Distributed Algorithms for Minimum Cost Multicast with Network Coding in Wireless Networks

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***Abstract*— We adopt the network coding approach to achieve minimum-cost multicast in interference-limited wireless networks where link capacities are functions of the signal-to-noise-plus-interference ratio (SINR). Since wireless link capacities can be controlled by varying transmission powers, minimum-cost multicast must be achieved by jointly optimizing network coding subgraphs with power control and congestion control schemes. To address this, we design a set of node-based distributed gradient projection algorithms which iteratively adjust local control variables so as to converge to the optimal power control, coding subgraph, and congestion control configuration. We explicitly derive the scaling matrices required in the gradient projection algorithms for fast, guaranteed global convergence, and show how the scaling matrices can be computed in a distributed manner.**

I. Introduction

The recent breakthrough in network coding [1], [2] extends the functionality of network nodes from traditional routing to performing algebraic or even random operations [3] on received data. In general, network coding techniques improve network throughput [1], network robustness [4], and the efficiency of network resource allocation [5], over those achievable by pure routing.

The advantage of network coding is most pronounced in establishing multicast connections. Li et al. [6] prove that linear coding suffices to obtain the optimal throughput of a multicast session, achieving the fundamental max-flow-min-cut upper bound. This result greatly facilitates the optimization of multicast flows based on network coding. In [7], throughput optimization in undirected coded networks is studied via a linear program. The problem of finding the minimum-cost multicast scheme using a network coding approach is addressed in [8]. It is shown [8] that the solution of this problem can be decomposed into two parts: finding the minimum-cost coding subgraphs and designing the code applied over the optimal subgraphs. A distributed solution for the second part was provided in [3]. To solve the first part, the work in [5] proposes a distributed algorithm for finding the optimal coding subgraphs via a primal-dual approach. In previous work [9], we solve the minimum-cost coding subgraph problem in wireline coded networks using simple node-based primal scaled gradient projection algorithms requiring no dual computations. By exploiting the intrinsic connection between the optimal coding subgraph problem and the optimal routing problem in traditional networks, we design a complete set of distributed solutions for the optimal multicast problem involving both congestion control and network coding [9].

Whereas network coding techniques have thus far been applied mostly to wireline networks, the performance gains offered by network coding point to their promising application in wireless networks, where multi-user interference, channel fading, energy constraints, and the lack of centralized coordination present new challenges. Initial studies on the application of network coding in wireless networks occur in [5], [10]. In [5], the minimum-energy multicast problem is studied by exploiting the "wireless multicast advantage." The work in [10] introduces a distributed protocol which supports multiple unicast flows efficiently by exploiting the shared nature of the wireless medium.

In this work, we extend the optimization framework and distributed algorithms in [9] to achieve minimum-cost multicast with network coding in interference-limited wireless networks. We consider wireless networks where link capacities are functions of the signal-to-interference-plus-noise ratio (SINR) at the receiver. In this context, wireless link capacities can be controlled by varying transmission powers. To achieve minimum-cost multicast, the coding subgraphs must now be jointly optimized with power control schemes at the physical layer. Moreover, this joint optimization must be carried out in the network without excessive control overhead. To solve this problem, we design a set of node-based scaled gradient projection algorithms which iteratively adjust local control variables at network nodes so as to converge to the optimal power control, coding subgraph, and congestion control configuration. These algorithms are distributed in the sense that network nodes can separately update their control variables after obtaining a limited number of control messages from their neighboring nodes. We explicitly derive the scaling matrices required in the gradient projection algorithms for fast, guaranteed global convergence, and show how the scaling matrices can be computed in a distributed manner.

II. Problem Formulation

We consider the problem of jointly optimal power control, congestion control, and network coding in wireless networks

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with multiple multicast sessions. Our optimization framework will yield a feasible set of transmission powers, link capacities, as well as a set of network coding subgraphs, one for each receiver of each multicast session.

Let the wireless network be modelled by a directed and connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ where $\mathcal{N}$ is the set of nodes and $\mathcal{E}$ the set of links. Each node $i \in \mathcal{N}$ models a wireless transceiver. We assume that the wireless network is *interference-limited*, so that the capacity of link (i, j), denoted by C_{ij}, is a nonnegative function of the signal-to-interference-plus-noise ratio (SINR) at the receiver of the link, i.e., $C_{ij} = C(SINR_{ij})$. We further assume $C(\cdot)$ is increasing, concave, and twice continuously differentiable. For $(i, j) \in \mathcal{E}$, $SINR_{ij}$ is given by

$$SINR_{ij}(\boldsymbol{P}) = \frac{G_{ij}P_{ij}}{G_{ij}\sum_{n\neq j} P_{in} + \sum_{m\neq i} G_{mj} \sum_{n} P_{mn} + N_j},$$

where P_{mn} is the transmission power on link (m, n), G_{mj} denotes the (constant) path gain from node m to j, N_j is the noise power at node j's receiver. For example, in a CDMA network using single-user decoding, the information-theoretic link capacity per unit bandwidth is $C_{ij} = \log(1 + SINR_{ij})$. Assume every node i is subject to an individual power constraint: $\sum_j P_{ij} \triangleq P_i \leq \bar{P}_i$. Denote the set of all feasible power vectors by $\Pi = \{\boldsymbol{P} \geq \boldsymbol{0} : \sum_j P_{ij} \leq \bar{P}_i, \forall i \in \mathcal{N}\}$.

Let $\mathcal{M}$ denote the set of multicast sessions. Each session $m \in \mathcal{M}$ is identified by the source-destination-set pair $(s(m), \mathcal{W}(m))$ where $s(m)$ is the source node and $\mathcal{W}(m)$ is the set of all receivers of session m. For each $w \in \mathcal{W}(m)$, we refer to $(s(m), w)$ as *sub-session* w of m. In this work, we assume network coding is applied to individual sessions such that data of different sessions are coded independently. In general, this restricted coding scheme is suboptimal. However, it provides a tractable framework for optimization. Moreover, it typically incurs little loss of optimality [7].

We adopt a flow model to analyze the transmission of the multicast sessions' data in the network. Let $F_{ij}(m)$ denote network-coded transmission rate of session m traffic on link (i, j). For simplicity, we refer to $F_{ij}(m)$ as the flow rate of session m traffic on (i, j). The flow rate of a sub-session $f_{ij}(w; m)$ represents the part of $F_{ij}(m)$ that is relevant for receiver $w \in \mathcal{W}(m)$. Thus, the vector $\boldsymbol{f}(w; m) = (f_{ij}(w; m))$ forms the *coding subgraph* [8] for the pair $(s(m), w)$. The flow rates of a session and its sub-sessions are related as follows:

$$F_{ij}(m) = \max_{w\in\mathcal{W}(m)} f_{ij}(w; m), \quad (1)$$

and the total flow rate on a link (i, j) is $F_{ij} = \sum_{m\in\mathcal{M}} F_{ij}(m)$. The flow of each sub-session is feasible if it satisfies the usual *flow conservation constraints* [1], [8]: for all $w \in \mathcal{W}(m)$, $f_{ij}(w; m) \geq 0$ and

$$\sum_{j\in\mathcal{O}_i} f_{ij}(w; m) = \begin{cases} r_m, & i = s(m), \\ 0, & i = w, \\ \sum_{j\in\mathcal{I}_i} f_{ji}(w; m) \triangleq t_i(w; m), & \text{otherwise.} \end{cases}$$

Here, r_m is the end-to-end flow rate of session m, $\mathcal{O}_i = \{j : (i, j) \in \mathcal{E}\}$ and $\mathcal{I}_i = \{j : (j, i) \in \mathcal{E}\}$ denote the set of next-hop neighbors and the set of immediately upstream neighbors of node i, respectively. In what follows, denote the set of feasible flows $(f_{ij}(w; m))_{(i,j)}$ by $\mathcal{F}(w; m)$. The concepts of session flow rates, sub-session flow rates, and coding subgraphs are illustrated in Figure 1.

Because the flow of any sub-session follows the same conservation constraints as a unicast session in traditional routed networks, it can be optimized by a *routing* methodology [9]. The main difference between the present problem and the traditional routing problem is that the session flow $F_{ij}(m)$ is the *maximum* (rather than the sum) of the sub-session flows $f_{ij}(w; m)$.

To assess the optimality of a multicast scheme, we first associate a utility function $U_m(r_m)$ with each session $m \in \mathcal{M}$. Assume that session m's maximum rate demand is R_m bits/sec, and $U_m(r_m)$ is strictly increasing, concave, and twice continuously differentiable in $r_m \in [0, R_m]$. Next, we assume a cost measured by the function $D_{ij}(C_{ij}, F_{ij})$ is incurred on link (i, j) when the total link flow rate is F_{ij} and the link capacity is C_{ij}. Assume $D_{ij}(\cdot, \cdot)$ is jointly convex, strictly decreasing (increasing), and twice continuously differentiable in C_{ij} (F_{ij}).

The optimal multicast scheme results from balancing the aggregate session utility and the total network:

$$\begin{aligned}
\text{maximize} \quad & \sum_{m\in\mathcal{M}} U_m(r_m) - \sum_{(i,j)\in\mathcal{E}} D_{ij}(C_{ij}, F_{ij}) && (2)\\
\text{subject to} \quad & 0 \leq r_m \leq R_m, \quad \forall m \in \mathcal{M}, && (3)\\
& C_{ij} = C(SINR_{ij}(\boldsymbol{P})), \ \forall (i, j) \in \mathcal{E} && (4)\\
& F_{ij} = \sum_{m\in\mathcal{M}} \max_{w\in\mathcal{W}(m)} f_{ij}(w; m), \ \forall (i, j) \in \mathcal{E}, &&\\
& \boldsymbol{P} \in \Pi, &&\\
& (f_{ij}(w; m)) \in \mathcal{F}(w; m), \ \forall w \in \mathcal{W}(m). && (5)
\end{aligned}$$

Note that the problem involves joint congestion control at the transport layer (cf. (3)), network coding at the network layer (cf. (5)), and capacity allocation through power control at the physical layer (cf. (4)) in the context of wireless networks.

By introducing *overflow rate* $F_m = R_m - r_m$ and *overflow cost* $D_m(F_m) = U_m(R_m) - U_m(r_m)$, we can convert (2) into a cost minimization problem [11]:

$$\text{minimize} \sum_{m\in\mathcal{M}} D_m(F_m) + \sum_{(i,j)\in\mathcal{E}} D_{ij}(C_{ij}, F_{ij}). \quad (6)$$

Since $D_m(F_m)$ is strictly increasing, convex, and twice continuously differentiable on $[0, R_m]$, it resembles ordinary link cost functions. Thus, one can think of the rejected flow F_m as being routed on a *virtual overflow link* connecting $s(m)$ directly to a virtual sink $s'(m)$. In this way, congestion control can be incorporated into the routing functionality of the source node. This idea is illustrated in Figure 1.

The objective function in (6) is convex in all flow variables. It is convex in $\boldsymbol{P}$ if every C_{ij} is concave in $\boldsymbol{P}$ for all (i, j). Unfortunately, given that $C_{ij} = C(SINR_{ij})$ is strictly

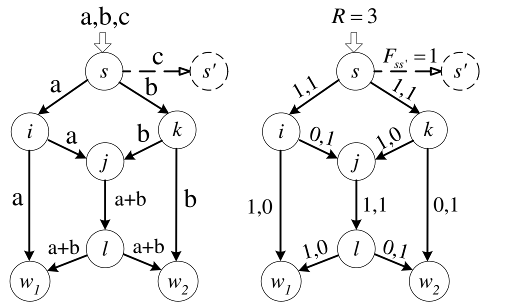

Fig. 1. Example of a coded multicast network with a single source. All links have unit capacity. The left figure characterizes a network coding scheme, showing the symbol stream on each link. The right figure shows the corresponding flow pattern where the two digits on each real link are flow rates of sub-sessions (s, w_1) and (s, w_2), respectively. An overflow link with traffic rejected from the network is shown in both figures.

increasing, $\nabla^2 C_{ij}(\boldsymbol{P})$ cannot be negative definite. However, if

$$C''(x) \cdot x + C'(x) \leq 0, \quad \forall x \geq 0, \tag{7}$$

then with changes of variables $S_{mn} = \ln P_{mn}$ [12], $\nabla^2 C_{ij}(\boldsymbol{S})$ is negative definite and the objective function is convex in $\boldsymbol{S}$. This observation is first made in [13], where the capacity function is required to satisfy $-xC''(x)/C'(x) \in [1, 2]$. Our results, however, indicate that the upper bound 2 can be removed. The detailed proof is omitted here for brevity. In what follows, we assume (7) and denote the set of feasible $\boldsymbol{S}$ by $\Pi_{\boldsymbol{S}} = \{\boldsymbol{S} \in \mathbb{R}^{|\mathcal{E}|} : \sum_j e^{S_{ij}} \leq \bar{P}_i, \forall i \in \mathcal{N}\}$.

With the desired convexity of the problem established, it remains to resolve the technical difficulty introduced by the non-differentiability of the maximum function in (1). As in [5], [14], we use the L^n-norm approximation

$$F_{ij}(m) = \max_{w \in \mathcal{W}(m)} f_{ij}(w; m) \approx \left(\sum_{w \in \mathcal{W}(m)} (f_{ij}(w; m))^n \right)^{1/n}.$$

With this, the derivative exists everywhere and is given by

$$\frac{\partial F_{ij}(m)}{\partial f_{ij}(w; m)} = \left(\frac{f_{ij}(w; m)}{F_{ij}(m)} \right)^{n-1}.$$

Thus, we obtain the following convex and twice continuously differentiable Jointly Optimal Power control, Network coding, and Congestion control (JOPNC) problem:

$$\begin{aligned}
\text{minimize} \quad & \sum_{m \in \mathcal{M}} D_m(F_m) + \sum_{(i,j) \in \mathcal{E}} D_{ij}(C_{ij}, F_{ij}) \qquad (8) \\
\text{subject to} \quad & F_m + r_m = R_m, \quad \forall m \in \mathcal{M}, \\
& C_{ij} = C(SINR_{ij}(\boldsymbol{S})), \quad \forall (i,j) \in \mathcal{E}, \\
& \boldsymbol{S} \in \Pi_{\boldsymbol{S}}, \\
& F_{ij} = \sum_{m \in \mathcal{M}} \left(\sum_{w \in \mathcal{W}(m)} (f_{ij}(w; m))^n \right)^{1/n}, \\
& (f_{ij}(w; m)) \in \mathcal{F}(w; m), \ \forall w \in \mathcal{W}(m).
\end{aligned}$$

III. Node-Based Control Variables and Optimality Conditions

In the previous work [9], we show that in wireline multicast networks with network coding, congestion control and coding subgraph optimization can be achieved using a routing methodology. We now extend this technique to wireless networks, where in contrast to wireline networks, link capacities can be further controlled by varying transmission powers.

Since large-scale wireless networks usually lack centralized coordination, it is desirable to distribute the control functionalities to individual nodes. For this purpose, we devise a set of node-based control variables. First, to permit each node to independently adjust the sub-session flow rates on its outgoing links, we adopt the *routing variables* introduced in [15]. For each $m \in \mathcal{M}$, define $\phi_m = \frac{F_m}{R_m}$ and

$$\phi_{ij}(w; m) = \begin{cases} \frac{f_{ij}(w;m)}{R_m}, & \text{if } i = s(m), \\ \frac{f_{ij}(w;m)}{t_i(w;m)}, & \text{if } i \neq s(m), w. \end{cases}$$

The routing variables must be nonnegative. In addition, $\phi_m + \sum_{j \in \mathcal{O}_i} \phi_{ij}(w; m) = 1$ for all $w \in \mathcal{W}(m)$ if $i = s(m)$, and $\sum_{j \in \mathcal{O}_i} \phi_{ij}(w; m) = 1$ for all $w \in \mathcal{W}(m)$ if $i \neq s(m), w$ and $t_i(w; m) > 0$.[2] Similarly, to achieve distributed power adjustment, define

$$\text{Power allocation variables: } \eta_{ij} = P_{ij}/P_i, \quad \forall (i,j) \in \mathcal{E},$$

$$\text{Power control variables: } \gamma_i = S_i/\bar{S}_i, \quad \forall i \in \mathcal{N}.$$

Here, $S_i = \ln P_i$ and $\bar{S}_i = \ln \bar{P}_i$. With appropriate scaling, we can always let $\bar{P}_i > 1$ so that $\bar{S}_i > 0$ for all i. Thus, (η_{ij}) and γ_i satisfy $\eta_{ij} \geq 0$, $\sum_{j \in \mathcal{O}_i} \eta_{ij} = 1$, and $\gamma_i \leq 1$.

The routing, power allocation, and power control variables defined above determine the transmission powers as well as all the sub-session flow rates on all links. The JOPNC problem in (8) can thus be posed in terms of these variables. To solve the resulting differentiable optimization problem, an iterative gradient projection method may be used. For distributed implementation, it is desirable for the cost gradients with respect to the local control variables of a node to be computable locally, after a possibly local exchange of information. Fortunately, for our problem, this turns out to be the case.

The first derivatives of the objective function, denoted by D, with respect to routing variables are [15]:

$$\frac{\partial D}{\partial \phi_m} = R_m \cdot \delta\phi_m,$$

$$\frac{\partial D}{\partial \phi_{ij}(w; m)} = \begin{cases} R_m \cdot \delta\phi_{ij}(w; m), & \text{if } i = s(m), \\ t_i(w; m) \cdot \delta\phi_{ij}(w; m), & \text{if } i \neq s(m), w. \end{cases}$$

Key information lies in the *marginal routing cost indicators*

$$\delta\phi_m = D'_m(F_m)$$

and

$$\delta\phi_{ij}(w; m) = \frac{\partial D_{ij}}{\partial f_{ij}(w; m)} + \frac{\partial D}{\partial r_j(w; m)},$$

[2]For those intermediate nodes having $t_i(w; m) = 0$, we let the routing variables $\phi_{ij}(w; m)$ assume arbitrary nonnegative values satisfying the same simplex constraint.

where

$$\frac{\partial D_{ij}}{\partial f_{ij}(w;m)} = \frac{\partial D_{ij}}{\partial F_{ij}} \left(\frac{f_{ij}(w;m)}{F_{ij}(m)} \right)^{n-1},$$

$$\frac{\partial D}{\partial r_j(w;m)} = \begin{cases} 0, & \text{if } j = w, \\ \sum_{k\in\mathcal{O}_j} \phi_{jk}(w;m)\delta\phi_{jk}(w;m), & \text{otherwise.} \end{cases}$$

From the previous recursive relation, we can see that the marginal routing cost indicators can be obtained via sequential marginal cost exchanges among neighboring nodes starting from the destination nodes. The finite termination of the sequential message passing relies on the routing pattern of the sub-session being loop-free. This, however, is guaranteed by the distributed routing algorithm discussed below.

We now turn to the derivatives with respect to the power variables. The first derivatives in the power allocation variables are

$$\frac{\partial D}{\partial \eta_{ij}} = P_i \left[-\sum_{(m,n)} \frac{\partial D_{mn}}{\partial C_{mn}} \frac{C'_{mn} G_{mn} G_{in} P_{mn}}{IN_{mn}^2} + \delta\eta_{ij} \right],$$

where the *marginal power allocation cost indicator* is

$$\delta\eta_{ij} = \frac{\partial D_{ij}}{\partial C_{ij}} \frac{C'_{ij} G_{ij}}{IN_{ij}} (1 + SINR_{ij}).$$

In above equations, C'_{mn} stands for $C'(SINR_{mn})$ and IN_{mn} denotes the interference plus noise on link (m,n): $G_{mn}(P_m - P_{mn}) + \sum_{l\neq m} G_{ln}P_l + N_n$.

Finally, the derivatives with respect to the power control variables are

$$\frac{\partial D}{\partial \gamma_i} = \bar{S}_i \delta\gamma_i,$$

where the *marginal power control cost indicator* is

$$\delta\gamma_i = P_i \left[-\sum_{(m,n)} \frac{\partial D_{mn}}{\partial C_{mn}} \frac{C'_{mn} G_{mn} G_{in} P_{mn}}{IN_{mn}^2} + \sum_{j\in\mathcal{O}_i} \delta\eta_{ij} \cdot \eta_{ij} \right]. \tag{9}$$

This formula for $\delta\gamma_i$ involves measures from all links in the network. We will introduce an efficient message exchange protocol for the computation of $\delta\gamma_i$ in the next section.

Theorem 1: For a feasible set of routing variables $(\phi_m), (\boldsymbol{\phi}_i(w;m))$ and power variables $(\boldsymbol{\eta}_i), (\gamma_i)$ to induce the jointly optimal sub-session flows and link capacities, the following conditions are necessary. For all $m \in \mathcal{M}$, $w \in \mathcal{W}(m)$ and $i \neq s(m), w$ with $t_i(w;m) > 0$, there exists a constant $\lambda_i(w;m)$ such that

$$\delta\phi_{ik}(w;m) \begin{cases} = \lambda_i(w;m), & \text{if } \phi_{ik}(w;m) > 0, \\ \geq \lambda_i(w;m), & \text{if } \phi_{ik}(w;m) = 0. \end{cases} \tag{10}$$

For the source node $i = s(m)$, define for every $w \in \mathcal{W}(m)$, $\lambda_i(w;m) = \min_{j\in\mathcal{O}_i} \delta\phi_{ij}(w;m)$, then $\delta\phi_{ik}(w;m)$ satisfies (10) and

$$\delta\phi_m \begin{cases} \geq \sum_{w\in\mathcal{W}(m)} \lambda_i(w;m), & \text{if } \phi_m = 0, \\ = \sum_{w\in\mathcal{W}(m)} \lambda_i(w;m), & \text{if } \phi_m \in (0,1), \\ \leq \sum_{w\in\mathcal{W}(m)} \lambda_i(w;m), & \text{if } \phi_m = 1. \end{cases} \tag{11}$$

For all $i \in \mathcal{N}$, there exists a constant ν_i such that

$$\delta\eta_{ik} \begin{cases} = \nu_i, & \text{if } \eta_{ik} > 0, \\ \geq \nu_i, & \text{if } \eta_{ik} = 0, \end{cases} \tag{12}$$

$$\frac{\delta\gamma_i}{P_i} \begin{cases} = 0, & \text{if } \gamma_i < 1, \\ \leq 0, & \text{if } \gamma_i = 1. \end{cases} \tag{13}$$

Moreover, the above set of conditions are sufficient if (10) holds at all intermediate nodes whether $t_i(w;m) > 0$ or not.[3]

IV. Node-Based Distributed Algorithms

Since the JOPNC problem in (8) involves the minimization of a convex objective over convex regions, the class of *scaled gradient projection* algorithms is appropriate for providing a distributed solution. Using this method, Gallager [15] and Bertsekas et al. [16] develop distributed routing algorithms for wireline networks supporting unicast sessions. In this section, we adapt this technique to design node-based algorithms for jointly optimal power control, network coding, and congestion control for wireless networks. These algorithms includes two kinds of routing algorithms implemented at the source and intermediate nodes respectively, as well as power allocation and power control algorithms implemented at individual nodes. Our algorithms use a new technique for computing scaling matrices which are amenable to distributed computation. With this technique, we show that our algorithms are guaranteed to converge from all initial conditions.[4]

A. Source Node Congestion Control/Routing Algorithm (CR)

This algorithm is implemented at the source node $s(m)$ of every session $m \in \mathcal{M}$. It adjusts the routing variables on all the outgoing links of $s(m)$ (including the virtual overflow link) and for all sub-sessions $w \in \mathcal{W}(m)$. We therefore call it the Congestion control/Routing (CR) algorithm. For conciseness, we suppress the session index m and use the short-hand notation $\boldsymbol{\phi}_s(w) = (\phi_{sj}(w))_{j\in\mathcal{O}_s}$. At the kth iteration, the feasible set of vector $\boldsymbol{\phi}_s = (\phi_m, (\boldsymbol{\phi}_s(w))_{w\in\mathcal{W}(m)})$ is

$$\begin{aligned} \mathcal{F}^k_{\boldsymbol{\phi}_s} \;\; = \;\; & \{\boldsymbol{\phi}_s \geq \mathbf{0} : \phi_m + \boldsymbol{\phi}_s(w)' \cdot \mathbf{1} = 1 \\ & \text{and } \phi_{sj}(w) = 0, \ \forall w \in \mathcal{W}, \ j \in \mathcal{B}^k_s(w)\}, \end{aligned}$$

where $'$ denotes the vector transpose, $\mathcal{B}^k_s(w)$ stands for the *blocked node set* of node s relative to sub-session w.[5]

Node s updates the current routing vector $\boldsymbol{\phi}^k_s$ via the following scaled gradient projection algorithm:

$$\boldsymbol{\phi}^{k+1}_s = CR(\boldsymbol{\phi}^k_s) = \left[\boldsymbol{\phi}^k_s - (M^k_s)^{-1} \cdot \delta\boldsymbol{\phi}^k_s\right]^+_{M^k_s}.$$

[3]When $P_i = 0$, define $\delta\gamma_i/P_i|_{P_i=0} = \lim_{P_i\to 0^+} \delta\gamma_i/P_i$.

[4]The work in [16] uses a more involved scheme to approximate the diagonal terms of the Hessian matrices with respect to the routing variables. Since the resulting scaling matrices do not always upper bound the Hessians, convergence may not happen for some initial conditions.

[5]This device is invented in [15], [16] to prevent the formation of loops in the routing pattern of sub-session w. For a node i, $\mathcal{B}^k_i(w)$ consists of its next-hop neighbor j with marginal cost $\frac{\partial D}{\partial r^k_j(w)}$ higher than $\frac{\partial D}{\partial r^k_i(w)}$, and neighbors that route positive flows to more costly downstream nodes. By blocking such nodes, we force each sub-session's traffic to flow through nodes in decreasing order of marginal costs, thus precluding the existence of loops. For an exact definition of $\mathcal{B}^k_i(w)$, see [16].

Here, the operator $[\cdot]^+_{M_s^k}$ denotes projection on the feasible set $\mathcal{F}^k_{\boldsymbol{\phi}_s}$ relative to the norm induced by matrix M_s^k. The vector $\delta\boldsymbol{\phi}_s^k$ consists of marginal cost indicators $(\delta\phi_m^k, (\delta\phi_s^k(w))_{j\in\mathcal{O}_s, w\in\mathcal{W}})$. The scaling matrix M_s^k is symmetric and positive definite. In particular, we choose M_s^k to be a diagonal matrix upper-bounding the Hessian evaluated at the kth step:

$$M_s^k = \tfrac{R_m}{2}\mathrm{diag}\left\{A_m(D^0), \left(|\mathcal{W}|\left[A_{sj}(D^0) + |\mathcal{AN}_s^k(w)|h_j(w)A(D^0)\right]\right)_{w\in\mathcal{W}, j\in\mathcal{AN}_s^k(w)}\right\}$$

where $\mathcal{AN}_s^k(w) \equiv \mathcal{O}_s \backslash \mathcal{B}_s^k(w)$, $h_j(w)$ is the maximum number of hops on a path from j to w, and

$$A_m(D^0) \equiv \max_{F:D_m(F)\le D^0} D''_m(F),$$

$$A_{ij}(D^0) \equiv \max_{F_{ij}:D_{ij}(C_{ij},F_{ij})\le D^0} \frac{\partial^2 D_{ij}}{\partial F_{ij}^2},$$

and

$$A(D^0) \equiv \max_{(m,n)\in\mathcal{E}} A_{mn}(D^0).$$

With this choice of M_s^k, the CR algorithm resembles a constrained Newton algorithm, which is known to have fast convergence properties [17]. Moreover, it is clear that M_s^k can be computed locally at s with a simple distributed protocol whereby the $h_j(w)$'s are determined. The scaling matrices we specify for subsequent algorithms share the same features.

B. Intermediate Node Routing Algorithm (RT)

Consider any session $m \in \mathcal{M}$ and for brevity omit the index m. Relative to a sub-session w, an intermediate node i changes the allocation of the sub-session's traffic on its outgoing links by adjusting its current routing vector $\boldsymbol{\phi}_i^k(w) = (\phi_{ij}^k(w))_{j\in\mathcal{O}_i}$ within the feasible set

$$\begin{aligned}\mathcal{F}^k_{\boldsymbol{\phi}_i(w)} &= \{\boldsymbol{\phi}_i(w) \ge \mathbf{0} : \boldsymbol{\phi}_i(w)' \cdot \mathbf{1} = 1 \\ &\quad \text{and } \phi_{ij}(w) = 0,\ \forall j \in \mathcal{B}_i^k(w)\}.\end{aligned}$$

Because $\boldsymbol{\phi}_i^k(w)$ affects only the routing pattern of sub-session w inside the network, we refer to the updating algorithm as a pure Routing algorithm (RT). Similar to CR, it has a scaled gradient projection form:

$$\begin{aligned}\boldsymbol{\phi}_i^{k+1}(w) &= RT(\boldsymbol{\phi}_i^k(w)) \\ &= \left[\boldsymbol{\phi}_i^k(w) - (M_i^k(w))^{-1}\cdot\delta\boldsymbol{\phi}_i^k(w)\right]^+_{M_i^k(w)}.\end{aligned}$$

Here, $\delta\boldsymbol{\phi}_i^k(w) = (\delta\phi_{ij}^k(w))$ and the diagonal scaling matrix $M_i^k(w)$ is chosen as

$$\frac{t_i^k(w)}{2}\mathrm{diag}\left\{\left(A_{ij}(D^0) + |\mathcal{AN}_i^k(w)|h_j^k(w)A(D^0)\right)_{j\in\mathcal{AN}_i^k(w)}\right\}$$

C. Power Allocation Algorithm (PA)

At the kth iteration, node i updates its power allocation vector $\boldsymbol{\eta}_i^k = (\eta_{ij}^k)_{j\in\mathcal{O}_i}$ within the feasible set $\mathcal{F}_{\boldsymbol{\eta}_i} = \{\boldsymbol{\eta}_i \ge \mathbf{0} : \boldsymbol{\eta}_i'\cdot\mathbf{1} = 1\}$ via the following scaled gradient projection:

$$\boldsymbol{\eta}_i^{k+1} = PA(\boldsymbol{\eta}_i^k) = \left[\boldsymbol{\eta}_i^k - (Q_i^k)^{-1}\cdot\delta\boldsymbol{\eta}_i^k\right]^+_{Q_i^k}.$$

We now specify the appropriate scaling matric Q_i^k. Assume the sum of the local link costs at node i before the kth iteration is $\sum_{j\in\mathcal{O}_i} D_{ij}^k = D_i^k$. The powers used by other nodes do not change over the iteration, and so C_{ij} depends only on η_{ij}:

$$C\left(\frac{G_{ij}P_i\eta_{ij}}{G_{ij}P_i(1-\eta_{ij}) + \sum_{m\ne i} G_{mj}P_m + N_j}\right) \triangleq C_{ij}(\eta_{ij}).$$

It can be shown that there exists a lower bound $\underline{\eta}_{ij}$ on the updated value of η_{ij} such that $\underline{C}_{ij} = C_{ij}(\underline{\eta}_{ij})$ and $D_{ij}(\underline{C}_{ij}, F_{ij}^k) = D_i^k$. Accordingly, the possible range of $SINR_{ij}$, abbreviated as x_{ij}, is

$$\begin{aligned} x_{ij}^{min} &= \frac{G_{ij}P_i\underline{\eta}_{ij}}{G_{ij}P_i(1-\underline{\eta}_{ij}) + \sum_{m\ne i}G_{mj}P_m + N_j} \le x_{ij} \\ &\le \frac{G_{ij}P_i}{\sum_{m\ne i}G_{mj}P_m + N_j} = x_{ij}^{max}.\end{aligned}$$

Define an auxiliary term

$$\begin{aligned}\beta_{ij} = \frac{1}{\underline{\eta}_{ij}^2}\Bigg[& B_{ij}(D_i^k)\max_{x_{ij}^{min}\le x\le x_{ij}^{max}}\{C'(x)^2x^2(1+x)^2\} \\ &+ \left.\frac{\partial D_{ij}}{\partial C_{ij}}\right|_{D_{ij}(C_{ij},F_{ij}^k)=D_i^k} \cdot \min_{x_{ij}^{min}\le x\le x_{ij}^{max}}\{C''(x)x^2(1+x)^2\}\Bigg],\end{aligned}$$

where $B_{ij}(D_i^k) = \max_{D_{ij}(C_{ij},F_{ij}^k)\le D_i^k}\frac{\partial^2 D_{ij}}{\partial C_{ij}^2}$. We choose the scaling matrix as

$$Q_i^k = \frac{1}{2P_i^k}\mathrm{diag}\left\{(\beta_{ij})_{j\in\mathcal{O}_i}\right\}.$$

For details of the derivation, please refer to [18].

D. Power Control Algorithm (PC)

At the kth iteration of the power control algorithm, the whole vector $\boldsymbol{\gamma} = (\gamma_i)$ is varied within the feasible set $\mathcal{F}_{\boldsymbol{\gamma}} = \left\{\boldsymbol{\gamma}\in\mathbb{R}^{|\mathcal{N}|} : \boldsymbol{\gamma}\le\mathbf{1}\right\}$. The update vector is given by the following scaled gradient projection:

$$\boldsymbol{\gamma}^{k+1} = PC(\boldsymbol{\gamma}^k) = \left[\boldsymbol{\gamma}^k - (V^k)^{-1}\cdot\delta\boldsymbol{\gamma}^k\right]^+_{V^k}.$$

Let the scaling matrix $V^k = \mathrm{diag}\{(v_i)_{i\in\mathcal{N}}\}$. Then PC can be decomposed into separate computations at individual nodes requiring only their own marginal cost indicators:

$$\gamma_i^{k+1} = PC(\gamma_i^k) = \min\left\{1, \gamma_i^k - \frac{\delta\gamma_i^k}{v_i}\right\}.$$

It remains to design a procedure to let every node i compute its own $\delta\gamma_i$ prior to the algorithm iteration. The following protocol is based on a convenient rearrangement of (9).

Power Control Message Exchange Protocol: Let each node n sum up the measures from all its incoming links (m, n) to form the power control message: $MSG_n \triangleq \sum_{m\in\mathcal{I}_n} \frac{\partial D_{mn}}{\partial C_{mn}} \frac{C'_{mn} SINR_{mn}}{IN_{mn}}$, which is then broadcast to the whole network. Upon obtaining MSG_n, node i processes it according to the following rule: if n is a next-hop neighbor of i, node i multiplies MSG_n with path gain G_{in} and subtracts the product from the value of local measure $\delta\eta_{in}\cdot\eta_{in}$; otherwise, node i multiplies MSG_n with $-G_{in}$. Finally, node i adds up all the processed messages, and this sum multiplied by P_i equals $\delta\gamma_i$. Note that this protocol requires *only one message from each node in the network*. For details, see [18].

By approximating the Hessian matrix, we find the appropriate diagonal terms of V^k:

$$v_i = \frac{\bar{S}_i}{2} |\mathcal{N}||\mathcal{E}| \left[\bar{B}(D^0)\kappa + \underline{B}(D^0)\varphi\right],$$

where

$$\bar{B}(D^0) = \max_{(m,j)\in\mathcal{E}} \max_{D_{mj}\le D^0} \frac{\partial^2 D_{mj}}{\partial C_{mj}^2},$$

$$\underline{B}(D^0) = \min_{(m,j)\in\mathcal{E}} \min_{D_{mj}\le D^0} \frac{\partial D_{mj}}{\partial C_{mj}},$$

$\kappa = \max_{0\le x\le\bar{x}} C'(x)^2 \cdot x^2$, and $\varphi = \max_{0\le x\le\bar{x}} C''(x) \cdot x^2$. Here, $\bar{x}$ is a finite upper bound for the achievable $SINR$ on all links, which must exist due to the peak power constraints.

E. Convergence of Algorithms

Applying the scaling matrices specified above for each of the algorithms, we have our central convergence result.

Theorem 2: Assume an initial set of loop-free routing variables (ϕ_m^0), $(\boldsymbol{\phi}_i^0(w;m))$, and power variables $(\boldsymbol{\eta}_i^0)$, (γ_i^0) such that the resulting network cost D^0 is finite, then the sequences generated by algorithms CR, RT, PA, and PC all converge, i.e., $\phi_m^k \to \phi_m^*$ for all $m \in \mathcal{M}$, $\boldsymbol{\phi}_i^k(w;m) \to \boldsymbol{\phi}_i^*(w;m)$ for all $w \in \mathcal{W}(m)$ and $i \neq w$, $\boldsymbol{\eta}_i^k \to \boldsymbol{\eta}_i^*$ for all $i \in \mathcal{N}$, and $\boldsymbol{\gamma}^k \to \boldsymbol{\gamma}^*$ as $k \to \infty$. Furthermore, (ϕ_m^*), $(\boldsymbol{\phi}_i^*(w;m))$, $\boldsymbol{\eta}_i^*$, and $\boldsymbol{\gamma}^*$ yield a jointly optimal solution to the JOPNC problem in (8).

Note that the theorem does not assume any order in running the algorithms CR, RT, PA, and PC at different nodes. For convergence to the joint optimum, every node only needs to iterate its own algorithms until its local variables satisfy the conditions in Theorem 1.[6] Thus, our algorithms provide a distributed method of finding the jointly optimal power control, network coding, and congestion control configuration for wireless multicast networks.[7]

[6]In practice, nodes may keep updating their local variables with the corresponding algorithms until further reduction in network cost by any one of the algorithms is negligible.

[7]Only the power control part of our algorithms requires a network-wide message exchange protocol. This scheme, however, requires only one message to be sent from every node, even in the presence of heavy mutual interference.

V. Conclusion

We adopt the network coding approach to achieve minimum-cost multicast in interference-limited wireless networks where link capacities are functions of the $SINR$. We develop a node-based framework within which transmission powers, network coding subgraphs, and admitted session rates are jointly optimized. We design a complete set of node-based distributed algorithms to achieve this joint optimization, and prove the convergence of the algorithms to the optimum wireless network configuration from all initial conditions.

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