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INTEGRAL EQUATIONS AND RESOLVENTS OF TOEPLITZ PLUS HANKEL KERNELS*

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ABSTRACT

Given a Fredholm integral equation with a Toeplitz plus Hankel kernel, we show that the solution may be described in terms of two coupled linear partial differential equations. These equations generalize the Levinson equations for the Toeplitz case and admit a fast numerical solution. We also obtain partial differential equations for the resolvent of the kernel and derive a representation of it. The study of Toeplitz plus Hankel kernels is motivated by a sufficiently long list of applications.

I. Introduction

The subject of this paper is the solution of the Fredholm integral equation

$$f(t) + \int_0^T K(t,s)\ f(s)\ ds = g(t) \quad 0 \leq t,\ s \leq T \tag{1.1}$$

when K(t,s) is the sum of a Toeplitz and a Hankel kernel, i.e.

$$K(t,s) = K_1(t-s) + K_2(t+s) \quad 0 \leq t,\ s \leq T\ . \tag{1.2}$$

The solution of equation (1.1) has many useful applications in such diverse fields as scattering theory, fluid dynamics and linear filtering theory. An extensive literature exists concerning the particular cases when K(t,s) is a Toeplitz kernel ($K(t,s) = K_1(t-s)$) [9, 11, 15] or a Hankel kernel ($K(t,s) = K(t+s)$) [2]. Results also exist for the case where $K(t,s) = K_1(t-s) + K_1(t+s)$ [7, 12, 16] i.e. the Toeplitz and the Hankel kernel are generated by the same function K_1. To motivate the present paper, we proceed by reviewing the range of applications and approaches related to our work.

In the inverse scattering problem a wave is scattered by a potential and the asymptotic phase of the wave, at $x = -\infty$ and $x = +\infty$ is measured, as a function of the frequency of the wave [4, 5]. This determines a so called scattering matrix or, differently said, the reflection and transmission coefficients of the wave. The problem then consists of determining the scattering potential from the knowledge of the scattering matrix. This problem is equivalent to the inverse Sturm-Liouville problem [17] in which one is asked to determine a second order differential operator from knowledge of the spectral distribution function of that operator. All known procedures start by transforming the scattering matrix coefficients from the frequency domain to the time domain; a certain kernel is defined and an integral equation is introduced. The solution of this integral equation determines in a straightforward way the desired potential. In the Gelfand-Levitan approach [4] the kernel $K(t,s)$ turns out to be a Toeplitz kernel whereas in the approach of Marchenko [1] it is a Hankel kernel.

Equation (1.1) also arises in linear filtering of stationary random processes with observations on a bounded interval. In that case K is a covariance (and in particular Toeplitz) kernel. Krein [13] has studied this problem by introducing a string associated to the spectral distribution of the kernel K, but this approach hasn't found its way to applications. Levinson's approach [15] reduces the problem to an appealing set of first order partial differential equations. These equations admit a recursive (as T increases) and "fast" numerical solution and have been of particular use in the field of fast signal processing [9].

If, in the above estimation problem, observations are obtained on an interval [-T, T] symmetric around the origin and one is interested in estimating symmetric functionals of the signal as well, we are led to equation (1.1) with $K(t,s) = K_1(t-s) + K_1(t+s)$, where K_1 is the covariance kernel. The symmetry of the problem may be exploited to yield an elegant solution. In [16] the solution is obtained by purely time domain manipulations and a correspondence is established with the string formalism of Krein [13] who solves it in a transformed domain by means of a symmetric (cosine-like) and an antisymmetric (sine-like) transform.

A solution of the symmetric estimation problem may be also obtained using the fact that solving (1.1) with $K(t,s) = K_1(t-s) + K_1(t+s)$ is equivalent to solving (1.1) with $K(t,s) = K_1(t-s)$, provided that T is changed to 2T and the forcing function is appropriately modified [12]. This fact is, however, no more true when K_1 and K_2 are generated by a different function. (This is the reason why the results of this paper cannot be derived from already known results on Toeplitz or Hankel kernels).

We should point out that this close similarity between inverse scattering and estimation problems is not accidental, nor entirely formal. In fact the inverse scattering problem for a quantum mechanical particle of zero angular momentum scattered by a central potential is identical to a linear estimation problem with anti-symmetric observations, as will be made precise in a forthcoming paper.

Toeplitz plus Hankel kernels also appear in the study of a circular punch penetrating a finitely thick elastic layer resting on a rigid foundation [14], in the study of atmospheric scattering [3] and in rarefied gas dynamics [7].

Equation (1.1) with $K(t,s) = K_1(|t-s|) + K_2(t+s)$ has been investigated in [7], under the assumption that $K_1(s) = \int_0^1 e^{-s/z} w(z)dz$ and $K_2(z) = \int_0^1 e^{-s/z} w(z)r(z)dz$, for some functions w, r. Our approach is, however, very different and leads to a different numerical algorithm. Finally, Friedlander and Morf have independently derived results in the same spirit as ours [6] in a discrete framework, where operators are replaced by finite matrices.

II. Solution for certain choices of the forcing function.

The solution of (1.1) may be obtained, for any $g(t)$ from the knowledge of the resolvent kernel H of K defined by $(I-H)(I+K) = I$. In that case, f is simply given by $(I-H)g$. However, in most applications it is not necessary to evaluate H because the forcing function $g(t)$ is related to the kernel K and this may be successfully exploited. Such is the case, in particular, in symmetric estimation problems and in the integral equations arising from the inverse scattering problem. We therefore start by investigating the particular cases where $g(t) = K(0,t)$ and $g(t) = K(T,t)$.

We pause for a moment to introduce certain conditions that will be assumed throughout this paper: (A1) $K_1(t)$ and $K_2(t)$ are twice continuously differentiable over all $\mathbb{R}$.

(A2) The operators $I+K$ and $I+\hat{K}$, mapping $L^2[0,T]$ into itself (where $\hat{K}$ is an integral operator with kernel $\hat{K}(t,s) = K_1(t-s) - K_2(t+s)$, $0 \leq s, t \leq T$) are invertible for all $T > 0$.

Assumption (A1) is rather restrictive and is far from being necessary for the results to be derived. Krein's solution of estimation problems, for example, is valid for any covariance kernel whose spectral distribution function W obeys $\int_{-\infty}^{\infty} (1+w^2)\,dW(w) < \infty$ [13]. Our exposition, however is greatly simplified with this smoothness assumption.

Assumption (A2) is guaranteed to be true in symmetric estimation problems where both K and $\hat{K}$ turn out to be covariance kernels. It is also always true for the Marchenko solution of the inverse scattering problem [2].

We investigate solutions of (1.1) as T increases. For this purpose, we make the dependence of $f(t)$ on T explicit by writing $f(T,t)$.

Consider the equations

$$f(T,t) + \int_0^T K(t,s)\ f(T,s)ds = K(t,0) \tag{2.1}$$

$$g(T,t) + \int_0^T K(t,s)\ g(T,s)ds = K(t,T) \quad . \tag{2.2}$$

To (2.1) and (2.2), we associate the auxiliary equations

$$\hat{f}(T,t) + \int_0^T \hat{K}(t,s)\ \hat{f}\ (T,s)ds = \hat{K}(t,0) \tag{2.3}$$

$$\hat{g}(T,t) + \int_0^T \hat{K}(t,s)\ \hat{g}\ (T,s)ds = \hat{K}(t,T) \quad . \tag{2.4}$$

Equations (2.1) - (2.4) have a unique solution, by assumption (A2). Moreover, their solutions are twice continuously differentiable, as a consequence of (A1). Observe that

$$\frac{\partial K}{\partial t}(t,s) + \frac{\partial \hat{K}}{\partial s}(t,s) = 0 \tag{2.5}$$

$$\frac{\partial K}{\partial s}(t,s) + \frac{\partial \hat{K}}{\partial t}(t,s) = 0 \quad . \tag{2.6}$$

We differentiate equations (2.1) and (2.3) with respect to T, to obtain

$$\frac{\partial f}{\partial T}(T,t) + K(t,T)\ \ f(T,T) + \int_0^T K(t,s)\ \frac{\partial f}{\partial T}(T,s)\ ds = 0 \tag{2.7}$$

$$\frac{\partial \hat{f}}{\partial T}(T,t) + \hat{K}(t,T)\ \ \hat{f}(T,T) + \int_0^T \hat{K}(t,s)\ \frac{\partial \hat{f}}{\partial T}(T,s)\ ds = 0 \quad . \tag{2.8}$$

Then, by uniqueness of solutions of equations (2.2) and (2.4) we obtain

$$\frac{\partial f}{\partial T}(T,t) = -\ f(T,T)\ g(T,t) \tag{2.9}$$

$$\frac{\partial \hat{f}}{\partial T}(T,t) = -\hat{f}(T,T)\ \hat{g}(T,t) \quad . \tag{2.10}$$

We differentiate equation (2.2) with respect to T and equation (2.4) with respect to t, to obtain

$$\frac{\partial g}{\partial T}(T,t) + K(t,T)\, g(T,T) + \int_0^T K(t,s)\, \frac{\partial g}{\partial T}(T,s)\, ds = \frac{\partial K}{\partial T}(t,T) \qquad (2.11)$$

$$\frac{\partial \hat{g}}{\partial t} + \int_0^T \frac{\partial \hat{K}}{\partial t}(t,s)\, \hat{g}(T,s)\, ds = \frac{\partial \hat{K}}{\partial t}(t,T) \quad . \qquad (2.12)$$

The integral in equation (2.12) is equal (using (2.6) and integration by parts) to

$$\int_0^T K(t,s)\, \frac{\partial}{\partial s}\hat{g}(T,s) ds + K(t,0)\, \hat{g}(T,0) - K(t,T)\, \hat{g}(T,T) \; . \qquad (2.13)$$

We now add (2.11), (2.12), using (2.13) and using (2.6) to cancel the right hand side. Then by uniqueness of solutions of equations (2.1), (2.2) we obtain

$$\frac{\partial g}{\partial T}(T,t) + \frac{\partial \hat{g}}{\partial t}(T,t) = -\,(g(T,T) - \hat{g}(T,T))\, g(T,t) - f(T,t)\, \hat{g}(t,0) \; . \qquad (2.14)$$

By a symmetrical argument, we obtain

$$\frac{\partial g}{\partial t}(T,t) + \frac{\partial \hat{g}}{\partial T}(T,t) = -\,(\hat{g}(T,T) - g(T,T))\, \hat{g}(T,t) - \hat{f}(T,t)\, g(T,0) \; . \qquad (2.15)$$

The differential equations (2.9), (2.10), (2.14), (2.15) together with initial conditions that can be obtained directly from the integral equations (2.1) - (2.4) provide a solution to our problem and lead to a natural and efficient numerical procedure to be discussed in the next section.

Let us now consider a few particular cases.

A) $K_2(t) \equiv 0$. In that case $K \equiv \hat{K}$, $f \equiv \hat{f}$, $g \equiv \hat{g}$. Equations (2.10) and (2.15) are replicas of (2.9) and (2.14) which in turn coincide with the well-known Krein-Levinson equations of linear filtering [9].

B) $K_1(-t) \equiv K_1(t) \equiv K_2(t)$. Here, $\hat{\hat{K}}(t,0) \equiv \hat{\hat{K}}(0,t) \equiv 0$ and by uniqueness of solutions to (2.3), we obtain $\hat{\hat{f}}(T,t) \equiv 0$. Moreover, letting $t = 0$ in equation (2.5) we obtain $\hat{\hat{g}}(T,0) = 0$. Then, equations (2.14) and (2.15) become

$$\frac{\partial g}{\partial T} + \frac{\partial \hat{g}}{\partial t} = - q(T)\; g(T,t)$$

$$\frac{\partial \hat{g}}{\partial t} + \frac{\partial \hat{g}}{\partial T} = q(T)\; \hat{g}(T,t)$$

with $q(T) = g(T,T) - \hat{g}(T,T)$ which are the equations that were derived in [16] for the symmetric estimation problem.

III. Numerical Solution

A straightforward procedure for the numerical solution of equation (2.1) is by discretizing the interval [0, T] to a set of points $0,\Delta,\dots,N\Delta$, where $N\Delta = T$. We then obtain from (2.1) a system of $N + 1$ linear algebraic equations to be solved. Although this procedure is very simple, it is rather inefficient since it requires $O(N^3)$ operations. Moveover, in signal processing and other applications one is often interested in a recursive solution as T increases. Such a solution is provided, for example, by the Levinson algorithm [8] for the inversion of Toeplitz matrices, which is the discrete time analog of the inversion of a Toeplitz operator $I + K_1(t-s)$. We now suggest a similar procedure for the problem of the last section.

We start by discretizing all functions involved. In particular, fix some $\Delta > 0$, let $G(m, n) = g(m\Delta, n\Delta)$ and define similarly $\hat{G}$, F, $\hat{F}$. Suppose that the values of G, $\hat{G}$, F, $\hat{F}$ have been obtained for $m = 0,1,\dots,M$ and $m = 0,1,\dots,M$. We discretize equations (2.14), (2.15) and obtain, for $n = 0,1,\dots,M-1$,

$$G(M+1,n) = G(M,n) + \hat{G}(M,n) - \hat{G}(M,n+1) - (G(M,M) - \hat{G}(M,M))\, G(M,n) - F(M,n)\, \hat{G}(M,0)$$

$$\hat{G}(M+1,n) = \hat{G}(M,n) + G(M,n) - G(M,n+1) - (\hat{G}(M,M) - G(M,M))\, \hat{G}(M,n) - \hat{F}(M,n)\, G(M,0) .$$

We may then compute G and $\hat{G}$ for $n = M$ and $n = M+1$ by discretizing equations

(2.2) and (2.4):

$$G(M+1,n) + \sum_{i=0}^{M+1} K(n\Delta,i\Delta)\, G(M+1,i) = K(n\Delta,\ (M+1)\Delta), \quad n=M,\ M+1$$

and similarly for $\hat{G}$. We finally update F and $\hat{F}$ by

$$F(M+1,n) = F(M,n) - F(M,M)\, G(M,n) \qquad n = 0,1,\ldots,M$$

$$\hat{F}(M+1,n) = \hat{F}(M+1,n) - \hat{F}(M,M)\, \hat{G}(M,n) \quad n = 0,1,\ldots,M$$

and obtain the boundary values F(M+1, M+1) and $\hat{F}$(M+1, M+1) by discretizing equations (2.1) and (2.3) and letting $T = t = (M+1)\Delta$.

It is not hard to see that the number of operations needed to update F, $\hat{F}$, G, $\hat{G}$ when M is increased by one is of the order of M. Therefore our recursive scheme requires only $O(M^2)$ operations which is a significant improvement over the simple procedure discussed at the beginning of this section.

IV. Generalization of the Sobolev Identity and Representation of the Resolvent.

In this section we consider the resolvents of K and $\hat{K}$ and obtain a generalization of the Sobolev identity, originally derived for the resolvent of a Toeplitz kernel [9]. We then integrate that identity appropriately to obtain a representation of the resolvent as a sum of products of operators.

The resolvents H and $\hat{H}$ of K and $\hat{K}$ are, respectively, the solutions of the integral equations

$$H(t,s;\ T) + \int_0^T K(t,u)\ H(u,s;\ T)\ du = K(t,s) \tag{4.1}$$

$$\hat{H}(t,s;\ T) + \int_0^T \hat{K}(t,u)\ \hat{H}(u,s;\ T)\ du = \hat{K}(t,s)\ . \tag{4.2}$$

Differentiating (4.1) and (4.2) with respect to t and s, respectively, we have

$$\frac{\partial H}{\partial t}(t,s;\ T) + \int_0^T \frac{\partial K}{\partial t}(t,u)\ H(u,s;\ T)\ du = \frac{\partial K}{\partial t}(t,s) \tag{4.3}$$

$$\frac{\partial \hat{H}}{\partial s}(t,s;\ T) + \int_0^T \hat{K}(t,u)\ \frac{\partial \hat{H}}{\partial s}(u,s;\ T)\ du = \frac{\partial \hat{K}}{\partial s}(t,s)\ . \tag{4.4}$$

The integral in (4.3) may be rewritten, using (2.5) and integration by parts as

$$\int_0^T \hat{K}(t,u)\ \frac{\partial H}{\partial u}(u,s;\ T)du + H(0,s;\ T)\ \hat{K}(t,0) - H(T,s;\ T)\ \hat{K}(t,T) \tag{4.5}$$

We now add (4.3) and (4.4) and use the expression (4.5). The right hand side of the resulting sum vanishes, therefore

$$(\frac{\partial H}{\partial t} + \frac{\partial \hat{H}}{\partial s})\ (t,s;\ T) + \int_0^T \hat{K}(t,u)\ (\frac{\partial H}{\partial u} + \frac{\partial \hat{H}}{\partial s})\ (u,s;\ T)\ du =$$

$$= H(T,s;\ T)\ \hat{K}(t,T) - H(0,s;\ T)\ \hat{K}(t,0)$$

and by uniqueness of solutions of (4.2) we obtain

$$(\frac{\partial H}{\partial t} + \frac{\partial \hat{H}}{\partial s})\ (t,s;\ T) = H(T,s;\ T)\ \hat{H}(t,T;\ T) - H(0,s;\ T)\ \hat{H}(t,0;\ T) \quad (4.6)$$

and by symmetry

$$(\frac{\partial \hat{H}}{\partial t} + \frac{\partial H}{\partial s})\ (t,s;\ T) = \hat{H}(T,s;\ T)\ H(t,T;\ T) - \hat{H}(0,s;\ T)\ H(t,0;\ T)\,. \quad (4.7)$$

In the case where $K_2 \equiv 0$, we have $H \equiv \hat{H}$ and any one of equations (4.6), (4.7) concides with the Sobolev identity derived for the Toeplitz case. Equations (4.6) and (4.7), viewed together, are a generalization of that identity.

Equations (4.6) and (4.7) as they now stand cannot be integrated to yield a representation for H, $\hat{H}$. More suitable expressions may be obtained by adding and substracting them. We also drop the argument T, for convenience, so that

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial s})\ (H+\hat{H})\ (t,s) = H(T,s)\ \hat{H}(t,T) - H(0,s)\ \hat{H}(t,0) + \\ + \hat{H}(T,s)\ H(t,T) - \hat{H}(0,s)\ H(t,0) \quad (4.8)$$

$$(\frac{\partial}{\partial t} - \frac{\partial}{\partial s})\ (H-\hat{H})\ (t,s) = H(T,s)\ \hat{H}(t,T) - H(0,s)\ \hat{H}(t,0) - \\ - \hat{H}(T,s)\ H(t,T) + \hat{H}(0,s)\ H(t,0)\,. \quad (4.9)$$

We define $H(t,s)$ and $\hat{H}(t,s)$ to be zero whenever $(t,s) \not\subset [0,T] \times [0,T]$. We integrate equation (4.8) to obtain

$$H(T-(s-t),\ T) + H(T,T-(t-s)) + \hat{H}(T-(s-t),\ T) + \hat{H}(T,T-(t-s)) =$$

$$= H(t,s) + \hat{H}(t,s) +$$

$$+ \int_0^T \hat{H}(t+u,T)\ H(T,s+u) - \hat{H}(t+u,0)\ H(0,s+u) +$$

$$+ H(t+u,T)\ \hat{H}(T,s+u) - H(T+u,0)\ \hat{H}(0,s+u)\ du\ . \qquad (4.10)$$

(Note that for $u > \min(T-t,T-s)$ the integrand is identically zero). We also integrate (4.9) to obtain

$$H(T-(t+s),\ 0) + H(T,\ t+s-T) - \hat{H}(T-(t+s),0) - \hat{H}(T,t+s-T)$$

$$= H(t,s) - \hat{H}(t,s) +$$

$$+ \int_0^T \hat{H}(t+u,T)\ H(T,s-u) - \hat{H}(t+u,0)\ H(0,s-u) -$$

$$- H(t+u,\ T)\ \hat{H}(T,s-u) + H(t+u,0)\ \hat{H}(0,s-u)\ du\ . \qquad (4.11)$$

Equations (4.10) and (4.11) provide a representation of $H + \hat{H}$ and $H - \hat{H}$ as a sum of convolutions of triangular Toeplitz and Hankel kernels. Using the fact that kernel convolution corresponds to operator multiplication, we will rewrite (4.10) and (4.11) in operator notation.

From now on we assume that the kernels $K(t,s)$ and $\hat{K}(t,s)$ are symmetric. Then, the resolvent kernels H and $\hat{H}$ are also symmetric. Let A_+, B_+, C_+, D_+, E_+ be the operators corresponding to the following kernels:

$$A_+(t,s) = H(T,T-(t-s))$$

$$B_+(t,s) = H(0,T-(t-s))$$

$$C_+(t,s) = H(T,t+s-T)$$

$$D_+(t,s) = H(0,T-(t+s))$$

$$E_+(t,s) = H(0,(t+s)-T)\ . \qquad (4.12)$$

We also define $A_-,\ldots,E_-$ in the same way except that we use $\hat{H}$ instead of H in (4.12). Finally, if an operator G is given by a kernel G(t,s), let G* denote the adjoint operator, given by the kernel $G^*(t,s) = G(s,t)$. Observe that $A_\pm$, $B_\pm$ are Toeplitz operators and $C_\pm$, $D_\pm$, $E_\pm$ are Hankel operators.

With this notation, (4.10) can now be written as

$$A_+ + A_+^* + A_- + A_-^* = H + \hat{H} + A_-^*A_+ + A_+^*A_- - B_-^*B_+ - B_+^*B_- \cdot \quad (4.13)$$

Rearranging terms in (4.13), we finally obtain

$$(I-H) + (I-\hat{H}) = (I-A_+^*)\ (I-A_+) + (I-A_-^*)\ (I-A_-) -$$
$$- (A_+ - A_-)^*\ (A_+ - A_-) - (B_+ + B_-)^*\ (B_+ + B_-) +$$
$$+ B_+^*B_+ + B_-^*B_- \quad . \quad (4.14)$$

Equation (4.11) can be also rewritten as

$$D_+ + C_+ - D_- - C_- = H - \hat{H} + A_-^*C_+ - A_+^*C_- - B_-^*E_+ + B_+^*E_- \quad (4.15)$$

which after rearrangement yields

$$(I-H) - (I-\hat{H}) = (I-A_+^*)\ C_- - (I-A_-^*)\ C_+ + D_- - D_+$$
$$- B_-^*E_+ + B_+^*E_- \quad \cdot \quad (4.16)$$

Finally, we may add (4.13) and (4.15) to write

$$H = \frac{1}{2}\ [A_+ + A_+^* + A_- + A_-^* + C_+ - C_- + D_+ - D_- - A_-^*\ (A_+ + C_+) -$$
$$- A_+^*\ (A_- - C_-) + B_-^*\ (B_+ + E_+) + B_+^*\ (B_- - E_-)] \quad . \quad (4.17)$$

Equation (4.17) shows that H is the sum of a) Toeplitz and Hankel operators, b) Products of Toeplitz with Toeplitz plus Hankel operators.

Moveover, all kernels involved are by definition (4.12) triangular. As a conclusion, Toeplitz plus Hankel operators have algebraic properties very similar to those of purely Toeplitz kernels. This agrees with the results obtained in [6] concerning Toeplitz plus Hankel matrices.

5. Concluding remarks

Kailath et. al. have shown [11] that the set of symmetric operators with kernel $\delta(t-s) + K(t-s)$ having the displacemenet property $\frac{\partial K}{\partial t} + \frac{\partial K}{\partial s} = \sum_{i=1}^{\alpha} \phi_i(t)\psi_i(s)$ for some finite integer α is an algebra (i.e. closed under addition, composition and inversion) which contains all Toeplitz operators. Analogous results also exist for the corresponding discrete case where operators are replaced by matrices [10].

Equations (4.6) and (4.7) provide us with insight as to how an algebra containing Toeplitz plus Hankel operators may be defined. Namely, we may define the elements of our algebra to consist of those symmetric kernels $\delta(t-s) + K(t,s)$ such that there exists a new symmetric kernel $\hat{K}(t,s)$ satisfying

$$\frac{\partial K}{\partial t} + \frac{\partial \hat{K}}{\partial s} = \sum_{i=1}^{\alpha} \phi_i(t)\,\psi_i(s)$$

$$\frac{\partial \hat{K}}{\partial t} + \frac{\partial K}{\partial s} = \sum_{i=1}^{\alpha} \phi_i(s)\,\psi_i(t)$$

We shall say that $I + K$ is invertible in the algebra if both operators $I + K$ and $I + \hat{K}$ are invertible. In that case, an approach similar to the derivation of the generalized Sobolev identity shows that our algebra is indeed closed under inversion. Closure under operator multiplication (kernel composition) is much easier to verify.

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