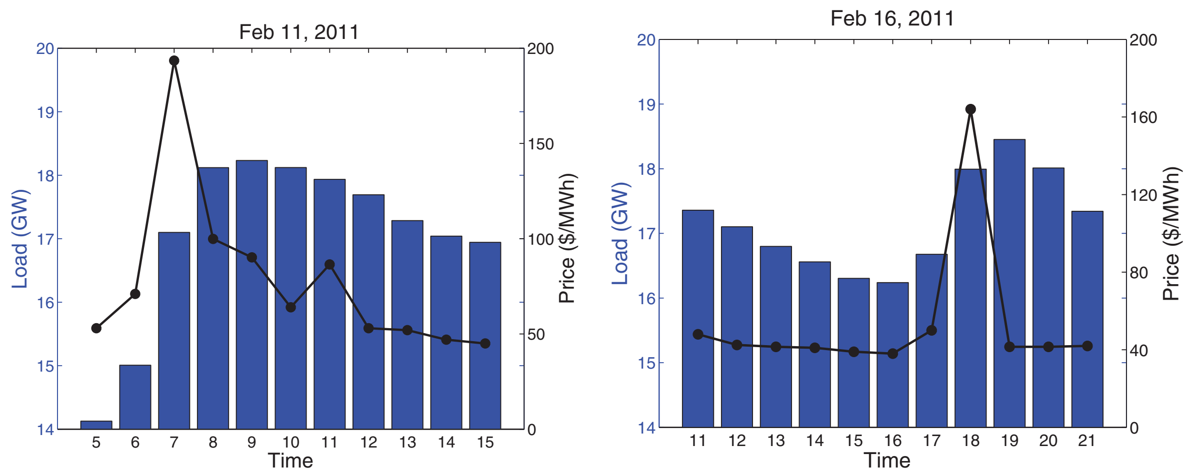

Fig. 1. Real-time prices and actual system load, ISO New England Inc. Blue bars represent the real-time system loads and the dots connected by a black line represent the hourly prices.

(ISO New England Inc., 2011). We observe that prices do not seem to be determined solely by A_t but that the changes in demand, $A_t - A_{t-1}$, also play a major role. In particular, the largest prices seem to occur after a demand surge, and not necessarily at the hour when the load is highest. We take this as evidence that the total cost over $T + 1$ periods is not of the form

$$\sum_{t=0}^{T} C(A_t),$$

but rather of the form

$$\sum_{t=0}^{T} \big(C(A_t) + H(A_{t-1}, A_t)\big), \tag{1}$$

for a suitable function H.

We take the form of Eq. (1) as our starting point and raise the question of the appropriate prices. We note that wholesale electricity prices set by an OPF (optimal power flow)-based approach is simply the highest marginal cost of active generating units (Sioshansi, Oren, & O'Neill, 2010; Wu, Rothleder, Alaywan, & Papalexopoulos, 2004): at time t, A_{t-1} has already been realized, and taking its value for granted, a consumer is charged a unit price equal to

$$C'(A_t) + \frac{\partial}{\partial A_t} H(A_{t-1}, A_t), \tag{2}$$

which is the supplier's marginal cost at stage t. We refer to this simple approach as "marginal cost pricing" (MCP), which is essentially the one used in the price calculation processes implemented by the California ISO (2009), New England ISO (Litvinov, 2011), and NYISO (cf. Section 17.1 of NYISO, 2012). However, a simple argument based on standard mathematical programming optimality conditions shows that for system optimality to obtain, the demand A_{t-1} should also incur (after A_t is realized) a unit price of (Sioshansi et al., 2010):

$$\frac{\partial}{\partial A_{t-1}} H(A_{t-1}, A_t), \tag{3}$$

This is in essence the pricing mechanism that we analyze in this paper.[3]

The actual model that we consider will be richer from the one discussed above in a number of respects. It includes an exogenous source of uncertainty (e.g., representing weather conditions) that has an impact on consumer utility and supplier cost, and therefore can incorporate the effects of volatile renewable electricity production[4]. It allows for consumers with internal state variables (e.g., a consumer's demand may be affected by how much electricity she has already used). It also allows for multiple consumer types (i.e., with different utility functions and different internal state dynamics). Consumers are generally modeled as price-takers, as would be the case in a model involving an infinity (a continuum) of consumers. However, we also consider the case of finite consumer populations and explore certain equilibrium concepts that are well-suited to the case of finite but large consumer populations. On the other hand, we ignore most of the distinctions between ex post and ex ante prices. Instead, we assume that at each time step, the electricity market clears. The details of how this could happen are important, but are generic to all electricity markets, hence not specific to our models, and somewhat orthogonal to the subject of this paper. (See however Appendix A for some discussion of implementation issues.)

The ancillary cost function $H(A_{t-1}, A_t)$ is a central element of our model. How can we be sure that this is the right form? In general, redispatch and reserve dynamics are complicated and one should not expect such a function to capture all of the complexity of the true system costs; perhaps, a more complex functional form such as $H(A_{t-2}, A_{t-1}, A_t)$ would be more appropriate. We believe that the form we have chosen is a good enough approximation, at least under certain conditions. To argue this point, we present in Appendix B an example that involves a more detailed system model (in which the true cost is a complicated function of the entire history of demands) and show that a function of the form $H(A_{t-1}, A_t)$ can capture most of the cost of ancillary services.

1.1. Summary and contributions

Before continuing, we provide here a roadmap of the paper together with a summary of our main contributions.

(a) We provide a stylized (yet quite rich) model of an electricity market, which incorporates the cost of ancillary services (cf. Section 2).
(b) We provide some justification of the form of the cost function in our model, as a reasonable approximation of more detailed physical models (cf. Appendix B).

[3] In current two-settlement systems, the real-time prices are charged only on the difference of the actual demand and the estimated demand at the day-ahead market. However, the two-settlement system provides the same real-time incentives to price-taking consumers, as if they were purchasing all of their electricity at the real-time prices (cf. Chapter 3–2 of Stoft, 2002).

[4] The value of demand response on mitigating the variability of renewable generation has received some recent attention (Rahimi & Ipakchi, 2010; Stadler, 2008).

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(c) We propose and analyze a pricing mechanism that properly charges for the effects of consumer actions on ancillary services (cf. Section 3).
(d) For a continuum model involving non-atomic price-taking consumers, we consider Dynamic Oblivious Equilibria (DOE), in which every consumer maximizes her expected payoff under the sequence of prices induced by a DOE strategy profile (Section 4). We show that (under standard convexity assumptions), a mechanism that properly charges for the effects of consumer actions on ancillary services maximizes social welfare (cf. Theorem 2 in Section 6).
(e) We carry out a game-theoretic analysis of the case of a large but finite number of consumers. We show that a large population of consumers who act according to a DOE (derived from an associated continuum game) results in asymptotically optimal (as the number of consumers goes to infinity) social welfare (cf. Theorem 2 in Section 6), and asymptotically maximizes every consumer's expected payoff (this is an "asymptotic Markov equilibrium" property; cf. Theorem 1 in Section 5).
(f) We illustrate the potential benefits of our mechanism through a simple numerical example. In particular, we show that compared with marginal cost pricing, the proposed mechanism reduces the peak load, and therefore has the potential to reduce the need for long-term investments in peaking plants (cf. Appendix E).

1.2. Related literature

There are two streams of literature, on electricity pricing and on game theory, that are relevant to our work, and which we now proceed to discuss, while also highlighting the differences from the present work.

Regarding electricity markets, the impact of supply friction on economic efficiency and price volatility has received some recent attention. Mansur (2008) shows that under ramping constraints, the prices faced by consumers may not necessarily equal the true supplier marginal cost. In a continuous-time competitive market model, Cho and Meyn (2010a) show that the limited capability of generating units to meet real-time demand, due to relatively slow ramping rates, does not harm social welfare, but may result in extreme price fluctuations. In a similar spirit, Kizilkale and Mannor (2010) construct a dynamic game-theoretic model to study the tradeoff between economic efficiency and price volatility. Cho and Meyn (2010b) construct a dynamic newsboy model to study the reserve management problem in electricity markets with exogenous demand. The supplier cost in their model depends not only on the overall demand, but also on the generation resources used to satisfy the demand. For example, a quickly increasing demand may require more responsive and more expensive resources (e.g., peaking generation plants). Wang, Negrete-Pincetic, Kowli, Shafieepoorfard, Meyn, and Shanbhag (2012) study a somewhat related dynamic competitive equilibrium that includes ancillary services with ramping constraints. Closer to the present paper, Sioshansi et al. (2010) suggest that wholesale electricity prices should explicitly account for intertemporal ramping constraints.

To study the impact of pricing mechanisms on consumer behavior and load fluctuations, we construct a dynamic game-theoretic model that differs from existing dynamic models for electricity markets and incorporates both the consumers' responses to real-time price fluctuations and the suppliers' ancillary cost incurred by load swings[5]. Our main results validate the suggestion made in Sioshansi et al. (2010), from a demand-response (DR) perspective: wholesale electricity prices should properly charge for the effects of consumer actions on ancillary services, because proper price signals will encourage consumers to adapt their consumption so as to offset the variability of demand on conventional units.

On the game-theoretic side, the standard solution concept for stochastic dynamic games is the **Markov perfect equilibrium** (MPE) (Fudenberg & Tirole, 1991; Maskin & Tirole, 1988), involving strategies where an agent's action depends on the current state of all agents. As the number of agents grows large, the computation of an MPE is often intractable (Doraszelski & Pakes, 2007). For this reason, alternative equilibrium concepts, for related games featuring a non-atomic continuum of agents (e.g., "oblivious equilibrium" or "stationary equilibrium" for dynamic games without aggregate shocks), have received much recent attention (Adlakha, Johari, Weintraub, & Goldsmith, 2011; Weintraub, Benkard, & Van Roy, 2008).

There is a large literature on a variety of approximation properties of non-atomic equilibria (Al-Najjar, 2004; 2008; Mas-Colell & Vives, 1993). Recently, Adlakha et al. (2011) have derived sufficient conditions for a stationary equilibrium strategy to have the **Asymptotic Markov Equilibrium** (AME) property, i.e., for a stationary equilibrium strategy to asymptotically maximize every agent's expected payoff (given that all the other agents use the same stationary equilibrium strategy), as the number of agents grows large. Their model includes random shocks that are assumed to be idiosyncratic to each agent. However, in the problem that we are interested in, it is important to incorporate aggregate shocks (such as weather conditions) that have a global impact on all agents. In this spirit, Weintraub, Benkard, and Van Roy (2010) consider a market model with aggregate profit shocks, and study an equilibrium concept at which every firm's strategy depends on the firm's current state and on the recent history of the aggregate shock. Finally, for dynamic oligopoly models with a few dominant firms and many fringe firms, Ifrachy and Weintraub (2012) propose and analyze a new equilibrium concept, moment-based Markov equilibrium (MME), in which each firm's action depends on the aggregate shock, the exact states of the dominant firms, and a few aggregate statistics on the distribution of fringe firm states.

For a general dynamic game model with aggregate shocks, Bodoh-Creed (2012) shows that a non-atomic counterpart of an MPE, which we refer to as a **Dynamic Oblivious Equilibrium (DOE)** in this paper, asymptotically approximates an MPE in the sense that as the number of agents increases to infinity, the actions taken in an MPE can be well approximated by those taken by a DOE strategy of the non-atomic limit game. However, without further restrictive assumptions on the agents' state transition kernel, the approximation property of the actions taken by a DOE strategy does not necessarily imply the AME property of the DOE, and we are not aware of any AME results for models that include aggregate shocks. Thus, our work is different in this respect: for a dynamic non-atomic model with aggregate shocks, which is a simplified variation of the general model considered in Bodoh-Creed (2012), we prove the AME property of a DOE.

The efficiency of non-atomic equilibria for static games has been addressed in recent research (Bodoh-Creed, 2011; Milchtaich, 2004; Roughgarden & Tardos, 2004). For a dynamic industry model with a continuum of identical producers and exogenous aggregate shocks, Lucas and Prescott (1971) show (under convexity assumptions) that the expected social welfare is maximized at a unique competitive equilibrium. In a similar spirit, in this paper we show (under convexity assumptions) that the proposed pricing mechanism maximizes the expected social welfare in a model involving a continuum of (possibly heterogeneous) consumers. We also consider the case of a large but finite number of consumers, and show that the expected social welfare is approximately maximized if all consumers act according to a non-atomic equilibrium (DOE). For large dynamic games, the asymptotic social optimality of non-atomic equilibria (DOEs) established in this paper seems to be new.

[5] Some major differences between our model and existing ones are discussed at the end of Section 2.

2. Model

We consider a $(T + 1)$-stage dynamic game with the following elements:

1. The game is played in **discrete time**. We index the time periods with $t = 0, 1, \ldots, T$. Each stage may represent a five minute interval in real-time balancing markets where prices and dispatch solutions are typically provided at five minute intervals.
2. There are n **consumers**, indexed by $1, \ldots, n$.
3. At each stage t, let $s_t \in \mathcal{S}$ be an **exogenous state**, which evolves as a Markov chain and whose transitions are not affected by consumer actions. The set $\mathcal{S}$ is assumed to be finite. In electricity markets, the exogenous state may represent time and/or weather conditions, which impact consumer utility and supplier cost. It may also represent the level of renewable generation.
4. For notational conciseness, for $t \geq 1$, let $\bar{s}_t = (s_{t-1}, s_t)$, and let $\bar{s}_0 = s_0$. We use $\overline{\mathcal{S}}_t$ to denote the set of all possible $\bar{s}_t$. We refer to $\bar{s}_t$ as the **global state** at stage t.
5. Given an initial global state s_0, the **initial states (types) of the consumers**, $\{x_{i,0}\}_{i=1}^n$, are independently drawn according to a probability measure η_{s_0} over a finite set $\mathcal{X}_0$. We use X to denote the cardinality of $\mathcal{X}_0$.
6. The **state of consumer** i at stage t is denoted by $x_{i,t}$. At $t = 0$, consumer i's initial state, $x_{i,0}$, indicates her type. For $t = 1, \ldots, T$, we have $x_{i,t} = (x_{i,0}, z_{i,t})$, where $z_{i,t} \in \mathcal{Z}$ and $\mathcal{Z} = [0, Z]$ is a compact subset of $\mathbb{R}$. The variables $\{z_{i,t}\}_{i=1}^n$ allow us to model intertemporal substitution effects in consumer i's demand.
7. We use $\mathcal{X}_t$ to denote a **consumer's state space** at stage t. In particular, at stage $t \geq 1$, $\mathcal{X}_t = \mathcal{X}_0 \times \mathcal{Z}$.
8. At stage t, consumer i takes an **action** $a_{i,t}$ and receives a nonnegative **utility**[6] $U_t(x_{i,t}, s_t, a_{i,t})$.
9. Each consumer's **action space** is $\mathcal{A} = [0, B]$, where B is a positive real number. (In the electric power context, B could reflect a local transmission capacity constraint.)
10. We use $A_t = \sum_{i=1}^n a_{i,t}$ to denote the **aggregate demand** at stage t.
11. Given consumer i's current state, $x_{i,t}$, and the next exogenous state s_{t+1}, the next state of consumer i is determined by her action taken at stage t, i.e., $x_{i,t+1} = (x_{i,0}, z_{i,t+1})$, where $z_{i,t+1} = r(x_{i,t}, a_{i,t}, s_{t+1})$, for a given function r.
12. Let $G_t = A_t + R_t$ be the **capacity** available at stage t, where R_t is the system **reserve** at stage t. For simplicity, we assume that the system reserve at stage t depends only on the current aggregate demand, A_t, and the current exogenous state s_t. That is, we have $R_t = g(A_t, s_t)$ for a given function of g that reflects the reserve policy of the system operator.
13. At stage t, let $\overline{C}(A_t, R_t, s_t)$ be the total conventional **generation cost**, that is, the sum of the supplier's cost to meet the aggregate demand A_t through its primary energy resources, e.g., coal-based and nuclear power plants, and the cost to maintain a system reserve R_t. Since R_t usually depends only on A_t and s_t, we can write $\overline{C}(A_t, R_t, s_t)$ as a function of A_t and s_t, i.e., there exists a **primary cost function** $C : \mathbb{R} \times \mathcal{S} \to [0, \infty)$ such that $C(A_t, s_t) = \overline{C}(A_t, R_t, s_t)$. We assume that for any $s \in \mathcal{S}$, $C(\cdot, s)$ is nondecreasing.
14. At stage $t \geq 1$, let $\overline{H}(A_{t-1}, A_t, R_{t-1}, R_t, s_t)$ denote the ancillary cost incurred by load swings[7]. Since R_t usually depends only on A_t and s_t, we can write $\overline{H}(A_{t-1}, A_t, R_{t-1}, R_t, s_t)$ as a function of A_{t-1}, A_t, s_{t-1}, and s_t, i.e., there exists an **ancillary cost function** $H : \mathbb{R}^2 \times \mathcal{S}^2 \to [0, \infty)$ such that $\overline{H} = H(A_{t-1}, A_t, \bar{s}_t)$. The ancillary cost at stage 0 is assumed to be a function of s_0 and A_0.
15. At stage 0, the total supplier cost is of the form
$$C(A_0, s_0) + H_0(A_0, s_0), \tag{4}$$
and for $t = 1, \ldots, T$, the total supplier cost at stage t is given by
$$C(A_t, s_t) + H(A_{t-1}, A_t, \bar{s}_t). \tag{5}$$

In contrast to existing dynamic models for electricity markets with an exogenous demand process (Cho & Meyn, 2010a; 2010b), our dynamic game-theoretic model incorporates the consumer reactions to price fluctuations, and allows us to study the impact of pricing mechanisms on consumer behavior and economic efficiency. Through a dynamic game-theoretic formulation, Kizilkale and Mannor (2010) study the tradeoff between economic efficiency and price volatility. Our model is different from the one studied in Kizilkale and Mannor (2010) in the following respects:

1. Our model allows the generation cost to depend on an exogenous state, and therefore can incorporate supply-side volatility due to uncertainty in renewable electricity generation. As an example, consider a case where the exogenous state, s_t, represents the electricity generation from renewable resources at stage t. Then the demand for conventional generation is $A_t - s_t$. For example, the system reserve can be set as the maximum of the minimum system reserve requirement[8] $\bar{R}(s_t)$ and a certain fraction of the current system load, say, δA_t for some constant $\delta > 0$. The cost function, $\overline{C}(A_t, R_t, s_t)$, then depends only on the output of conventional generating units, $A_t - s_t$, and the system reserve, $\max\{\bar{R}(s_t), \delta A_t\}$. Furthermore, the ancillary cost incurred at stage t depends on the system reserve and the outputs of conventional generating units at stages $t - 1$ and t, and is therefore a function of A_{t-1}, A_t, s_{t-1}, and s_t.
2. More important, instead of penalizing each consumer's attempt to change her own action across time, the ancillary cost function in our model penalizes the change in the aggregate demand by all consumers. The change in a single consumer's action may harm or benefit the social welfare, while aggregate demand volatility is usually undesirable.

The main feature of our model is the ancillary cost function H, which makes the supplier cost non-separable over time. In an electric power system, the ancillary cost function models the costs associated with the variability of conventional thermal generator output, such as the energy cost of peaking plants. Note that the ancillary cost is not necessarily zero when $A_t \leq A_{t-1}$, because thermal generating units have ramping-down constraints, and because a decrease in renewable electricity production may lead to an increase of the system reserve, even if $A_t \leq A_{t-1}$. The presence of the ancillary cost function makes conventional marginal cost pricing inefficient (cf. Example 1 in Section 3).

To keep the model simple, we do not incorporate any idiosyncratic randomness in the consumer state evolution. Thus, besides the randomness of consumer types (initial states), the only source of stochasticity in the model is the exogenous state s_t.

To effectively highlight the impact of pricing mechanisms on consumer behavior, as well as on economic efficiency and demand volatility, we have made the following simplifications and assumptions for the power grid:

(a) As in Cho and Meyn (2010a), we assume that the physical production capacity is large enough so that the possible changes of the generation capacity are not constrained.

[6] At $t = 0$, U_0 is a mapping from $\mathcal{X}_0 \times \mathcal{S} \times \mathcal{A}$ to $[0, \infty)$, while for $t \geq 1$, U_t is a mapping from $\mathcal{X}_0 \times \mathcal{Z} \times \mathcal{S} \times \mathcal{A}$ to $[0, \infty)$.

[7] In general, the supplier ancillary cost may depend on the entire history of system load and global states. However, ancillary cost functions with the simple form $\overline{H}(A_{t-1}, A_t, R_{t-1}, R_t, s_t)$ can serve as a good approximation of the supplier's true ancillary cost (cf. Appendix B).

[8] Conventionally, the minimum system reserve is usually defined by the biggest online generator (Wang, Redondo, & Galiana, 2003). Since renewable generation causes important fluctuations, reserves should also be allowed to depend on the level of renewable generation (that is incorporated by the global state s_t).

(b) Transmission capacity is large enough to avoid any congestion. We also assume that the cost of supplying electricity to consumers at different locations is the same. Therefore, a common price for all consumers is appropriate.
(c) We use a simplified form of ancillary cost functions, $H(A_{t-1}, A_t, \bar{s}_t)$, to approximate the supplier ancillary cost. In Appendix B, we discuss this approximation and present a numerical example to justify it.

Actual power systems and markets are quite complex and our model does not necessarily capture all relevant aspects in a realistic manner. For example, the assumption of two types of generators may not be satisfied in real power systems. However, we make such assumptions for specificity, and in order to avoid an exceedingly complex model, while still being able to develop our main argument, which is somewhat orthogonal to such issues.

3. The Pricing mechanism

The marginal cost pricing mechanism discussed in Section 1 charges a time-varying unit price on each consumer's demand. As demonstrated in the following example, a time-varying price that equals the supplier's instantaneous marginal cost may not achieve social optimality in a setting that includes ancillary costs. For this reason, we propose a new pricing mechanism that takes into account the ancillary cost associated with a consumer's demand at the **previous** stage.

Example 1. This example shows that at an efficient competitive equilibrium (between one supplier and one consumer), the per-unit price charged to the consumer does not equal the supplier's instantaneous marginal cost. Consider a two-stage deterministic model with one consumer and one supplier. At stage t, the consumer's utility function is U_t: $[0, \infty) \to [0, \infty)$. Let a_t denote the demand at stage t, and let $\mathbf{a} = (a_0, a_1)$. Let g_t denote the actual generation at stage t, and let $\mathbf{g} = (g_0, g_1)$. Two unit prices, p_0 and p_1, are charged on the consumption at stage 0 and 1, respectively. Let $\mathbf{p} = (p_0, p_1)$. The consumer's payoff-maximization problem is

$$\underset{\mathbf{a}}{\text{Maximize}} \quad U_0(a_0) - p_0 a_0 + U_1(a_1) - p_1 a_1. \tag{6}$$

Let H_0 be identically zero, and let the ancillary cost function at stage 1 depend only on the difference between the supply at the two stages. That is, the ancillary cost at stage 1 is of the form $H(g_1 - g_0)$. The supplier's profit-maximization problem is

$$\underset{\mathbf{g}}{\text{Maximize}} \quad p_0 g_0 + p_1 g_1 - C(g_0) - C(g_1) - H(g_1 - g_0). \tag{7}$$

The social planner's problem is

$$\begin{aligned} &\underset{(\mathbf{a},\mathbf{g})}{\text{Maximize}} && U_0(a_0) + U_1(a_1) - C(g_0) - C(g_1) - H(g_1 - g_0) \\ &\text{subject to} && \mathbf{a} = \mathbf{g}. \end{aligned} \tag{8}$$

Now consider a **competitive equilibrium**, $(\mathbf{a}, \mathbf{g}, \mathbf{p})$, at which the vector $\mathbf{a}$ solves the consumer's optimization problem (6), the vector $\mathbf{g}$ solves the supplier's optimization problem (7), and the market clears, i.e., $\mathbf{a} = \mathbf{g}$. Suppose that the utility functions are concave and continuously differentiable, and that the cost functions C and H are convex and continuously differentiable. We further assume that $H'(0) = 0$, and that for $t = 0, 1$, $U_t'(0) > C'(0)$, $U_t'(B) < C'(B)$. Then, there exists a competitive equilibrium, $(\mathbf{a}, \mathbf{g}, \mathbf{p})$, which satisfies the following conditions:

$$\begin{cases} U_0'(a_0) = p_0, \\ U_1'(a_1) = p_1, \end{cases} \quad \begin{cases} C'(a_0) - H'(a_1 - a_0) = p_0, \\ C'(a_1) + H'(a_1 - a_0) = p_1. \end{cases} \tag{9}$$

We conclude that the competitive equilibrium solves the social welfare maximization problem in (8), because it satisfies the following (sufficient) optimality conditions:

$$\begin{aligned} &U_0'(a_0) = C'(a_0) - H'(a_1 - a_0), \quad U_1'(a_1) = C'(a_1) + H'(a_1 - a_0), \\ &a_0 = g_0, \quad a_1 = g_1. \end{aligned} \tag{10}$$

However, we observe that the socially optimal price p_0 does not equals the supplier's instantaneous marginal cost at stage 0, $C'(a_0)$. Hence, by setting the price equal to $C'(a_0)$, as would be done in a real-time balancing market, we may not achieve social optimality. More generally, marginal cost pricing need not be socially optimal because it does not take into account the externality conferred by the action a_0 on the ancillary cost at stage 1, $H(a_1 - a_0)$. At a socially optimal competitive equilibrium, the consumer should pay

$$(C'(a_0) - H'(a_1 - a_0))a_0 + (C'(a_1) + H'(a_1 - a_0))a_1,$$

i.e., the price on a_0 should be the sum of the supplier marginal cost at stage 0, $C'(a_0)$, and the marginal ancillary cost associated with a_0, $-H'(a_1 - a_0)$, which is determined at the next stage, after a_1 is realized.

Before describing the precise pricing mechanism we propose, we introduce a differentiability assumption on the cost functions.

Assumption 1. For any $s \in \mathcal{S}$, $C(\cdot, s)$ and $H_0(\cdot, s)$ are continuously differentiable on $[0, \infty)$. For any $(A', \bar{s}) \in \mathcal{A} \times \mathcal{S}^2$, $H(A, A', \bar{s})$ and $H(A', A, \bar{s})$ are continuously differentiable in A on $[0, \infty)$.[9]

Inspired by Example 1, we introduce prices

$$p_t = C'(A_t, s_t), \quad t = 0, \dots, T, \tag{11}$$

and

$$q_t = \frac{\partial H(A_{t-1}, A_t, \bar{s}_t)}{\partial A_{t-1}}, \quad w_t = \frac{\partial H(A_{t-1}, A_t, \bar{s}_t)}{\partial A_t}, \quad t = 1, \dots, T. \tag{12}$$

At stage 0, we let $q_0 = 0$ and $w_0 = H_0'(A_0, s_0)$. Under the proposed pricing mechanism, consumer i's payoff at stage t is given by

$$U_t(x_{i,t}, s_t, a_{i,t}) - (p_t + w_t)a_{i,t} - q_t a_{i,t-1}. \tag{13}$$

Note that $p_t + w_t$ is the supplier marginal cost at stage t (including the marginal ancillary cost). The proposed pricing mechanism charges consumer i an additional price q_t on her previous demand, equal to the marginal ancillary cost with respect to $a_{i,\,t-1}$.

We now define some of the notation that we will be using. For $t = 1, \dots, T$, let $y_{i,\,t} = (a_{i,\,t-1}, x_{i,\,t})$ be the **augmented state** of consumer i at stage t. At $t = 0$, let $y_{i,\,0} = x_{i,\,0}$. For stage t, let $\mathcal{Y}_t$ be the set of all possible augmented states. In particular, we have $\mathcal{Y}_0 = \mathcal{X}_0$, and $\mathcal{Y}_t = \mathcal{A} \times \mathcal{X}_t$, for $t = 1, \dots, T$.

Let $\Delta_n(D)$ be the set of empirical probability distributions over a given set D that can be generated by n samples from D. (Note that empirical distributions are always discrete, even if D is a continuous set.) Let $f_t \in \Delta_n(\mathcal{Y}_t)$ be the empirical distribution of the augmented state of all consumers at stage t, and let $f_{-i,t} \in \Delta_{n-1}(\mathcal{Y}_t)$ be the empirical distribution of the augmented state of all consumers (excluding consumer i) at stage t. We refer to f_t as the **population state** at stage t. Let $u_t \in \Delta_n(\mathcal{A})$ denote the empirical distribution of all consumers' actions at stage t, and let $u_{-i,t} \in \Delta_{n-1}(\mathcal{A})$ be the empirical distribution of all consumers' (excluding consumer i) actions at stage t.

For a given n, it can be seen from (11) and (12) that the prices, and thus the stage payoff in (13), are determined by the current global state, $\bar{s}_t$, consumer i's current augmented state, $y_{i,\,t}$, and current action, $a_{i,\,t}$, as well as the empirical distributions, $f_{-i,\,t}$ and $u_{-i,\,t}$ of other consumers' current augmented state and action. Hence, for a certain function $\pi(\cdot)$, we can write the stage payoff in (13) as

$$\pi(y_{i,t}, \bar{s}_t, a_{i,t}, f_{-i,t}, u_{-i,t}) = U_t(x_{i,t}, s_t, a_{i,t}) - (p_t + w_t)a_{i,t} - q_t a_{i,t-1}. \tag{14}$$

[9] At the boundary of the domain, 0, we require continuity of the right-derivatives of C, H_0, and H.

4. A continuum model and dynamic oblivious strategies

To study the aggregate behavior of a large number of consumers, we consider a non-atomic game involving a continuum of infinitesimally small consumers, indexed by $i \in [0, 1]$. We assume that (under state s_0) a fraction η_{s_0} of the consumers has initial state x. In a non-atomic model, any single consumer's action has no influence on the aggregate demand and the prices. We consider a class of strategies (dynamic oblivious strategies) in which a consumer's action depends only on the history of past exogenous states, $h_t = (s_0, \ldots, s_t)$, and her own current state[10], i.e., of the form

$$a_{i,t} = \overline{\nu}_t(x_{i,t}, h_t).$$

Suppose that consumer i uses a dynamic oblivious strategy $\overline{\nu} = (\overline{\nu}_0, \ldots, \overline{\nu}_T)$. Since there is no idiosyncratic randomness, given a history h_t, the state $x_{i,t}$ of consumer i at stage t depends only on her initial state $x_{i,0}$. That is, there is a mapping $l_{\overline{\nu},h_t} : \mathcal{X}_0 \to \mathcal{X}_t$, such that $x_{i,t} = l_{\overline{\nu},h_t}(x_{i,0})$. Therefore, we can specify the action taken by a dynamic oblivious strategy in the alternative form

$$a_{i,t} = \nu_t(x_{i,0}, h_t) \triangleq \overline{\nu}_t(l_{\overline{\nu},h_t}(x_{i,0}), h_t). \tag{15}$$

We refer to $\nu = (\nu_0, \ldots, \nu_T)$ as a **dynamic oblivious strategy**, and let $\mathfrak{V}$ be the set of all such strategies.

An alternative formulation involving strategies that depend on consumer expectations on future prices would lead to a Rational Expectations Equilibrium (REE), an equilibrium concept based on the rational expectations approach pioneered by Muth (1961). In our continuum model, since the only source of stochasticity is from the exogenous state s_t, future prices under any given strategy profile, are completely determined by the history h_t. Therefore, it is reasonable to expect that strategies of the form (15) will lead to an equilibrium concept that is identical in outcomes with a REE (cf. the discussion in Section 4.2).

Before formally defining a Dynamic Oblivious Equilibrium (DOE), we first provide some of the intuition behind the definition. In a continuum model, if all consumers use a common dynamic oblivious strategy ν, the aggregate demand and the prices at stage t depend only on the history of exogenous states, $h_t = (s_0, \ldots, s_t)$. A dynamic oblivious strategy ν is a DOE (cf. the formal definition in Section 4.2) if it maximizes every consumer's expected total payoff, under the sequence of prices that ν induces. In Section 4.3, we associate a continuum model with a sequence of n-consumer models ($n = 1, 2, \ldots$), and specify the relation between the continuum model and the corresponding n-consumer model.

4.1. The sequence of prices induced by a dynamic oblivious strategy

Let $h_t = (s_0, \ldots, s_t)$ denote a history up to stage t, and let $\mathcal{H}_t = \mathcal{S}^{t+1}$ denote the set of all possible such histories. Recall that in a continuum model, given an initial global state s_0, the distribution of consumers' initial states is η_{s_0}. Therefore, under a history h_t, if all consumers use the same dynamic oblivious strategy ν, then the average demand is

$$\widetilde{A}_{t|\nu,h_t} = \sum_{x \in \mathcal{X}_0} \eta_{s_0}(x) \cdot \nu_t(x, h_t). \tag{16}$$

We now introduce the cost functions in a continuum model. Let $\widetilde{C} : \mathbb{R} \times \mathcal{S} \to [0, \infty)$ be a primary cost function. Let $\widetilde{H} : \mathbb{R}^2 \times \mathcal{S}^2 \to [0, \infty)$ be an ancillary cost function at stage $t \geq 1$, and let $\widetilde{H}_0 : \mathbb{R} \times \mathcal{S} \to [0, \infty)$ be an ancillary cost function at the initial stage 0.

Given the cost functions in a continuum model, we define the sequence of prices induced by a dynamic oblivious strategy as follows (cf. Eqs. (11)–(12)):

$$\widetilde{p}_{t|\nu,h_t} = \widetilde{C}'(\widetilde{A}_{t|\nu,h_t}, s_t), \quad \widetilde{q}_{0|\nu,h_0} = 0, \quad \widetilde{w}_{0|\nu,h_0} = \widetilde{H}_0'(\widetilde{A}_{0|\nu,h_0}, s_0), \tag{17}$$

and for $t \geq 1$,

$$\widetilde{q}_{t|\nu,h_t} = \frac{\partial \widetilde{H}\left(\widetilde{A}_{t-1|\nu,h_{t-1}}, \widetilde{A}_{t|\nu,h_t}, \bar{s}_t\right)}{\partial \widetilde{A}_{t-1|\nu,h_{t-1}}},$$
$$\widetilde{w}_{t|\nu,h_t} = \frac{\partial \widetilde{H}\left(\widetilde{A}_{t-1|\nu,h_{t-1}}, \widetilde{A}_{t|\nu,h_t}, \bar{s}_t\right)}{\partial \widetilde{A}_{t|\nu,h_t}}. \tag{18}$$

4.2. Equilibrium strategies

In this subsection we define the concept of a DOE. Suppose that all consumers other than i use a dynamic oblivious strategy ν. In a continuum model, consumer i's action does not affect the prices. If all consumers except i use a dynamic oblivious strategy ν, consumer i's **oblivious stage value** (the stage payoff in a continuum model) under a history h_t and an action $a_{i,t}$, is

$$\widetilde{\pi}_{i,t}(y_{i,t}, h_t, a_{i,t} \mid \nu) = U_t(x_{i,t}, s_t, a_{i,t}) - (\widetilde{p}_{t|\nu,h_t} + \widetilde{w}_{t|\nu,h_t})a_{i,t} - \widetilde{q}_{t|\nu,h_t} a_{i,t-1}, \tag{19}$$

where the prices, $\widetilde{p}_{t|\nu,h_t}$, $\widetilde{w}_{t|\nu,h_t}$, and $\widetilde{q}_{t|\nu,h_t}$, are defined in (17) and (18). Since a single consumer's action cannot influence $\widetilde{q}_t$, the last term in (19) is not affected by the action $a_{i,t}$, and the decision $a_{i,t}$ at stage t need not take $a_{i,t-1}$ into account, but should take $\widetilde{q}_{t+1}$ into account.

Consumer i's oblivious stage value under a dynamic oblivious strategy $\hat{\nu}$, is[11]

$$\widetilde{\pi}_{i,t}(y_{i,t}, h_t \mid \hat{\nu}, \nu) \triangleq \widetilde{\pi}_{i,t}(y_{i,t}, h_t, \hat{\nu}_t(x_{i,0}, h_t) \mid \nu). \tag{20}$$

In particular, we use $\widetilde{\pi}_{i,t}(y_{i,t}, h_t \mid \nu, \nu)$ to denote the oblivious stage value of consumer i at stage t, if all consumers use the strategy ν. Given an initial global state s_0 and an initial state of consumer i, $x_{i,0}$, her **oblivious value function** (total future expected payoff function in a continuum model) is

$$\widetilde{V}_{i,0}(x_{i,0}, s_0 \mid \hat{\nu}, \nu) = \mathbb{E}\left\{\sum_{\tau=0}^{T} \widetilde{\pi}_{i,\tau}(y_{i,\tau}, h_\tau \mid \hat{\nu}, \nu)\right\}, \tag{21}$$

where the expectation is over the future global states, $\{s_\tau\}_{\tau=1}^{T}$.

Definition 1. A strategy ν is a **Dynamic Oblivious Equilibrium** (DOE) if

$$\sup_{\hat{\nu} \in \mathfrak{V}} \widetilde{V}_{i,0}(x_{i,0}, s_0 \mid \hat{\nu}, \nu) = \widetilde{V}_{i,0}(x_{i,0}, s_0 \mid \nu, \nu), \quad \forall x_{i,0} \in \mathcal{X}_0, \quad \forall s_0 \in \mathcal{S}.$$

A DOE is guaranteed to exist, under suitable assumptions, and this is known to be the case for our model (under our assumptions), and even for a more general model that includes idiosyncratic randomness (Bergin & Bernhardt, 1992). The DOE, as defined above, is essentially the same concept as the "dynamic competitive equilibrium" studied in Bodoh-Creed (2012), which is defined as the non-atomic equivalent of an MPE, in a continuum model. At a DOE, the beliefs of all consumers on future prices are consistent with the equilibrium outcomes. Therefore, a DOE is identical in outcomes with a Rational Expectations Equilibrium (REE).

Note that to compute a DOE one has to keep track of the entire history of shocks, and that consumers may not have the rationality

[10] Note that a dynamic oblivious strategy depends only on the consumer's current state, instead of her augmented state. As we will see in Section 4.2, in a continuum model, since any single consumer i has no influence on the prices, a best response or equilibrium strategy need not take into account the action $a_{i,t-1}$ taken at the previous stage.

[11] Recall that the initial state (the type) of consumer i, $x_{i,0}$, is included in its state $x_{i,t}$, as well in its augmented state $y_{i,t}$, for any t.

or computational power to calculate a DOE. However, in future electricity markets, consumers may receive price estimates from utilities and/or the independent system operator (ISO) through advanced metering infrastructures. (In Appendix A, we provide some discussion of a possible implementation of the proposed real-time pricing mechanism.) If utilities and/or the ISO could make accurate estimates (contingent on future history of the exogenous state) of the prices[12], a REE (equivalently, a DOE) will be a plausible outcome of such a market. Furthermore, we will show (Theorem 2) that under the proposed pricing mechanism, and under certain convexity assumptions, a DOE is socially optimal for the continuum model.

4.3. The n-consumer model associated with a continuum model

We would like to take the cost functions in a continuum model to approximate the cost functions in an n-consumer model. Since the continuum of consumers is described by distributions over [0, 1], the demand given in (16) can be regarded as the average demand per consumer. To capture this correspondence, we assume the following relation between the cost functions in a continuum model and their counterparts in a corresponding n-consumer model.

Assumption 2. For any $n \in \mathbb{N}$, any $s \in \mathcal{S}$, and any $\bar{s}$ in $\mathcal{S}^2$, we have

$$C^n(A,s) = n\widetilde{C}\left(\frac{A}{n}, s\right), \quad H_0^n(A,s) = n\widetilde{H}_0\left(\frac{A}{n}, s\right),$$

and

$$H^n(A, A', \bar{s}) = n\widetilde{H}\left(\frac{A}{n}, \frac{A'}{n}, \bar{s}\right),$$

where the superscript n is used to indicate that these are the cost functions associated with an n-consumer model.

Assumption 2 implies that

$$(C^n)'(A,s) = \widetilde{C}'(A/n, s), \quad (H_0^n)'(A,s) = \widetilde{H}_0'(A/n, s), \quad s \in \mathcal{S},$$

and for any $\bar{s} \in \mathcal{S}^2$,

$$\frac{\partial H^n(A, A', \bar{s})}{\partial A} = \frac{\partial \widetilde{H}(A/n, A'/n, \bar{s})}{\partial (A/n)},$$

$$\frac{\partial H^n(A, A', \bar{s})}{\partial A'} = \frac{\partial \widetilde{H}(A/n, A'/n, \bar{s})}{\partial (A'/n)},$$

so that there is a correspondence between the marginal cost in the continuum model (evaluated at the average demand) and in the corresponding n-consumer model.

5. Approximation in large games

In this section, we consider a sequence of dynamic games, and show that as the number of consumers increases to infinity, a DOE strategy for the corresponding continuum game is asymptotically optimal for every consumer (i.e., an approximate best response), if the other consumers follow that same strategy. In the rest of the paper, we often use a superscript n to indicate quantities associated with an n-consumer model.

Suppose that all consumers except i use a dynamic oblivious strategy ν. Given a history h_t and an empirical distribution $f^n_{-i,t}$, we use $\nu(h_t, f^n_{-i,t}, \nu)$ to denote the empirical distribution, $u^n_{-i,t}$, of the actions taken by consumers excluding i. In an n-consumer model, suppose that consumer i uses a **history-dependent strategy** $\kappa^n = \{\kappa^n_t\}_{t=0}^T$ of the form

$$a_{i,t} = \kappa_t^n(y_{i,t}, h_t, f^n_{-i,t}), \tag{22}$$

while the other consumers use a dynamic oblivious strategy ν. Let $\mathfrak{K}_n$ denote the set of all possible history-dependent strategies κ^n for the n-consumer model. Note that since all other consumers use the oblivious strategy ν, $f^n_{-i,t}$ is completely determined by ν, $f^n_{-i,0}$, and h_t.

The stage payoff received by consumer i at time t is

$$\pi^n_{i,t}(y_{i,t}, h_t, f^n_{-i,t} \mid \kappa^n, \nu) = \pi^n\left(y_{i,t}, \bar{s}_t, a_{i,t}, f^n_{-i,t}, \nu(h_t, f^n_{-i,t}, \nu)\right), \tag{23}$$

where $a_{i,t} = \kappa^n_t(y_{i,t}, h_t, f^n_{-i,t})$, and the stage payoff function on the right-hand side is given in (14). Given an initial global state, s_0, and consumer i's initial state, $x_{i,0}$, consumer i's expected payoff under the strategy κ^n is

$$V^n_{i,0}(x_{i,0}, s_0 \mid \kappa^n, \nu) = \mathbb{E}\left\{\sum_{t=0}^{T} \pi^n_{i,t}(y_{i,t}, h_t, f^n_{-i,t} \mid \kappa^n, \nu)\right\}, \tag{24}$$

where the expectation is over the initial distribution $f^n_{-i,0}$ and over the future global states, $\{s_t\}_{t=1}^T$. In particular, we use $V^n_{i,0}(x_{i,0}, s_0 \mid \nu, \nu)$ to denote the expected payoff obtained by consumer i if all consumers use the strategy ν.

Definition 2. A dynamic oblivious strategy ν has the **asymptotic Markov equilibrium (AME)** property (Adlakha et al., 2011), if for any initial global state $s_0 \in \mathcal{S}$, any initial consumer state $x_{i,0} \in \mathcal{X}_0$, and any sequence of history-dependent strategies $\{\kappa^n\}$, we have

$$\limsup_{n\to\infty}\left(V^n_{i,0}(x_{i,0}, s_0 \mid \kappa^n, \nu) - V^n_{i,0}(x_{i,0}, s_0 \mid \nu, \nu)\right) \le 0.$$

We will show that every DOE has the AME property, under the following assumption, which strengthens Assumption 1.

Assumption 3. We assume that:

3.1. The following four families of functions, of A, $\{\widetilde{C}'(A,s) : s \in \mathcal{S}\}$, $\{\widetilde{H}_0'(A,s) : s \in \mathcal{S}\}$, $\{\partial\widetilde{H}(A, A', \bar{s})/\partial A : (A', \bar{s}) \in \mathcal{A} \times \mathcal{S}^2\}$, and $\{\partial\widetilde{H}(A', A, \bar{s})/\partial A : (A', \bar{s}) \in \mathcal{A} \times \mathcal{S}^2\}$, are uniformly equicontinuous on $[0, \infty)$.[13]

3.2. The marginal costs are bounded from above, i.e.,

$$|\widetilde{C}'(A,s)| \le P, \quad |\widetilde{H}_0'(A,s)| \le P, \quad \forall (A,s) \in \mathcal{A} \times \mathcal{S},$$

and

$$\left|\frac{\partial \widetilde{H}(A, A', \bar{s})}{\partial A}\right| \le P, \quad \left|\frac{\partial \widetilde{H}(A', A, \bar{s})}{\partial A}\right| \le P, \quad \forall (A', \bar{s}) \in \mathcal{A} \times \mathcal{S}^2,$$

where P is a positive constant.

3.3. The utility functions, $\{U_t(x, s, a)\}_{t=0}^T$, are continuous in a and bounded from above, i.e.,

$$U_t(x,s,a) \le Q, \quad t = 0, \ldots, T, \quad \forall (x,s,a) \in \mathcal{X}_t \times \mathcal{S} \times \mathcal{A},$$

where Q is a positive constant.

Combining with Assumption 2, Assumption 3.1 implies that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any positive integer n, if $|A - \bar{A}| \le n\delta$, then for every $s \in \mathcal{S}$,

$$\left|(C^n)'(A,s) - (C^n)'(\bar{A},s)\right| \le \varepsilon, \quad \left|(H_0^n)'(A,s) - (H_0^n)'(\bar{A},s)\right| \le \varepsilon, \tag{25}$$

[12] We note that several recent works show that a reasonably accurate approximation could be achieved even if the agents' strategy depends only on the recent history (as opposed to the full history) of the aggregate shock (Bodoh-Creed, 2012; Weintraub et al., 2010). These results suggest that utilities may be able to obtain accurate price estimates by taking into account only the recent history of the aggregate shock.

[13] A sufficient condition for this assumption to hold is to require a universal bound on the derivatives of the functions in each family.

and for any $(A', \bar{s}) \in \mathcal{A} \times \mathcal{S}^2$,

$$\left| \frac{\partial H^n(A, A', \bar{s})}{\partial A} - \frac{\partial H^n(\bar{A}, A', \bar{s})}{\partial \bar{A}} \right| \le \varepsilon, \quad \times \left| \frac{\partial H^n(A', A, \bar{s})}{\partial A} - \frac{\partial H^n(A', \bar{A}, \bar{s})}{\partial \bar{A}} \right| \le \varepsilon. \tag{26}$$

Note that the boundness of the cost function derivatives implies the Lipschitz continuity of the cost functions. Combining with Assumption 2, for any pair of real numbers $(A, \bar{A})$, any positive integer n, and for every $s \in \mathcal{S}$ we have

$$\left| C^n(A, s) - C^n(\bar{A}, s) \right| \le P|A - \bar{A}|, \quad \left| H_0^n(A, s) - H_0^n(\bar{A}, s) \right| \le P|A - \bar{A}|, \tag{27}$$

and for any $(A', \bar{s}) \in \mathcal{A} \times \mathcal{S}^2$,

$$\begin{aligned} \left| H^n(A, A', \bar{s}) - H^n(\bar{A}, A', \bar{s}) \right| &\le P|A - \bar{A}|, \\ \left| H^n(A', A, \bar{s}) - H^n(A', \bar{A}, \bar{s}) \right| &\le P|A - \bar{A}|. \end{aligned} \tag{28}$$

The following theorem states that a DOE strategy approximately maximizes a consumer's expected payoff (among all possible history-dependent strategies) in a dynamic game with a large but finite number of consumers, if the other consumers also use that strategy.

Theorem 1. *Suppose that Assumptions 2–3 hold. Every DOE has the AME property.*

Theorem 1 is proved in Appendix C. Various approximation properties of non-atomic equilibrium concepts in a continuum game have been investigated in previous works. Sufficient conditions for a stationary equilibrium (an equilibrium concept for a continuum game without aggregate uncertainty) to have the AME property are derived in Adlakha et al. (2011). For a continuum game with both idiosyncratic and aggregate uncertainties, Bodoh-Creed (2012) shows that as the number of agents increases to infinity, the actions taken in an MPE can be well approximated by some DOE strategy of the non-atomic limit game. Note, however, that in a general n-consumer game, even if all consumers take an action that is close to the action taken by a DOE strategy of the non-atomic limit game, the population states and the prices in the n-consumer game can still be very different from their counterparts in the non-atomic limit game. Therefore, without further assumptions on the consumers' state transition kernel (e.g., continuous dependence of consumer states on their previous actions), the approximation property of a DOE on the action space does not necessarily imply the AME property of the DOE.

6. Asymptotic social optimality

In Section 6.1, we define the social welfare associated with an n-consumer model and with a continuum model. In Section 6.2, we show that for a continuum model, the social welfare is maximized (over all symmetric dynamic oblivious strategy profiles) at a DOE, and that for a sequence of n-consumer models, if all consumers use the DOE strategy of the corresponding continuum model, then the social welfare is asymptotically maximized, as the number of consumers increases to infinity.

6.1. Social welfare

In an n-consumer model, let $\mathbf{x}_t = (x_{1,t}, \ldots, x_{n,t})$ and $\mathbf{a}_t = (a_{1,t}, \ldots, a_{n,t})$ be the vectors of consumer states and actions, respectively, at stage t. Let $\bar{\mathbf{a}}_t = (\mathbf{a}_{t-1}, \mathbf{a}_t)$ for $t \ge 1$, and $\bar{\mathbf{a}}_0 = \mathbf{a}_0$. For $t = 1, \ldots, T$, the social welfare realized at stage t is

$$W_t^n(\mathbf{x}_t, \bar{s}_t, \bar{\mathbf{a}}_t) = -C^n(A_t, s_t) - H^n(A_{t-1}, A_t, \bar{s}_t) + \sum_{i=1}^{n} U_t(x_{i,t}, s_t, a_{i,t}), \tag{29}$$

and at stage 0, the social welfare is

$$W_t^n(\mathbf{x}_0, s_0, \bar{\mathbf{a}}_0) = -C^n(A_0, s_0) - H_0^n(A_0, s_0) + \sum_{i=1}^{n} U_0(x_{i,0}, s_0, a_{i,0}). \tag{30}$$

Because of the symmetry of the problem, the social welfare at stage t depends on $\mathbf{x}_t$ and $\mathbf{a}_t$ only through the empirical distribution of state-action pairs. In particular, under a symmetric history-dependent strategy profile $\boldsymbol{\kappa}^n = (\kappa^n, \ldots, \kappa^n)$ (cf. the definition of a history-dependent strategy in Eq. (22)), we can write the social welfare at time t (with a slight abuse of notation) as $W_t^n(f_t^n, h_t \mid \kappa^n)$. Given an initial global state s_0 and an initial population state f_0^n, the expected social welfare achieved under a symmetric history-dependent strategy profile $\boldsymbol{\kappa}^n$ is given by

$$\mathcal{W}_0^n(f_0^n, s_0 \mid \kappa^n) = W_0^n(f_0^n, s_0 \mid \kappa^n) + \mathbb{E}\left\{ \sum_{t=1}^{T} W_t^n(f_t^n, h_t \mid \kappa^n) \right\}, \tag{31}$$

where the expectation is over the future global states $\{s_t\}_{t=1}^T$. In particular, we use $\mathcal{W}_0^n(f_0^n, s_0 \mid \nu^n)$ to denote the expected social welfare achieved by the "symmetric dynamic oblivious strategy profile", $\boldsymbol{\nu}^n = (\nu, \ldots, \nu)$.

In a continuum model, suppose that all consumers use a common dynamic oblivious strategy ν. Given an initial global state s_0, the expected social welfare is

$$\widetilde{\mathcal{W}}_0(s_0 \mid \nu) = \widetilde{W}_0(s_0 \mid \nu) + \mathbb{E}\left\{ \sum_{t=1}^{T} \widetilde{W}_t(h_t \mid \nu) \right\}, \tag{32}$$

where the expectation is over the future global states, $\{s_t\}_{t=1}^T$. Here, $\widetilde{W}_t(h_t \mid \nu)$ is the stage social welfare under history h_t:

$$\begin{aligned} \widetilde{W}_t(h_t \mid \nu) = &-\widetilde{C}(\widetilde{A}_{t|\nu,h_t}, s_t) - \widetilde{H}(\widetilde{A}_{t-1|\nu,h_{t-1}}, \widetilde{A}_{t|\nu,h_t}, \bar{s}_t) \\ &+ \sum_{x \in \mathcal{X}_0} \eta_{s_0}(x) U_t\left(l_{\nu,h_t}(x), s_t, \nu_t(x, h_t)\right), \quad t = 1, \ldots, T, \end{aligned} \tag{33}$$

where l_{ν,h_t} maps a consumer's initial state into her state at stage t, under the history h_t and the dynamic oblivious strategy ν. The social welfare at stage 0 is given by

$$\begin{aligned} \widetilde{W}_0(s_0 \mid \nu) = &-\widetilde{C}(\widetilde{A}_{0|\nu,h_0}, s_0) - \widetilde{H}_0(\widetilde{A}_{0|\nu,h_0}, s_0) \\ &+ \sum_{x \in \mathcal{X}_0} \eta_{s_0}(x) U_0\left(x, s_0, \nu_0(x, s_0)\right). \end{aligned} \tag{34}$$

6.2. Asymptotic social optimality of a DOE

We now define some notation that will be useful in this subsection. Since there is no idiosyncratic randomness, given a history h_t, the state of consumer i at stage t depends only on her initial state $x_{i,0}$, and her actions taken at $\tau = 0, \ldots, t-1$. At stage $t \ge 1$, the history h_t and the transition function $z_{i,t+1} = r(x_{i,t}, a_{i,t}, s_{t+1})$ define a mapping $k_{h_t} : \mathcal{X}_0 \times \mathcal{A}^t \to \mathcal{Z}$:

$$z_{i,t} = k_{h_t}(x_{i,0}, a_{i,0}, \ldots, a_{i,t-1}), \quad t = 1, \ldots, T. \tag{35}$$

Given an initial state $x_{i,0}$, consumer i's total utility under a history h_t can be written as a function of her actions taken at stages $\tau = 0, \ldots, t$:

$$\begin{aligned} \overline{U}_{h_t}(x_{i,0}, a_{i,0}, \ldots, a_{i,t}) = &\ U_0(x_{i,0}, s_0, a_{i,0}) \\ &+ \sum_{\tau=1}^{t} U_\tau(x_{i,0}, k_{h_\tau}(x_{i,0}, a_{i,0}, \ldots, a_{i,\tau-1}), s_\tau, a_{i,\tau}). \end{aligned} \tag{36}$$

Before proving the main result of this section, we introduce a series of assumptions on the convexity and differentiability of the cost and utility functions.

Assumption 4. We assume the following:

4.1. For any $s \in \mathcal{S}$, $\widetilde{C}(\cdot, s)$ is convex; for any $\bar{s} \in \mathcal{S}^2$, $\widetilde{H}(A, A', \bar{s})$ is convex in (A, A').
4.2. For any $h_T \in \mathcal{H}_T$ and any $x_{i,0} \in \mathcal{X}_0$, the function defined in (36) is concave with respect to the vector $(a_{i,0}, \ldots, a_{i,T})$.
4.3. For any $t \geq 1$, any $h_t \in \mathcal{H}_t$, and any $x_{i,0} \in \mathcal{X}_0$, the function k_{h_t} defined in (35) is monotonic in $a_{i,\tau}$, for $\tau = 0, \ldots, t-1$; further, its left and right derivatives with respect to $a_{i,\tau}$ exist, for $\tau = 0, \ldots, t-1$.
4.4. For $t \geq 1$, and for any $(x, s, a) \in \mathcal{X}_0 \times \mathcal{S} \times \mathcal{A}$, the left and right derivatives of the utility function $U_t(x, z, s, a)$ in z exist.

We note that supplier cost functions are generally not convex (because, for example, of start-up costs), and that there is a substantial literature on the pricing of non-convexities in electricity markets (Araoz & Jörnsten, 2011; Motto & Galiana, 2004; O'Neill, Sotkiewicz, Hobbs, Rothkopf, & Stewart, 2005). The convexity assumption (Assumption 4.1) results in a tractable analytical setting that can be used to provide some theoretic results, and is often used in the literature on optimal power flow (OPF) (Lavaei & Low, 2012; Wu et al., 2004) and on energy market economics (Baldick, Grant, & Kahn, 2004; Sioshansi & Oren, 2007). We finally note that even for non-convex cost functions, the proof of Theorem 2 (in Appendix D) shows that in a non-atomic game, a DOE corresponds to a local optimum of the social welfare. This is the best one could wish for since a local optimal cannot be improved unless a significant fraction of the consumers change their action simultaneously.

For concave utility functions (with respect to a), Assumption 4.2 requires that the transition function k_{h_t} preserves concavity (a linear function would be an example). Assumptions 4.1 and 4.2 guarantee that in both models (a dynamic game with a finite number of consumers, and the corresponding continuum game), the expected social welfare (consumer i's expected payoff) is concave in the vector of actions taken by all consumers (respectively, by consumer i). Assumptions 4.3 and 4.4 ensure the existence of left and right derivatives of the expected social welfare given in (32), with respect to the actions taken by consumers. An example where Assumptions 4.2–4.4 hold is given next.

Example 2. We show in this example that Assumptions 4.2–4.4 hold for a large category of deferrable electric loads. For appliances such as Plug-in Hybrid Electric Vehicles (PHEVs), dish washers, or clothes washers, a customer usually only cares whether a task is completed before a certain time.

Given an initial state (type) of consumer i, $x_{i,0}$, let $D(x_{i,0})$ and $T(x_{i,0})$ indicate her total desired demand and the stage by which the task has to be completed, respectively. Under a given history h_t, the total utility accumulated by consumer i until time t is assumed to be of the form

$$\overline{U}_{h_t}(x_{i,0}, a_{i,0}, \ldots, a_{i,t}) = Z\left(x_{i,0}, \min\left\{D(x_{i,0}), \sum_{\tau=0}^{\min\{T(x_{i,0}),t\}} a_{i,\tau}\right\}\right),$$

for some function Z. If for every $x_{i,0} \in \mathcal{X}_0$, $Z(x_{i,0}, \cdot)$ is nondecreasing and concave, then Assumption 4.2 holds. At stage $t = 0$, we have

$$U_0(x_{i,0}, s_0, a_{i,0}) = Z\left(x_{i,0}, \min\left\{D(x_{i,0}), a_{i,0}\right\}\right).$$

For $t = 1, \ldots, T(x_{i,0})$, we let $z_{i,t} = \sum_{\tau=0}^{t-1} a_{i,\tau}$, and

$$U_t(x_{i,0}, z_{i,t}, s_t, a_{i,t}) = Z\left(x_{i,0}, \min\left\{D(x_{i,0}), a_{i,t} + z_{i,t}\right\}\right) - Z\left(x_{i,0}, \min\left\{D(x_{i,0}), z_{i,t}\right\}\right).$$

For $t \geq T(x_{i,0}) + 1$, we let $z_{i,t} = D(x_{i,0})$, and let $U_t(x_{i,t}, s_t, a_{i,t})$ be identically zero. Suppose that for every $x_{i,0} \in \mathcal{X}_0$, the right and left derivatives of $Z(x_{i,0}, \cdot)$ exist. Then, Assumptions 4.3 and 4.4 hold.

Theorem 2. *Suppose that Assumptions 2–4 hold. Let ν be a DOE of the continuum game. Then, the following hold.*

(a) *In the continuum game, the social welfare is maximized (over all dynamic oblivious strategy profiles) at the DOE, i.e.,*[14]

$$\widetilde{\mathcal{W}}_0(s_0 \mid \nu) = \sup_{\vartheta \in \mathfrak{V}} \widetilde{\mathcal{W}}_0(s_0 \mid \vartheta), \quad \forall s_0 \in \mathcal{S},$$

where $\mathfrak{V}$ is the set of all dynamic oblivious strategies.

(b) *For a sequence of n-consumer games, the symmetric DOE strategy profile, $\boldsymbol{\nu}^n = (\nu, \ldots, \nu)$, asymptotically maximizes the expected social welfare, as the number of consumers increases to infinity. That is, for any initial global state s_0, and any sequence of symmetric history-dependent strategy profiles $\{\boldsymbol{\kappa}^n\}$, we have*[15]

$$\limsup_{n\to\infty} \mathbb{E}\left\{\frac{\mathcal{W}_0^n(f_0^n, s_0 \mid \kappa^n) - \mathcal{W}_0^n(f_0^n, s_0 \mid \nu^n)}{n}\right\} \leq 0,$$

where the expectation is over the initial population state, f_0^n.

The proof of Theorem 2 is given in Appendix D.

7. Conclusion and future directions

In an electric power system, load swings may result in significant ancillary cost to suppliers. Motivated by the observation that marginal cost pricing may not achieve social optimality in electricity markets, we propose a new dynamic pricing mechanism that takes into account the externality conferred by a consumer's action on future ancillary costs. Besides proposing a suitable game-theoretic model that incorporates the cost of load fluctuations and a particular pricing mechanism for electricity markets, a main contribution of this paper is to show that the proposed pricing mechanism achieves social optimality in a dynamic non-atomic game, and approximate social optimality for the case of finitely many consumers, under certain convexity assumptions.

To compare the proposed pricing mechanism with marginal cost pricing, we have studied a numerical example in which demand increases sharply at the last stage (cf. Appendix E). In this example, the proposed pricing mechanism creates a stronger incentive (compared to marginal cost pricing) for consumers to shift their peak load, through an additional negative price charged on off-peak consumer demand. As a result, compared with marginal cost pricing, the proposed pricing mechanism achieves a higher social welfare, and at the same time, reduces the peak load, and therefore has the potential to reduce the need for long-term investments in peaking plants.

We believe that the constructed dynamic game-theoretic model, the proposed pricing mechanism, and more importantly, the insights provided by this work, can be applied to more general markets with friction. As an extension and future work, one can potentially develop and use variations of our framework to a market of a perishable product/service where demand fluctuations incur significant cost to suppliers. Examples include data centers implementing cloud services that suffer from the switching costs to toggle a server into and out of a power-saving mode (Lin, Wierman, Andrew, & Thereska, 2011), and large organizations such as hospitals that use on-call staff to meet unexpected demand.

[14] Note that we are only comparing the social welfare under different **symmetric** dynamic oblivious strategy profiles, where all consumers are using the same dynamic oblivious strategy (ν or ϑ). This is no loss of generality because under Assumption 4, the social welfare in a continuum game is a concave function of the collection of consumer actions taken under the different histories. Hence, it can be shown that the optimal social welfare can be achieved by a symmetric dynamic oblivious strategy profile.

[15] Under Assumption 4, the social welfare in an n-consumer game is a concave function of the collection of consumer actions taken under the different histories. Therefore, $\sup_{\kappa^n \in \mathfrak{K}_n} \mathcal{W}_0^n(f_0^n, s_0 \mid \kappa^n)$ is also the maximum social welfare that can be achieved by a (possibly non-symmetric) history-dependent strategy profile.

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Supplementary material

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