69-9411

KIANG, Wan-lin, 1937-
CONTROLLABILITY OF TWO-LEVEL LINEAR DYNAMICAL SYSTEMS.

Case Western Reserve University, Ph.D., 1968
Engineering, general

University Microfilms, Inc., Ann Arbor, Michigan

CONTROLLABILITY OF
TWO-LEVEL LINEAR DYNAMICAL SYSTEMS

by

WAN-LIN KIANG

Submitted in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

School of Engineering
Case Western Reserve University
June 1968

CASE WESTERN RESERVE UNIVERSITY

GRADUATE STUDIES

We hereby approve the thesis of

Wan-Lin Kiang

candidate for the Ph. D.

degree.

Signed: (Chairman)

Date 5/28/68

CONTROLLABILITY OF

TWO-LEVEL LINEAR DYNAMICAL SYSTEMS

by

WAN-LIN KIANG

ABSTRACT

The mathematical model of a two-level linear dynamical system is constructed. Controllability of such systems is then defined and studied. The relevance of the model to possible applications is demonstrated by examples.

The problem of coordination is the primary concern in the present research. The schemes of coordination are to be conducted by the superemal using image, constraint, goal, and interaction interventions. These terms are defined and their relations with above mentioned mathematical model and controllability of the system are then explored.

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to Dr. S. K. Mitter, under whose direction this thesis was written, for his guidance, criticism and encouragement. Thanks are also extended to Dr. M. D. Mesarovic for his guidance during the initial phases of the research.

Deep appreciation is also extended to Assumpta, the author's wife, for her typing of the manuscript, her patience, cooperation and encouragement.

The research reported here was supported in part by Office of Naval Research under contract No. NONR - 1141 - 12.

TABLE OF CONTENTS

Page

Page

LIST OF COMMON SYMBOLS

<u>Matrices</u>

A, A_i, $\mathcal{A}$, Λ	System matrix in various representations
B_i	Weighting matrix for the supremal
C, C_i, $\mathcal{C}$	Control matrix in various representations
D_{ij}, $\mathcal{D}$	Interaction matrix for the infimals
H_i, $\mathcal{H}$	Feedback matrix for the infimals
$V(t, T)$	equivalent to $\Phi(t, T)C(t)$
$W(T)$	equivalent to $\int_0^T V(t, T)V'(t, T)dt$
$W^{+}(T)$	Psendo-inverse of $W(T)$
Φ, Φ_i, Φ_I	Systems fundamental matrix

<u>Vectors</u>

f, f_i	Disturbance function
m, m_i, $\mathcal{M}$	Control vector in various representations
x, x_i, $\overline{x}_i$, x_{iH}, x_{im}, $\mathcal{X}$, $\mathcal{X}_d$, $\mathcal{X}_m$	State vector in various representations
x_o, x_{io}, $\mathcal{X}_o$	Initial state
x_d, x_{id}, $\mathcal{X}_d$	Desired state
$Z(T)$, $Z_i(T)$	Constructed state at time T
$\omega(T)$, $\omega_i(T)$	Element of $\Omega(T, F)$

Vectors (cont'd)

$\theta(T)$	Element of $L(T, M)$

Scalars

$b_1, b_2, b_{i1}, b_{i2}, b_{i3}, b_{I1}, b_{I2}$	Constants related to b.i.b.o. stable systems
k, k_i, k'	Constants defining the sets M, M_i, M'
k_o, k_{io}	Constants defining the sets X_o, X_{io}
K	Constant defining the set F
$\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_{i1}, \alpha_{i2}, \alpha_{i3}$	Constants related to exponential-asymptotically stable functions.
β	equal to $\max_i(\beta_i)$
β_i	equal to $\sup_{t \geq o} B_i(t)$
δ_i	Constants defining f_i
ε	Constant defining ε-controllability
σ	Constant defining the set Ω_T
ρ_L	Constant defining the size of $L(T, M)$
ρ_Ω	Constant defining the size of $\Omega(T, F)$

Sets

F	Set of admissible disturbance functions
J, J_o	Time interval $[0, T]$ and $[0, T_o]$
$L(T, M)$	Set of attainable states

Sets (cont'd)

$L_\varepsilon(T, M)$	Enlarged set of attainable states
$M, M', M'', M_i, \mathcal{M}$	Spaces of admissible controls
X_o, X_{io}	Sets of admissible initial states
X_d, X_{id}	Sets of target states
$\Omega(T, F)$	Set of perturbed states
Ω_T	Set of disturbed states

Notations

E	Control energy
L(T)	Linear operator
R^n	n-dimensional real Euclidean space
R^+	Positive real line
X	State space
$\dot{x}$	Equivalent to $\frac{dx}{dt}$
X^*	Conjugate space of X
A'	Transpose of A
$\mathcal{L}_2, \mathcal{L}_\infty$	Spaces of Lebesque integrable functions
λ_i	Eigenvalues
∂S	Set of boundary points for the set S
$\Vert \cdot \Vert$	Euclidean norm
$\Vert \cdot \Vert_2, \Vert \cdot \Vert_\infty$	Norms for $\mathcal{L}_2$ and $\mathcal{L}_\infty$ respectively
$\langle\ ,\ \rangle$	Inner product

Notations (cont'd)

b.i.b.o.	Bounded-input bounded-output
inf.	Infimum
max	Maximum
min	Minimum
sup	Supremum

CHAPTER I

INTRODUCTION

1.1. Motivations and Objectives

The notion of multi-level system has been a subject of study for some time. In its broadest sense, the concept of multi-level system has been associated with the description of human organization, for instance, the corporate stucture of a large business organization. Also, the concept of multi-level system has been associated with large complex physical systems, such as an integrated steel mill. These systems are usually too complex to be analyzed as a whole. Consequently, the problem of decentralization of control has become a very active center of investigation. In orther words, it is the problem of decomposition for large scale systems that has received most attention. This is by no means not justified, but we still feel that contributions to multi-level systems theory can be made via a different approach.

In order to explain the rationale behind the present research, we shall first try to answer several questions:

(i) What is a multi-level system? (ii) What structure characterizes a multi-level system? (iii) What are the special properties of a multi-level system? (iv) How should a multi-level system be studied?

To answer the first question, we shall understand a <u>multi-level system</u> as a group of interacting goal-seeking subsystems arranged in an hierarchial order. For the moment, we shall only say that such a hierarchial order exists if the activities of certain subsystems are supervised by other subsystems. At the same time, a goal-seeking activity is also assigned to the group of subsystems as a whole. Therefore, a multi-level system is itself a decision-maker.

The answer to the second question is closely related to the first one. Since a multi-level system is understood as a group of goal-seeking subsystems, we may say that the division of labor or decision-making capacity among the subsystems is the special feature which distinguishes the multi-level systems. This is very much like the situation studied in team decision theory if not for the word "level". A goal-seeking subsystem is said to situate on the highest level whenever its decision-making capacity is directly restraining the activities of all other goal-seeking subsystems. A goal-seeking subsystem is said to situate on an intermediate level whenever its decision-making capacity is defined by higher level subsystems, and at the same time, must restrain the decision-making capacity of lower level systems. A goal-seeking subsystem is said to situate on lowest level whenever it imposes no direct constraint on any other subsystem.

The answer to third question is "interaction". However, the special kind of interaction in a multi-level system originates primarily from the fact that each composing subsystem is itself a goal seeking system. For goal-seeking subsystems situated on the same level, the decision made by one subsystem is assumed not predictable by other subsystems. The resulting interaction thus presents as an unknown influence to other sybsystems on the same level. Since this type of disturbance arises as an internal force in a multi-level system, it may be called as <u>internal disturbance</u>. This type of internal disturbance will always present in a multi-level system. Consequently, study of multi-level systems must incorporate a successful treatment of internal disturbance. One way of doing this is that interaction of goal-seeking subsystems situated on a lower level be supervised and regulated by higher level subsystems. We shall call such function <u>coordination</u>.

There are perhaps many possible ways of studying the structures and behaviors of multi-level systems. First of all, we may divide the study of multi-level systems into two different but complimentary parts of synthesis and analysis. From a functional view-point both problems of decomposition and coordination must be studied. However, it is felt that emphasis on decomposition or on coordination should be weighted differently depending upon whether the analysis or the synthesis of multi-level systems is of major concerned.

In the analysis of an existing system that might be modelled as a multi-level system, for instance, the nerve system of a high intelligence animal, one usually starts from the consideration of an integrated system. The first step is to decompose the integrated system into a group of subsystems according to certain criteria. The next step is the coordination of behavior of the subsystems. The success of coordination depends heavily on how system decomposition is conceived and devised. In this respect, the problems of decomposition would be equally as important as the problems of coordination.

In the synthesis of multi-level systems, particularly physical and engineering systems, the problem of decomposition seems to be a minor one as compared with the problem of coordination. The reasons for this conjecture are extracted from the following observations: (1) In the construction of a complex system, the over-all system structure is often not immediately known. Rather, there usually are individual subsystems of known nature which need to be integrated. The natural boundaries of such subsystems could then be used as a division line for the decomposition in the synthesized multi-level system. (2) In many complex systems, the addition or deletion of a particular subsystem is always a distinct possibility. Consequently, it would be very convenient to take the natural boundaries of known subsystems as the decomposition from the outset.

The above analysis illustrates why the viewpoint of coordination, which is to be formulated as a mathematical problem in Chapter III, will be adopted as the basis of the present study.

In previous studies on multi-level system, emphasis has been mostly placed on the part of quantitative theory. The study of qualitative behavior of multi-level systems is quite scarce and ought to be supplemented. Since controllability is one of the most fundamental qualitative property of any control system, the controllability of a class of multi-level systems will be defined and used as a basic goal to be achieved from the synthesis viewpoint. Within the framework of a given mathematical model, we shall try to answer the following questions:

(i) What are the decision problems of the various levels of a control hierarchy and how are they related between levels?

(ii) How coordinations can be carried out in order to achieve a predetermined over-all goal?

1.2 Problem Statement

Consider the two-level linear dynamical system represented by

Supremal:

$$\dot{x} = A(t)x(t) + C(t)m(t) + \Sigma B_i(t)x_i(t) \qquad (1.2\text{-}1)$$

$$\dot{x} = A(t)x(t) + C(t)m(t) \qquad (1.2\text{-}1')$$

Infimals:

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) + \sum_{j \neq i} D_{ij}(t)x_j(t) \qquad (1.2\text{-}2)$$

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) \qquad i = 1, 2, \ldots, p \tag{1.2-2'}$$

The precise definitions of the above equations will be given in Chapters II and III.

The basic problem may be formulated in the following manner.

(i) Suppose that the supremal (1.2-1') is controllable at some finite time T with respect to a given space M of admissible controls and a given pair of states (x_o, x_d), where x_o being the initial state and x_d the desired state. Under what conditions will supremal (1.2-1) be controllable or ε-controllable at T or some $t \not\leq T$ with respect to the same M and (x_o, x_d)?

(ii) Suppose that the above problem is solvable and that a set of constraints is established on the time function $\Sigma B_i(t)x_i(t)$ where $x_i(t)$ are generated by the infimals. Can the supremal use coordinative interventions to adjust $A_i(t)$, $m_i(t)$, x_{io}, x_{id}, $D_{ij}(t)$ so that the above mentioned set of constraints can be met?

(iii) Suppose the answer to problem (ii) is affirmative. What are the possible schemes of coordination to be performed by the supremal so that to lead to the goal mentioned in problem (i)?

1.3 Nature of Results

The major contributions of this thesis are listed below to correspond to the problem statements in the previous section.

(i) Some new results on the controllability of linear dynamical systems are obtained in section 2.3. The concept of uniform controllability for linear dynamical systems is introduced in the same section and some related results are derived.

(ii) The concept of controllability under disturbance is introduced in section 2.4. For linear dynamical systems subjected to additive disturbances, some results are obtained on the above concept by correlating system dynamics, control space M, constraints on the additive disturbance, and the concept of uniform controllability. The concept of redundant control energy is also introduced which improves the results obtained above.

(iii) In Chapter III, two-level linear dynamical systems which allow no direct interaction among the infimals are studied. The problems posed in section 1.2 are answered affirmatively using the results derived in Chapter II. In Chapter IV, the same problems are answered for the two-level linear dynamical systems which allows direct interaction among the infimals.

CHAPTER II

CONTROLLABILITY OF LINEAR DYNAMICAL SYSTEMS

2.1 Introduction

In this thesis, the study on the behavior of two-level linear dynamical system will be restricted to the notion of controllability. Since controllability is a kind of qualitative property, it would be more meaningful to conduct the study on a system of definite structure.

As we shall see later in next chapter, the proposed two-level linear dynamical system is essentially an aggregation of ordinary linear dynamical systems which are describable by ordinary linear differential equation systems. The interaction among systems belonging to this aggregation will be characterized by additive linear operators.

The principal approach in tackling the problem is the use of "parametric coordination", which reduces the mathematical model of the two-level linear dynamical system into a group of independent linear dynamical subsystems described by ordinary linear differential equation systems. The study of the integrated system, i.e., the original two-level linear dynamical system, is then carried out based on the understanding of the behaviors of the subsystems. Therefore, the knowledge on the behavior of subsystems, such as controllability, is extremely important.

Since the formal introducing in around 1960 by Kalman[16], the property of controllability of dynamical systems has been extensively studied. For linear deterministic dynamical systems describable by ordinary differential equation systems, the results were summarily given in Kalman, Ho, and Narendra[18].

Many of the known results can be used in our present research. Nevertheless, these are by no means complete and exhaustive. In order to make present analysis successful, it became quite clear that new knowledge on controllability must be obtained. Therefore, it is the primary objective of this chapter to carve out the results which are necessary for later development in the subsequent chapters.

In the present study, we shall understand by a linear dynamical system simply as a system which is describable by linear ordinary differential equations. In section 2 of this chapter, we shall give a brief survey of past researches on the notion of controllability for linear dynamical systems. In section 3 we shall continue research on the same notion for linear dynamical systems with deterministic structure. The concept of uniform controllability, which is of vital importance for later developments, is then introduced. In section 4 of this chapter, we shall focus our attention on linear dynamical systems subject to additive disturbances. In this latter case, the linear dynamical will be assumed to be controllable when the additive disturbance is absent.

2.2 Survey of Existing Results on Controllability for Linear Dynamical Systems

The important notion of controllability for dynamical systems was first formally introduced by Kalman[15] in 1959. Subsequently, in the first IFAC Congress held in Moscow , Kalman[16] presented a more complete study on controllability and its dual property -- observability for linear control systems. It should be understood that, based on different requirement, controllability can be considered either as structural property or behavioral property for a given dynamical system. This differentiation of viewpoint appears to be nonessential in the formal sense. Yet, its influence of the choice of tools for analysis should not be slighted.

In this section, exhaustive collection of existing results is not intended, we shall present only those results which are closely related to our present study. We follow closely the definitive publication of Kalman, Ho, and Narendra[18], and those of Antosiewicz[1] and Mitter[28].

For all subsequent analysis, an axiomatic approach of defining generalized dynamical systems (see for instance, Roxin[31]), is not attempted. It is understood that, by a linear dynamical system we mean a system having a mathematical model described by linear ordinary differential equation systems defined on a finite dimensional Euclidean space and the real line.

Let us consider the genral model of the system under study as described by

$$\dot{x} = A(t)x(t) + C(t)m(t) + f(t) \qquad (2.2\text{-}1)$$
$$x(0) = x_o$$

where x is an n-dimensional vector defined for all $t \geq 0$ such that x(t) at each time instant t is an element in the n-dimensional Euclidean space R^n; A(t) and C(t) are matrices of continuous functions with dimensions n x n and n x r respectively; m(t) is a vector in R^m defined on a compact interval $J = [0, T]$ in $R^+ = [0, \infty)$ such that $t \to m(t)$ is a vector in the normed linear space $\mathcal{L}_2$ or $\mathcal{L}_\infty$ of Lebesque integrable functions with norms defined respectively by

$$\|m\|_2 = \left(\int_0^T \|m(t)\|^2 dt\right)^{\frac{1}{2}}$$
$$\|m\|_\infty = \max_{1 \leq i \leq r} \left(\sup_{t \in J} |m_i(t)|\right) \qquad (2.2\text{-}2)$$

where $\|.\|$ denotes the usual Euclidean norm such that $\|x\| = (\sum_i x_i^2)^{\frac{1}{2}}$; f(t) is a vector in R^n defined on J and $t \to f(t)$ is a vector in the linear spaces $\mathcal{L}_2$ or $\mathcal{L}_\infty$. With these definitions the existance of a unique solution of (2.2-1), for any given triple (x_0, m, f), is quaranteed. And the fact that the solution x(t) defined on J is a vector in $\mathcal{L}_2$ or $\mathcal{L}_\infty$ becomes evident.

It is well known that the general solution of (2.2-1) is given by

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t, s)C(s)m(s)ds + \int_0^t \Phi(t, s)f(s)ds \qquad (2.2\text{-}3)$$

where $\Phi(t)$ is the fundamental matrix of the autonomous system $\dot{x} = A(t)x$ satisfying the matrix differential equation $\dot{X} = A(t)X$ with the initial condition $X(0) = I$.

To conform with usual terminology of control system theory, let us call R^n the state space, denoted by X, of the system, $x(t) \in X$ state of the system, m(t) the control function, and f(t) the disturbance function.

In words, the study of controllability is the effort in answering the fundamental question: "Starting from a given initial state $x_o \in X$, can the system (2.2-1) be steered to some desired state $x_d \in X$ in finite time by the application of some appropriate control function m(t)?". We give the formal definition on this concept as follows:

Definition 2.2-1 Given a desired state $x_d \in X$ and an inition state $x_o \in X$ at $t = 0$, the linear dynamical system (2.2-1) is said to be controllable if there exists some finite time $T > 0$ and some control function m(t) which transfer the system (2.2-1) from x_o to x_d at T.

Definition 2.2-2 Given an initial state $x_o \in X$, the linear dynamical system (2.2-1) is said to be completely controllable if there exists some finite $T > 0$ such that the system can be transferred from $x_o \in X$ to any desired state $x_d \in X$ at T.

Definition 2.2-3 Given a desired state $x_d \in X$ and an initial state $x_o \in X$, the linear dynamical system (2.2-1) is said to be <u>approximately controllable</u> or <u>ε-controllable</u> if there exists some $T > 0$ and some control function $m(t)$ such that $\|x(T) - x_d\| \leq \varepsilon$, where $\varepsilon > 0$ is a positive constant.

The concept of complete controllability is essentially a structural property for a given dynamical system. Even though its mathematical implication is quite important, its significance in engineering application can not be over-emphasized. As we shall see from the various theorems listed subsequently, the tool of analysis in the study of complete controllability has been mostly algebraic, which is not very useful in extending present results to dynamical systems other than the linear ones. Furthermore, it seems to be a reasonable doubt that such global property can not be meaningfully established for any dynamical system.

On the other hand, the concept of controllability may be considered to be a local property as well as a behavioral property. In the present research, it is more fruitful to look at this concept as a behavioral property. The available tools for studying the problems are varied, but follow categorically two basic lines, namely, algebraic methods and topological methods.

The inclusion of the concept of approximate controllability in the present study is most significant. First of all, we feel that this concept is closely associated with the real world because engineering applications invariably involve some degree of tolerance or approximation. Furthermore, the concept helps significantly to reduce difficulties encountered in mathematical analysis.

Following the algebraic approach, we have these fundamental results: The classical theorem was obtained for a linear time-invariant dynamical system under no additive disturbance.

Theorem 2.2-1 Let $A(t)$, $C(t)$ be constant matrices and $f(t) = 0$ for $t \geq 0$. Then system (2.2-1) is completely controllable if and only if the matrix

$$Q = [A, AC, \ldots, A^{n-1}C]$$

is of rank n. Moreover, if the system is completely controllable, any state can be transferred to any other state in an arbitrarily short interval of time.

For convenience, let us introduce the symmetric, non-negative definite matrix

$$W(T) = \int_0^T \Phi(t)C(t)C'(t)\Phi'(t)dt \qquad (2.2\text{-}4)$$

Then, for a time-varying system, we have

Theorem 2.2-2 Let $f(t) = 0$ for $t \geq 0$. The system (2.2-1) is completely controllable if and only if, for some finite $T > 0$, the matrix $W(T)$ is positive definite.

Theorem 2.2-3 Let $f(t) = 0$ for $t \geq 0$. The system (2.2-1) is controllable if and only if the vector $\Phi(T)x_d - x_o$ is an element in the range of $W(T)$.

Now, if we let $W^{\#}(T)$ to denote the pseudo-inverse of the matrix $W(T)$, which reduces to the ordinary inverse $W^{-1}(T)$ when $W(T)$ is nonsingular. Then,

Theorem 2.2-4 Let $f(t) = 0$ for $t \geq 0$. The system (2.2-1) is controllable if and only if the condition

$$[I - W(T)W^{\#}(T)][\Phi(T)x_d - x_o] = 0 \qquad (2.2\text{-}5)$$

is satisfied, in which case the control

$$m^o(t) = C'(t)\Phi'(t)W^{\#}(T)[\Phi(T)x_d - x_o] \qquad (2.2\text{-}6)$$

will accomplish the desired transfer.

Theorem 2.2-5 The minimum control energy E required to transfer the system (2.2-1), when $f(t) = 0$ for $t \geq 0$, from x_o to x_d at time $T < \infty$ is given by

$$E = \int_0^T \|m^o(t)\|^2 dt = \|\Phi(T)x_d - x_o\|_{W^{\#}(T)}$$

$$= (\Phi(T)x_d - x_o)'W^{\#}(T)(\Phi(T)x_d - x_o) \qquad (2.2\text{-}7)$$

By following the methods of topological analysis, different conditions were obtained which quarantee similar results. First, let us introduce the following notations

$$z_1(T) = x_d - \Phi(T)x_o + \int_o^T \Phi(T,t)f(t)dt \qquad (2.2\text{-}8)$$

Let us also define the n x r matrix

$$V(t,T) = \Phi(T,t)C(t)$$

It is now appropriate to introduce the space of admissible controls M by defining

$$M = \{m(t): \|m\|_{2 \text{ or } \infty} \leq k, \ t \in J\} \qquad (2.2\text{-}9)$$

where $k > 0$ is a constant.

Let X^* be the conjugate space of X. Then, by using the separation property of convex sets in a separable space such as Euclidean spaces, the fundamental theorem was obtained.

<u>Theorem 2.2-6</u> Given initial state x_o and desired state x_d, the system (2.2-1) is approximately controllable (ε-controllable) with respect to the space M of admissible controls if and only if, for every $x^* \in X^*$, the following inequality holds

$$\langle z_1(T), x^* \rangle - \varepsilon\|x^*\| \leq k\left(\int_o^T \|x^* V(t,T)\|^2 dt\right)^{\frac{1}{2}}$$

<u>Corollary 2.2-7</u> If the system (2.2-1) is approximately controllable with respect to M and (x_o, x_d), then there

exists a least compact time interval $J_o = [0, T_o]$ in R^+ such that system (2.2-1) is approximately controllable with respect to M and (x_o, x_d).

Theorem 2.2-6 apparently includes the effects of the disturbace function f(t). However, explicit utilization of this theorem would be very difficult when f(t) is not known a priori. A theorem which would estimate the effects of additive disturbance on a deterministic system appears to be much useful in our analysis. So, let us introduce the disturbance set Ω_T, which is necessary for the statement of the following theorem, by defining:

$$\Omega_T = \{\omega(T)\epsilon X: \|\omega(T)\| \leq \sigma \text{ and } \omega(T) = \int_0^T \Phi(T,t)f(t)dt\} \tag{2.2-10}$$

<u>Theorem 2.2-8</u> Let system (2.2-1) be approximately controllable with respect to M and (x_o, x_d) when $f(t) = 0$ for $t \geq 0$. Then the perturbed system is ε-controllable with respect to M and (x_o, x_d) if and only if

$$\sigma \leq \varepsilon - \max(0, \sup\{|\langle x_d - x_o, x^*\rangle - k(\int_0^T \|x^* V(t,T)\|^2 dt)^{\frac{1}{2}};\ \|x^*\| = 1\})$$

2.3 Controllability for Disturbance-Free Linear Dynamical Systems

In order to answer certain questions on controllability of two-level linear dynamical systems, which is still to be defined in next chapter, we found that there are several questions to be answered on controllability of linear dynamical systems such as (2.2-1). The problems can be put into two main categories: (1) further understandings on controllability of disturbance-free systems, which we shall try to answer in this section; and (2) the effects of disturbance on controllability of deterministic linear dynamical systems, which we shall try to answer in next section.

Therefore, we shall assume throughout this section that $f(t) = 0$ for $t \geq 0$, and system (2.2-1) reduces to the form

$$\dot{x} = A(t)x(t) + C(t)m(t) \qquad x(0) = x_o \qquad (2.3\text{-}1)$$

As we have mentioned previously, the concept of complete controllability is essentially a structural property for a given linear dynamical system. In other words, the conditions of complete controllability depends only on system structure characterized by system

Matrix A(t) and control matrix C(t). From theorem 2.2-2, we know that the system is completely controllable if and only if there is some $0 < T < \infty$ such that the matrix W(T) is positive definite. A very difficult question to answer is how to determine the least of such T. Another open question is this: For some given fixed time $T < \infty$ and fixed system matrix A(t), can system (2.3-1) be made completely controllable at T by manipulating the control matrix C(t)? If this is so, is there a simple algorithm of finding an appropriate control matrix C(t) such that (2.3-1) is completely controllable at T? We shall try to answer the latter question.

We emphasize particularly on the manipulation of the control matrix C(t) only because it would not always be permissible in practical applications to change the system characteristics by altering A(t).

<u>Proposition 2.3-1</u> Let system (2.3-1) be given. A sufficient condition for positive-definiteness of the matrix W(T) is that the control matrix C(t) be square and nonsingular for some compact interval $[t_1, t_2]$ in R^+ where $0 \le t_1 < t_2 \le T$.

<u>Proof</u> It is well known that the fundamental matrix $\Phi(t)$ is nonsingular for all $t > 0$. Thus, as a consequence of the assumption, the matrix $\Phi(t)C(t)C'(t)\Phi'(t)$ is nonsingular on $[t_1, t_2]$. Since $\Phi(t)C(t)C'(t)\Phi'(t)$

is also symmetric, this implies that the matrix is positive-definite for each $t \in [t_1, t_2]$ and positive-semidefinite for all other $t \geq 0$. Now we must show that $\langle x, W(T)x \rangle > 0$ for any $x \neq 0$. Clearly

$$\begin{aligned}\langle x, W(T)x\rangle &= \int_0^T \langle x, \Phi(t)C(t)C'(t)\Phi'(t)x\rangle dt \\ &= \int_0^{t_1} \langle x, \Phi(t)C(t)C'(t)\Phi'(t)x\rangle dt \\ &\quad + \int_{t_1}^{t_2} \langle x, \Phi(t)C(t)C'(t)\Phi'(t)x\rangle dt \\ &\quad + \int_{t_2}^{T} \langle x, \Phi(t)C(t)C'(t)\Phi'(t)x\rangle dt\end{aligned}$$

The second term in this equality is apparently greater than zero while the other two terms are non-negative. This completes the proof.

The fact that $C(t)$ be square is not necessary for the positive-definiteness of $W(T)$ is easily disclosed by following reasoning: Let P, Q be any nonsingular square matrices and A be any square matrix of appropriate dimensions. It is well known that the matrices PAQ and A has the same rank. It is also known that, for any rectangular matrix A, the ranks of the matrices A, A'A, and AA' are the same. As a consequence, let $C(t)$ be a rectangular matrix of dimension $n \times r$ with $n > r$. Then the rank ρ of the matrix $\Phi(t)C(t)C'(t)\Phi'(t)$ is given by $\rho(\Phi(t)C(t)C'(t)\Phi'(t)) = \rho(C(t)C'(t)) = \rho(C(t)) \leq r < n$. In other words, the matrix $\Phi(t)C(t)C'(t)\Phi'(t)$ can not be positive-definite. However, we do know that there are linear dynamical systems of the form (2.3-1) with rectangular C(t) which are completely controllable.

A different scheme appears also useful in manipulating the control matrix C(t) so that a resulting W(T) would be assured to be positive-definte. Let us first prove the following result.

<u>Lemma 2.3-2</u> Given any symmetric matrix A there exists a scalar α such that the matrix $A + \alpha I$ is positive definite.

<u>Proof</u>: By definition, we must show that there exists an α such that $\langle x, (A + \alpha I)x\rangle > 0$ for any $x \neq 0$. Since $A = A'$, we know there is an orthogonal matrix P with the properties $PP' = I$ and $P'AP = \Lambda$, where Λ is a diagonal matrix. Let $x = Py$, then

$$\begin{aligned}\langle x, (A + \alpha I)x\rangle &= \langle Py, (A + \alpha I)Py\rangle \\ &= \langle y, (\Lambda + \alpha I)y\rangle \\ &= \langle y, \Lambda y\rangle + \alpha\langle y, y\rangle \\ &= \sum_{i=1}^{n} \lambda_i y_i^2 + \alpha \sum_{i=1}^{n} y_i^2 \\ &\geq \lambda^* \sum_{i=1}^{n} y_i^2 + \alpha \sum_{i=1}^{n} y_i^2\end{aligned}$$

Where

$$\lambda^* = \min_{1\le i\le n} (\lambda_i)$$

Let $\alpha > |\lambda^*|$, we have the proof.

Now we shall develope the scheme by considering two different cases.

Case 1. $C(t)$ is a square matrix but is singular on some subset $S \in [0, T]$ of non-zero measure.

Let us define a new control matrix

$$C_1(t) = C(t) + \alpha I \tag{2.3-2}$$

Where α is constant to be determined. Then

$$C_1(t)C_1'(t) = C(t)C'(t) + \alpha(C(t) + C'(t)) + \alpha^2 I$$

And

$$\begin{aligned} W_1(T) &= \int_0^T \Phi(t)C_1(t)C_1'(t)\Phi'(t)dt \\ &= W(T) + \alpha\int_0^T \Phi(t)(C(t) + C'(t))\Phi'(t)dt \\ &\quad + \alpha^2 \int_0^T \Phi(t)\Phi'(t)dt \end{aligned} \tag{2.3-3}$$

From the way we define the new control matrix $C_1(t)$, the fact that matrix $W_1(T)$ can be made positive-definite is quite obvious because it is always possible to select a constant α such that $C_1(t)$ is nonsingular. And the positive-definiteness of $W_1(T)$ follows from proposition 2.3-1. This fact is also enhanced by looking at (2.3-3), we realized that the last term is always positive-definite, the first term is non-negative-definite, and the second term is symmetric. Thus, if α is made to be sufficiently large, the matrix $W_1(T)$ will be positive-definite.

<u>Case 2</u> The control matrix C(t) is a general $n \times r$ continuous matrix.

As a direct application of Lemma 2.3-2 a quite trivial algorithm, which would yield a positive-definite matrix, can be established. The essential scheme is to modify the original control matrix C(t) in such a way that a new positive-definite matrix $C_1(t)$ is obtained on $[0, T] \subset R^+$.

Let us define

$$C_1(t) = \begin{pmatrix} C(t)_{n\times r} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{pmatrix}_{n\times n} + \begin{pmatrix} C(t)_{n\times r} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{pmatrix}_{n\times n} + \alpha I \qquad (2.3\text{-}4)$$

Let us denote the eigenvalues of matrix $C_1(t) - \alpha I$ by $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$. Let

$$\lambda^*(t) = \max_{1\leq i\leq n} |\lambda_i(t)| \tag{2.3-5}$$

And, on some finite time interval $[t_1, t_2]$, let

$$\alpha = \lambda^* = \sup_{t\in[t_1,t_2]} |\lambda^*(t)| \tag{2.3-6}$$

Then, by Lemma 2.3-2, the matrix $C_1(t)$ is positive-definite and nonsigular for all $t \in [t_1, t_2]$. As we can see, Case 1 is in fact a special case of the above.

To summarize, we have

Proposition 2.3-3 For a given linear system (2.3-1) it can always be made completely controllable by modfying the control matrix C(t) in a way described by formulii (2.3-4,5,6).

As we have repeatedly mentioned, complete controllability is a structural property, thus, this property is not affected by outside disturbances such as control function m(t). It is well known that, in order to transfer the system (2.3-1), when it is completely controllable, from any point in the state space to the origin, infinite control energy will be required. When available control energy is limited, the part of state space can be reached by a system of the form (2.3-1) from a given initial state is also limited. Based on geometrical consideration, it would be interesting and meaningful to answer some of the questions relating the definitions of control energy, attainable states, etc.

Let us first introduce the definitions

<u>Definition 2.3-4</u> <u>The space M of admissible controls</u> is the set

$$M = \{ m(t) : \|m\|_{2 \text{ or } \infty} \leq k, \; t \in J \}$$

such that $M \subset \mathcal{L}_2$ or $\mathcal{L}_\infty$ is a proper subset of $\mathcal{L}_2$ or $\mathcal{L}_\infty$.

Most of the results to follow will be true when M is a subset of either $\mathcal{L}_2$ or $\mathcal{L}_\infty$. Therefore, if it is not particularly mentioned, the stated results will be true for both cases.

<u>Definition 2.3-5</u> Let the system (2.3-1) be controllable. Let the space M of admissible controls be given. <u>The space X_o of admissible initial states</u> is the collection of points in the state space such that the system can be transferred from any point in X_o to the origin in finite time T using only the control functions in M.

Some of the interesting and valuable properties of X_o can easily be deduced.

<u>Proposition 2.3-6</u> The set X_o has the following properties:

(i) X_o is convex

(ii) X_o is symmetric with respect to origin.

(iii) Let $X_o(T_1)$ and $X_o(T_2)$ be respectively the spaces of admissible initial states for T_1 and T_2, and $T_2 > T_1$. Then, $x_o \in X_o(T_1)$ implies $x_o \in X_o(T_2)$.

Proof: We shall only show the case when M is a subset of $\mathcal{L}_2$.

(i) Let $x_{o1} \in X_o$ and $x_{o2} \in X_o$ correspond to some $m_1(t) \in M$ and $m_2(t) \in M$ respectively. Let $m_o(t) = \alpha m_1(t)+(1-\alpha)m_2(t)$, where $0 \le \alpha \le 1$. Then,

$$
\begin{aligned}
\left(\int_o^T \|m_o(t)\|^2 dt\right)^{\frac{1}{2}} &= \left(\int_o^T \|\alpha m_1(t) + (1-\alpha)m_2(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\le \left(\int_o^T (\alpha \|m_1(t)\| + (1-\alpha)\|m_2(t)\|)^2 dt\right)^{\frac{1}{2}} \\
&\le \alpha \int_o^T \|m_1(t)\|^2 dt + (1-\alpha)\int_o^T \|m_2(t)\|^2 dt \\
&\le \alpha k + (1-\alpha)k = k
\end{aligned}
$$

Thus $m_o(t)$ is admissible. By hypothesis, we know

$$0 = \Phi(T)x_{o1} + \int_o^T \Phi(T,s)C(s)m_1(s)ds$$

$$0 = \Phi(T)x_{o2} + \int_o^T \Phi(T,s)C(s)m_2(s)ds$$

Let $x_{oo} = \alpha x_{o1} + (1-\alpha)x_{o2}$. Then, at time T and some $m^* \in M$, the solution of (2.3-1) gives

$$x(T) = \Phi(T)x_{oo} + \int_o^T \Phi(T,s)C(s)m^*(s)ds$$

By picking $m^*(t) = m_o(t)$ on $[0, T]$, we have

$$x(T) = \Phi(T)(\alpha x_{o1} + (1-\alpha)x_{o2})\int_o^T \Phi(T,s)(\alpha m_1(s) + (1-\alpha)m_2(s))ds$$

$$= \alpha(\Phi(T)x_{o1} + \int_o^T \Phi(T,s)C(s)m_1(s)ds) + (1-\alpha)(\Phi(T)x_{o2} + \int_o^T \Phi(T,s)C(s)m_2(s)ds)$$

$$= 0$$

Thus we have shown that X_o is convex, i.e., x_{oo} ia also an element of X_o.

To show (ii), we notice for any $x_o \in X_o$ corresponding to some $m_o(t) \in M$ there is another admissible control function $-m_o(t) \in M$, corresponding to which there is $-x_o \in X_o$. Thus, the conclusion follows from the definition of a symmetric set.

To show (iii), let $m_o(t) \in M$ be chosen such that system (2.3-1) is transferred from $x_o \in X_o$ to the origin at time T_1. Defining

$$m_o^*(t) = \begin{cases} m_o(t) & 0 \leq t \leq T_1 \\ 0 & T_1 < t \leq T_2 \end{cases}$$

it is obvious that $m_o^*(t)$ is admissible. It is well known that the fundatmental matrix can be written in the form $\Phi(t, s) = \Psi(t)\Psi^{-1}(s)$. Thus, by hypothesis,

$$x(T_1) = \Psi(T_1)(x_o + \int_o^{T_1} \Psi^{-1}(t)C(t)m_o(t)dt) = 0$$

Since $\Psi(T_1)$ is nonsingular, this implies that

$$x_o + \int_o^{T_1} \Psi^{-1}(t)C(t)m_o(t)dt = 0$$

Consequently, since

$$x(T_2) = \Psi(T_2)x_o + \Psi(T_2)\int_o^{T_2} \Psi^{-1}(t)C(t)m_o^*(t)dt$$

We have

$$\Psi^{-1}(T_2)x(T_2) = x_o + \int_o^{T_1} \Psi^{-1}(t)C(t)m_o(t)dt = 0$$

i.e. $x_o \in X_o(T_2)$.

This completes the proof.

For any given space of admissible controls, the space of admissible initial states will in general not assume a regular geometric form such as a circular disk in a two-dimensional case. In order to eliminate many of the details which may cause formidable difficulties in analysis, we might as well assume a well defined geometric shape for X_o in the state space, and in turn try to estimate the minimum control energy required to transfer the system to the origin from any initial state in this assumed X_o. Therefore, let us define, from now on, the space of admissible initial states as

$$X_o = \{ x_o : \|x_o\| \leq k_o,\ x_o \in X \} \tag{2.3-7}$$

Let the control energy be defined by

$$E = \int_o^T \|m(t)\|^2 dt \tag{2.3-8}$$

From Proposition 2.2-5, we know that, if the system (2.3-1) is transferrable from an initial state x_o to a desired state $x_d \in X$ in finite time, then the minimum control energy required to do the transfer is given by

$$E = \int_o^T \|m(t)\|^2 dt = \|\Phi(T)x_d - x_o\|^2_{W^{\#}(T)} \tag{2.3-9}$$

where $W^{\#}(T)$ is the generalized inverse of the matrix $W(T)$. For the sake of simplicity, let us assume that system (2.3-1) is completely controllable. This assumption reduces $W^{\#}(T)$ into its ordinary form $W^{-1}(T)$, which is also symmetric and positive-definite. Let us also define the target set as

$$X_d = \{ x_d \in X : \|x_d\| \leq \varepsilon \} \tag{2.3-10}$$

where $k_o > \varepsilon$. We shall now say that system (2.3-1) is controllable if it is transferrable from an initial state to the target set in finite time. Clearly, if the initial state chosen is also an element of X_o, this requirement is fulfilled automatically. Thus, we shall assume that x_o is taken from $X_o - X_d$. When X_o and X_d are given, the task remained is the estimation of minimum control energy required to transfer system (2.3-1) from any $x_o \in X_o$ to the target set in T. Let ∂S denotes the set of boundary points of a given set S. Then, we have:

<u>Proposition 2.3-7</u> Let system (2.3-1) be completely controllable when M is not bounded. Let X_o and X_d be given. Then system (2.3-1) is transferrable from any $x_o \in X_o$ to the target set X_d if and only if the space of admissible controls $M = \{m(t): \|m\|_2 \leq k,\ t \in J\}$ satisfies

$$k \geq \max_{x_o \in \partial X_o} \ \min_{x_d \in \partial X_d} \|\Phi(T)x_d - x_o\|_{W^{-1}(T)}$$

<u>Proof Necessity</u> Pick any $x_o \in X_o - X_d$. Let m(t) be an admissible control function chich transfer the system from x_o to the target set at T. Since X_d is convex and compact and the trajectory x(t) of (2.3-1) is continuous, x(t) will first touch X_d on its boundary ∂X_d. Thus, picking any point x_d on ∂X_d, $E = \|\Phi(T)x_d - x_o\|_{W^{-1}(T)}$ is

the minimum energy required to transfer (2.3-1) from x_o to x_d at T. Since E is a continuous function of x_d, by taking infimum over ∂X_d on E, we know that $E_1 = \min_{x_d \in \partial X_d} \| \Phi(t)x_d - x_o \|_{W^{-1}(T)}$ is the minimum control energy required to transfer (2.3-1) from x_o to X_d at T. Obviously, $E_2 = \max_{x_o \in X_o} (E_1)$ is the minimum control energy required to transfer (2.3-1) from any $x_o \in X_o$ to X_d at T. Now, we must show that $E_2 = \max_{x_o \in \partial X_o} E_1$ Suppose not, then there exists some $\bar{x}_o \in X_o$ and $\bar{x}_o \notin \partial X_o$ such that $E_2 = \| \Phi(T)x_d - \bar{x}_o \|_{W^{-1}(T)}$. But this is impossible because X_o is also a convex and compact set in X. This completes the proof of necessity. Now the sufficiency part is evident, because we know that the worst case, i.e. the case when most control energy is need, will happen only when $x_o \in \partial X_o$.

Equiped with this proposition, an estimate of required control energy when x_o and X_d are given is possible, which constitutes an indispensable basis for coordination to be studied in the later chapters.

The notion of controllability considered so far is basically an on-off property in the sense that a single time instant is of importance. Thus, it is possible for an unstable system to be controllable, which is apparantly undesirable in practice. This liberty is particularly objectionable for some of the problems to be considered later. In order to eliminate this fallibility we shall do as follows:

Definition 2.3-8 Let system (2.3-1) be controllable at time T with respect to a given space M of admissible controls and a pair of states (x_o, x_d). Then it is said to be uniformly controllable for $t \geq T$ if it is controllable for all $t \geq T$, i.e., there exists m(t) in M such that the corresponding solution $x(t) = x_d$ of (2.3-1) for all $t \geq T$.

Clearly, the above property will not be obtained for any unstable system. Kalman in an early study of linear filtering problem [17] used the concept of uniformly complete controllability to serve a similar purpose. For the sake of continuity and completeness, several well-known definitions will be introduced in the following.

Definition 2.3-9 The solution $x(t) = 0$ on R^+ of the autonomous system $\dot{x} = A(t)x$, denoted by $\bar{0}$, is called the trivial solution.

Definition 2.3-10 The zero state $x = 0$ is said to be asymptotically stable for the linear system (2.3-1) if (i) given any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x_o\| \le \delta \rightarrow \|x(t)\| \le \epsilon$ for all $t \in R^+$, and (ii) for a sufficiently small $\delta > 0$, $\|x_o\| \le \delta$ implies $x(t) \rightarrow 0$ as $t \rightarrow \infty$. It is uniformly asymptotically stable if it is asymptotically stable for all $x_o \in X$.

Definition 2.3-11 The autonomous system $\dot{x} = A(t)x$ is said to be asymptotically stable whenever its zero state is asymptotically stable. It is uniformly asymptotically stable if the zero state is uniformly asymptotically stable.

Definition 2.3-12 The nonautonomous system (2.3-1) is said to be bounded-input bounded-output stable (b.i.b.o. stable) if, for all $x_o \in X$ and for all uniformly bounded input $m(t)$ defined on R^+, the state function solution $x(t)$ of (2.3-1) is uniformly bounded on R^+.

In addition, we shall state without proof the following well known theorems on stability for linear differential systems:

Lemma 2.3-13 The linear autonomous system $\dot{x} = A(t)x$ is uniformly asymptotically stable if and only if there exists constants $\alpha_1 > 0$, $\alpha_2 > 0$ such that $\|\Phi(t)\| \le \alpha_1 e^{-\alpha_2 t}$

for $t \geq 0$.

<u>Lemma 2.3-14</u> The linear system (2.3-1) is b.i.b.o. stable if and only if

(i) there exists a constant b_1 such that $\|\Phi(t)\| \leq b_1$ for $t \geq 0$.

(ii) there exists a constant b_2 such that $\int_0^t \|\Phi(t,s)C(s)\| ds \leq b_2$ for $t \geq 0$.

<u>Lemma 2.3-15</u> The linear system (2.3-1) is b.i.b.o. stable if and only if its autonomous part is uniformly stable.

These theorems in fact tell us that the conditions of b.i.b.o. stability, asymptotical stability, and uniformly asymptotical stability are all equivalent for linear system (2.3-1). As a consequence of this fact, it is now not unduly difficult to prove the theorems on uniform controllability for linear systems such as those described by (2.3-1).

<u>Proposition 2.3-16</u> Let the linear system (2.3-1) be controllable at time T_o with respect to a given space M of admissible controls and the pair of states $(x_o, 0)$. Then it is uniformly controllable if and only if it is b.i.b.o. stable.

<u>Proof:</u> Necessity is evident. For sufficiency, it is known by hypothesis that there is a $m(t) \in M$ defined on $J_o = [0, T_o]$ which will transfer the system from x_o to the origin at time T_o. We claim that the control

$$m_1(t) = \begin{cases} m(t) & \text{for } t \in J_o \\ 0 & \text{for } t > T_o \end{cases}$$

will make the system uniformly controllable. This is true because the assumption of b.i.b.o. stability implies that the complimentary solution of (2.3-1) (i.e., the solution of $\dot{x} = A(t)x$) will be the trivial solution for $t > T_o$. Therefore we have $x(t) = 0$ for $t > T_o$. Notice also that $m_1(t)$ is admissible. This completes the proof.

Since the assumption of b.i.b.o. stability for system (2.3-1) implies that its homogeneous part is uniformly asymptotical stable, the requirement that $x_d = 0$ seems to be quite artificial. It would look much natural if we could extend the above results to cases where $x_d = 0$ is not necessary. One way to accomplish the stated intention is the use of some well known geometrical properties of the set $L(T, M)$ of attainable states, which is defined as follows:

<u>Defintion 2.3-17</u> For system (2.3-1) with a given space M of admissible controls the <u>set of attainable states</u> at some time T is defined as:

$$L(T, M) = \{\theta(T) \in X: \theta(T) = \int_0^T \Phi(T, s)C(s)m(s)ds, \ m \in M\}$$

<u>Lemma 2.3-18</u> (i) the set $L(T, M)$ is symmetric with respect to the origin and convex.

(ii) $L(T_1, M) \subset L(T_2, M)$ if $T_1 < T_2$.

This lemma leads to the following results.

<u>Proposition 2.3-19</u> Let the linear system (2.3-1) be controllable at T with respect to a given space M of admissible controls and pair of states (x_o, x_d). Then it is umiformly controllable if

(i) $\|x_d - \Phi(t)x_o\|$ is a monotonically decreasing function of t.

(ii) $\|x_d - \Phi(T)x_o\| \leq \min\limits_{\theta(T) \in \partial L(T, M)} \|\theta(T)\|$

<u>Proof:</u> We must show that, by definition, $x_d - \Phi(t)x_o \in L(T, M)$ for $t \geq T$. Since $L(T, M) \subset L(T_1, M)$ whenever $T < T_1$ by the above lemma, it suffices to show that $x_d - \Phi(t)x_o \in L(T, M)$ for $t \geq T$. Conditions (i) and (ii) ensure that $\|x_d - \Phi(t)x_o\| \leq \min\limits_{\theta(T) \in \partial L(T, M)} \|\theta(T)\|$

for all $t \geq T$. Since $L(T, M)$ is symmetric with respect to the origin, this implies that

$x_d - \Phi(t)x_o \in L(T, M) \subset L(t, M)$ for all $t \geq T$, which completes the proof.

<u>Corollary 2.3-20</u> Let system (2.3-1) be controllable at

time T with respect to given space M of admissible controls and a pair of states (x_o, x_d). Let the solutions of (2.3-1) be uniformly bounded on R^+. Then system (2.3-1) is uniformly controllable if

$$\|x_d\| + \sup_{t \geq T} \|\Phi(t)x_o\| \leq \min_{\theta(T) \in \partial L(T, M)} \|\theta(T)\|.$$

Proof: Similar to the proof of Proposition 2.3-19, it suffices to show that

$$\|x_d - \Phi(t)x_o\| \leq \min_{(T) \in \partial L(T, M)} \| (T)\| \text{ for all } t \geq T.$$

But $\|x_d - \Phi(t)x_o\| \leq \|x_d\| + \|\Phi(t)x_o\| \leq \min_{\theta(T) \in \partial L(T,M)} \|\theta(T)\|$

by hypothesis. This completes the proof.

It can be shown that the above condition can not be improved in general without imposing further constraints on the behavior of the state functions x(t) of system (2.3-1).

Corollary 2.3-21 Let system (2.3-1) be controllable at time T with respect to a given M and (x_o, x_d). Then it is uniformly controllable for all $t \geq T$ if

(i) the system is b.i.b.o. stable.

(ii) $\|x_d\| + \alpha_1\|x_o\| \leq \min_{\theta(T) \in \partial L(Y, M)} \|\theta(T)\|$, where

α_1 is a constant depending on the matrix A(t).

Where condition (i) is also necessary.

Proof: The assumption of b.i.b.o. stability for system (2.3-1) ensures that the solution x(t) of $\dot{x} = A(t)x$ has the property $\|x(t)\| \le \alpha_1 \|x_o\| e^{-\alpha_2 t}$, for all $t \ge 0$ where α_1, α_2 are positive constants depending on A(t). Clearly, $\sup_{t \ge T} \|\Phi(t)x_o\| \le \alpha_1 \|x_o\|$. This completes the proof.

One of the common constraints which might be imposed on the state functions of system (2.3-1) is the non-oscillatory behavior. Let us introduce the following:

Definition 2.3-22 A time function x(t) taking its value in R^n is said to non-oscillatory if it satisfies the following conditions:

(i) the absolute values of its component functions $|x_i(t)|$, i = 1, 2, ..., n are monotonically decreasing functions of time.

(ii) $\mathrm{sgn}\,(x_i(t)) = \mathrm{sgn}\,(x_i(0))$ for all $t \ge 0$, $i = 1,2,\ldots, n$.

Where sgn denotes the usual sign function, i.e., for $a \in R$, $\mathrm{sgn}\,(a) = -1$ if $a < 0$, $\mathrm{sgn}\,(a) = 1$ if $a > 0$.

Corollary 2.3-23 Let system (2.3-1) be controllable at time T with respect to a given M and (x_o, x_d). Then it is uniformly controllable for $t \ge T$ if

(i) the solution of $\dot{x} = A(t)$ is non-oscillatory.

(ii) $\|x_d\| + \|\Phi(T)x_o\| \le \min_{\theta(T) \in \partial L(T, M)} \|\theta(T)\|$

When x_d is not the origin, we would intuitively expect that such departure from the origin might be corrected by employing excessive control energy. This intuition would be true at least for many linear systems. In here, we shall say that a linear system has some excessive control energy available to him when it is controllable with respect to a subset of a given space M of admissible controls.

Proposition 2.3-24 Let system (2.3-1) be controllable at T with respect to a given M and (x_o, x_d). Then it is uniformly controllable for $t \geq T$ if

(i) the system is b.i.b.o. stable.

(ii) $\Phi(t)x_o \in L(t, M_1)$ for $t \geq T$ and $x_d \in L(T, M_2)$, where

$$M \supset M_1 = \{m(t) : \|m\|_{2 \text{ or } \infty} \leq k_1\},$$

$$M \supset M_2 = \{m(t) : \|m\|_{2 \text{ or } \infty} \leq k_2\} \text{ and } k_1 + k_2 \leq k.$$

Where condition (i) is also necessary.

Proof; Let $m_o(t) \in M$ be a control function which transfer system (2.3-1) from x_o to x_d at time T. Let $m_1(t) \in M_1$ be such that $-\Phi(t)x_o = \int_o^t \Phi(t, s)C(s)m_1(s)ds$ for $t \geq T$, which is possible by hypothesis (ii). Since $L(T, M_2) \subset L(t, M_2)$ for any $t \geq T$ and $x_d \in L(T, M_2)$, we may select $m_2(t) \in M_2$ such that

$x_d = \int_o^t \Phi(t, s)C(s)m_2(s)ds$ for $t \geq T$. The control function

$$m(t) = \begin{cases} m_o(t) & t \in J \\ m_1(t) + m_2(t) & \text{for } t \geq T \end{cases}$$ which is admissible

to M, will accomplish the goal of uniform controllability.

<u>Corollary 2.3-25</u> Let system (2.3-1) be controllable at time T with respect to a given M and (x_o, x_d). Then it is uniformly controllable for $t \geq T$ if

(i) the system is b.i.b.o. stable.

(ii) $\sup_{t \geq T} \|\Phi(t)x_o\| \leq \min_{\theta(T) \in \partial L(T, M_1)} \|\theta(T)\|$

and $\|x_d\| \leq \min_{\theta(T) \in \partial L(T, M_2)} \|\theta(T)\|$ and $k_1 + k_2 \leq k$.

2.4 Controllability of Linear Dynamical Systems under Additive Disturbances

The theory of controllability for disturbance free linear dynamical systems has been extensively studied. Many important known results were summerized in section 2 of this chapter. There has been also studies on controllability of systems other than deterministic linear dynamic systems. For instance, there are studies on non-linear systems by Hermes[14] and Marcus[24] and on stochastic systems by Connors[9]. Yet, there seems to have very little published studies on controllability on dynamic systems which are deterministic but are under the influence of disturbances of finite magnitude not known a priori. Since it will become clear in the subsequent chapters that such problems are to play a central role in the study on controllability of two-level linear dynamical systems, at least a partial solution of the problem mentioned is eminent.

Let us first pose the problem to be studied. Consider the linear dynamical system:

$$\dot{x} = A(t)x(t) + C(t)m(t) + f(t) \qquad (2.4\text{-}1)$$

$$M = \{m(t) : \|m\|_{2 \text{ or } \infty} \leq k,\ t \in J\} \quad x(0) = x_o$$

$$\dot{x} = A(t)x(t) + C(t)m(t) \qquad (2.4\text{-}1')$$

The above system is assumed to have the same definition as in section 2.2.

It is important at this moment to emphasize that we consider the additive function f(t) as perturbation on the disturbance-free system (2.4-1'). Consequently, we shall assume in the beginning that system (2.4-1') possesses certain properties outlined in previous sections and in turn ask what are the effects of the function f(t) on those properties.

A very interesting question to be answered is whether a completely controllable disturbance free linear dynamical system, such as (2.4-1), is still completely controllable under the influence of f(t) which is uniformly bounded on R. As we have mentioned earlier, complete controllability is primarily a property concerning the nature of systems structure. A necessary condition for a system such as (2.4-1') to have such property is that the space of admissible controls be unbounded. In other words, regarding to our definition, the space of admissible controls will have to be either of the entire function spaces $\mathcal{L}_2$ or $\mathcal{L}_\infty$.

In order to answer this question let us assume that f(t) is also a vector in $R^n = X$ while $f : t \to f(t)$

is a function in either one of the spaces of Lebesque integrable functions $\mathcal{L}_2$ or $\mathcal{L}_\infty$ with corresponding norms defined by formula (2.2-1). We shall now say that the function f(t) is uniformly bounded whenever it is norm-bounded. By so doning, we see that the set of disturbance functions will have the same definition as the space of admissible controls. This is of course done intentionally to simplify the problems.

Proposition 2.4-1 Let system (2.4-1') be completely controllable at time T when the space M of admissible controls is the whole space $\mathcal{L}_2$ or $\mathcal{L}_\infty$. Then the system (2.4-1) is also completely controllable at T when f(t) is uniformly bounded.

Proof: We shall prove this proposition via constructive procedure. The general solution of (2.4-1) is given by

$$x(t) = \Phi(t)x_o + \int_o^t \Phi(t, s)C(s)m(s)ds + \int_o^t \Phi(t, s)f(s)ds$$

It is known that, following the assumption of complete controllability, for any given pair of states (x_o, x_d), there is an admissible control function $m_o(t)$ defined on J such that $x_d = \Phi(T)x_o + \int_o^T \Phi(T, s)C(s)m_o(s)ds$.

Since f(t) is also a function in $\mathcal{L}_2$ or $\mathcal{L}_\infty$, the last term $x_f = \int_0^T \Phi(T,s)f(s)ds$ is nothing but an element in X. Since system (2.4-1') is assumed to be completely conrollable, it is always possible to pick an $m_1(t)$ defined on J such that $-x_f = \int_0^T \Phi(T,s)C(s)m_1(s)ds$. In fact, the choice of $m_1(s)$ such that $C(s)m_1(s) = f(s)$ a. e., on J will do. This is possible because C(s) is a continuous operator by assumption. Clearly, given any (x_o, x_d), the control function $m(t) = m_o(t) + m_1(t)$ defined on J, which is admissible, will do the desired transfer. This completes the proof.

When attention is turned onto the problem of complete controllability of two-level linear dynamical systems in the later chapters, the above proposition will provide a tool to solve most questions concerning the specific problem of complete controllability.

However, as we have mentioned, we are interested more on the behavior of two-level linear dynamical systems. This dictates us to adopt the point of view that controllability be considered as behavioral property for a given dynamical system. Therefore, it will become a prerequisite to solve some of the problems on controllability and approximate controllability when the system (2.4-1') is under additive disturbances.

Before we attempt to give a fundamental geometrical interpretation on controllability based on certain topological properties of the set of attainable state, a few simple results can be derived from known results on algebraic arguments.

Let us recall that the main result derived by Antosievicz (Theorem 2.2-6) states that the necessary and sufficient condition for approximate controllability for system (2.4-1) is: for every $x^* \in X^*$

$$|\langle z_1(T), x^*\rangle| - \varepsilon \|x^*\|_{X^*} \leq k\left(\int_0^T \|x^* V(t, T)\|^2 dt\right)^{\frac{1}{2}} \quad (2.4\text{-}2)$$

The above condition apparently includes controllability as a special case by setting $\varepsilon = 0$.

Let us assume that system (2.4-2') is ε- conrollable with respect to a given space M of admissible controls and a pair of states (x_o, x_d). Let

$$z(T) = x_d - \Phi(T)x_o \quad (2.4\text{-}3)$$

It follows from the above assumption and Theorem (2.2-6) that, for all $x^* \in X^*$

$$|\langle z(T), x^*\rangle| - \varepsilon \|x^*\|_{X^*} \leq k\left(\int_0^T \|x^* V(t, T)\|^2 dt\right)^{\frac{1}{2}} \quad (2.4\text{-}4)$$

When the additive disturbance f(t) is no longer identically zero on J, we have

$$z_1(T) = x_d - \Phi(T)x_o - \int_o^T \Phi(T,t)f(t)dt \qquad (2.4\text{-}5)$$

Apparently, if we can establish inequality (2.4-2) for a class of disturbance functions subject to inequality (2.4-4), the ε- controllability of system (2.4-1) at time T with respect to M and (x_o, x_d) is immediately assured. Comparing inequalities (2.4-2) and (2.4-4), it is clear that a sufficient condition for (2.4-2) to be true subject to (2.4-4) is the satisfaction of

$$|\langle z_1(T), x^*\rangle| \leq |\langle z(T), x^*\rangle| \text{ for all } x^* \in X^* \qquad (2.4\text{-}6)$$

In other words, a sufficient condition for system (2.4-1) to remain ε - controllable at T with respect to M and (x_o, x_d) is the satisfaction of inequality (2.4-6). With this oberservation, the following is established:

<u>Proposition 2.4-2</u> A necessary and sufficient condition for inequality (2.4-6) to hold is the simultaneous satisfaction of

(i) $\langle x_d, \omega(T)\rangle > \langle \Phi(T)x_o, \omega(T)\rangle$

(ii) $2|\langle z(T), \Phi(T)x_o\rangle| \geq \langle \omega(T), \omega(T)\rangle$

Where $\omega(T) = \int_o^T \Phi(T, s)f(s)ds$.

Proof: Sufficiency: Let (i) and (ii) be satisfied, we shall show that $|\langle z_1(T), \Phi(T)x_o\rangle| \leq |\langle z(T), x^*\rangle|$ for all $x^* \in X^*$. Since X is a reflexive space, by choosing the ordinary Euclidean norm for X, we have $X = X^*$. Thus, it is equivalent to show that

$|\langle z(T), x\rangle - \langle \omega(T), x\rangle| \leq |\langle z(T), x\rangle|$ for all $x \in X$.

Let $\psi = \{a_1, a_2, \ldots, a_n\}$ be an orthogonal basis for the state space X with $a_1 = \omega(T)$. Let the one dimensional linear space spanned by a_1 be denoted by S_1 and the linear space spanned by $\psi - a_1$ be denoted by S_{n-1}. It is obvious that $S_1 \oplus S_{n-1} = X$. If $\langle \omega(T), x\rangle = 0$, which is true for all $x \in S_{n-1}$, inequality (2.4-6) will always hold. Thus it suffices to show that $|\langle z(T), x\rangle - \langle \omega(T), x\rangle| \leq |\langle z(T), x\rangle|$ for all $x \in S_1$. Or, if we write $x = \alpha\omega(T)$, $\alpha \in R$, we must show that $|\langle z(T), \omega(T)\rangle| - \langle \omega(T), \omega(T)\rangle| \leq |\langle z(T), \omega(T)\rangle|$

From (i), (ii) and the fact $\langle \omega(T), \omega(T)\rangle > 0$ whenever $\omega(T) \neq 0$, we have

$$|\langle z(T), \omega(T)\rangle - \langle \omega(T), \omega(T)\rangle| \leq |\langle z(T), \omega(T)\rangle - 2\langle z(T), \omega(T)\rangle|$$
$$= |\langle z(T), \omega(T)\rangle|$$

Necessity: Let inequality (2.4-6) be given, which is equivalent to $|\langle z(T), \omega(T)\rangle - \langle \omega(T), \omega(T)\rangle| \leq |\langle z(T), \omega(T)\rangle|$. Suppose $\langle z(T), \omega(T)\rangle = 0$, we must have $\langle \omega(T), \omega(T)\rangle = 0$. So let us exclude this singular case. Suppose

$\langle z(T), \omega(T)\rangle < 0$. Then

$\langle z(T), \omega(T)\rangle - \langle \omega(T), \omega(T)\rangle < \langle z(T), \omega(T)\rangle < 0$. By taking absolute value, we have arrived at a contradiction. Thus condition (i) must be true. Suppose

$2\,|\langle z(T), \omega(T)\rangle| < \langle \omega(T), \omega(T)\rangle$. Then

$$|\langle z(T), \omega(T)\rangle - \langle \omega(T), \omega(T)\rangle| > |\langle z(T), \omega(T)\rangle - 2\langle z(T), \omega(T)\rangle|$$
$$= |\langle z(T), \omega(T)\rangle|$$

which is a contradiction. Thus condition (ii) must also be true. This completes the proof.

<u>Corrollary 2.4-3</u> Let system (2.4-1') be ε - controllable at time T with respect to a given space M of admissible controls and a pair of states (x_o, x_d). Then system (2.4-1) will be ε - controllable at T with respect to M and (x_o, x_d) if the disturbance function f(t) satisfies the conditions imposed by Proposition 2.4-2.

An interesting point to notice is that the vector z(T) should not be an element in S_1. Because if it is so, we would have $\langle \omega(T), \omega(T)\rangle = 0$. In other words, no disturbance with non-zero amplitude might be allowed without affecting the property of ε - controllability.

However, the usefulness of this proposition cannot be overstated. First of all, it is at most a sufficient condition on ε - controllability for a known system.

Secondly, it appears difficult to establish a broad range of disturbance functions, under which the property of ε-controllability for given systems is not affected, by using the conditions in Proposition 2.4-2 alone. In order to obtain a general and unified approach to solve this problem, we shall rely heavily on geometrical motivation. Before doing so, let us define the following:

<u>Definition 2.4-3</u> Let the system (2.4-1) be given. The <u>space of admissible disturbance functions</u> is the set

$$F = \{f(t); \ \|f\|_{2 \text{ or } \infty} \leq K, \quad t \in J\}$$

where K is a finite constant, and the norms are same as those defined in (2.2-1).

The general solution of equation (2.4-1) is given by

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t, s)C(s)m(s)ds + \int_0^t \Phi(t, s)f(s)ds \qquad (2.4\text{-}7)$$

By definition, x(T) is a vector in the n-dimensional Euclidean space $R^n = X$. As we can see from (2.4-7), x(T) is in fact the vector sum of three vectors in state space. When a target state x_d is defined,

ε-controllability is equivalent to the statement that the Euclidean distance measure between $x(T)$ and x_d satisfies the inequality

$$\|x_d - x(T)\| \leq \varepsilon$$

or, rewriting by using previous notations:

$$\left\| \int_0^T V(t, T)m(t)ds - z_1(T) \right\| \leq \varepsilon \qquad (2.4\text{-}8)$$

Alternatively, in terms of geometry, ε-controllability of (2.4-1) is also equivalent to say that the set of attainable states $L(T, M)$ has a non-empty intersection with the closed ball $S(z_1(T), \varepsilon)$ with center at $z_1(T)$ and radius ε in the state space X. Thus the basic problem is reduced to the study of the properties of the set $L(T, M)$ and its relations with $S(z_1(T), \varepsilon)$ in X.

Now, Let us use a simple sketch in the 2-dimensional case to illustrate our notivation as in Figure 2.1.

When the ε-controllability of system (2.4-1') is assumed, it is clear that the sets $L(T, M)$ and $S(z(T), \varepsilon)$ are not disjoint. The effect of disturbance is represented as the translation of the ball $S(z(T), \varepsilon)$ to a new position centered at $z_1(T)$ by a vector $\omega(T)$, which is solely due to $f(t)$. When we

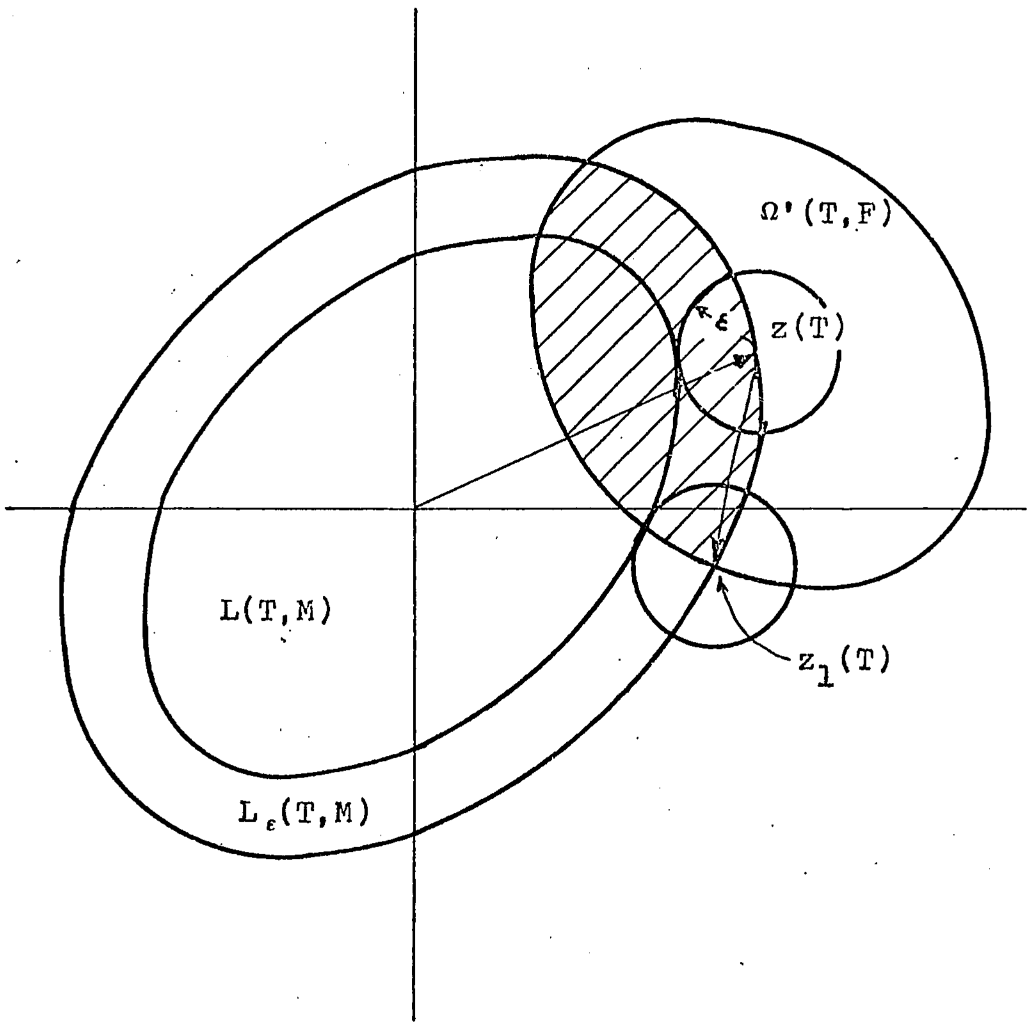


Figure 2.1

require that the system to remain ε-controllable under disturbance, we must therefore find conditions which would assure us that the shifed ball $S(z_1(T), \varepsilon)$ has a non-empty intersection with $L(T, M)$. Let us now introduce

Definition 2.4-4 Let the system (2.4-1) be given. For each given space F of admissible disturbance functions, the set of perturbed states is the set

$$\Omega(T, F) = \{\omega(T) = \int_0^T \Phi(T, s)f(s)ds, \quad f \in F\} .$$

Let us also enlarge the set of attainable state by defining a new set

$$L_\varepsilon(T, M) = \{\cup S(\theta(T), \varepsilon), \quad \theta(T) \in L(T, M)\} \qquad (2.4\text{-}9)$$

where $S(\theta(T), \varepsilon)$ is a closed ball in X with center at $\theta(T)$ and radius ε. Intuitively, by looking at the sketch, it is clear that the system will remain ε-controllable at T whenever the vector $\omega(T)$ is contained in the shaded region, i.e., the intersection of the sets $L_\varepsilon(T, M)$ and $\Omega(T, F)$. The fullfilment of this conclusion will naturally depend on certain topological properties of the sets $L_\varepsilon(T, M)$ and $\Omega(T, F)$ and their relative positions to $z(T)$.

In addition, a crucial remark to be noticed is that some fixed time instant T is implicityly assumed throughout the whole argument. In the mean time, the superposition principle for linear systems has been used for interpretation.

Since the assumption has been that system (2.4-1') is ε-controllable at T, it would be meaningless to evaluate the effect of disturbance on the property of ε-controllability at T for a different time instant. Because, in general, there is no guarantee that the set of attainable states will not change at an instant other than T. Furthermore, as part of the trajectory of system (2.4-1), the position of the state z(T) will certainly change with time. Therefore, in the formal presentation to follow, it is to be understood that we are in effect dealing with a fixed time problem for the moment.

The following lemma due to Antosiewicz [1] is the basis for the following analysis.

Lemma 2.4-5 Let the system (2.4-1) be given. Let the space M of admissible controls and the space F of admissible disturbance functions be also given. Then the sets L(T, M) and Ω(T, F) are compact convex set in X.

If we say a set S is symmetric with respect to the origin when $x \in S$ implies $-x \in S$, then it is easy to show that both L(T, M) and Ω(T, F) are symmetric with respect to the origin. We may introduce two linear transformations as follows:

$$L(T): M \to L(T, M) \quad \theta(T) = L(T)m \tag{2.4-10}$$

$$\Omega(T): F \to \Omega(T, F) \quad \omega(T) = \Omega(T)f \tag{2.4-11}$$

By definition, the mappings are onto.

Let ρ_L and ρ_Ω denote the sizes of the sets L(T, M) and Ω(T, F) respectively. We define

$$\rho_L = \sup_{\theta(T) \in L(T, M)} \|\theta(T)\| \tag{2.4-12}$$

$$\rho_\Omega = \sup_{\omega(T) \in \Omega(T, F)} \|\omega(T)\| \tag{2.4-13}$$

Let us also define a norm for the continuous linear mappings L(T) and Ω(T) as

$$L(T) = \sup\|L(T)m\| \quad \|m\|_{2 \text{ or } \infty} \leq 1 \tag{2.4-14}$$

$$\Omega(T) = \sup\|\Omega(T)f\| \quad \|f\|_{2 \text{ or } \infty} \leq 1 \tag{2.4-15}$$

Then

$$\rho_L = \sup_{\theta(T) \in L(T,\ M)} \|\theta(T)\|$$

$$= \sup_{m \in M} \|L(T)m\| \quad \|m\|_{2 \text{ or } \infty} \leq k$$

$$= k \sup_{m \in M} \|L(T)m\| \quad \|m\|_{2 \text{ or } \infty} \leq 1$$

$$= k\|L(T)\| \tag{2.4-16}$$

Similarly

$$\rho_\Omega = K\|\Omega(T)\| \tag{2.4-17}$$

It is clear that, in general, the sizes of the sets $L(T,\ M)$ and $\Omega(T,\ F)$ will not be zero when $k \neq 0$ and $K \neq 0$.

Again, from observation of the sketch, there will exist elements in F, under the influence of which, the property of ε-controllability for system (2.4-1) may not be retained unless $L_\varepsilon(T,\ M) \cup \Omega(T,\ F) = L_\varepsilon(T,\ M)$, i.e., the set $\Omega(T,\ F)$ is a subset of $L_\varepsilon(T,\ M)$.

We know from Corollary 2.2-7, if the system (2.4-1) is ε-controllable with respect to M, then there exists a least compact time interval $J_o = [0,\ T_o]$

in R^+ such that the system is ε-controllable. Let us call this particular time instant T_o the optimal time. For most cases to follow, we shall designate this T_o as our fixed time instant.

It is well known then, following the above designation, that the closed ball $S(z(T), \varepsilon)$ will barely contact the set $L(T, M)$. Formally, this means that the sets $S(z(T), \varepsilon)$ and $L(T, M)$ are not disjoint, and in addition, there exists a supporting hyperplan which separates the convex sets $S(z(T), \varepsilon)$ and $L(T, M)$ at their point of contact.

Equiped with these observations and knowledge, we have arrived at the following conclusions. For which, we shall first introduce the definition:

Definition 2.4-6 Let the space F, with $K > 0$, of admissible disturbance functions be given. A dynamical system, such as (2.4-1), is said to be ε-controllable under disturbance with respect to given M and (x_o, x_d) if, for every $f \in F$, the system is ε-controllable. The system is said to be completely controllable under disturbance if it is completely controllable.

Proposition 2.4-7 Let system (2.4-1') be given. Let the system be ε-controllable at T_o with respect to given M and (x_o, x_d). Then, if F is given with $K > 0$, the system (2.4-1) can not be ε-controllable under disturbance at T_o with respect to M and (x_o, x_d).

<u>Proof:</u> Let $K > 0$, it suffices to show that there exists $f \in F$ such that system (2.4-1) is not ε-controllable at time T_o. It can be shown that the set $L_\varepsilon(T, M)$ is also compact. Thus, $z(T_o)$ will be a boundary point of $L_\varepsilon(T_o, M)$. On the other hand, $z(T_o)$ serves as the center of the translated set $\Omega'(T_o, F) = \{x + z(T_o) : z(T_o) \in \Omega(T_o, F)\}$ and is thus an interior point of $\Omega'(T_o, F)$. Then, by definition of an interior point and a boundary point, there is a $\varepsilon > 0$ such that the ε-neighborhood $S(z(T_o), \varepsilon)$ is a proper subset of $\Omega'(T_o, F)$ and $S(z(T_o), \varepsilon) \cup L(T_o, M) \not\subset L_\varepsilon(T_o, M)$. But, since the mapping $\Omega(T_o)$ is continuous, $S(z(T_o), \varepsilon)$ is the range of $\Omega(T_o)$ corresponding to some subset of F. By definition of ε-controllability, it is required that $z_1(T_o)$ be element of $L_\varepsilon(T_o, M)$ corresponding to every $f \in F$, which is clearly impossible whenever $K > 0$. This completes the proof.

<u>Proposition 2.4-8</u> Let system (2.4-1') be given. Let

the system be ε-controllable at some time $T \neq T_o$ with respect to given M and (x_o, x_d). Let z(T) be an interior point of $L_\varepsilon(T, M)$. Then there exists F with $K > 0$ depending on z(T) such that the system (2.4-1) will be ε-controllable under disturbance at T with respect to M and (x_o, x_d).

Proof: Following the definition of ε-controllability, it suffices to show that the set $\Omega'(T, F)$ with center at z(T) is a subset of $L_\varepsilon(T, M)$. The size of $\Omega'(T, F)$ is given by $\rho_\Omega = K\|\Omega(T)\|$, or $K = \rho_L/\|\Omega(T)\|$. By hypothesis, there exists $\varepsilon > 0$ such that the ε-neighborhood of z(T), $S(z(T), \varepsilon)$, is contained in $L_\varepsilon(T, M)$. Since $\Omega'(T, F)$ is campact and convex, and norm of $\Omega(T)$ is fixed, we may choose $K = \varepsilon/\|\Omega(T)\|$ so that $\Omega'(T, F)$ is contained in $L_\varepsilon(T, M)$. This comletes the proof.

Corollary 2.4-9 Let system (2.4-1') be given. Let the system be controllable at T_o with respect to given M and (x_o, x_d). Then, for every $\varepsilon > 0$, there exists $K > 0$ such that the system (2.4-1) will be ε-controllable under disturbance at T_o with respect to M and (x_o, x_d).

Proof: This is only a special case of Proposition 2.4-8.

Using different phrasing, we have the following corollary which is important to our subsequent analysis.

<u>Corollary 2.4-10</u> Let the system (2.4-1') be given. Let the system be controllable at T_o with respect to given M and (x_o, x_d). Then, for a given $\varepsilon > 0$, a sufficient condition for system (2.4-1) to be ε-controllable under disturbance at T_o with respect to M (x_o, x_d) is the satisfaction of the inequality

$$K \leq \varepsilon / \|\Omega(T_o)\| \qquad (2.4\text{-}18)$$

One remark to be made here, the choice of the optimal time T_o is really not important. The crucial point is that we have chosen a control function such that z(T) is a boundary point of $L_\varepsilon(T, M)$. The choice of optimal time simply enhances this requirement.

The above condition in Corollary 2.4-10 is apparently too restrictive since inequality (2.4-18) imposes a uniform bound on the disturbance function f(t) over J_o where the sketch depicts only the instantaneous situation of system behavior at T_o. With this understanding, we have derived the following result, which is in essence equivalent to Proposition 2.2-8.

<u>Proposition 2.4-11</u> Let the system (2.4-1') be given. Let the system be controllable at T_o with respect to

given M and (x_o, x_d). Then, for a given $\varepsilon > 0$, a necessary and sufficient condition for the system (2.4-1) to be ε-controllable under disturbance at T_o with respect to M and (x_o, x_d) is the satisfaction of the inequality

$$\sigma \leq \varepsilon \qquad (2.4\text{-}19)$$

where σ is given in formula (2.2-10).

<u>Proof:</u> Sufficiency is a direct consequence of the definition of ε-controllability under disturbance and the hypothesis. For necessity, let $\sigma \geq \varepsilon$ and assume (2.4-1) be ε-controllable under disturbance at T_o, we must show a contradiction. Since both sets $L_\varepsilon(T_o, M)$ and Ω_{T_o} are convex and compact, $\sigma \geq \varepsilon$ implies that there exists $z_1(T_o) \notin L_\varepsilon(T_o, M)$. This obvious contradicts the requirement of ε-controllability. The proof is completed.

Following the above development, an observation is to be made here. It is clear from the sketch that, when the system (2.4-1') is controllable at some T with respect to given M and (x_o, x_d), then system (2.4-1) will be ε-controllable under disturbance for given $\varepsilon > 0$ at T when appropriate choice of F is made. We notice also the some of the disturbance functions in F may even improve the system performance, if we mean by improvement that $z_1(T)$ is shifted to become an interior point of $L_\varepsilon(T, M)$ owing to effect of disturbance. However, in order to know which disturbance function would produce such improvisation, we must know more about the characteristics of both $\Omega(T)$ and f(t). It will become clear from our subsequent considerations that we may not be willing or able to impose such specific constraints on f(t) other than the bounds on amplitude as depicted by the definition of F. Based on this consideration, the above propositions appear to be eminent.

Nevertheless, the uniform bound imposed on f(t) appears definitely too stringent in many cases when the present study is transformed into the context of studing two-level linear dynamical systems. In order to alleviate some of the restrictions, we will have to impose other kinds of constraints on the disturbance f(t), which must in itself not be unrealistic.

Let x(t) be the general solution of system (2.4-1). Let $x_c(t)$ denotes the complimentary function of (2.4-1), i.e., the solution of its homogenous part

$\dot{x} = A(t)x$. Let $x_p(t)$ denotes the particular solution of (2.4-1), which results from the forcing function $C(t)m(t) + f(t)$. It is well known that $x_c(t) \to 0$ as $t \to \infty$ when system (2.4-1) is b.i.b.o. stable. Therefore, if we could find $m(t)$ and $f(t)$ such that $x_p(t) \to 0$ as $t \to \infty$ to combine with the concept of uniform controllability, it might be possible to say some thing about controllability under disturbance and strenghthen the previous results. First of all, let us introduce the following defintion.

Definition 2.4-12 A vectorial function $f(t)$ is said to have the exponential-asymptotically stable property if there exists positive constants a_1, a_2 such that

$\|f(t)\| \leq a_1 e^{-a_2 t}$ for all $t \geq 0$.

Proposition 2.4-13 Let us consider a b.i.b.o. stable system $\dot{x} = A(t)x + f(t)$. The trivial solution of the system is uniformly asymptotically stable if $f(t)$ is exponential-asymptotically stable.

Proof: By means of the principle of superposition, it suffices to show that $x_p(t) \to 0$, which results from $f(t)$ only, as $t \to \infty$. Let

$\|f(t)\| \leq a_1 e^{-a_2 t}$ for $t \geq 0$. It is known that

$x_p(t) = \int_0^t \Phi(T, s)f(s)ds$, where $\Phi(t, s) = U(t)U^{-1}(s)$ and $U(t)$ is the fundamental matrix of $\dot{X} = A(t)X$.

Uniform asymptotical stability of the system implies that there exist positive constants α_1, α_2 such that $\|\Phi(t, 0)\| = \|U(t)\| \leq \alpha_1 e^{-\alpha_2 t}$ for $t \geq 0$, while $\|U^{-1}(s)\| \leq \alpha_1 e^{\alpha_2 t}$. There, $x_p(t) = U(t)\int_0^t U^{-1}(s)f(s)ds$

and $\|x_p(t)\| \leq \alpha_1 e^{-\alpha_2 t} \int_0^t (\alpha_1 e^{\alpha_2 s})(a_1 e^{-a_2 s})ds$

$= (a_1 \alpha_1 \alpha_1)e^{-\alpha_2 t} \int_0^t e^{-(a_2 - \alpha_2)s} ds$. Suppose $a_2 = \alpha_2$, it is obvious that $x_p(t) \to 0$ as $t \to \infty$.

Otherwise, $\|x_p(t)\| \leq (a_1 \alpha_1 \alpha_1)(1 - \frac{1}{a_2 - \alpha_2} e^{-(a_2 - \alpha_2)t})$.

Clearly, $x_p(t) \to 0$ as $t \to \infty$. This completes the proof.

As a consequence of this proposition we have derived the following intersting result.

<u>Proposition 2.4-14</u> Let system (2.4-1') be controllable at some time T with respect to given M and $(x_o, 0)$. then, given $\varepsilon > 0$, the system (2.4-1) will be ε-controllable under disturbance at some time $t > T$ with respect to M and $(x_o, 0)$ if

(i) F is the set of exponential-asymptotical stable functions.

(ii) the system is b.i.b.o. stable.

Proof: Since the desired state is the origin, we must show that there exists a control function $m(t) \in M$ corresponding to which $\|x(t)\| \leq \varepsilon$ for some $t \geq T$. Condition (ii) and Proposition 2.3-16 assure that system (2.4-1') is uniformly controllable. In other words there exists $m_o(t)$ in M such that the solution $x_f(t)$ of (2.4-1') satisfies $x_f(t) = 0$ for all $t \geq T$. Let us pick $m(t) = m_o(t)$, it suffices to show that $\|x(t) - x_f(t)\| = \|\int_o^t \Phi(t, s)f(s)ds\| \leq \varepsilon$ for some $t \geq T$. But the conclusion is assured by condition (i), (ii) and Propostion 2.4-13. This completes the proof.

We notice here that a very stringent requirement, which exists in previous analysis of ε-controllability under disturbance for system (2.4-1) has been removed. Namely, the disturbance function in F must be uniformly bounded by a specific bound. Instead, the set of admissible disturbances functions now read as a set of functions defined on R^+ which have the exponential-asymptotical stable property. Although the new constraint still requires that the admissible disturbance functions be uniformly bounded, at least a specific bound need not be specified a priori in order to fulfill the conditions for ε-controllability dictated in previous proposition. It will become more evident later that the new constraint can often be met in practical considerations for two-level linear dynamical systems.

Again, we notice that the requirement $x_d = 0$ is quite artificial and should be removed. In fact, the following result established a general condition we shall need for our subsequent analysis.

Proposition 2.4-15 Let system (2.4-1') be uniformly controllable for $t \geq T$ with respect to a given M and (x_o, x_d). Then, given $\varepsilon > 0$, the system (2.4-1) will be ε-controllable under disturbance at some time $t \geq T$ with respect to M and (x_o, x_d) if the set F is the set of exponential-asymptotical stable functions.

Proof: Let x(t) be the solution of (2.4-1). We must show that there is a control function $m(t) \in M$ such that $\|x(t) - x_d\| \leq \varepsilon$ for some time t. Since the system is assumed to be uniformly controllable, there must exist $m_o(t) \in M$ such that the solution $x_f(t)$ of (2.4-1') satisfies $x_f(t) = x_d$ for all $t \geq T$. Let $\omega(t)=\int_o^t \Phi(t, s)f(s)ds$ then it suffices to show that, after picking $m(t) = m_o(t)$ as the control function for (2.4-1), $\| x(t) - x_f(t)\| = \|\omega(t)\| \le \varepsilon$ for some $t \ge T$. and all $f(t) \in F$. Since uniform controllability implies that (2.4-1) is b.i.b.o. stable, Proposition 2.4-13 assures that the required conclusion is true, i.e., $\|\omega(t)\| \le \varepsilon$ for some $t \ge T$ and all $f(t) \in F$. This completes the proof.

There is another problem of substantial importance. That is, could the effects of additive disturbances be offset by excessive control energy? The answer is affirmative at least for linear systems, as disclosed by the following proposition.

Propostion 2.4-16 Let the system (2.4-1') be controllable at time T with respect to given M and (x_o, x_d). Then system (2.4-1) will be controllable under disturbance at T with respect to an enlarged space of admissible controls $M' = \{m(t) : \|m\|_{2 \text{ or } \infty} \le k', \ t \in J\}$ if the space F of admissible disturbance functions is defined as in Definition 2.4-3 and if $\gamma(k' - k) \le K$, where $\gamma = \sup_{t \in J}\|C(t)\|$.

Proof: Let us consider only the case $\mathcal{L}_2$, the case of $\mathcal{L}_\infty$ can be easily extended. It suffices to show that

$F \subset M_c = \{C(t)m(t) \ ; \ \|C(t)m\|_2 \le \gamma(k' - k), \ t \in J\}$,

while $m(t) \in M' = \{m(t) : \|m\|_2 \le k' - k, \ t \in J\}$.

Because, corresponding to each $\omega(T) = \int_0^T \Phi(T, s)f(s)ds$ with $f(t) \in F$, there exists $\bar{m}(t) \in M'$ such that $-\omega(T) = \int_0^T \Phi(T, s)C(s)m(s)ds$ when the above is true.

$$\int_0^T \|C(t)m(t)\|^2 ds \le \int_0^T \|C(t)\|^2 \|m(t)\|^2 dt \le \gamma^2 \int_0^T \|m(t)\|^2 dt$$

$\le \gamma^2 (k' - k)^2$. Since $K \geq \gamma(k' - k)$ by assumption, this implies $F \subset M_c$. This completes the proof.

In effect, this proposition tells us that outside disturbances can counteracted by the use of additional effort in the case of a linear system. Nevertheless, a strong condition is imposed on the nature of disturbance, namely the disturbances must at least be measurable in the Lebesque sense. For the study on two-level linear dynamical systems to follow in the next chapeters, we shall see that this requirement will invariably be satisfied.

CHAPPTER III

THE CASE WHEN NO DIRECT INTERACTION BETWEEN INFIMALS PRESENTS

3.1 Introduction

In this report, only qualitative property of multilevel system will be concerned. In order to carry out such study it is necessary that a definite mathematical model be formulated as the basis of study. In section 3.2, the completion of this task will be the main objective.

Controllability of a dynamical system has been viewed either as a structural or behavioral property. Since we shall take the later viewpoint in this report, mainly section 3.3 is used to discuss the problem of complete controllability, which is a structural property, to complete the discussion.

In section 3.4, two references of coordination will be developed. For later analysis, these two references will be the principal guideline.

In sections 3.5 and 3.6, the coordination problems of two-level system, which was defined in section 3.2 will be studied for the case when no direction interaction between the infimals presents.

In section 3.7, the concept of redundant control energy will be explored to improve those results obtained in sections 3.5 and 3.6.

3.2 Statement of Problem and Formulation of Two-level linear Dynamical system

As we have said in Chapter I, the two-level linear dynamical system to be studied in this thesis is, in general, a multi-level system. The fundamental charateristics that distinguish a multi-level system from a system of usual understanding are interaction, intervention, and internal uncertainty. A mathematical formulation, or rather an indealized representation of actual situation, of such systems must preserve and identify these characteristics.

Before we attempt to formulate the mathematical structure of the two-level linear dynamical system to be studied, we shall give in the following a few informal definitions which would help to clarify the meaning of our subsequent analysis.

Definition 3.2-1 A goal-seeking system is a general system[25] having the following attributes:

(i) A system $x = \psi(m, f)$, where x, m, and f are state function, control function, and disturbance function respectively, and ψ is mapping of the imput space M x F into the state space X, where x, m, and f are elements of X, M, and F respectively.

(ii) A set G of objectives or goals.

(iii) A set M of admissible alternative actions.

(iv) A set F of uncertainties.

As we have mentioned earlier in this thesis, the linear dynamical system to be considered is simply understood as mathematical system describalbe by a set of ordinary linear differential equations (2.2-1). When controllability requirement is imposed, the system (2.2-1) becomes naturally a goal-seeking system as the attributes are indentified as follows: (i) system: equation (2.2-1); (ii) Goal: controllability with respect to M and (x_o, x_d); (iii) Alternative actions: the space M of admissible controls; (iv) Uncertainties: the set F of admissible disturbance functions.

In this thesis, we take the point of view that a multi-level system is a collection of interacting goal-seeking subsystems. Since the problem under consideration will be restricted to a class of systems with assumed mathematical structure, no formal definition on the notion of subsystem is attempted. But a schematic diagram showing the general structure of the system to be studied would help us to have some insight to the problem being studied.

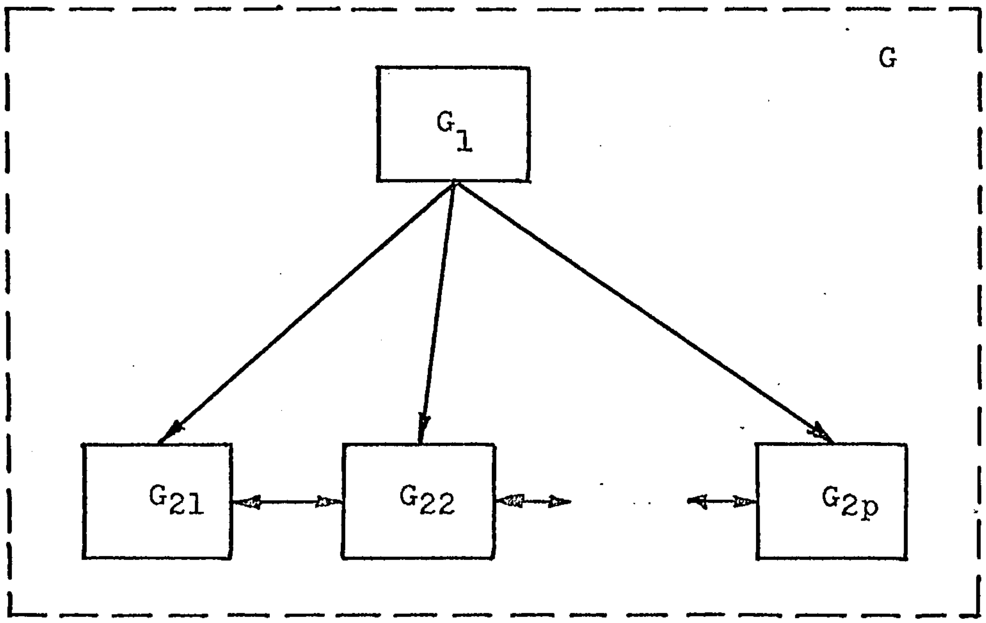


Figure 3.1

The system G is assumed to be a goal seeking system, which is composed of a collection of interacting smaller goal-seeking systems G_{ij}, or subsystems. The systems G_{ij} are smaller only in the sense that they are part of the over-all system G. In the above diagram, each block is thus assumed to represent one of the smaller goal-seeking systems where the two-way arrows indicate that the units are interacting with each other. Based on this structure, we may have the following informal definitions.

<u>Definition 3.2-2</u> A <u>Multi-level goal-seeking system</u> G is a goal-seeking system such that it is composed of a collection of interacting goal-seeking systems G_{ij}, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, r_i$, while the decisions made by the goal-seeking subsystem G_{ij} directly affect and are binding on the activity of subsystems G_{1j}, where $i > 1$, but the decisions made by G_{1j} influence the activity of G_{ij} only indirectly via the over-all goal of the system G. The subscripts i distinguish the subsystems to be situated on a different <u>level</u> i.

<u>Definition 3.2-3</u> Let us consider a two-level goal-seeking system G $(i = 1, 2)$ in which $j = 1$ when $i = 1$, $j = 1, 2, \ldots, p$ when $i = 2$. Subsystem G_{11} will be called the <u>supremal</u> and subsystems G_{2j}, $j = 1, 2, \ldots, p$, will be called <u>infimal</u>.

Following this definition, the supremal will clearly have priority of action over the infimals. Besides the duty of satisfying its own goal, the supremal must also overlook the activities of the infimals in order to satisfy the over-all goal of G. We may call this kind of supervising the control problem of the supremal.

Definition 3.2-4 Let us consider the two-level goal-seeking system defined in Definition 3.2-3. The control problem of the supremal is called coordination. The coordinative actions to be performed by the supremal are called intervention.

There are several forms of intervention a supremal could use, namely: (i) Goal intervention, in which the supremal affects the goal or objective which is the basis for the decision of the infimal. (ii) Image intervention, in which the supremal modifies the model of the system which the infimal uses. (iii) Constraints intervention, in which the supremal restricts the domain on the control action of the infimals. (iv) Interaction intervention, in which the supremal controls the communication channels between the infimals.

As we have said, we are interested in knowing some qualitative properties of a two-level goal-seeking system, to be exact the property of controllability of such system. It is therefore necessary that a definite mathematical model of the system be specified. This we shall do as follows:

The supremal:

$$\dot{x} = A(t)x(t) + C(t)m(t) + \sum_i B_i(t)x_i(t) \qquad (3.2\text{-}1)$$

$$M = \{ m(t): \|m\|_{2 \text{ or } \infty} \le k, \quad t \in J\}$$

The infimals:

$$\dot{x} = A_i(t)x_i(t) + C_i(t)m_i(t) + \sum_{j \neq i} D_{ij}(t)x_j(t) \qquad (3.2\text{-}2)$$

$$M_i = \{ m_i(t): \|m_i\|_{2 \text{ or } \infty} \le k_i, \quad t \in J_i\}$$

where each differential equation system is assumed to have the general definition as system (2.2-1); and $B_i(t)$, $D_{ij}(t)$ are matrices of continuous functions. To be noted here, the subscript i denotes individual subsystems and should not be confused with the components of the supremal. We shall write for components of each items as $x = (x_1, x_2, \ldots, x_n)$; $x_i = (x_{i1}, x_{i2}, \ldots, x_{in_i})$; $A(t) = (a_{ij}(t))$; $A_i(t) = (a_{ijk}(t))$, $j,k = 1, 2, \ldots, n_i$; $D_{ij}(t) = (d_{ijkl}(t))$, $k = 1, 2, \ldots, n_i$, $l = 1, 2, \ldots, n_j$. The term $\sum_i B_i(t)x_i(t)$ in (3.2-1) will reflect the interaction between the supremal and the infimals while the term $\sum_{j \neq i} D_{ij}(t)x_j(t)$ in (3.2-2) represents the communication among the infimals. It is also clear that, if we consider the terms $\sum_i B_i(t)x_i(t)$ and $\sum_{j \neq i} D_{ij}(t)x_j(t)$ as additive disturbances imposed on each respective system, the study of controllability problems will utilize very heavily the results obtained in the previous chapter.

Within the framework of the above mathematical description, it is clear that the system (3.2-1)-(3.2-2) is a collection of interacting goal-seeking subsystems, which have the attributes: systems (3.2-1) or (3.2-2), sets of alternative actions M or M_i, sets of uncertainties F (represented by $\sum_i B_i(t)x_i(t)$) or F_i (represented by $\sum_{j \neq i} D_{ij}(t)x_{ij}(t)$), and sets of goals G_1 or G_{2j} which are to be defined.

<u>Definition 3.2-5</u> The system (3.2-1)(3.2-2) is called a <u>two-level linear dynamical system</u> in the sense that the supremal and infimals are all described by linear differential equation systems.

<u>Definition 3.2-6</u> The <u>over-all goal</u> of the two-level linear dynamical system (3.2-1)(3.2-2) is defined to be the ε-controllability with $\varepsilon \geq 0$ of the supremal (3.2-1), i.e., given M and (x_o, x_d) there exists $m \in M$ such that the solution $x(t, x_i(t))$ of (3.2-1) at some

$T < \infty$ satisfies the inequality $\| x(T, x_i) - x_d \| \leq \varepsilon$.

One crucial observation to mentioned is: why do we consider the terms $\sum_i B_i(t)x_i(t)$ and $\sum_{j \neq i} D_{ij}(t)x_j(t)$ as uncertainties when they are genearated by deterministic systems? As we have repeatedly said, each infimal is considered as a goal-seeking subsystem. Therefore, when the space M_i of admissible controls and the goal is defined, each infimal will behave like an independent goal-seeking system, i.e., they will choose their own course of action within th bound of imposed constraint. Consequently, the solution $x_i(t)$ of (3.2-2) as a function of $m_i \in M_i$ will vary in a certain range. This kind of variation will usually not known to the supremal a priori. Since solution $x(t)$ of (3.2-1) is clearly a function of the solutions $x_i(t)$ of (3.2-2), and thus also denoted by $x(t, x_i)$, the supremal is then forced to consider the functions $x_i(t)$ as uncertainties. This kind of uncertainty has been called internal uncertainty[34] which is inherent to any true multi-level system. For the present study, one way to cope with the problem of internal uncertainty is to impose bounds on the functions $x_i(t)$ or to require certain characteristics be satisfied by $x_i(t)$. The purpose of coordination to be performed by the supremal is exactly to carry out this kind of regulation.

In the present study, we shall restrict the coordinative actions to have the following forms:

(i) Goal intervention: by setting the desired states x_{id} for each infimal.

(ii) Image intervention: by requiring the imfimals to possess certain properties, such as stability,

(iii) Constraints intervention: by specifying the spaces M_i of admissible controls and initial states x_{io}.

(iv) Interaction intervention: by imposing restrictions on the matrices $D_{ij}(t)$.

Definition 3.2-7 It is required that the coordinative actions performed by supremal be representable by constants or in simple terms. These coordinative actions will be called parametric coordination.

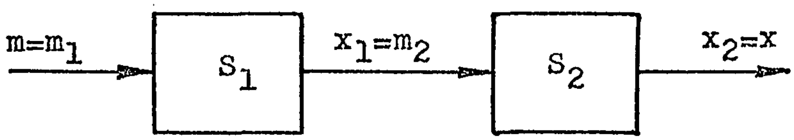
The advantages of employing parametric coordination are two fold: (i) the coordinative actions can be clearly specified, and (ii) when interventions have been imposed, the infimals and supremal can then operate as isolated goal-seeking system without worrying about deteriorating over-all performance of system (3.2-1)(3.2-2).

Equiped with the above definitions we may attempt to give the following statement.

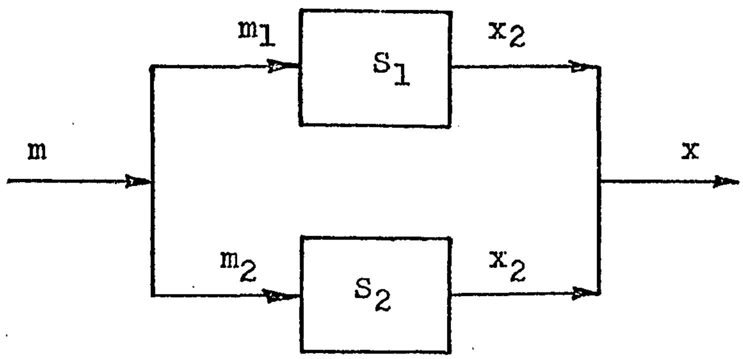
Statement of Problem: It is required to know whether and how the supremal (3.2-1) could use parametric coordination to regulate the behaviors of the infimals (3.2-2) so that the over-all goal of the two-level linear dynamical system (3.2-1)(3.2-2), as defined by Definition 3.2-6, can be attained.

3.3 The Problem of Complete Controllability

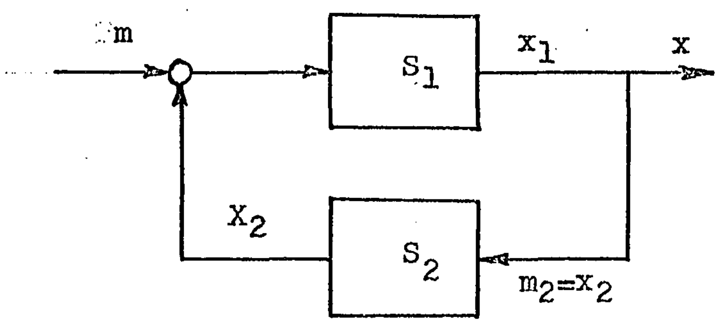
When looking at the schematic representation of the two-level linear dynamical system we had in last section, one is inclined to compare it with linear composity systems. The problem of complete controllability of composite systems was first considered by Gilbert[12] and quite recently by Chen and Desoer[7] . The composite systems studied by Gilbert have in general the following schematic representations:



series



parallel



closed

Figure 3.2

In the diagrams, S_1 and S_2 were assumed to be linear transformation. We also notice in each of the three representations that there is only one input m and one output x of interests, which are both vectors of course. But, at least in principle, these composite systems can be reduced to a single system of the following representation

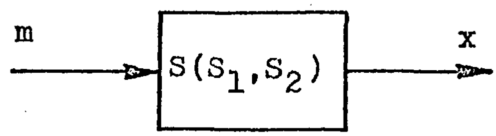


Figure 3.3

The problem of complete controllability for system $S(S_1, S_2)$ was then studied when certain properties of subsystems S_1, S_2 were assumed to be known. However, no control problem was associated with each subsystems.

As we have mentioned, controllability of a dynamical systems can either be considered as a structural property or a behavioral property of the system. In the previous study of composite systems, clearly it was conceived as structural. However, it would not be so fruitful if it be so considered in the present study of two-level linear dynamical systems. The primary rationale for this differentiation is that the problem of controllability for our present system will be considered from a viewpoint of decision-making. By associating with each infimal a decision-making problem, the structural interaction among the subsystems of two-level system becomes less rigid than those of the composite systems. Consequently, the problem of complete controllability for the two-level linear dynamical system (3.2-1)(3.2-2) will not be so important and interesting in itself.

In this section, we shall only consider the case when $D_{ij}(t) \equiv 0$ for all $t \geq 0$ and all i, j. We shall also need some known preliminary results in order to prove a final result on complete comtrollability for system (3.2-1)(3.2-2). Let us consider the linear system

$$\dot{x} = A(t)x(t) + f(t) \qquad (3.3\text{-}1)$$

where it has the usual definition as system (2.2-1), The general solution of (3.3-1) with $x(0) = x_o$ is given by

$$x(t) = \Phi(t)x_o + \int_o^t \Phi(t, s)f(s)ds \qquad (3.3\text{-}2)$$

At any finite time instant T, we have

$$x(T) = \Phi(T)x_o + \int_o^t \Phi(T, s)f(s)ds$$

$$= \Gamma x_o + \Omega(T)f$$

where $\Gamma : x_o \rightarrow \Phi(T)x_o$

and $\Omega(T): f \rightarrow \omega(T) = \int_o^T \Phi(T, s)f(s)ds$

are completely continuous linear operators. Therefore there exist constants K_1, K_2 depending only on Γ and $\Omega(T)$ such that the following are true:

$$\|\Gamma x_o\| \leq K_1\|x_o\|$$

$$\|\Omega(T)f\| \leq K_2\|f\|$$

In other words, we must have

$$\|x\| \leq K_1\|x_o\| + K_2\|f\| \quad \text{for } 0 \leq t \leq T \qquad (3.3\text{-}3)$$

Using this fact, it is possible to prove the following proposition.

<u>Proposition 3.3-1</u> Let us consider the two-level linear dynamical system (3.2-1)(3.2-2). Let $D_{ij}(t) = 0$ for all $t \geq 0$. Let the supremal (3.2-1), with $B_i(t) = 0$, for

$t \geq 0$ and all i, be completely controllable at some time T when the space M of admissible controls is the entire $\mathcal{L}_2$ or $\mathcal{L}_\infty$ space. Then the two-level linear system is completely controllable at T.

Proof: As a consequence of Proposition 2.4-1, it suffices to show that the function $\sum_i B_i(t)x_i(t)$ is uniformly bounded on J. But this required fact is guaranteed by the above observation (3.3-3) since the matrices $B_i(t)$ are also assumed to be continuous. This completes the proof.

As pointed out by Kalman[19], a general linear dynamical system can be decomposed algebraically into four parts which are completely controllable and completely observable, completely controllable but unobservable, uncontrollable but completely observable, and uncontrollable and unobservable, respectively. In order to avoid any unnecessary pitfall, we shall assume from now on that all subsystems in the two-level system (3.2-1)(3.2-2) are completely controllabe when $B_i(t) = D_{ij}(t) = 0$ for all $t \geq 0$ and all i, j.

3.4 The Fundmental Inequalities

The basic assumption that the supremal is controllable at the optimal time T_o with respect to a given space of admissible controls has greatly reduce the mathematical difficulties in dealing with problems associated with the assumed two-level linear dynamical system, because we are now in a position to solve the problems by considering only the relationships between the system matrix A(t) of the supremal and the disturbances functions which are in fact generated by the infimals. Since we are in essence considering a linear problem for which the principle of superposition prevails, we notice that the assumption of a fixed time, at T_o in particular, does not reduce the significance of the results obtained previously and those to follow. Because, as we have observed in section 2.4, only the topological properties of the sets L(T, M), Ω(T, F), and z(T) and their relative position in the state space are important, which, in the case of linear system, can certainly be adjusted by the use of the principle of superposition.

Corollary 2.4-10 and Proposition 2.4-11 give us two basic conditions which can be applied to the study of controllability of the assumed two-level linear dynamical system. From the very beginning of this thesis we have emphsized the viewpoint that each infimal shall operate as independent goal-seeking system after appropriate constraints are being imposed by the supremal. Attending to this guideline, we must reduce the inequalities given by Corollary 2.4-10 and Proposition 2.4-11 to a form suitable for the implementation of parametric coordination.

Let us write down again the mathematical model of the two-level linear dynamical system.

The supremal:

$$\dot{x} = A(t)x(t) + C(t)m(t) + \sum_i B_i(t)x_i(t) \qquad (3.4\text{-}1)$$

$$\dot{x} = A(t)x(t) + C(t)m(t) \qquad (3.4\text{-}1')$$

$$x(0) = x_o, \qquad M = \{m(t): \quad \|m\|_{2 \text{ or } \infty} \leq k, \quad t \in J\}$$

The infimals:

$$\dot{x} = A_i(t)x_i(t) + C_i(t)m_i(t) \qquad i = 1, 2, \ldots, p$$

$$(3.4\text{-}2)$$

$$x_i(0) = x_{io}, \; M_i = \{m_i(t): \|m_i\|_{2 \text{ or } \infty} \leq k, \quad t \in J\}$$

We shall understand that the subsystems are of appropriate dimensions n, n_1, $n_2, \ldots, n_p$, and have the usual definition as system (2.2-1). For any chosen admissible control functions, the general solutions for the sub-systems are respectively:

$$x(t) = \Phi(t)x_o + \int_o^t \Phi(t, s)C(s)m(s)ds + \sum_i \int_o^t \Phi(t,s)B_i(s)x_i(s)ds \tag{3.4-3}$$

and

$$x_i(t) = \Phi_i(t)x_{io} + \int_o^t \Phi_i(t, s)C_i(s)m_i(s)ds \tag{3.4-4}$$

Let us define

$$f(t) = \sum_i B_i(t)x_i(t) \tag{3.4-5}$$

Then, the norm of f(t) will satisfy the following:

$$\|f(t)\| \leq \|\sum_i B_i(t)x_i(t)\|$$

$$\leq \sum_i \|B_i(t)\| \cdot \|x_i(t)\| \qquad \text{for all } t \geq 0$$

Here, $\|B_i(t)\|$ is the Euclidean matrix norm of the matrix $B_i(t)$ and $\|B_i(t)\| = (\mathrm{tr}\, B_i(t)B'_i(t))^{\frac{1}{2}} = \sum_j \sum_k (b_{ijk}^2)^{\frac{1}{2}}$ at each time instant t. Since $B_i(t)$ are assumed to be known matrices, the norms are uniformly bounded for all t and we may define the following constants:

$$\beta_i = \sup_{t \in J} \|B_i(t)\|, \qquad i = 1, 2, \ldots, p \tag{3.4-6}$$

Following this definition, we have

$$\|f(t)\| \leq \sum_i \beta_i \|x_i(t)\| \qquad t \in J \qquad (3.4\text{-}7)$$

With this knowledge, we have

Proposition 3.4-1 Let the supremal (3.4-1') be controllable at the optimal time T_o with respect to given M and (x_o, x_d). Then, given $\varepsilon > 0$, the supremal (3.4-1) will be ε-controllable at T_o with respect to M and (x_o, x_d) if the following is true:

$$\sum_i \beta_i \|x_i(t)\| \leq \varepsilon / \|\Omega(T_o)\|, \qquad t \in J_o \qquad (3.4\text{-}8)$$

Proof: This proposition is a direct consequence of Corollary 2.4-10 and formula (3.4-7).

On the other hand, we may use the inequality provided by Proposition 2.4-11 to derive another criterion of significant importance to the subsequent development.

By incorporating the definition of f(t) as in (3.4-5) and the set $\Omega(T, F)$ of perturbed states as in Definition 2.4-4, we have

$$\Omega(T, F) = \{\omega(T): \quad \omega(T) = \sum_i \int_o^T \Phi(T, s) B_i(s) x_i(s) ds\} \qquad (3.4\text{-}9)$$

Then, proposition 2.4-11 depicts that the following inequality is a sufficient condition to achieve the over-all goal:

$$\|\omega(T)\| \leq \varepsilon \qquad \text{for all} \quad \omega(T) \in \Omega(T, F)$$

Again the norm of $\omega(T)$ will satisfy

$$\|\omega(T)\| \leq \sum_i \left\| \int_0^T \Phi(T, s) B_i(s) x_i(s) ds \right\|$$

Let us denote

$$\omega_i(T) = \int_0^T \Phi_i(T, s) B_i(s) x_i(s) ds \tag{3.4-10}$$

Then

$$\|\omega(T)\| \leq \sum_i \|\omega_i(T)\|$$

Therefore, we have the following proposition which provides another fundamental inequality.

<u>Proposition 3.4-2</u> Let the sumpremal (3.4-1') be controllable at the optimal time T_o with respect to given M and (x_o, x_d). Then, given $\varepsilon > 0$, the supremal (3.4-1) will be ε-controllable at T_o with respect to M and (x_o, x_d) if the following is true:

$$\sum_i \|\omega_i(T_o)\| \leq \varepsilon \tag{3.4-11}$$

Inequality (3.4-11) will provide an important clue for coordination. Nevertheless, this condition alone does not provide us a powerful tool to achieve fruitful analysis, because it only stipulates an instantaneous situation at T_o. Unless the properties of $\Phi(t)$, $B_i(t)$, and $x_i(t)$ and their combined effect as represented by $\omega_i(t)$ are fully known and completely adjustable, it is almost an impossible task to guarantee the satisfaction of this inequality at that particular time instant T_o. Undoubtedly, this is beyond the scope of parametric coordination which we intend to use exclusively in the present study. Consequently, certain compromises must be made in order to solve the problem. By reviewing the results obtained in section 2.4, we see that Propositions 2.4-14 and 2.4-15 do provide us with the necessary tool.

The guidelines of coordination provided by inequalities (3.4-8) and (3.4-11) are undoubtedly too restrictive to achieve the over-all goal intended. The reason lies of course on the use of norm which simply supresses certain nice properties of the time functions $x_i(t)$, $B_i(t)$, or $\omega_i(t)$ that might be used to improve the sufficiency conditions. Nevertheless, these guidelines are not likely to be improvable in general because of the viewpoint we have taken in treating the functions $x_i(t)$ as internal uncertainties, In which case the bounds or certain simple characteristics set on th state functions

$x_i(t)$ seem to be the only reasonable constraints.

Nevertheless, by summerizing the results in this section and previous observation, a few conclusions could still emerge. That is, there are several schemes of coordination the supremal could use to regulate the system behavior and to achieve the over-all goal. The following cases will be investigated in the subsequent sections.

(i) The supremal may use constrains intervention to set for each infimal the space X_{io} of admissible initial states and the space M_i of admissible controls so that inequality (3.4-8) might be satisfied.

(ii) The supremal may use goal intervention to set the target set for each infimal, constraints interventions as in (i), and image intervention so that inequality (3.4-11) might be satisfied.

(iii) The employment of redundant control energy to improve the results obtained in (i) and (ii).

Since the interaction among infimals is assumed to be completely supressed in the study of this chapter, the use of interaction intervention will be discussed in the next chapter.

3.5 Coordination by Use of Image and Constraints Interventions

Let us write down again the two-level liner dynamical system to be considered in this section. The supremal

$$\dot{x} = A(t)x(t) + C(t)m(t) + \sum_i B_i(t)x_i(t) \qquad (3.5\text{-}1)$$

$$x(0) = x_o, \quad M = \{ m(t): \ \|m\|_{2 \text{ or } \infty} \leq k, \quad t \in J\}$$

$$\dot{x} = A(t)x(t) + C(t)m(t) \qquad (3.5\text{-}1')$$

The infimals

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) \qquad i = 1, 2, \ldots, p \qquad (3.5\text{-}2)$$

$$x_i(0) = x_{io}, \quad M_i = \{ m_i(t): \ \|m_i\|_{2 \text{ or } \infty} \leq k_i, \quad t \in J_i \}$$

Let us assume that the unperturbed supremal (3.5-1') is controllable at time T_o with respect to M and (x_o, x_d). The problem to be solved is then: how the supremal can use interventions to coordinate the activities of the infimals so that the supremal (3.5-1) itself can remain controllable or ϵ-controllable under the influence of the state functions $x_i(t)$.

In last section, one of the fundamental inequalities developed was

$$\sum_i \beta_i \|x_i(t)\| \leq \epsilon/\|\Omega(T_o)\| \qquad \text{for } t \in J_o \qquad (3.4\text{-}8)$$

We shall use this inequality as a basis to develop some schemes of coordination which the supremal may use in order to regulate the activities of the infimals.

It is quite obvious that inequality (3.4-8) is a more restrictive version of the following inequality, which also serves as a sufficient condition for the stated problem:

$$\sum_i \|B_i(t)x_i(t)\| \leq \epsilon/\|\Omega(T_o)\| \qquad \text{for } t \in J_o \qquad (3.5\text{-}3)$$

Suppose that the weighting matrices $B_i(t)$ can be designed as desired, inequality (3.5-3) can always be satisfied no matter what the state functions $x_i(t)$ are. For instance, the weighting matrices $B_i(t)$ might be the representation of a saturation mechanism such that

$$\|B_i(t)x_i(t)\| \leq \alpha_i, \qquad \text{for } t \in J_o$$

Then, let the saturation mechanism be so designed that $\sum_i \alpha_i \leq \epsilon/\|\Omega(T_o)\|$, the achievement of over-all goal for system (3.5-1)(3.5-2) will be accomplished. Or, if the state functions $x_i(t)$ are uniformly bounded over R^+ and $B_i(t)$ are time-varying, there always exists a set of $B_i(t)$ such that the above inequality will be satisfied. However, if we consider $B_i(t)$ to bear certain physical meaning which is usually the case in the real world, then we do not have the liberty of arbitrarily designing $B_i(t)$. Thus, we shall assume in this section and those to follow that matrices $B_i(t)$ are fixed on R^+. In order to achieve the over-all goal, the supremal must then use interventions to restrict the amplitude of the state functions $x_i(t)$.

The fact that the state functions $x_i(t)$ will be bounded on any finite time interval was clearly established in section 3.3. However, the above observation is usually too crude to be useful. Let us take a look of the fundamental lemma on differential inequality.

Lemma 3.5-1 [13] Let $x(t)$ defined on R^+ be a solution of $\dot{x} = A(t)x(t) + f(t)$. Then

$$\|x(t)\| \leq \left\{ \|x_o\| + \int_o^t \|f(s)\| ds \right\} \exp \int_o^t \|A(s)\| ds$$

Applying this lemma to the solution of any infimal (3.5-2) we have

$$\|x_i(t)\| \leq \{\|x_{io}\| + \int_o^t \|C_i(s)m_i(s)\| ds\} \exp \int_o^t \|A_i(s)\| ds$$

If we let

$$\alpha_i = \sup_{t \in J_o} \|A_i(t)\|$$

$$\gamma_i = \sup_{t \in J_o} \|C_i(t)\|$$

Clearly,

$$\|x_i(t)\| \leq \{\|x_{io}\| + \gamma_i k_i T_o\} \exp(\alpha_i T_o) \quad \text{for } t \in J_o \tag{3.5-4}$$

Thus, if we select x_{io}, γ_i, k_i, α_i such that the right-hand side of (3.5-4) is less than some constant a_i, this would establish a sufficient condition for ϵ-controllability of the supremal (3.5-1). However, since inequality (3.5-4) is exponentially growing as a function of T_o, the bounds established is again too restrictive. The reason for this inconvenience is that no restriction has been imposed on system structures of the infimals. In other words,more refined inequality might be obtained when the infimals are required to possess certain properties, which constitutes the employment of image intervention.

Let us consider the case when the infimals are required to be bounded-input bounded-output stable. Lemma 2.3-14 assures that the following property holds for each infimal:

$$\|\Phi_i(t)\| \leq b_{i1} \qquad \text{for all } t \geq 0 \tag{3.5-5}$$

$$\int_0^t \|\Phi_i(t, s)c_i(s)\| ds \leq b_{i2} \qquad \text{for all } t \geq 0 \tag{3.5-6}$$

where b_{i1}, b_{i2} are positive constants. Consequently, we have

$$\|x_i(t)\| \leq \|\Phi_i(t)\|\|x_{io}\| + \int_0^t \|\Phi_i(t,s)C_i(s)\| \|m_i(s)\| ds$$

$$\leq b_{i1}\|x_{io}\| + k_i b_{i2} \tag{3.5-7}$$

Thus, if the constants b_{i1}, x_{io}, k_i, b_{i2} can be so chosen that the right-hand side of (3.5-7) is less than some constant a_i, this will establish a sufficient condition for the ε-controllability of the supremal, i.e., the achievement of the over-all goal. We may summerize the above analysis to give the following proposition.

<u>Proposition 3.5-2</u> Let the supremal (3.5-1') be controllable at time T_o with respect to (x_o, x_d) and a given M. Then, for some given $\varepsilon > 0$, the supremal (3.5-1) will be

ε-controllable at time T_o with respect to M and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable.

(ii) $\sum_i \beta_i (b_{i1} \|x_{io}\| + k_i b_{i2}) \le \varepsilon / \|\Omega(T_o)\|$ (3.5-8)

Based on this theorm, a scheme of coordination, which reduces the two-level linear dynamical system into a group of independently operating subsystems, can be divised. It should be pointed out, however, that inequality (3.5-8) may not be satisfied for a given $\varepsilon > 0$ when the infimals have fixed systems structure and must start from some fixed initial states. In this case, the supremal can only use the allocation of control energy represented by the spaces of admissible controls as a means of intervention. For convenience, let us give

Definition 3.5-3 For some given $\varepsilon > 0$, the inequality (3.5-8) is said to be consistent if

$$\sum_i \beta_i (b_{i1} \|x_{io}\|) \le \varepsilon / \|\Omega(T_o)\| \qquad (3.5\text{-}9)$$

When the consistancy condition is satisfied, the supremal will have some freedom in the use of constraint intervention. Suppose that the initial states of each infimal can be adjusted, then the supremal may assign the spaces x_{io} of admissible initial states by giving the constants k_{io}. In this case inequality (3.5-8) becomes

$$\sum_i \beta_i (b_{i1} k_{io} + b_{i2} k_i) \leq \varepsilon / \| \Omega(T_o) \| \qquad (3.5\text{-}8')$$

Obviously, the consistancy condition can always be satisfied by some appropriate choice of k_{io}.

Based on the above proposition, we may formulate a scheme of coordination as follows.

Scheme of Coordination 3.5-4

(i) the supremal exercises image intervention by requiring that each infimal be b.i.b.o. stable.

(ii) the supremal commands the infimals to send in informations concerning the constants b_{i1} and b_{i2}.

(iii) the supremal exercises its constraint intervention by selectin x_{io} for each infimal so that the consistancy condition is satisfied.

(iv) the supremal exercises constraint intervention by selecting M_i for each infimal so that inequality (3.5-8') is satisfied.

In this scheme of coordination, the flow of information is completely representable by real constants.

In orther words, it does conform to the definition of "parametric coordination" as defined in section 3.2.

When the assumption of b.i.b.o. stability is lifted, it would be difficult to assess explicit bounds on the state functions using the differential inequality in Lemma 3.5-1 as has been demonstrated. However, using a method due to Rosen[30], it is possible to discern the existance of such bounds. Let us first introduce

<u>Definition 3.5-5</u> The differential control system $\dot{x} = f(t, x, m)$ with its control space M is <u>ρ-stable</u>, if for every $t_o \in R^+$ and $m \in M$, $x(t_o) \in R(\rho)$ $R(\rho) = \{x : \|x\| \le \rho\}$ implies $x(t) \in R(\rho)$ for $t_o \le t \le T$.

<u>Definition 3.5-6</u> The differential system $\dot{x} = f(t,x,m)$ is <u>controllable ρ-stable</u> if there exists a control function $m_o \in M$ such that $\dot{x} = f(t, x, m_o)$ is ρ-stable.

Rosen was successful to obtain result for linear constant systems. So let us consider

$$\dot{x} = Ax(t) + Cm(t) \qquad (3.5\text{-}1'')$$

This is only a special case of (3.5-1').

<u>Definition 3.5-7</u> The matrix C is a <u>control matrix</u> for A if there exist no vector x, such that the conditions

$$x'(A+A')x > 0 \quad \text{and} \quad \|C'x\| = 0$$

hold simultaneously.

Then, for system (3.5-1") he gave

Lemma 3.5-8 There exists a $\rho > 0$, such that the system (3.5-1") is controllable ρ-stable if and only if C is a control matrix for A.

In order to apply this theorem to the proposed two-level linear system, let us assume that all the infimals are constant systems. Thus

$$\dot{x}_i = A_i x_i(t) + C_i m_i(t), \quad i = 1, 2, \ldots, p \qquad (3.5\text{-}2')$$

and we have

Proposition 3.5-9 Let the perturbation matrices $B_i(t)$ be fixed. For a given $\varepsilon > 0$, in order to satisfy inequality (3.4-8) it is necessary that C_i are control matrices for A_i respectively.

Proof: By hypothesis, it is required that $\|x_i(t)\| \leq a_i$ $0 \leq t \leq T$, $i = 1, 2, \ldots, p$, i.e., the infimals (3.5-2') are at least controllable ρ-stable. Thus, the proof follows as a consequence of Lemma 3.5-8.

The presentation of above analysis here is only for its aesthetic value. Since no explicit bound can be established based on Proposition 3.5-9 alone, the practical value of this theorem in the present context is limited.

Another class of systems of general interests is the class of systems which employ feedback type control. That is , the control function is representable by some weighted function of the state vector. In the following case, we shall assume that all infimals employ exclusively linear continuous feedback type control, i.e.

$$m_i(t) = H_{i1}(t)x_i(t), \quad 0 \leq t \leq T, \quad i = 1, 2, \ldots, p \tag{3.5-10}$$

where $H_{i1}(t)$ are continuous matrices of appropriate dimensions. Without loss of generality, we may write

$$\begin{aligned} C_i(t)m_i(t) &= C_i(t)H_{i1}(t)x_i(t) \qquad (3.5\text{-}10') \\ &= H_i(t)x_i(t), \quad i = 1, 2, \ldots, p \end{aligned}$$

where $H_i(t)$ are necessarily square matrices of dimension n_i. Therefore

$$\dot{x}_i = A_i(t)x_i(t) + H_i(t)x_i(t), \quad i = 1, 2, \ldots, p \tag{3.5-11}$$

where $H_i(t)$ are to be designed. Again, we want the solutions of (3.5-11) to satisfy the criteria

$$\|x_i(t)\| \le a_i, \quad 0 \le t \le T.$$

There are several approaches could be utilized to solve the problem stated previously.

As we have said previously in section 2.3, for a linear dynamical system, the property of bounded-input bounded-out stability has a very close relationship with the property of uniformly asymptotical stability. Since the satisfaction of inequality (3.4-8) dictates a uniform bound on the state function $x_i(t)$, it would be natural to require that the infimal to have the bound-input bounded-output stable property. We shall also apply this idea to the feedback control system stipulated above. One could write system (3.5-11) also as

$$\dot{x}_i = (A_i(t) + H_i(t))x_i(t) \qquad (3.5\text{-}11')$$

Then, formally, one could consider system (3.5-11') as an autonomous system, i.e., those systems without external forcing functions. In order to maintain the solution $x_i(t)$ of (3.5-11') within a uniform bound on R^+, the matrix $(A_i(t) + H_i(t))$ must have certain properties. In order to satisfy the required condition, it is well known[5] that $H_i(t)$ can not be any arbitrary matrix, when system (3.5-2) is assumed to be b.i.b.o. stable. On the other hand, for any given $A_i(t)$, there do exist some feedback matrix $H_i(t)$ which quarantees the asymptotical stability of (3.5-11). This fact is established in following lemma:

Lemma 3.5-10 Let the system (3.5-2) be bounded-input bounded-output stable. Then there exists a feedback matrix $H_i(t)$ such that system (3.5-11) is asymptotically stable, provided that $\|H_i(t)\| \leq \alpha_3$ for $t \geq 0$, where α_3 is a constant depending upon $A_i(t)$.

For the proof of this lemma, we shall need the following well-known result:

Lemma 3.5-11 If $u, v \geq 0$, if α_1 is a positive constant, and if

$$u \leq \alpha_1 + \int_0^t uvds$$

Then $$u \leq \alpha_1 \exp(\int_0^t vds)$$

Proof of Lemma 3.5-10 The solution of (3.5-11) is

$$x_i(t) = \Phi_i(t)x_{io} + \int_0^t \Phi_i(t, s)H_i(s)x_i(s)ds$$

From the proof of Proposition 2.4-13, we know that

$$\|x_i(t)\| \le \alpha_{i1} e^{-\alpha_{i2}t} + \alpha_{i1} e^{-\alpha_{i2}t} \int_0^t e^{\alpha_{i2}s} \|H_i(s)\| \cdot \|x_i(s)\| ds$$

or,

$$\|x_i(t)\| e^{\alpha_2 t} \le \alpha_1 + \alpha_1 \alpha_3 \int_0^t e^{\alpha_{12}s} \|x_i(s)\| ds$$

From Lemma 3.5-11, we have

$$\|x_i(t)\| e^{\alpha_2 t} \le \alpha_1 e^{\alpha_1 \alpha_3 t} \qquad \text{for } t \ge 0$$

If $\alpha_1 \alpha_3 < \alpha_2$, we can conclude that $\|x_i(t)\| \to 0$ as $t \to \infty$. Since the constants α_{i1}, α_{i2} depend upon $A_i(t)$, it is clear that α_{i3} depend upon $A_i(t)$.

Using the terminology we have used so far, the asymptotical stability of system (3.5-11) is equivalent to say that the state function $x_i(t)$ has the exponential-asymptotically stable property. Then, by the application of Lemma 3.5-10, the following proposition is proved.

Propostion 3.5-12 Let the supremal (3.5-1') be controllable at T_o with respect to a given M and (x_o, x_d). Then, given $\varepsilon > 0$, the supremal (3.5-1) will be

ε-controllable at T_o with respect to M and (x_o, x_d) if

(i) the infimals (3.5-2) are b.i.b.o. stable

(ii) the infimals use feedback control with $\|H_i(t)\| \leq \alpha_{i3}$

(iii) $\sum_{i=1} \beta_i \alpha_i \|x_{io}\| \leq \varepsilon / \|\Omega(T_o)\|$

where α_{i3} & α_i are constants depending on $A_i(t)$.

<u>Proof</u> As a consequence of Lemma 3.5-10, it is possible to select the feedback matrices $H_i(t)$ with $\|H_i(t)\| < \alpha_{i3}$ such that infimals (3.5-11) are uniformly asymptotically stable. Therefore, $\|x_i(t)\| \leq \alpha_i \|x_{io}\| e^{-a_i t}$ for $t \geq 0$, where α_i and a_i are positive constants. Clearly $\|x_i(t)\| \leq \alpha_i \|x_{io}\|$ which depends on $A_i(t)$. Then, condition (iii) guarantees the conclusion which follows from inequality (3.4-8). This completes the proof.

For practical purposes, the compulation of constants α_i would present a problem. However, there are cases in which these constants can readily be computed, for instant, time-invariant linear systems and non-oscillatory type systems. Suppose this problem is solvable by an algorithm, then a general scheme of coordination can be devised as follows:

Scheme of Coordination 3.5-13

(i) The supremal commands the infimals to send up informations on the constants b_{i1}.

(ii) The supremal computes constants α_i by selecting appropriate feedback matrices $H_i(t)$ (or limitations on $H_i(t)$) with $\|H_i(t)\| \leq \alpha_{i3}$.

(iii) The supremal selects judicial constants α_i and initial states x_{io} (or the space x_{io} of admissible initial states) such that inequality (3.5-13) is satisfied.

(iv) The supremal commands the infimals to operate within the constraints on $H_i(t)$ and X_{io}.

Let us now investigate the case when the infimals are time invariant systems. The infimals are now described by

$$\dot{x}_i = A_i x_i(t) + C_i m_i(t) \tag{3.5-2"}$$

The foundamental matrices are $\Phi_i(t) = \exp(A_i + H_i)t$.

Let the eigenvalues of the matrices A_i be all distinct. The assumption of b.i.b.o. stability ensures that all the eigenvalues are negative real constants.

By a standard technique in matrix analysis, we may choose orthogonal matrices G_i such that $G'_i G_i = I$ and $G_i^{-1} A_i G_i = \Lambda_i$ where Λ_i are diagonal matrices with $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}$, $i = 1, 2, \ldots, p$ on the diagonal. Let $x_i = G_i y_i$, (3.5-2") reduce to

$$\dot{y}_i = (\Lambda_i + G_i^{-1} H_i G_i) y_i(t), \quad y_{io} = G_i^{-1} x_{io} \qquad (3.5\text{-}14)$$

and

$$y_i(t) = \Phi_i(t) G_i^{-1} x_{io} \qquad (3.5\text{-}15)$$

where $\Phi_i(t) = \exp(\Lambda_i + G_i^{-1} H_i G_i)t$. Suppose $A_i H_i = H_i A_i$ for $i = 1, 2, \ldots, p$, it is easy to show that $\Phi_i = \exp(\Lambda_i t)\exp(G_i^{-1} H_i G_i t)$, because $\Lambda_i (G_i^{-1} H_i G_i) = (G_i^{-1} H_i G_i)\Lambda$. Under this hypothesis

$$y_i(t) = G_i^{-1} x_i(t) = \exp(\Lambda_i t)\exp(G_i^{-1} H_i G_i t) G_i^{-1} x_o$$

One sufficient condition for $y_i(t)$ to be non-oscillatory is that $\exp(G_i^{-1} H_i G_i t)$ be non-oscillatory, because $\exp(\Lambda_i t)$ is non-oscillatory. This requirement will be assured if the matrices H_i have distint negative real constants as eigenvalues. This is so because the matrices $G_i^{-1} H_i G_i$ has the same eigenvalues as H_i. To summerize, we have

<u>Proposition 3.5-14</u> Let the conditions of Proposition 3.5-12 be satisfied. Let the infimals be constant systems described by (3.5-2). Let all the system matrices A_i, $i = 1, 2, \ldots, p$ have distinct real negative eigenvalues. Let $A_i H_i = H_i A_i$, $i = 1, 2, \ldots, p$. Then the conclusion in Proposition 3.5-12 holds if

$$\sum_{i=1}^{p} \beta_i n_i \|x_{io}\| \leq \epsilon / \|\Omega(T_o)\| \qquad (3.5\text{-}17)$$

<u>Proof:</u> If suffices to show that $\|x_i(t)\| \leq n_i \|x_{io}\|$ $i = 1, 2, \ldots, p$, $0 \leq t$. In fact

$$\begin{aligned}\|x_i(t)\| &\leq \|G_i \exp(\Lambda_i t) \exp(G_i^{-1} H_i t) G_i^{-1}\| \cdot \|x_{io}\| \\ &= \|\exp(\Lambda_i t) \exp(G_i^{-1} H_i G_i t)\| \cdot \|x_{io}\|\end{aligned}$$

because G_i is also an isometry. Thus the above reduces to $\|x_i(t)\| \leq \|\exp(\Lambda_i t)\| \cdot \|\exp(G_i^{-1} H_i G_i t)\| \cdot \|x_{io}\|$. But $\|\exp \Lambda_i t\| \leq \sqrt{n_i}$, $\|\exp(G_i^{-1} H_i G_i t)\| \leq \sqrt{n_i}$ because of the negativity in eigenvalues. Then $\|x_i(t)\| \le n_i \|x_{io}\|$ as required.

It is known that, if $A_i H_i = H_i A_i$ and if A_i and H_i are diagonalizable, then there exists a normal matrix G_i of the property $G_i G_i' = G_i' G_i$ such that $G_i^{-1} H_i G_i$ and $G_i^{-1} A_i G_i$ are both diagonal matrix. Let this assumption be made and let $y_i = G_i^{-1} x_i$. Then $\Phi_i(t) = \exp(\Lambda_i + \Delta_i)t$ where Λ_i and Δ_i are diagonal matrices composed of eigenvalues.

<u>Proposition 3.5-15</u> Let the conditions of Proposition 3.5-12 be satisfied. Let the infimals be constant systems. Let the system matrices have distinct real neqative eigenvalues. Then the conclusion of Proposition 3.5-12 holds if

(i) $|\lambda_{ij}| > |\delta_{ij}|$, $i = 1, 2, \ldots, p; j=1, 2, \ldots, n_i$ where λ_{ij} and δ_{ij} are eigenvalues of A_i and H_i respectively.

(ii) $\sum_{i=1}^{p} \beta_i \sqrt{n_i} \|x_{io}\| \le \epsilon / \|\Omega(T_o)\|$ (3.5-18)

<u>Proof:</u> It suffices to show that $\|x_i(t)\| \le \sqrt{n_i} \|x_{io}\|$ for $i = 1, 2, \ldots, p$. In fact

$$\|x_i(t)\| \le \|G_i \exp(\Lambda_i + \Delta_i) \times G_i^{-1}\| \cdot \|x_{io}\|$$

$$= \|\exp(\Lambda_i + \Delta_i)t\| \cdot \|x_{io}\|$$

Let $|\lambda_{ij}| > |\delta_{ij}|$ be satisfied, the matrix $\Lambda_i + \Delta_i$ will have negative eigenvalues. Thus $\|\exp(\Lambda_i + \Delta_i t)\| \leq \sqrt{n_i}$. Therefore, $\|x_i(t)\| \leq \sqrt{n_i}\|x_{io}\|$ as required.

Similar bounds on $x_i(t)$ could also be established when commutative property is not assumed between A_i and H_i. Let us consider the time-invariant linear infimals (3.5-2"). By the fundamental differential inequality used previously we have

$$\|x_i(t)\| \leq \|x_{io}\| \exp\left[\int_o^t \|A_i + H_i\| ds\right] \tag{3.5-19}$$

or

$$\|x_i(t)\| \leq \|x_{io}\| \exp\|A_i\| t \exp\|H_i\| t \tag{3.5-20}$$

Again, if we assume that A_i have distinctive eigenvalues, we may take the same transformation $x_i = G_i y_i$. Then

$$\|y_i(t)\| \leq \|G_i^{-1} x_{io}\| \exp\|\Lambda_i\| t \exp\|H_i\| t, \ 0 \leq t$$

By requiring $\|x_i(t)\| \leq \|G_i\| \|y_i(t)\| \leq a_i, \ 0 \leq t \leq T$, a sufficient condition would then be

$$\exp\|H_i\| t \leq \frac{a_i}{\|G_i\| \|G_i^{-1}\| \|x_{io}\| \exp\|\Lambda_i\| t} \tag{3.5-21}$$

Since Λ_i are diagonal, $\|\Lambda_i\| = (\sum_i \lambda_{ji}^2)^{\frac{1}{2}}$. We must establish an explicit estimation of $\|H_i\|$. In order to do so, let us state the following lemma without proof:

<u>Lemma 3.5-16</u> [11] Let A be a matrix with eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_s$, $s \leq n$ not necessarily distinct .

Then $\|A^k\| \leq c_o k^{m-1} (\sum_{j=1}^{n} |\lambda_j|^{2k})^{\frac{1}{2}}$

where $\|A\| = (\sum_{i,j} |a_{ij}|^2)^{\frac{1}{2}} = (\mathrm{tr}\, AA')^{\frac{1}{2}}$

$$c_o = \prod_{i > j}^{1,s} (\lambda_i - \lambda_j)^{m_i m_j}$$

m_i, m_j are the multiplicity of any eigenvalue and $m = \max_{1 \leq t \leq s} (m_i)$. Therefore, for any H_i with distinct eigenvalues, we shall have

$$\|H_i\| \leq c_{io} (\sum_j |\bar{\lambda}_{ji}|^2)^{\frac{1}{2}} \tag{3.5-22}$$

Where $c_{io} = \prod_{j>k} (\bar{\lambda}_{ji} - \bar{\lambda}_{ki})$, $\bar{\lambda}_{ji}$ are the eigenvalues of H_i. With this derivation, formula (3.5-21) is reduced to

$$\exp\left(c_{io}\left(\sum_j |\bar{\lambda}_{ji}|^2\right)^{\frac{1}{2}} T\right) \leq \frac{a_i}{\|G_i\| \, \|G_i^{-1}\| \cdot \|x_{io}\| \exp\left(\sum_j \lambda_{ji}^2\right)^{\frac{1}{2}} T} \qquad (3.5\text{-}23)$$

To summerize, we have

Proposition 3.5-17 Let the condition of Proposition 3.5-12 be satisfied. Let the infimals be constant systems described by (3.5-2"). Let the matrices A_i, H_i have distince negative real eigenvalues. Then the conclusion of Proposition 3.5-12 holds if

(i) $\sum_{i=1}^{p} \beta_i a_i \leq \varepsilon / \|\Omega(T_o)\|$

(ii) inequality (3.5-23) is satisfied for $i = 1, 2, \ldots, p$.

The above analysis could equally be carried out for time-varying linear systems. Using a technique developed by Bernstein[6], $A_i(t)$ could theoretically be reduced to a tri-diagonal form, which facilitates an explicit estimation of $\|A_i(t)\|$. Nevertheless, since there exists no algorithm which computes the eigen-function $\bar{\lambda}_{ji}(t)$ of an arbitrary matrix $H_i(t)$, it is practically useless to have a formula like (3.5-23).

A similar formula could also be established for a constant system by using an inequality due to

Wazewski[38]. Let us state it without proof as a lemma

Lemma 3.5-18 Let $\dot{x} = A(t)x(t)$. Let $B(t) = \frac{1}{2}(A(t) + A'(t))$. let $\lambda(t)$ and $\sigma(t)$ be the minimum and maximum eigenvalues of $B(t)$. Then

$$\|x_0\| \exp \int_0^t \lambda(s)ds \leq \|x(t)\| \leq \|x_0\| \exp \int_0^t \sigma(s)ds$$

Let us again consider the time-invariant infimals described by (3.5-2") which employ feedback-type controls. Let

$$P_i = \tfrac{1}{2}(A_i + H_i + A_i' + H_i')$$

Let the eigenvalues of A_i, H_i be λ_{ji}, $\bar{\lambda}_{ji}$ respectively. For physical consideration, we shall restrict that the matrices $A_i + H_i$ have distinct eigenvalues σ_{ji} with negative real parts. Let σ_i^* denote $\max_j(\sigma_{ji})$. Then, since

$$\sum_j(a_{ijj} + h_{ijj}) = \sum_j \sigma_{ji} < 0, \quad i = 1, 2, \ldots, p$$

we have

$$|\sigma_i^*| < |\sum_j(a_{ijj} + h_{ijj})|$$

where a_{ijj}, h_{ijj} are diagonal elements of matrices A_i, H_i

respectively. Therefore,

$$|\lambda_i^*| < 2|\sum_j (a_{ijj} + h_{ijj})| \leq 2(|\sum_j a_{ijj}| + |\sum_j |h_{ijj}|)$$

where λ_i^* are the maximum eigenvalues of matrices P_i. Using Lemma 3.5-19

$$\|x_i(t)\| \leq \|x_{io}\| \exp\left[\int_o^t \lambda_i^* ds\right]$$

$$\leq \|x_{io}\| \exp\left[\int_o^t |\lambda_i^*| ds\right]$$

$$\leq \|x_{io}\| \exp|\lambda_i^*| t$$

$$\leq \|x_{io}\| \exp 2|\sum_j a_{ijj}| t \ \exp 2|\sum_j h_{ijj}| t$$

By requiring $\|x_i(t)\| \leq a_i$, $0 \leq t \leq T_o$, we have another sufficient condition

$$\exp 2|\sum_j h_{ijj}| T_o \leq \frac{a_i}{\|x_{io}\| \exp 2|\sum_j a_{ijj}| T_o} \qquad (3.5\text{-}24)$$

To summerize, we have

<u>Proposition 3.5-19</u> Let the conditions of Propostion 3.5-12 be satisfied. Let the infimals be constant systems described by (3.5-2"). Let the matrices A_i, H_i have the properties assumed above. Then the conclusion of Proposition 3.5-12 holds if

(i) $\sum_{i=1}^{p} \beta_i a_i \leq \varepsilon / \|\Omega(T_o)\|$

(ii) inequality (3.5-24) be satisfied for i = 1, 2, ..., p.

Combining the utilization of Scheme of Coordination 3.5-13 and Propositions 3.5-14, 3.5-15, 3.5-17 and 3.5-19, the supremal has several coordination schemes at his disposal to guide the performance of the two-level linear dynamical system.

We have explored quite extensively when the infimals are commanded to use feedback type controls. The primary rationale in the above analysis is the use of exponential-asymptotically stable property of the state functions $x_i(t)$. This is by no means unreasonable in the case of linear systems. This property could certainly be enhanced by the use of norm-bounded controls, which will be studied in the next section.

3.6 Coordination Using Image, Constraint, and Goal Interventions

In Propostion 3.4-2, a sufficient condition was established for the ε-controllability of supremal. The condition

$$\sum_{i=1}^{p} \|\omega_i(T)\|^2 \le \varepsilon^2 \tag{3.4-11}$$

where $\omega_i(T_o) = \int_o^{T_o} \Phi(T_o, s)B_i(s)x_i(s)ds$, is essentially an end point condition. Therefore, in order to achieve the over-all goal defined previously, restrictive properties must be imposed on $\Phi(t)$, $B_i(t)$, and $x_i(t)$ so that inequality (3.4-11) can be guaranteed. The main theme of this section is to investigate such possibilities.

Let us again write down the system structure.

The supremal:

$$\dot{x} = A(t)x(t) + C(t)m(t) + \sum_{i=1}^{p} B_i(t)x_i(t), \quad x(0) = x_o \tag{3.6-1}$$

$$\dot{x} = A(t)x(t) + C(t)m(t) \qquad x(0) = x_o \tag{3.6-1'}$$

The infimals:

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t), \quad x_i(0) = x_{io} \tag{3.6-2}$$

Let us assume that the unperturbed supremal (3.6-1') is controllable at time T_o with respect to a given space M of admissible controls and a pair of states (x_o, x_d), where $x_d = 0$ when-ever unspecified.

The problem to be solved is then: how the supremal can use interventions to coordinate the behaviors of the infimals so that the supremal itself can remain controllable or ε-controllable for a given $\varepsilon > 0$ with respect to the fixed control space M and the pair of states (x_o, x_d).

In order to conform with the general scheme of "parametric coordination", we want to find a criterion which is suitable for use on any typical infimal. From formula (3.4-11), it appears that such a scheme is possible if we can have $\|\omega_i(T)\| \leq b_i$ for all $i = 1, 2, \ldots, p$, while $\sum_i b_i^2 \leq \varepsilon^2$.

In this section we shall assume that the matrices $B_i(t)$ are known and fixed. Therefore, we may let

$$y_i(t) = B_i(t)x_i(t), \quad i = 1, 2, \ldots, p \qquad (3.6\text{-}3)$$

Thus

$$\omega_i(T) = \int_o^T \Phi(T,s)y_i(s)ds, \; i = 1, 2, \ldots, p \qquad (3.6\text{-}4)$$

Clearly, the points $\omega_i(T) \in R^n$ will be completely determined whenever $\Phi(t)$ and $y_i(t) = B_i(t)x_i(t)$ are known. We shall apply some of the results obtained in section 2.3

and section 2.4 to solve the coordination problem.

In last section, the essential restriction placed on the infimals was the requirement that the state functions $x_i(t)$ be uniformly norm-bounded on a time interval $[0, T]$. This kind of constraint may be too restrictive for practical purposes. As a consequence, we shall try in this section to substitute the boundedness requirement by conditions commonly assumed for practical systems, such as stability and controllability.

Throughout this section, we shall assume that the supremal is b.i.b.o. stable. Furthermore following the practice of section 2.3, the fixed time requirement will be relinquished as we shall see its necessity.

<u>Proposition 3.6-1</u> Let the supremal (3.6-1') be controllable at time T with respect to a given M and $(x_o, 0)$. Then the supremal (3.6-1) will be ε-controllable for a given $\varepsilon > 0$ at some time $t \geq T$ if the infimals with zero control are asymptotically stable.

<u>Proof:</u> As a consquence of Proposition 2.4-14, it suffices to show that $\|\Sigma y_i(t)\| \leq \alpha_1 e^{-\alpha_2 t}$, where α_1 and α_2 are positive constants. Since

$\|\sum_i y_i(t)\| \leq \sum_i \|y_i(t)\|$, it suffices in turn to show that

$\|y_i(t)\| \leq \alpha_{i1} e^{-\alpha_{i2}t}$ where $\alpha_{i1}, \alpha_{i2} > 0$. The matrices $B_i(t)$ are assumed to be continuous, thus we may show instead that $\|x_i(t)\| \leq \alpha'_{i1} e^{-\alpha'_{i2}t}$, $\alpha'_{i1}, \alpha'_{i2} > 0$. But this is exactly the case following the hypothesis and Lemma 2.3-13. This completes the proof.

As it is well known, for a linear dynamical system, the term asymptotic stability is practically synonymous to stability. Since this is one of the most common properties assumed for a practical system, the above proposition does induce a profound implication. One short coming to be overcome is how to estimate the particular t at which the supremal (3.6-1) will be ε-controllable under the perturbation of infimals.

<u>Proposition 3.6-2</u> Let the supremal (3.6-1') be controllable at time T with respect to a given M and $(x_o, 0)$. Then the supremal (3.6-1) will be ε-controllable for a given $\varepsilon > 0$ at some time $t \geq T$ with respect to M and $(x_o, 0)$ if the infimals are

(i) b.i.b.o. stable

(ii) controllable with respect to M_i and $(x_{io}, 0_i)$.

<u>Proof:</u> Following the proof of last proposition, it

suffices to show that $\|x_i(t)\| \leq \alpha_{i1} e^{-\alpha_{i2} t}$ for for $i = 1, 2, \ldots, p$. Let $\bar{m}_i(t) \in M_i$ be the control functions which transfer the respective infimals from x_{io} to 0. Clearly, the selection of control functions

$$m_i(t) = \begin{cases} \bar{m}_i(t) & 0 \leq t \leq T_i \\ 0 & T_i < t \end{cases}$$ will accomplish the proof,

because $x_i(t) = 0$ for $t \geq T_i$ following the property of b.i.b.o. stability. Since $\|x_i(t)\|$ are bounded over a finite time interval and zero otherwise, it is always possibe to find positive constant α_{i1}, α_{i2} such that $\|x_i(t)\| \leq \alpha_{i1} e^{-\alpha_{i2} t}$. This completes the proof.

One thing to be noted in above proposition is that the time instant T_i is not required to be fixed. Clearly, this is a consequence of the assumption that the subsystems be asymptotically stable.

Corollary 3.6-3 Let the conditions of Proposition 3.6-2 be satisfied. Then the same conclusion holds if the infimals are b.i.b.o. stable and controllable with respect to M_i and (x_{io}, x_{id}) where $x_{id} \neq 0$.

Proof: Let $\bar{m}_i(t) \in M_i$ be the control function which transfers the respective infimal from x_{io} to x_{id}.

Clearly, the selection of control function

$$m_i(t) = \begin{cases} \bar{m}_i(t) & 0 \le t \le T_i \\ 0 & T_i < t \end{cases}$$ will accomplish the proof.

Because, let $x_i(T_i) = x_{id}$, the function $\|x_i(t)\|$ for $t \geq T_i$ exhibits the exponential-asymptotic stable preperty. Following the argument of previous proposition, the complete proof follows.

As we can see, all the above propositions depend on the assumptions of asymptotical stability and uniform controllability. The requirement that the supremal be transferrable from an initial state to the origin is nothing but an assurance of uniform controllability. Since we have shown in section 2.3 that the property of uniform controllability is attainable in less restrictive circumtances, the above requirement should be reducible, which is evident in the following proposition. First, let us recall that

$$L(T, M) = \{ \theta(T): \theta(T) = \int_0^T \Phi(T, s)C(s)m(s)ds, \quad m \in M\}$$

<u>Proposition 3.6-4</u> Let supremal (3.6-1') be controllable at T with respect to a given M and (x_o, x_d). Then supremal (3.6-1) will be ε-controllable for a given $\varepsilon > 0$ at some time instant $t \geq T$ with respect to M and (x_o, x_d) if

(i) $\|x_d\| + \sup_{T \le t} \|\Phi(t)x_o\| \le \min_{\theta(T) \in \partial L(T,M)} \|\theta(T)\|$.

(ii) the infimals are b.i.b.o. stable and controllable with respect to M_i and $(x_{io}, 0)$.

Proof: Condition (i) and the assumption that supremal (3.6-1') is b.i.b.o. stable guarantee that sumpremal (3.6-1') is uniformly controllable, which follows from Corollary 2.3-20. The remain of the proof follows from the proof of Proposition 3.6-2.

As a consequence, we have also

Corollary 3.6-5 Let the conditions of Proposition 3.6-4 be satisfied. Then the same conclusion holds if

(i) $\|x_d\| + \sup_{T \le t} \|\Phi(t)x_o\| \le \min_{\theta(T) \in \partial L(T,M)} \|\theta(T)\|$

(ii) the infimals are b.i.b.o. stable and controllable with respect to M_i and (x_{io}, x_{id}), where $x_{id} \ne 0$.

Proof: Same as in Corollary 3.6-3.

As we have seen in above propositions, the condition that the infimals be controllable is quite essential. Otherwise the state functions $x_i(t)$ will not necessarily exhibit the exponential-asymptotically stable property. A natural question arises: Can the fixed time requirement be retained if additional constraints are imposed onto the controllability condition of the infimals? The answer appears to be yes. But, since a formal proof can not be provided, we shall only state it as

Conjecture 3.6-6 Let the supremal (3.6-1') be controllable at T with respect to a given M and (x_o, x_d). Then supremal (3.6-1) will be ϵ-controllable at T with respect to M and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable.

(ii) the infimals are controllable at time T_i, which are sufficiently small, with respect to M_i and $(x_{io}, 0)$.

Proof(heuristic): The proof relies upon whether we could obtain $\omega_i(T)$ by regulating T_i such that

$\sum_i \|\omega_i(T)\|^2 \leq \epsilon^2$. The extreme case is $x_i(t) \equiv 0$ which automatically satisfies the condition. On the other hand, we know that the infimals are uniformly controllable. Thus $x_i(t) \equiv 0$ for $t \geq T_i$. The state functions have

the form $x_i(t) = \begin{cases} x_i(t) & 0 \leq t \leq T_i \\ 0 & T_i < t \end{cases}$

Since $\omega_i(T) = \int_0^T \Phi(T, t)B_i(t)x_i(t)dt = \int_0^{T_i} \Phi(T,t)B_i(t)x_i(t)dt$

in which both $\Phi(T, t)$ and $x_i(T)$ have the exponential-asymptotically stable property, it should be possible to select T_i so small that $\sum_i \|\omega_i(T)\|^2 \leq \varepsilon^2$.

The above propositions provide a very broad basis for the coordination to be performed by supremal in order to achieve the over-all goal. We may formulate a general guide-line as follows.

<u>Scheme of Coordination 3.6-7</u>: The uniformly controllable supremal will retain its ε-controllability by attending to these procedures:

(i) Requiring that the infimals to have b.i.b.o. stable property.

(ii) Determining for each infimal the spaces of admissible controls, the space of admissible initial conditions, and the target states.

(iii) Requiring that the infimals be transferred to the target states at some pre-selected time T_i.

Another legitimate question to ask at this moment is: Since $\omega_i(T) \to 0$ as $T \to \infty$ in the present case, can the supremal use its control action to offset the effects of $\omega_i(T)$ so that to make itself strictly controllable at some finite time interval? The answer seems to be negative in general. For instance, we do not exclude the case when the supremal must use all the available control energy in order to maintain its uniform controllability. In this case, it is clearly impossible to obtain extra control energy to offset the effect of $\omega_i(T)$. We shall further consider this question in next section.

Now we shall examine the special case when the infimals employ feedback type controls. Following previous practices, we may rewrite equation (3.6-2) as

$$\dot{x}_i = A_i(t)x_i(t) + H_i(t)x_i(t) \tag{3.6-4}$$

where $H_i(t)$ are feedback matrices to be determined. For this type of control, we have the fundamental result.

<u>Proposition 3.6-8</u> Let supremal (3.6-1') be controllable at T with respect to a given M and $(x_o, 0)$. Then, for a given $\varepsilon > 0$, supremal (3.6-1) is ε-controllable at some $t \geq T$ with respect to M and $(x_o, 0)$ if

(i) the infimals (3.6-2) are b.i.b.o. stable

(ii) the feedback matrices satisfy $\|H_i(t)\| \leq \alpha_{i3}$, while α_{i3} is a constant depending on $A_i(t)$.

<u>Proof</u>: Following the proof of previous propositions, it suffices to show that the solutions of (3.6-4) have

the exponential-asymptotically stable property. Since $H_i(t)$ are adjustable to needs, Lemma 3.5-10 and condition (i) ensure the last requirement. This completes the proof.

A synthesis algorithm of obtaining such feedback matrices for general linear systems is not an easy task. However, when the infimals are restricted to be time-invariant linear systems, a procedure of design is possible as demonstrated in the previous section.

In order to improve system performance in the sense that supremal (3.6-1) may achieve its goal of ε-controllability for a least time interval, one would intuitively feel that an increasing in the decaying rate of the state function $x_i(t)$ should help. This could certainly be done by the judicial choice of feedback matrices $H_i(t)$. The supporting motive for this observation is no different from that of conjecture 3.6-6.

3.7 Coordination by Using Redundant Control Energy

The ideas of using redundant control energy to counteract the effects of disturbances is a natural one.

Based on this conception, the fundamental result Proposition 2.4-16 was derived. In the propsed two-level linear dynamical system, interaction among the subsystems is considered as additive disturbance. We would then expect that the above conception should provide us a powerful tool in dealing with the coordination problem.

Let us consider the following system:

The supremal:

$$\dot{x} = A(t)x(t) + C(t)m(t) + \sum_i B_i(t)x_i(t) \qquad (3.7\text{-}1)$$

$$\dot{x} = A(t)x(t) + C(t)m(t) \qquad (3.7\text{-}1')$$

$$M = \{ m(t): \ \|m\|_{2 \text{ or } \infty} \leq k, \ t \in J \}$$

$$M' = \{ m(t): \ \|m\|_{2 \text{ or } \infty} \leq k', \quad t \in J \}, \quad k < k'$$

The infimals:

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) \qquad (3.7\text{-}2)$$

$$M_i = \{ m_i(t): \ \|m_i\|_{2 \text{ or } \infty} \leq k_i, \quad t \in J \}$$

We shall assume that supremal (3.7-1') is controllable with respect to given M and (x_o, x_d). As a consequence of Proposition 2.4-16, the controllability of supremal (3.7-1) with respect to M and (x_o, x_d) will be guaranteed provided that the following inequality is satisfied:

$$\sum_i \beta_i \|x_i(t)\| \le \gamma(k' - k) \qquad \text{for } t \in J \qquad (3.7\text{-}3)$$

This requires that all state functions $x_i(t)$ of the infimals be uniformly bounded on some finite time interval. This possibility has been shown previously. However, in order to get more explicit and useful results, it is necessary that the infimals possess certain properties.

Let us assume that the infimals are all b.i.b.o. stable. Then, there exist positive constants, b_{i1}, b_{i2}, $i = 1, 2, \ldots, p$ such that

$$\|\Phi_i(t)\| \le b_{i1} \qquad \text{for} \quad t \ge 0$$

$$\int_0^t \|\Phi_i(t, s)C_i(s)\| ds \le b_{i2} \qquad \text{for} \quad t \ge 0$$

For every solution $x_i(t)$ of (3.7-2) we have

$$\|x_i(t)\| \le b_{i1}\|x_{io}\| + b_{i2}k_i \qquad (3.7\text{-}4)$$

Consequently, the following result is established

<u>Proposition 3.7-1</u> Let supremal (3.7-1') be controllable at T with respect to a given M and (x_o, x_d). Then supremal (3.7-1) will be controllable at T with respect to $M' \supset M$ and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable.

(ii) $\sum_i \beta_i (b_{i1}\|x_{io}\| + b_{i2}k_i) \le \gamma(k' - k)$

<u>Proof</u> This proposition is a direct consequence of Proposition 2.4-16 and formula (3.7-3)(3.7-4).

When the two-level system (3.7-1)(3.7-2) is given, the constants β_i, b_{i1}, b_{i2}, γ are fixed constants while $\|x_{io}\|$ and k_i are variables. Therefore, the supremal may carry out its coordinative actions by adjusting these variables. A scheme of coordination based on Proposition 3.7-1 may be constructed as follows

<u>Scheme of Coordination 3.7-2</u> Suppose supremal (3.7-1') is controllable at T with respect to a given M and (x_o, x_d). Then the over-all goal of the two-level linear system (3.7-1)(3.7-2) will be achieved if the following coordinative procedure is followed:

(i) the supremal uses image intervention to ensure that the infimals are b.i.b.o. statble.

(ii) the supremal commands the infimals to send up information concerning the constants b_{i1} and b_{i2}.

(iii) the supremal computes the values k_{io} and k_i so that condition (ii) in Proposition 3.7-1 is satisfied.

(iv) the supremal uses constraints interventions by assigning the spaces M_i and X_{io} for each infimal. This is done by giving the constants, k_{io} and k_i.

The above result can certainly be refined when additional conditions are imposed on the activities of the infimals. For instance, we have

Proposition 3.7-3 Let the supremal (3.7-1') be controllable at T with respect to a given M and (x_o, x_d). Then, supremal (3.7-1) will be controllable at T with respect to $M' \supset M$ and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable.

(ii) the infimals are controllable with respect to given M_i and $(x_{io}, 0)$.

(iii) $\Sigma \beta_i \alpha_{i1} \|x_{io}\| \le \gamma(k' - k)$, where α_i are constants depending on $A_i(t)$.

Proof Let the solution of $\dot{x}_i = A_i(t)x_i$ be $\bar{x}_i(t)$. As a consequence of condition (i), we know that there exist positive constants α_{i1}, α_{i2} such that

$\|\bar{x}_i(t)\| \le \alpha_{i1} e^{-\alpha_{i2}t}$ for $t \ge 0$. Let $x_i(t)$ denote the solution of (3.7-2). Then condition (ii) ensures that there exists a $m_i \in M_i$ such that $\|x_i(t)\| \le \alpha_{i1} e^{-\alpha_{i2}t} \le \alpha_{i1}$. The conclusion follows from Proposition 2.4-16 and condition (iii). This completes the proof.

Since condition (iii) in the above Proposition is quite similar to condition (ii) of Proposition 3.7-1, the Scheme of Coordination 3.7-2 needs only minor modification to give:

Scheme of Coordination 3.7-4

(i) the supremal uses image intervention to ensure that the infimals are b.i.b.o. stable.

(ii) the supremal commands the infimals to send up informations concerning the constants α_{il}.

(iii) the supremal computes the values k_{io} so that condition (iii) in Proposition 3.7-3 is satisfied.

(iv) the supremal uses constraint intervention to determine M_i for each infimal and commands the infimals by using goal intervention to behave in such a way that condition (ii) of Proposition 3.7-3 is satisfied.

When the analysis for the general case of norm-bounded controls has been studied as in the above, it is natural to turn our attention to the case when the infimals are linear feedback control systems. Therefore, the infimals will have in general the following mathematical representation:

$$x_i = A_i(t)x_i(t) + H_i(t)x_i(t) \qquad (3.7\text{-}2')$$

where $H_i(t)$ are continuous matrices to be designed. Since $H_i(t)$ is a realization of the control action of system (3.7-2), we shall still call the restrictions imposed on $H_i(t)$ as constraint interventions. The system $x_i = A_i(t)x_i(t)$ are still assumed to be unifornly asymptotically stable. Therefore, after we denote by

$\bar{x}_i(t)$ the solutions of $\dot{x}_i = A_i(t)x_i(t)$, $\bar{x}_i(t)$ will satisfy the following property:

$$\|\bar{x}_i(t)\| \leq \alpha_{i1}\, x_{io}\, e^{-\alpha_{i2}t}, \quad \text{for} \quad t \geq 0 \qquad (3.7\text{-}5)$$

where α_{i1}, α_{i2} are positive constants.

<u>Proposition 3.7-5</u> Let supremal (3.7-1') be controllable at some T with respect to a given M and (x_o, x_d). Then supremal (3.7-1) will be controllable at T with respect to $M' \supset M$ and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable.

(ii) $\|H_i(t)\| \leq \alpha_{i3}$ for $t \geq 0$ where $\alpha_{i3} > 0$ depending on $A_i(t)$.

(iii) $\Sigma \beta_i \alpha_i \|x_{io}\| \leq \gamma(k' - k)$, where α_i are positive constants depending on $A_i(t)$ and $H_i(t)$.

<u>Proof</u> Condition (i) guarantees that there are positive constants α_{i1}, α_{i2} such that inequality (3.7-5) is satisfied. Let $\alpha_{i1}\alpha_{i3} < \alpha_{i2}$, Lemma 3.5-10 ensures that systems (3.7-2') are uniformly asymptotically stable. Therefore, when an appropriate feedback matrix $H_i(t)$ is chosen, the solution $x_i(t)$ of system(3.7-2') must satisfy $\|x_i(t)\| \leq \alpha_i\, x_{io}\, e^{\pm\nu_i t}$ for $t \geq 0$ and $\alpha_i > 0$, $\nu_i > 0$. Or $\|x_i(t)\| \leq \alpha_i \|x_{io}\|$ for $t \geq 0$. Then, the conclusion of this proposition follows from condition (iii) and Proposition 2.4-16. This completes the proof.

Again, the scheme of coordination previously stated can be used with minor modification. We shall not repeat the scheme of coordination here.

Obviously, the analysis for general time-varying linear dynamical systems, as we have in the above, provides only a guideline for the design and coordination problems. Any detailed algorithm for design and coordination must be provided when a specific system is given. Referring to the present status of control systems theory, this appears to be possible only when the systems considered are strictly time-invariant. For such cases, some results were obtained in section 3.5 for the linear feedback control systems. It should be obvious, however, that most results obtained therein are adaptable to fit into the study of this section. In fact, most propositions of section 3.5 are valid when two modifications are made in the statement of those propositions, namely: (i) the statement "ε-controllable at T with respect to M and (x_o, x_d)" is substituted by "controllable at T with respect to $M' \supset M$ and (x_o, x_d)"; and (ii) the constant $\varepsilon / \|\Omega(T)\|$ is substituted by $\gamma(k' - k)$ in the corresponding inequalities. Therefore, we shall not pursue further the improvisation of system performance by using redundant control energy for the case when inequality (3.4-8) is used as a basis of coordination. Rather, it is interesting to see how the same problem can be solved when inequality (3.4-11) is to be used as the basis of coordination as was done in section3.6.

In last section we have seen the possibility of maintaining the ε-controllability of a uniformly controllable supremal when it is subject to a disturbance function having the exponential-asymtotically stable property. The limitation on the conclusion of ε-controllability rather than controllability was the joint consequence of two causes, namely: (i) the possibility of exhausting all available control energy in order to maintain the uniform controllability of the supremal; and (ii) the effect of disturbance will die down only as time t approaching infinite. Intuitively, one would expect that the conclusion of ε-controllability could be improved to that of strict controllability if either one of the above two causes is eliminated. Clearly, the introduction of redundant control energy has eliminated the first cause because the control energy available to the supremal is by definition not exhaustible in the sense mentioned above. In fact, the introduction of redundant control energy leads to the expected conclusion:

<u>Proposition 3.7-6</u> Let supremal (3.7-1') be uniformly controllable for $t \geq T$ with respect to a given M and

(x_o, x_d). Let supremal (3.7-1) be subjected to the disturbance function $f(t)$ which has the exponential-asymptotically stable property. Then supremal (3.7-1) will be controllable at some $t \geq T$ with respect to $M' \supset M$ and (x_o, x_d).

Proof By hypothesis we know that $k' - k > 0$. As a consequence of the assumption of uniform controllability, it suffices to show that there exists a control function $m_o(t)$ in $M'' = \{m(t): \|m(t)\|_{2 \text{ or } \infty} \leq k' - k\}$ such that $\omega(t) = \int_o^t \Phi(t,s)f(s)ds = \int_o^t \Phi(t,s)C(s)m_o(s)ds$ for some $t \geq T$. Similar to previous analysis, we know that $\|\omega(t)\| \to 0$ as $t \to \infty$. On the other hand, it is known that $L(t, M'')$ is a closed symmetric convex set in X which has the property $L(t_1, M'') \subset L(t_2, M'')$ for any $t_2 > t_1 > 0$. Clearly, $\omega(t') \in L(t', M'')$ for some $t' \geq 0$ and remains so for all $t > t'$. This completes the proof.

Equiped with this result, the findings of last section can now be strengthened. In the sequel, we shall assume that supremal (3.7-1') is b.i.b.o. stable.

Proposition 3.7-7 Let supremal (3.7-1') be controllable at T with respect to a given M and $(x_o, 0)$. Then supremal (3.7-1) will be controllable at some $t \geq T$ with respect to $M' \supset M$ and $(x_o, 0)$ if

(i) the infimals are b.i.b.o. stable.

(ii) $m_i(t) = 0$ for $t \geq 0$, $i = 1, 2, \ldots, p$.

Proof From Proposition 2.3-16 and the hypothesis, we know that supremal (3.7-1') is uniformly controllable. As a consequence of Proposition 3.7-6, it then suffice to show that the function $\sum_i B_i(t)x_i(t)$ has the exponential-asymptotically stable property. But this requirement was proved for the present case as in the proof of Proposition 3.6-1. This completes the proof.

Proposition 3.7-8 Let supremal (3.7-1') be controllable at T with respect to a given M and $(x_o, 0)$. Then supremal (3.7-1) will be controllable at some $t \geq T$ with respect to $M' \supset M$ and $(x_o, 0)$ if

(i) the infimals are b.i.b.o. stable.

(ii) the infimals are controllable with respect to given M_i and $(x_{io}, 0)$.

Proof Similar to the proof of the previous proposition, it suffices to show that the function $\sum_i B_i(t)x_i(t)$, or the state functions $x_i(t)$, has the exponential-asymptotically stable property. From Proposition 2.3-16, conditions (i) and (ii) ensure that each infimal is uniformly controllable with respect to M_i and $(x_{io}, 0)$. This implies that $x_i(t) = 0$ at some T_i and all $t \geq T_i$, $i = 1, 2, \ldots, p$. Thus, $x_i(t)$ must have the exponential-asymptotically stable property as required. This completes the proof.

This proposition can be strengthened for the case when the supremal (3.7-1') is controllable with a desired state $x_d \neq 0$. We have

<u>Proposition 3.7-9</u> Let supremal (3.7-1') be controllable at T with respect to a given M and (x_o, x_d). Then supremal (3.7-1) will be controllable at some $t \geq T$ with respect to $M' \supset M$ and (x_o, x_d) if

(i) $\|x_d\| + \sup_{T \leq t} \|\Phi(t)x_o\| \leq \min_{\theta(T) \in \partial L(T,M)} \|\theta(T)\|$.

(ii) the infimals are b.i.b.o. stable.

(iii) the infimals are controllable with respect to given M_i and $(x_{io}, 0)$.

<u>Proof</u> From Corollary 2.3-20, condition (i) ensures that supremal (3.7-1') is uniformly controllable for $t \geq T$. Thus, a complete proof follows from the proof of the previous proposition.

For the class of infimals in which feedback type controls are employed, which have the mathematical representation (3.7-2'), we have:

<u>Proposition 3.7-10</u> Let supremal (3.7-1') be controllable at T with respect to a given M and (x_o, x_d). Then supremal (3.7-1) will be controllable at some $t \geq T$ with respect to $M' \supset M$ and (x_o, x_d) if

(i) $\|x_d\| + \sup_{T \leq t} \|\Phi(t)x_o\| \leq \min_{\theta(T) \in \partial L(T,M)} \|\theta(T)\|$.

(ii) the infimals (3.7-2') are uniformly asymptotically stable.

Proof By condition (ii), solution $x_i(t)$ of (3.7-2') has the exponential-asymptotically stable property. The proof follows as a consequence of the fundamental Proposition 3.7-6.

After some thinking on the nature of the problem, it becomes quite obvious that the implications of Conjecture 3.6-6 could equally be applied to the present case. In fact, by increasing the size of the space M', which leads to an enlarged set L(T, M"), the same consequence can be reached. Hopefully, many improvements to the above results can then be obtained.

CHAPTER IV

THE CASE WHEN THERE IS DIRECT INTERACTION AMONG INFIMALS

4.1 Introduction

In this chapter the same problem defined and studied in Chapter III will be studied. The two-level system concerned here, however, contains a set of directly interacting infimals.

In sections 4.2 and 4.3, coordination is studied using respectively the two basic guidelines developed in section 3.4.

In section 4.4, the concept of using redundant control energy will be explored to strengthen those results obtained in sections 4.2 and 4.3.

4.2 Coordination and Norm-Bounded State Functions

When the fundamental inequality (3.4-8) is used as a basis of coordination, as will be done in this section, a uniform bound on the state functions $x_i(t)$ of the infimals must be established. Because of the existance of interaction among the infimals, the task of establishing such bounds is more complicated than before. First of all, let us write down the mathematical model of the two-level system to be studied in this section. We have

The supremal:

$$\dot{x} = A(t)x(t) + C(t)m(t) + \Sigma B_i(t)x_i(t) \tag{4.2-1}$$

$$\dot{x} = A(t)x(t) + C(t)m(t) \tag{4.2-1'}$$

The infimals:

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) + \sum_{j\neq i} D_{ij}(t)x_i(t) \tag{4.2-2}$$

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) \tag{4.2-2'}$$

where the equations will have the usual definitions as previously defined.

Similar to the practice in section 3.5, we shall again assume that the unperturbed supremal (4.2-1') is controllable at some time T with respect to a given space M of admissible controls and a pair of states (x_o, x_d). Then the major question to be answered in this section is the solution to the same control problem of the supremal using inequality (3.4-8) as a basis. Namely: Given $\varepsilon > 0$, how the supremal can use different forms of interventions to coordinate the activities of the infimals in order that the system (4.2-1) is ε-controllable at T with respect to M and (x_o, x_d).

From a mathematical viewpoint, the matrices $B_i(t)$ and $D_{ij}(t)$ would have the same meaning, i.e., they represent the interaction among systems. However, since the activities of the infimals are to be limited by the coordinative actions of the supremal, we shall assume that the matrices $D_{ij}(t)$ are to be designed to fit the command of the supremal. This practice indicates the use of interaction intervention.

There are two possible approaches of designing the interactions, which stipulate entirely different implication in the design philosophy. One of the approaches is to condister the term $\Sigma D_{ij}(t)x_j(t)$ in equation (4.2-2) as an outside disturbance acted on the particular infimal i. In this case, the objective of interaction intervention is to place a uniform bound on this disturbance function, for instance

$$\|\sum_{j\neq i} D_{ij}(t)x_j(t)\| \leq \delta_i, \quad t \in J, \quad i = 1, 2, \ldots, p \tag{4.2-3}$$

By so doning, the design of interaction matrices $D_{ij}(t)$ becomes a quite simple matter. In fact, the physical realization of the matrices $D_{ij}(t)$ in this particular situation will be some saturation devices. We notice that this kind of interaction intervention is often used in the real world. For instance, the protective circuit-breaker in an electricity transmission line is usually designed to handle a fixed amount of load.

In addition to the advantage mentioned above, this approach will reduce quite significantly the difficulties in analysis. For instance, we may introduce the following new function without loss of generality.

$$f_i(t) = \sum_{j \neq i} D_{ij}(t)x_j(t) \qquad t \in J$$

$$\|f_i(t)\| \leq \delta_i \qquad (4.2\text{-}4)$$

In this case, $f_i : t \rightarrow f_i(t)$ is clearly a Lebesque integrable function which is uniformly bounded on a a compact time interval J. Consequently, its effect on the behavior of the infimal can be assessed using the results obtained in sections 2.3 and 2.4.

In many cases, however, the above approach is not realizable, i.e., the interaction between the infimals cannot be realized as saturation devices. For instance, the transmission tie line of two electricity pooling areas is usually a device of continuous flow. Under these circumstances, the interaction matrices $D_{ij}(t)$

will be operators satisfying certain constraints. The influence of interaction on system behavior is now due to the state functions $x_j(t)$ under the transformations $D_{ij}(t)$. Therefore, the design of interaction in this approach becomes more or less a structural problem. In other words, the design criteria of $D_{ij}(t)$ is closely associated with the system matrices $A_i(t)$ and control matrices $C_i(t)$. We shall limit ourself by restricting $D_{ij}(t)$ to be matrices with continuous time functions as elements.

If the second approach is necessary, we may rewrite the equations of the infimals by introducing the following notations:

$$\mathcal{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdots\cdots \\ x_p(t) \end{pmatrix}, \quad \mathcal{A}(t) = \begin{pmatrix} A_1(t) & 0 & \cdots & 0 \\ 0 & A_2(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_p(t) \end{pmatrix}$$

$$\mathcal{C}(t) = \begin{pmatrix} C_1(t) & 0 & \cdots\cdots & 0 \\ 0 & C_2(t) & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & C_p(t) \end{pmatrix}$$

$$\mathcal{D}(t) = \begin{pmatrix} 0 & D_{12}(t) & \cdots\cdots & D_{1p}(t) \\ D_{21}(t) & 0 & \cdots\cdots & D_{2p}(t) \\ \cdots & \cdots & \cdots & \cdots \\ D_{p1}(t) & D_{p2}(t) & \cdots\cdots & 0 \end{pmatrix}$$

$$\mathcal{X}_o = \begin{pmatrix} x_{1o} \\ x_{2o} \\ \cdots \\ x_{po} \end{pmatrix} \qquad \mathcal{M}(t) = \begin{pmatrix} m_1(t) \\ m_2(t) \\ \cdots\cdots \\ m_p(t) \end{pmatrix} \tag{4.2-5}$$

Then, the infimals become a single integrated system represented by

$$\dot{\mathcal{X}} = \mathcal{A}(t)\mathcal{X}(t) + \mathcal{C}(t)\mathcal{M}(t) + \mathcal{D}(t)\mathcal{X}(t) \tag{4.2-6}$$

Since $\|\mathcal{M}\| \leq \sum_i \| m_i \|$, it is clear that the admissible controls $\mathcal{M}(t)$ for system (4.2-6) will be elements in some subset of the following set:

$$\mathcal{M} = \{ \mathcal{M}(t): \quad \|\mathcal{M}\|_{2 \text{ or } \infty} \leq \sum_i k_i, \quad t \in J \} \tag{4.2-7}$$

First, let us investigate the situation when the first approach is permissible. In this case, the interaction among the infimals is represented by equation (4.2-4). The mathematical description for each infimal is now

$$\dot{x} = A_i(t)x_i(t) + C_i(t)m_i(t) + f_i(t) \tag{4.2-8}$$

where $f_i(t)$ is subject to the constraint (4.2-4)

Proposition 4.2-1 Let supremal (4.2-1') be controllable at some T with respect to a given M and (x_o, x_d). Then, for some given $\varepsilon > 0$, supremal (4.2-1) is ε-controllable at time T with respect to M and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable.

(ii) $\Sigma \beta_i (b_{i1} x_{io} + b_{i2} k_i + b_{i3} \delta_i) \leq \varepsilon / \|\Omega(T)\|$

where b_{i1}, b_{i2}, b_{i3} are positive constants depending on $A_i(t)$ and $C_i(t)$.

Proof As a consequence of condition (i), we know that there are positive constants b_{i1}, b_{i2} such that $\|\Phi_i(t)\| \leq b_{i1}$ and $\int_o^t \|\Phi_i(s) C_i(s)\| ds \leq b_{i2}$ for all $t \geq 0$. This in turn implies that there is positive constant b_{i3} such that $\int_o^t \|\Phi_i(s)\| ds \leq b_{i3}$. Folowing Proposition 3.4-1, the claim will hold if $\sum_i \beta_i x_i(t) \leq \varepsilon / \|\Omega(T)\|$ for $t \in J$. Since

$$\|x_i(t)\| \leq \|\Phi_i(t)\| \cdot \|x_{io}\| + \int_o^t \|\Phi_i(s) C_i(s)\| \cdot \|m_i(s)\| ds + \int_o^t \|\Phi_i(s)\| \cdot \|f_i(s)\| ds$$

$$\leq b_{i1} \|x_{io}\| + b_{i2} k_i + b_{i3} \delta_i \quad \text{for all } t \geq 0,$$

condition (ii) ensures the claim. This completes the proof.

The above result has the same meaning as Proposition 3.5-2. Therefore, in order that the supremal may exercise its coordinative actions, the consistancy condition as defined in Definition 3.5-3 will have to be satisfied. In the mean time, Scheme of Coordination 3.5-4 can be employed in the present case with only minor modification. It is clear that, in the above proposition, condition (i) is a realization of image intervention while condition (ii) contains the use of both constraint intervention and interaction intervention.

The condition revealed by Proposition 4.2-1 is possibly too restrictive for many applications. However, the result can be greatly improved when the use of goal interaction interaction is exercised by the supremal. As before, the goal intervention is to be understood as the assignment of desired target set and some controllability requirement to be fulfilled by the infimals. The precise formulation is given in the proposition to follow.

<u>Proposition 4.2-2</u> Let supremal (4.2-1') be controllable at some T with respect to a given M and (x_o, x_d). Then, given $\varepsilon > 0$, the supremal (4.2-1) will be ε-controllable at T with respect to M and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable.

(ii) the infimals (4.2-2') are controllable with respect to given M_i and $(x_{io}, 0)$.

(iii) $\Sigma\beta_i(\alpha_{i1}\|x_{io}\| + b_{i3}\delta_i) \leq \varepsilon / \|\Omega(T)\|$

where α_i and b_{i3} are positive constants depending on $A_i(t)$.

Proof Similar to the proof of Proposition 4.2-1, it suffices to show that $\Sigma\beta_i\|x_i(t)\| \leq \varepsilon / \|\Omega(T)\|$ for $t \in J$. From condition (i) and Lemma 2.3-15, we know that the solutions $\bar{x}_i(t)$ of $\dot{x}_i = A_i(t)x_i$ satisfy the inequality $\|\bar{x}_i(t)\| \leq \alpha_{i1}\|x_{io}\| e^{-\alpha_{i2}t}$ for all $t \geq 0$. Let the solutions of (4.2-2') be $x_{im}(t)$. Then, condition (ii) guarantees that there exists some $m_i \in M_i$ such that $\|x_{im}(t)\| \leq \|\bar{x}_i(t)\|$ for all $t \geq 0$. In other words $\|x_{im}(t)\| \leq \alpha_i\|x_{io}\|$ for $t \geq 0$. Since the solutions $x_i(t)$ of (4.2-2) satisfy

$$\|x_i(t)\| \leq \|x_{im}(t)\| + \int_0^t \|\Phi_i(s)\| \cdot \|f_i(s)\| ds$$

$$\leq \alpha_i\|x_{io}\| + b_{i3}\delta_i \quad \text{for all } t \geq 0,$$

condition (iii) ensures the satisfaction of the required inequality. This completes the proof.

The above proposition clearly demonstrated the usefulness of goal intervention in the present context. However, the requirement that the desired target for each infimal should be the origin in the corresponding state space of the infimal is again restrictive in some cases. We may thus strengthen the above result by modifying condition (ii) in Proposition 4.2-2.

<u>Proposition 4.2-3</u> Let supremal (4.2-1') be controllable at some T with respect to a given M and (x_o, x_d). Then, given $\varepsilon > 0$, the supremal (4.2-1) will be ε-controllable at T with respect to M and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable.

(ii) the infimals (4.2-2') are controllable with respect to given M_i and (x_{io}, x_{id}).

(iii) $\Sigma\beta_i(\alpha_{i1}\|x_{io}\| + b_{i3}\delta_i) \leq \varepsilon/\|\Omega(T)\|$ if $\alpha_{i1}\|x_{io}\| \geq \|x_{id}\|$

$\Sigma\beta_i(\|x_{id}\| + b_{i3}\delta_i) \leq \varepsilon/\|\Omega(T)\|$ if $\alpha_{i1}\|x_{io}\| \leq \|x_{id}\|$

<u>Proof</u> Again it suffices to show that $\Sigma\beta_i\|x_i(t)\| \leq \varepsilon/\|\Omega(T)\|$ for $t \in J$. Using the notation in the proof of last proposition, it suffices to show that

$\|x_{im}(t)\| \leq \alpha_{i1}\|x_{io}\|$ when $\alpha_{i1}\|x_{io}\| \geq \|x_{id}\|$ or

$\|x_{im}(t)\| \leq \|x_{id}\|$ when $\alpha_{i1}\|x_{io}\| \leq \|x_{id}\|$. Let

$\alpha_{i1}\|x_{io}\| \geq \|x_{id}\|$, then condition (i) and (ii) ensure that the trajectory $x_{im}(t)$ is contained in the tube $R^+ \times \bar{X}_i$ where $\bar{X}_i = \{x_i \in X_i : \|x_i\| \leq \alpha_i\|x_{io}\|\}$, i.e., $\|x_{im}(t)\| \leq \alpha_{i1}\|x_{io}\|$ for all $t \geq 0$. Similarly, if $\alpha_{i1}\|x_{io}\| \leq \|x_{id}\|$, then there is some $m_i \in M_i$ such that $\|x_{im}(t)\| \leq \|x_{id}\|$ for all $t \geq 0$. Consequently, condition (iii) assures the claim. This completes the proof.

Since the situation considered in Proposition 4.2-3 is the most general case studied so far, a scheme of coordination can be proposed using Proposition 4.2-3 as a basis.

Scheme of Coordinates 4.2-4 The over-all goal of the two-level linear dynamical system (4.2-1)(4.2-2) will be achieved under the presumed conditions if the supremal uses interventions to coordinate the behaviors of the infimals in the following way:

(i) the supremal exercises its image intervention by requiring that the infimals (4.2-2') be b.i.b.o. stable.

(ii) the supremal commands the infimals to report the constants α_i.

(iii) the supremal exercises the constraint intervention by selecting for each infimal the set X_{io} of admissible initial states (by giving the constants k_{io}) so that the consistancy condition is satisfied.

(iv) the supremal exercises its interaction intervention by selecting constants δ_i so that condition (iii) in Proposition 4.2-3 is satisfied.

(v) the supremal exercises its goal intervention by selecting the target state x_{id} for each infimal and requiring that the infimals be controllable with respect to (X_{io}, x_{id}).

(vi) the supremal exercises its constraint intervention by selecting the appropriate space M_i of admissible controls, which is accomplished by the assignment of constants k_i.

After the above development, it is a natural consequence to consider the special class of infimals in which feedback type control function is used. As we have done previously, the mathematical model for the infimals is given by

$$\dot{x}_i = A_i(t)x_i(t) + H_i(t)x_i(t) + f_i(t) \qquad (4.2\text{-}9)$$

where $H_i(t)$ are unknown continuous matrices to be designed.

We notice in this particular case that the only difference between the analysis to follow and those in section 3.5 is caused by the interaction function $f_i(t)$. Therefore, the results obtained in section 3.5 for the linear feedback control case can be directly applied to the studies on behaviors of systems (4.2-9). Since the previous results were centered around the requirement that trivial solution of the system $\dot{x}_i = A_i(t)x_i(t) + H_i(t)x_i(t)$ be uniformly asymptotically stable, we may use the following general proposition to conclude the study of the two-level linear dynamical system when the first approach of designing interaction is employed.

Proposition 4.2-5 Let supremal (4.2-1') be controllable at some time T with respect to a given M and (x_o, x_d). Then, given $\varepsilon > 0$, supremal (4.2-1) will be ε-controllable at time T with respect to M and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable

(ii) the feedback matrices $H_i(t)$ are so selected that the infimals (4.2-2') are uniformly asymptotically stable.

(iii) $\Sigma\beta_i(\alpha_{i1}\|x_{io}\| + b_{i3}\delta_i) \leq \varepsilon/\|\Omega(T)\|$. Where α_{i1} are positive constants depending on $A_i(t)$ and $H_i(t)$.

Proof: Conditions (i) and (ii) ensure that $\|x_{im}(t)\| \leq \alpha_{i1}\|x_{io}\|$. The proof follows immediately from the proof of Proposition 4.2-2.

When the second approach of designing interaction for the infimals is adopted, the estimation of the amplitude of the state functions $\|x_i(t)\|$ appears more complicated because the matrices $D_{ij}(t)$ are continuous. Let us also write down a similar mathematical representation for the imfimal when direct interaction does not present:

$$\dot{\mathcal{X}} = \mathcal{A}(t)\mathcal{X}(t) + \mathcal{C}(t)\mathcal{m}(t) \qquad (4.2\text{-}6')$$

Let us denote by $x_i(t)$ the solutions of (4.2-2) and by $\mathcal{X}(t)$ the solution of (4.2-6). Then

$$\|\mathcal{X}(t)\| = \left(\sum_i \|x_i(t)\|^2 \right)^{\frac{1}{2}} \leq \left(\sum_i \|x_i(t)\| \right) \qquad (4.2\text{-}10)$$

Let us denote by $\mathcal{X}_m(t)$ the solution of (4.2-6') corresponding to some control function m and by $x_{im}(t)$ the solutions of (4.2-2') corresponding to some control functions m_i. Then similar to (4.2-10), we have

$$\|\mathcal{X}_m(t)\| \leq \sum_i \|x_{im}(t)\| \qquad (4.2\text{-}10')$$

Suppose that supremal (4.2-1') is controllable at some T with respect to a given M and (x_o, x_d). Then, given $\varepsilon > 0$, Proposition 3.4-1 assumes that supremal (4.2-1) is ε-controllable at T, with respect to M and (x_o, x_d) provided that

$$\sum_i \beta_i \|x_i(t)\| \leq \varepsilon / \|\Omega(T)\| \quad \text{for } t \in J \qquad (3.4\text{-}8)$$

Let $\beta = \max_i(\beta_i)$. Then, using relation (4.2-10) inequality (3.4-8) would become the following which provides a similar basis for coordination.

$$\beta \|\mathcal{X}(t)\| \leq \varepsilon / \|\Omega(T)\| \quad \text{for } t \in J \qquad (4.2\text{-}11)$$

Comparing the system equations (4.2-6) and (4.2-6'), we notice that the introduction of interaction matrix $\Delta(t)$ is in effect a provision of additional feedback control to the system (4.2-6'). With the understanding that the matrices $\mathcal{A}(t)$, $\mathcal{C}(t)$, $\mathcal{D}(t)$ are all continuous, the system (4.2-6) is well-behaved. Thus, as it was known in section 3.3, there exist constants κ_1 and κ_2, which depond only upon $\mathcal{A}(t)$, $\mathcal{C}(t)$ and $\mathcal{D}(t)$, such that

$$\|\mathcal{X}(t)\| \leq \kappa_1 \|\mathcal{X}_o\| + \kappa_2 \|\mathcal{M}(t)\| \qquad \text{for } t \in J$$

Although the above inequality is too crude to useful, at least it has shown that $\mathcal{X}(t)$ could be uniformly bounded on a give finite time interval. In order to proceed further, it is not unreasonable to assume that the infimals (4.2-2') are all b.i.b.o. stable. With this assumption, it is clear that the differential system (4.2-6') is also b.i.b.o. stable because (4.2-6') is simply a direct product of the infimals (4.2-2'). By Lemma 3.5-10, an interaction matrix $\mathcal{D}(t)$ can be designed so that system (4.2-6) is also b.i.b.o. stable, or equivalently, that the autonomous system $\dot{\mathcal{X}} = (\mathcal{A}(t) + \mathcal{D}(t))\mathcal{X}$ is uniformly asymptotically stable. And

<u>Proposition 4.2-6</u> Let the supremal (4.2-1') be controllable at some T with respect to a given M and (x_o, x_d). Then, given $\varepsilon > 0$, the supremal (4.2-1) will be ε-controllable at T with respect to M and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable

(ii) an interaction matrix $\mathcal{D}(t)$ could be designed with the property $\|\mathcal{D}(t)\| \leq \alpha_{i4}$ for $t \geq 0$ such that system (4.2-6) is b.i.b.o. stable.

(iii) $\beta(b_{I1}\|x_o\| + b_{I2}(\sum_i k_i)) \leq \varepsilon/\|\Omega(T)\|$ where b_{I1} and b_{I2} are positive constants depending on $\mathcal{A}(t)$ and $\mathcal{D}(t)$.

Proof: It suffices to show that inequality (4.2-10) is satisfied under the given conditions, condition (i) assures that system (4.2-6') is also b.i.b.o. stable. Therefore, denoting by $\Phi_I(t)$ the fundamental matrix of $\dot{x} = (\mathcal{A}(t) + \mathcal{D}(t))x$, condition (ii) assure that there are positive constants b_{I1} and b_{I2} such that $\|\Phi_I(t)\| \leq b_{I1}$ and $\int_o^t \|\Phi_I(s)\mathcal{C}(s)\|ds \leq b_{I2}$ for $t \geq 0$. Consequently, $\|x(t)\| \leq b_{I1}\|x_o\| + b_{I2}\|m(t)\|$. Since $m(t)$ is an element of some subset of that set $\mathcal{M}$ as defined by (4.2-7), it is clear that $\|x(t)\| \leq b_{I1}\|x_o\| + b_{I2}\sum_i k_i$. Obviously, condition (iii) assures the satisfaction inequality (4.2-11). This complete the proof.

The above results was obtained by the use of image intervention, constraint intervention, and interaction intervention. Similar to the first approach of designing interaction, the use of goal intervention might be employed to strenghen the above result.

Proposition 4.2-7 Let the supremal (4.2-1') be controllable at some T with respect to a given M and (x_o, x_d). Then, given $\varepsilon > 0$, the supremal (4.2-1) will be ε-controllable at T with respect to M and (x_o, x_d) if

(i) the infimals are b. i.b.o. stable.

(ii) $\mathcal{D}(t)$ can be designed such that

$\dot{\mathcal{X}} = \mathcal{A}(t)\mathcal{X}(t) + \mathcal{D}(t)\mathcal{X}(t)$ is uniformly asymptotically stable.

(iii) the infimals (4.2-2) are controllable with respect to given M_i and $(x_{io}, 0)$.

(iv) $\beta\,\alpha_{I1}\|\mathcal{X}_o\| \leq \varepsilon/\|\Omega(T)\|$, where $\alpha_{I1} > 0$ is a constant depending on $\mathcal{A}(t)$ and $\mathcal{D}(t)$.

Proof: It suffices to show that $\beta\|\mathcal{X}(t)\| \leq \varepsilon/\|\Omega(T)\|$ for $t \in J$. Let us denote the solution of the autonomous system $\dot{\mathcal{X}} = (\mathcal{A}(t) + \mathcal{D}(t))\mathcal{X}$ by $\mathcal{X}_D(t)$. Condition (ii) ensures that $\|\mathcal{X}_D(t)\| \leq \alpha_{I1}\|\mathcal{X}_o\| e^{-\alpha_{I2}t}$, for all $t > 0$ where α_{I1}, α_{I2} are positive constants depending on $\mathcal{A}(t)$ and $\mathcal{D}(t)$. This implies that there are positive constants $\bar{\alpha}_{i1}$, $\bar{\alpha}_{i2}$ such that

$\|x_{iD}(t)\| \leq \bar{\alpha}_{i1}\|x_{io}\| e^{-\bar{\alpha}_{i2}t}$ for $t \geq 0$, where $x_{iD}(t)$ are solutions of $\dot{x}_i = A_i(t)x_i(t) + \sum_{j\neq i} D_{ij}(t)x_j(t)$.

$x_{iD}(t) = \Phi_i(t)x_{io} + \int_o^t \Phi_i(t, s) \sum_{j\neq i} D_{ij}(s)x_j(s)ds$. Let the solutions of $\dot{x}_i = A_i(t)x_i$ be $\bar{x}_i(t)$. Then condition (i) ensures that there are positive constants α_{i1}, α_{i2} such that $\|\bar{x}_i(t)\| \leq \alpha_{i1}\|x_{io}\|e^{-\alpha_{i2}t} \leq \alpha_{i1}\|x_{io}\|$ for $t \geq 0$. In the mean time, condition (iii) ensures that there are control functions $m_i(t) \in M_i$ such that $\|x_{im}(t)\| \leq \|\bar{x}_i(t)\| \leq \alpha_{i1}\|x_{io}\|$ for $t \geq 0$, where $x_{im}(t)$ are solutions of (4.2-2'). Let $\mathcal{D}(t)$ be chosen such that $\|\bar{x}_i\|(t)\| \leq \|x_{iD}(t)\|$ for all $t \geq 0$ and $i = 1, 2, \ldots, p$. Then, by choosing $\mathcal{M}(t) = (m_1(t), m_2(t), \ldots, m_p(t))'$, we have $\|x_i(t)\| \leq \alpha_{i1}\|x_{io}\|$ for $i = 1, 2, \ldots, p$ and $t \geq 0$. In other words $\|\mathcal{X}(t)\| \leq \alpha_{I1}\|\mathcal{X}_o\|$. Thus, condition (iv) ensures the satisfaction of inequality (4.2-10). This completes the proof.

Using procedure similar to the proof of Proposition 4.2-3 and the above result, we may modify conditions (iii) and (iv) to give a stronger result, which we shall state as a corollary without proof.

Corollary 4.2-8. Let the supremal (4.2-1') be controllable at some T with respect to a given M and (x_o, x_d). Then, given $\varepsilon > 0$, supremal (4.2-1) will be ε-controllable at T with respect to M and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable

(ii) $\mathcal{D}(t)$ can be designed so the $\dot{\mathcal{X}} = (\mathcal{A}(t) + \mathcal{D}(t))$ is uniformly asymptotically stable.

(iii) the infimals (4.2-2') are controllable with respect to given M_i and (x_{io}, x_{id})

(iv) $\beta \dot{\alpha}_{I1} \|\mathscr{x}_o\| \leq \varepsilon / \|\Omega(T)\|$ if $\alpha_{I1} \|\mathscr{x}_o\| \geq \|\mathscr{x}_d\|$ or $\beta \|\mathscr{x}_d\| \leq \varepsilon / \|\Omega(T)\|$ if $\alpha_{I1} \|\mathscr{x}_o\| \leq \|\mathscr{x}_d\|$ where $\mathscr{x}_d = (x_{1d}, x_{2d}, \ldots, x_{pd})'$, $\alpha_{I1} > 0$ depending on $\mathscr{A}(t)$ and $\mathscr{D}(t)$.

When the infimals are assumed to use only feedback type controls, modification of system (4.2-6) leads to

$$\dot{\mathscr{x}} = (\mathscr{A}(t) + \mathscr{H}(t) + \mathscr{D}(t))\mathscr{x} \tag{4.2-12}$$

$$\mathscr{H}(t) = \begin{pmatrix} H_1(t) & 0 & \cdots & 0 \\ 0 & H_2(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & H_p(t) \end{pmatrix}$$

Following the line of analysis depicted in the above, it is essential to design both $\mathscr{H}(t)$ and $\mathscr{D}(t)$ in such a way that the trivial solution of (4.2-12) is uniformly asymptotically stable. We shall rule out the possibility that $\mathscr{H}(t) \to 0$ and $\mathscr{D}(t) \to 0$ as $t \to \infty$, because this case is unlikely to occur in practice. As a consequence, Lemma 3.5-10 seems to be the only tool which is general enough for our purpose. We shall use the following proposition to conclude the study in this section.

Proposition 4.2-9 Let the supremal (4.2-1') be controllable at some T with respect to given M and (x_o, x_d). Then, given $\varepsilon > 0$, supremal (4.2-1) will be ε-controllable at T with respect to M and (x_o, x_d) if

(i) the infimals are b.i.b.o. stable

(ii) the feedback matrices $H_i(t)$ in (4.2-9) are chosen so that systems $\dot{x}_i = (A_i(t) + H_i(t))x_i$ are uniformly asymptotically stable.

(iii) $\mathcal{D}(t)$ is chosen so that system (4.2-12) is uniformly stable.

(iv) $\beta\alpha_{I1}\|\mathcal{X}_o\| \leq \varepsilon/\|\Omega(T)\|$ where $\alpha_{I1} > 0$ depends on $\mathcal{A}(t)$, $\mathcal{H}(t)$, $\mathcal{D}(t)$.

Proof: It suffices to show that inequality (4.2-10) holds. Let solutions of $\dot{x}_i = A_i(t)x_i$ be denoted by $\bar{x}_i(t)$. Condition (i) ensures that there are positive constants α_{i1}, α_{i2} such that

$\|\bar{x}_i(t)\| \leq \alpha_{i1}\|x_{io}\|e^{-\alpha_{i2}t}$ for $t \geq 0$. By using Lemma 3.5-10, we may choose $H_i(t)$ such $\|H_i(t)\| \leq \alpha_{i3}$ for $t \geq 0$ and $\alpha_{i1}\alpha_{i3} < \alpha_{i2}$ to give

$\|x_{ih}(t)\| \leq \alpha_{i1}\|x_{io}\|e^{-(\alpha_{i1}\alpha_{i3}-\alpha_{i2})t}$ for $t > 0$ where

$x_{ih}(t)$ are solutions of $\dot{x}_i = (A_i(t) + H_i(t))x_i$

Using these $H_i(t)$, it is clear that

$\|\mathcal{X}_{H}(t)\| \le \Sigma\|x_{ih}(t)\| \le \Sigma\alpha_{i1}\|x_{io}\|$ for $t \ge 0$ where $\mathcal{X}_{H}(t)$ is solution of $\dot{\mathcal{X}} = (\mathcal{A}(t) + \mathcal{H}(t))\mathcal{X}$. Let $\alpha_{1} = \max_{i}(\alpha_{i1})$, we have $\|\mathcal{X}_{H}(t)\| \le \alpha_{1}\|\mathcal{X}_{o}\|$. By using Lemma 3.5-10 again, a judicial choice of $\mathcal{D}(t)$ can be made so that $\|\mathcal{X}(t)\| \le \alpha_{I1}\|\mathcal{X}_{o}\|$ for $t \ge 0$. Thus condition (iv) ensures that inequality (4.2-11) is satisfied. This completes the proof.

A few remarks are pertinent at the conclusion of this section: (i) Although the heuristics of devising an appropriate scheme of coordination is clearly contained in the above results, the actual design procedure for the selection of $\mathcal{D}(t)$ and/or $\mathcal{H}(t)$ is quite a different matter. Even if the individual subsystems considered are time-invariant systems, a general algorithm of design would present itself as a major task. (ii) In this section, $\mathcal{D}(t)$ is simply assumed to be some continuous matrix. In other words, we have not studied the problem how different forms of interaction among the infimals, which are represented by special classes of $\mathcal{D}(t)$, would affect the system behaviors. In fact, different forms of interconnection among the infimals do have significant effects on system behavior. Partial solution will be given in the next section.

4.3 Coordination and Uniform Controllability

As we have seen previously, the concept of uniform controllability is quite useful in developing a scheme of coordination. The basic requirement for the success of employing such concept lies primarily on the restriction that the state functions of the infimals have the exponential-asymptotically stable property. When the general case, in which interaction among the infimals is present, is considered, it is therefore necessary to investigate closely how this kind of interaction would relate to the exponential-asymptotically stable property. To some extent, one would expect that the configuration of interconnection among the infimals will have some effect on the fullfilment of the above mentioned requirement. For convenience, we shall again write down the system equations as follows:

The supremal:

$$\dot{x} = A(t)x(t) + C(t)m(t) + \sum_i B_i(t)x_i(t) \qquad (4.3\text{-}1)$$

$$\dot{x} = A(t)x(t) + C(t)m(t) \qquad (4.2\text{-}1')$$

The infimals:

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) + \sum_{j\neq i} D_{ij}(t)x_j(t) \qquad (4.3\text{-}2)$$

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) \qquad (4.3\text{-}2')$$

where the equations have the usual definitions as were defined previously.

The number of possible configurations of interconnection in the infimals are many. In order to limit our commitment, we shall only consider the following three general types. The simplest might be called tandem is described in Figure 4.1

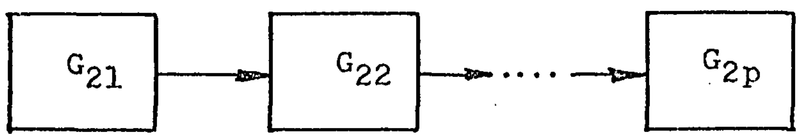


Figure 4.1

In this case, the infimals are described by

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) + D_{i,i-1}(t)x_{i-1}(t) \qquad (4.3\text{-}3)$$

The second configuration might be called closed-loop is described in Figure 4.2

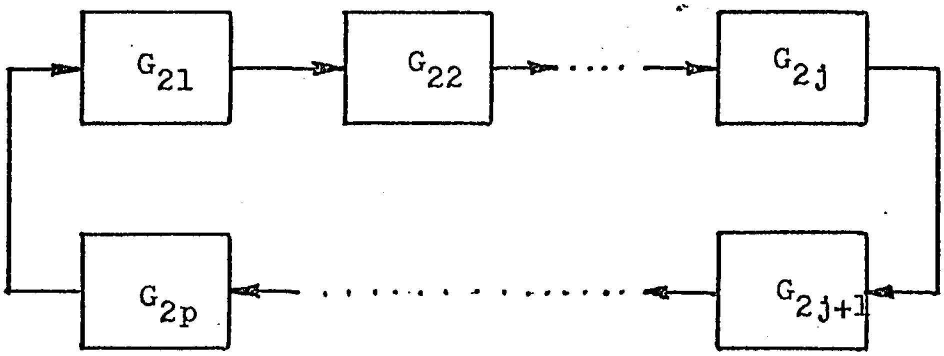


Figure 4.2

The system equations which describe the infimals are the same as in Equation (4.3-3) except the case when i = 1. In this case, we have

$$\dot{x}_1 = A_1(t)x_1(t) + C_1(t)m_1(t) + D_{1p}(t)x_p(t) \tag{4.3-4}$$

The third is the general configuration where the infimals are described by (4.3-2) in general.

Using the notation defined in the previous section, the infimals may be combined to give the following system equation

$$\dot{\mathcal{X}}(t) = \mathcal{A}(t)\mathcal{X}(t) + \mathcal{C}(t)\mathcal{M}(t) + \mathcal{D}(t)\mathcal{X}(t) \tag{4.3-5}$$

$$\dot{\mathcal{x}}(t) = \mathcal{A}(t)\mathcal{x}(t) + \mathcal{C}(t)\mathcal{m}(t) \tag{4.3-5'}$$

For the three different configurations mentioned above, the interaction matrix $\mathcal{D}(t)$ will have consecutively there representations:

Type 1 $$\mathcal{D}(t) = \begin{pmatrix} 0 & 0 & \cdots\cdots & 0 \\ D_{21}(t) & 0 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & D_{pp-1}(t) & 0 \end{pmatrix} \tag{4.3-6}$$

Type 2 $$\mathcal{D}(t) = \begin{pmatrix} 0 & 0 & \cdots\cdots & & D_{1p}(t) \\ D_{21}(t) & 0 & \cdots\cdots & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & D_{p,p-1}(t) & 0 \end{pmatrix}$$

(4.3-7)

Type 3 $$\mathcal{D}(t) = \begin{pmatrix} 0 & D_{12}(t) & \cdots\cdots & D_{1p}(t) \\ D_{21}(t) & 0 & \cdots\cdots & D_{2p}(t) \\ \cdots & \cdots & \cdots & \cdots \\ D_{p1}(t) & D_{p2}(t) & \cdots\cdots & 0 \end{pmatrix}$$

(4.3-8)

As we have seen previously, the design of matirx $\mathcal{D}(t)$ will greatly affect the stable property of system (4.3-5) when the system (4.3-5') is itself b.i.b.o. stable. We would ask the question: if the matrix $\mathcal{D}(t)$ is described by either (4.3-6) or (4.3-7), would the situation be a little different? to answer this question, we shall state the following lemmas due to Bailey [3].

<u>Lemma 4.3-1</u> Let system (4.3-2') be b.i.b.o. stable. Then the system (4.3-5) is also b.i.b.o. stable if the matrix $\mathcal{D}(t)$ has the representation (4.3-6).

When the systems considered above are time-invariant, explicit result can also be found. The following lemma is well-known.

<u>Lemma 4.3-2</u> Consider the system $\dot{x} = f(x, t)$. Then the trivial solution of the system is exponential-asymptotically stable if and only if there is a positive definite function $v(x, t)$ and positive constants c_1, c_2, c_3, c_4 such that

(i) $c_1\|x\|^2 \leq v(x, t) \leq c_2\|x\|^2$

(ii) $\dot{v}(x, t) \leq -c_3\|x\|^2$

(iii) $\|\nabla v\| \leq c_4\|x\|$

where $\nabla v = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \ldots, \frac{\partial v}{\partial x_n} \right)$

Let us define a gain constant for any b.i.b.o. stable system (4.3-2) as follows by neglecting the transients

$$\eta_i = \frac{\sup_{t\geq 0} \| x_i \|}{\sup_{t\geq 0} \| m_i \|} \tag{4.3-9}$$

<u>Lemma 4.3-3</u> [3] The b.i.b.o. stable system $\dot{x} = f(x, t) + Dm(t)$ where m being the input, has a gain

$$\eta \leq \left(\frac{c_4}{c_3}\right)\left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} \| D \|$$

where c_1, c_2, c_3, c_4 are the constants in Lemma 4.3-2.

<u>Lemma 4.3-4</u> [3] Consider a set of time-invariant infimals as described by (4.3-2'). Supose that these infimals are b.i.b.o. stable with gain constants η_i as defined in (4.3-9). Then the system (4.3-5) is also b.i.b.o. stable if

(i) $\Delta(t)$ is given by (4.3-7)

(ii) $\prod_i \eta_i < 1$

Using the above lemmas it is now possible to obtain the following results.

<u>Proposition 4.3-5</u> Let the supremal (4.3-1') be controllable at T with respect to a given M and $(x_o, 0)$. Then, the supremal (4.3-1) will be ε-controllable for a given $\varepsilon > 0$ at some $t \geq T$ with respect to M and $(x_o, 0)$ if

(i) the infimals (4.3-2') are b.i.b.o. stable

(ii) $m_i(t) \equiv 0$ for $i = 1, 2, \ldots, p$, $t \geq 0$

(iii) $\mathcal{D}(t)$ is given by (4.3-6)

(iv) the supremal is b.i.b.o. stable.

Proof: As a consequence of Proposition 2.4-14, it suffices to show that the state functions $x_i(t)$ all have the exponential-asymptotically stable property. It is knowm from Lemma 4.3-1 that condtions (i) and (ii) and (iii) imply that the solution $\mathcal{X}(t)$ of system (4.3-5) has the exponential-asymptotically stable property. This in turn ensures that its components $x_i(t)$ have the same property as required. This completes the proof.

Propostion 4.3-6. Let the supremal (4.3-1') be controllable at some T with respect to a given M and $(x_o, 0)$. Then the supremal (4.3-1) will be ϵ-controllable for a given $\epsilon > 0$ at some $t \geq T$ with respect to M and $(x_o, 0)$ if

(i) the infimals (4.3-2') are b.i.b.o. stable and time-invariant.

(ii) $m_i(t) \equiv 0$, for $i = 1, 2, \ldots, p$ and $t \geq 0$

(iii) $\mathcal{D}(t)$ is given by (4.3-7) and is constant.

(iv) $\prod_i \eta_i < 1$

(v) the supremal is b.i.b.o. stable.

Proof: Again it suffices to show that the state functions

$x_i(t)$ have the exponential-asymptotically stable property. It is known from Lemma 4.3-4 that conditions (i), (ii), (iii) and (iv) guarantee the solution $\mathcal{X}(t)$ of (4.3-5) to be exponential-asymptotically stable. This ensures that $x_i(t)$ have the same property as required. This completes the proof.

Proposition 4.3-7 Let the supremal (4.3-1') be controllable at some T with respect to a given M and $(x_o, 0)$. Then, the supremal (4.3-1) will be ε-controllable for a given $\varepsilon > 0$ at some $t \geq T$ with respect to M and $(x_o, 0)$ if

(i) the infimals (4.3-2') are b.i.b.o. stable

(ii) $m_i(t) \equiv 0$ for $i = 1, 2, \ldots, p$ and $t \geq 0$

(iii) $\mathcal{D}(t)$ is given by (4.3-8) and $\|\mathcal{D}(t)\| \leq \alpha_3$ for $t \geq 0$ where α_3 is a constant depending on $\mathcal{A}(t)$

(iv) the supremal is b.o.b.o. stable.

Proof: It suffices to show that $x_i(t)$ have the exponential asymptotically stable property. Condition (i) implies that the system (4.3-5') is also b.i.b.o. stable. Let the solution of $\dot{\mathcal{X}} = \mathcal{A}(t)\mathcal{X}$ be denoted by $\tilde{\mathcal{X}}(t)$. Then there are positive constants α_1, α_2 such that

$\|\tilde{\mathcal{X}}(t)\| \leq \alpha_1 \|\mathcal{X}_o\| e^{-\alpha_2 t}$ for $t \geq 0$, where α_1, α_2 depend only on $\mathcal{A}(t)$. Let α_3 be so chosen that $\alpha_1 \alpha_3 < \alpha_2$, Lemma 3.5-10 and condition (ii) ensures that the solution $\mathfrak{x}(t)$ of (4.3-5) has the exponential-asymptotically stable property, which in turn implies that $x_i(t)$ have the same properties as required. This completes the proof.

The three propositions stated above have encompassed the assumed configurations of interconnections among the infimals. As we can see, the design of interaction, which must be done by the supremal, presents no problem when type 1 configuration is used. In this case the interaction matrices $D_{i,\ i-1}(t)$ can be practically any continuous matrix. When type 2 configuration must be used, in which the infimals must be time-invariant systems in order to apply the proposition, the design of interaction is still relatively easy. For this particular case, we may devise a scheme of coordination for the supremal as follows.

Scheme of Coordination 4.3-8

(i) The supremal commands the infimals to sent in informations concerning the constants c_1, c_2, c_3, c_4 as given in Lemma 4.3-2.

(ii) The supremal designs the interaction matrices D_{1p}, D_{21}, ..., $D_{p,p-1}$ so that the inequality $\prod_i \eta_i < 1$ is satisfied by using the information that

$$\eta_i < \left(\frac{c_{i4}}{c_{i3}}\right)\cdot\left(\frac{c_{i2}}{c_{i1}}\right)^{\frac{1}{2}}\cdot\|D_{i,i-1}\| \text{ for any typical infimal.}$$

When the general configuration type 3 is necessary, the design of interaction to be performed by the supremal may become very involved. We may proceed as follows in order to provide a solution

$$\|\mathcal{X}(t)\| \leq \sum_i \|x_i(t)\|$$

$$\leq \sum \alpha_{i1}\|x_{io}\| e^{-\alpha_{i2}t} \qquad \text{for } t \geq 0$$

Let

$$\bar{\alpha}_1 = \max_i(\alpha_{i1}\|x_{io}\|)$$

$$\bar{\alpha}_2 = \min_i(\alpha_{i2})$$

Then

$$\|\mathcal{X}(t)\| \leq n\bar{\alpha}_1 e^{-\bar{\alpha}_2 t} \tag{4.3-10}$$

Clearly, the constants $n\bar{\alpha}_1$ and $\bar{\alpha}_2$ are those required in determining an appropriate $\mathcal{D}(t)$ which satisfies Proposition 4.3-3. Based on inequality (4.3-10), the supremal could design $D_{ij}(t)$ for the infimals following a procedure essentially similar to Scheme of Coordination 4.3-8.

In the above scheme of coordination, only two types of interventions are employed by the supremal, namely image intervention which requires the infimals to be b.i.b.o. stable and interaction intervention which is done by choices of $D_{ij}(t)$. If the additional goal intervention and constraints intervention are used we should obtain results similar to those obtained in section 3.6.

<u>Proposition 4.3-9</u> Let supremal (4.3-1') be controllable at some T with respect to a given M and $(x_o, 0)$. Then, given $\varepsilon > 0$, supremal (4.3-1) will be ε-controllable at some $t \geq T$ with respect to M and $(x_o, 0)$ if

(i) the infimals (4.3-2') are b.i.b.o. stable.

(ii) the infimals (4.3-2') are controllable with respect to given M_i and $(x_{io}, 0)$.

(iii) $\mathcal{D}(t)$ is given by (4.3-6)

(iv) the supremal is b.i.b.o. stable.

<u>Proof</u>: As a consequence of Proposition 2.4-14 and condition (iv), it suffices to show that $x_i(t)$ have the exponential-asymptotically stable property. From conditions (i) and (ii), we may choose a control function m_i for each infimal such that $x_i(t) = 0$ for some T_i and $t > T_i$ and all i. Let $\mathcal{X}_m(t)$ denotes the solution of (4.3-5'). This fact implies that $\mathcal{X}_m(t) = 0$ for some $\overline{T}$ and all $t > \overline{T}$. In other words $\mathcal{X}_m(t)$ has the exponential-

asymptotically stable property. Consequently, as a result of Lemma 4.3-1 and condition (iii), this property is retained by the solution $\mathcal{X}(t)$ of (4.3-5), which implies that the state functions $x_i(t)$ have the same property. This completes the proof.

Using essentially the same proof, we may prove the following two statements.

Propostion 4.3-10 Let supremal (4.3-1') be controllable at some T with respect to a given M and $(x_o, 0)$. Then, given $\varepsilon > 0$, supremal (4.3-1) will be ε-controllable at some $t \geq T$ with respect to M and $(x_o, 0)$ if

(i) the infimals (4.3-2') are b.i.b.o. stable and time-invariant,

(ii) the infimals (4.3-2') are controllable with respect to given M_i and $(x_{io}, 0)$,

(iii) $\Delta(t)$ is constant and is given by (4.3-7),

(iv) $\prod_i \eta_i < 1$,

(v) the supremal is b.i.b.o. stable.

Propostion 4.3-11 Let the supremal (4.3-1') be controllable at some T with respect to a given M and $(x_o, 0)$. Then, given $\varepsilon > 0$, supremal (4.3-1) will be ε-controllable at some $t \geq T$ with respect to M and $(x_o, 0)$ if

(i) the infimals are b.i.b.o. stable,

(ii) the infimals (4.3-2') are controllable with respect to given M_i and $(x_{io}, 0)$.

(iii) $\mathcal{D}(t)$ is given by (4.3-8) such that $\|\mathcal{D}(t)\| \leq \alpha_3$ for $t \geq 0$, where α_3 is a constant depending on (t),

(iv) the supremal is b.i.b.o. stable.

The basic idea underline the above propositions is still the concept of uniform controllability. It becomes therefore necessary to extend the results to more general cases. In the mean time, we notice that the major requirement in the design of interaction is to ensure that the resulting system (4.3-5) be b.i.b.o. stable. Clearly, type 1 and type 3 configurations of interconnection are only special cases of the general type 3. Therefore, from now on, we shall only deal with the general case where $\mathcal{D}(t)$ is described by (4.3-8)

<u>Propostion 4.3-12</u> Let supremal (4.3-1') be controllable at some T with respect to given M and (x_o, x_d). Then, given $\varepsilon > 0$, supremal (4.3-1) will be ε-controllable at some $t \geq T$ with respect to M and (x_o, x_d) if

(i) the infimals (4.3-2') are b.i.b.o. stable

(ii) the infimals (4.3-2') are controllable with respect to given M and $(x_{io}, 0)$,

(iii) $\|\mathcal{D}(t)\| \leq \alpha_3$ for $t \geq 0$ where α_3 is a constant depending on $\mathcal{A}(t)$,

(iv) $\|x_d\| + \sup_{T \le t} \|\Phi(t)x_o\| \le \min_{\theta(T) \in \partial L(T,M)} \|\theta(T)\|$

(v) the supremal is b.i.b.o. stable.

Proof: Conditions (iv) and (v) ensures that the supremal (4.3-1') is uniformly controllable for $t \ge T$ with respect to M and (x_o, x_d) as a consequence of Corollary 2.3-20. Therefore, by Proposition 2.4-15, it suffices to show that the state functions $x_i(t)$ all have the exponential-asymptotically stable property. But this fact was demonstrate in Proposition 4.3-11 following conditions (i), (ii) and (iii). This completes the proof.

The case when infimals use feedback type controls is a logical step to be studied next. However, since it is essentially the same as we have studied previously, we shall not repeat the same study here.

4.4 Coordination and Redundant Control Energy

As was demonstrated in the last chapter, the idea of using redundant control energy is a powerful tool to improve system performance considered in this report. When the general case, in which interaction among infimals is not severed, is studied, one would also expect the same merit of using redundant control energy. This is the primary objective to be achieved in this section. For convenience, we shall write down again the system equations for the two-level linear dynamical system being considered.

The supremal:

$$\dot{x} = A(t)x(t) + C(t)m(t) + \sum_i B_i(t)x_i(t) \qquad (4.4\text{-}1)$$

$$\dot{x} = A(t)x(t) + C(t)m(t) \qquad (4.4\text{-}1')$$

$$M = \{m(t) : \|m\|_{2 \text{ or } \infty} \leq k, \quad t \in J\}$$

$$M' = \{m(t) : \|m\|_{2 \text{ or } \infty} \leq k', \quad t \in J\}, \quad k' > k$$

The infimals:

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) + \sum_{j \neq i} D_{ij}(t)x_j(t) \qquad (4.4\text{-}2)$$

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) \qquad (4.4\text{-}2')$$

It well be assumed in the above definitions that $k < k'$.

For the study to be conducted in this section, we should consider the two basic types of designing interaction for the infimals namely: the saturation approach for which the interaction term in the system equation (4.4-2) is given by formula (4.2-3), and the approach in which the interaction is representable by a set of continuous matrices $D_{ij}(t)$.

In the first approach, the interaction term $\sum_{j\neq i} D_{ij}(t)x_j(t)$ can be considered as additive disturbance to any particular infimal. Therefore, when the fundamental inequality (3.4-8) is used as the reference of coordination, one would expect the same results as derived in the first portion of section 3.7. Consequently, we shall only consider the case when the second approach is taken. In this case, system equations (4.4-1) and (4.4-2) completely represent the system to be studied.

When the system equations of the infimals are combined, as it was done in section 4.2, we have

$$\dot{\mathcal{X}} = \mathcal{A}(t)\mathcal{X}(t) + \mathcal{C}(t)\mathcal{M}(t) + \mathcal{D}(t)\mathcal{X}(t) \qquad (4.4\text{-}3)$$

$$\dot{\mathcal{X}} = \mathcal{A}(t)\mathcal{X}(t) + \mathcal{C}(t)\mathcal{M}(t) \qquad (4.4\text{-}3')$$

where $\mathcal{X}(t)$, $\mathcal{A}(t)$, $\mathcal{C}(t)$, $\mathcal{D}(t)$, $\mathcal{M}(t)$ have the usual definition.

Using the notations defined in section 4.2, we have the following basic statement.

<u>Proposition 4.4-1</u> Let supremal (4.4-1') be controllable at some T with respect to a given M and (x_o, x_d). Then, the supremal (4.4-1) will be controllable at T with

respect to $M' \supset M$ and (x_o, x_d) if

$$\beta \|\mathcal{X}(t)\| \leq \gamma(k' - k) \quad \text{for} \quad t \in J$$

Proof: As a consequence of Proposition 2.4-16, we need only to show that $\|\Sigma B_i(t)x_i(t)\| \leq \gamma(k' - k)$ for $t \geq J$. But $\|\Sigma B_i(t)x_i(t)\| \leq \sum_i \beta_i \|x_i(t)\| \leq \beta \Sigma \|x_i(t)\|$. And $\|\mathcal{X}(t)\| \leq \Sigma \|x_i(t)\|$. Therefore, if $\beta\|\mathcal{X}(t)\| \leq \gamma(k' - k)$, the requirement is satisfied. This completes the proof.

This proposition leads to the following:

Proposition 4.4-2 Let supremal (4.4-1') be controllable at some T with respect to a given M and (x_o, x_d). Then, supremal (4.4-1) will be controllable at T with respect to $M' \supset M$ and (x_o, x_d) if

(i) the infimals (4.4-2') are b.i.b.o. stable,

(ii) an interaction matrix $\mathcal{D}(t)$ can be chosen such that the system (4.4-3) is also b.i.b.o. stable,

(iii) $\beta(b_{I1}\|\mathcal{X}_o\| + b_{I2}(\sum_i k_i)) \leq \gamma(k' - k)$ where b_{I1}, b_{I2} are positive constants depending on $\mathcal{A}(t)$ and $\mathcal{D}(t)$.

Proof: Condition (i) implies that the system (4.4-3') is also b.i.b.o. stable. Combining this claim and condition (ii), two positive constants b_{I1}, b_{I2} can be found such that $\|\Phi_I(t)\| \le b_{I1}$ for $t \ge 0$ and $\int_0^t \|\Phi_I(s)\mathcal{C}(s)\| ds \le b_{I2}$ for $t \ge 0$, where $\Phi_I(t)$ denotes the fundamental matrix of the system $\dot{x} = (\mathcal{A}(t) + \mathcal{D}(t))\mathcal{X}$. Consequently $\|\mathcal{X}(t)\| \le b_{I1}\|\mathcal{X}_o\| + b_{I2}(\Sigma k_i)$. Thus condition (iii) ensures that $\beta\|\mathcal{X}(t)\| \le \gamma(k' - k)$. By Proposition 4.4-1, the conclusion follows.

The above result was obtained by the use of image intervention, constraint intervention, and interaction intervention. The additional use of goal intervention might be employed to strengthen the above result.

<u>Proposition 4.4-3</u> Let supremal (4.4-1') be controllable at some T with respect to a given M and (x_o, x_d). Then supremal (4.4-1) will be controllable at T with respect to $M' \supset M$ and (x_o, x_d) if

(i) the infimals (4.4-2') are b.i.b.o. stable,

(ii) the infimals (4.4-2') are controllable with respect to given M_i and $(x_{io}, 0)$,

(iii) the interaction matrix $\mathcal{D}(t)$ is designed so that the system (4.4-3) is b.i.b.o. stable,

(iv) $\beta\alpha_{I1}\|\mathcal{X}_o\| \le \gamma(k' - k)$, where α_{I1} is a positive constant depending on $\mathcal{A}(t)$ and $\mathcal{D}(t)$.

<u>Proof</u>: This proposition is the consequence of Proposition 4.2-7 and Proposition 4.4-1.

<u>Corollary 4.4-4</u> Let supremal (4.4-1') be controllable at some T with respect to a given M and (x_o, x_d). Then supremal (4.4-1) will be controllable at T with respect to $M' \supset M$ and (x_o, x_d) if

(i) the infimals (4.4-2') are b.i.b.o. stable,

(ii) the infimals (4.4-2') are controllable with respect to given M_i and (x_{io}, x_{id}),

(iii) $\delta(t)$ is designed so that the system (4.4-3) is b.i.b.o. stable,

(iv) $\beta \alpha_{I1} \|x_o\| \leq \gamma(k' - k)$ if $\alpha_{I1} \|x_o\| \geq \|x_d\|$ or $\|x_d\| \leq \gamma(k' - k)$ if $\alpha_{I1}\|x_o\| \leq \|x_d\|$.

<u>Proof</u>: This proposition is a consequence of Corollary 4.2-8 and Proposition 4.4-1.

When feedback type controls are employed by the infimals, similar results will be obtained. Thus, we shall not treat this case here.

When the concept of uniform controllability is combined with the idea of using redundant control energy, similar results as those obtained in section 3.7 are possible in the present case. The basic statement is the following

<u>Proposition 4.4-5</u> Let supremal (4.4-1') be uniformly

controllable for $t \geq T$ with respect to a given M and (x_o, x_d). Then, supremal (4.4-1) will be controllable at some $t \geq T$ with respect to $M' \supset M$ and (x_o, x_d) if $\mathcal{X}(t)$ has the exponential-asymptotically stable property.

Proof: Let $\mathcal{X}(t)$ has the required property. This implies that the state functions $x_i(t)$ have the same property. The conclusion follows as consequence of Proposition 3.7-6.

The following results are direct consequences of the above statement.

Corollary 4.4-6 Let supremal (4.4-1') be controllable at T with respect to a given M and $(x_o, 0)$. Then, supremal (4.4-1) will be controllable at some $t \geq T$ with respect to $M' \supset M$ and $(x_o, 0)$ if

(i) the infimals (4.4-2') are b.i.b.o. stable,

(ii) the infimals (4.4-2') are controllable with respect to given M_i and $(x_{io}, 0)$,

(iii) $\mathcal{D}(t)$ is designed so that system (4.4-3) is also b.i.b.o. stable,

(iv) the supremal is b.i.b.o. stable.

Proof: Conditions (i), (ii), (iii) ensure that $\mathcal{X}(t)$ has the exponential-asymptotically stable property, as was demonstrated in Proposition 4.3-11. Condition (iv) and Proposition 2.3-16 ensure that the supremal is uniformly controllable. The conclusion follows from Proposition 4.4-5.

Corollary 4.4-7 Let supremal (4.4-1') be controllable at T with respect to a given M and (x_o, x_d). Then, supremal (4.4-1) will be controllable at some $t \geq T$ with respect to $M' \supset M$ and (x_o, x_d) if

(i) the infimals (4.4-2') are b.i.b.o. stable.

(ii) the infimals (4.4-2') are controllable with respect to given M_i and $(x_{io}, 0)$.

(iii) $\Delta(t)$ is designed in a such a way that system (4.4-3) is b.i.b.o. stable.

(iv) $\|x_d\| + \sup_{t \geq T} \|\Phi(t)x_o\| \leq \min_{\theta(T) \in \partial L(T,M)} \|\theta(T)\|$

(v) the supremal is b.i.b.o. stable.

Proof Condition (v) and Corollary 2.3-20 ensure that the supremal (4.4-1') is uniformly controllable. The rest of the proof follows from the previous corollary.

CHAPTER V

DISCUSSIONS

5.1 Introduction

The objective of inclusion of this chapter is primarily to correlate the present research to some of the previous studies on the same subject of multi-level control systems. In order to do so, it would be most fruitful to list categorically the differences and similarities which exist between the present research and previous studies. Hence, we shall divide this chapter further into three section.

In section 5.2, we shall use a classification of multi-level systems, which was previously proposed as a basis of comparison and we list three principal differences between the present research and previous studies.

In section 5.3, a previously proposed approach of solving the control problem of multi-level system will be explained. We also demonstrate that this approach is in essense the basic method used for solving the control problem of two-level linear dynamical system in the present research.

In section 5.4, a non-iterative scheme of co-ordination, which was used in one of the previous studies, will be adapted to fit the present problems.

5.2 Discussions with Reference to Previous Studies

The study on the theory of multi-level systems has been carried on for some time. Takahara's doctoral dessertation[34] more or less summarized previous studies on this subject, and it will be used here as a basis of comparison.

First of all, we shall discuss how the two-level linear dynamical system studied in this report can be categorized using Takahara's classification.

Since multi-level systems are characterized by the existences of internal disturbances and interactions, Takahara classified the class of multi-level systems into four sub-classes according to its interactions among the subsystems. For simplicity, we shall not use the mathematical notations he used. In-stead, an equivalent verbal classification is given as follows:

Type I: In this sub-class the multi-level system is a collection of independent control subsystems. No coordination is needed.

Type II: In this sub-class the multi-level systems have their lowest level composed of independent control subsystems which are isolated from each other but there may be interactions through the over-all performance criterion or goal. Coordination is necessary.

Type III: In this sub-class, the subsystems may not be isolated but the over-all performance criterion has a simple relation to the performance criteria of the subsystems. Coordintion is necessary.

Type IV: In this sub-class, the multi-level systems are composed of control subsystems which interact directly with each other and also through the over-all performance criterion.

Using this classification, we observe that the two-level linear dynamical systems studied in this report fall in two sub-classes. The systems studied in Chapter III, in which no direct interaction among infimals is allowed, are those belonging to Type II. The systems studied in Chapter IV, in which direct interactions among infimals are represented by additive time functions, are those belonging to Type IV. One might feel a little discontented about why multi-level systems of Type III were not studied. The reason lies in the differentiation of definitions between Type III and Type IV systems. We notice that Type III systems is only a special class of Type IV systems in the sense that the systems of Type III must satisfy an addtional specification. In this present research, the performance criteria we chose for the infimals did not have the special property required for Type III systems defined by Takahara. Consequently, the study of Type III systems was in fact embeded in the study of Type IV systems.

Most of past studies on multi-level systems were concerned with Type III systems. One of the principal reasons for this limitation was that the over-all performance criterion chosen for those studies could be decomposed to accommodate the simple mathematical relationship defined by Takahara for Type III systems. This is clearly impossible in the present research.

After some consideration, we may observe several fundamental differences between the present research and previous studies on the same subject.

Perhaps the most significant difference is that qualitative properties of multi-level control system have been the primary concern of the present research, while the previous stduies concerned with themselves primarily on quantitative properties of multi-level control systems.

The second significant differnce lies in the general philsophy of viewing any multi-level systems. In the past, the studies of multi-level systems started usually from the consideration of a single integrated system problem. The needs for a mult-level modelling for such problem then arised because of the requirement in division of labor. This requirement led naturally to the decompostion of the integrated system into a group of subsystems. Coordination was then introduced, and the result was a multi-level system. In the present research, we feel that the decomposition step is artificial and quite unnecessary in many cases. For instance, the natural boundaries between subsystems of many physical systems are quite evident which may very well be considered as lines of decomposition. Consequently, we started from the outset by developing the requirements of coordination.

The third significant difference is in the mathematical structure of multi-level systems considered in this research and previous works. In general, the multilevel systems studied in previous works have the schematic structure as shown in Figure 5.1.

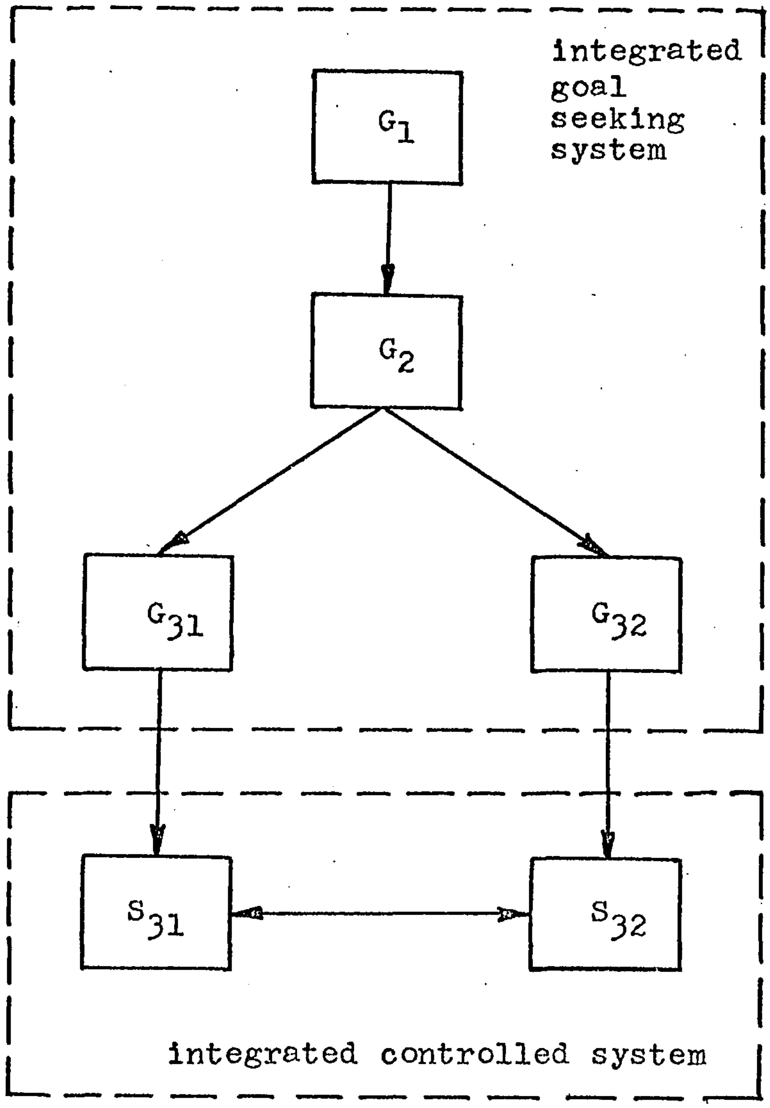


Figure 5.1

A hierarchy is defined among the conrollers following the indicated directions. We notice that only the first level controllers have direct contact with the controlled subsystems. On the other hand, the two-level linear dynamical systems studied in this report have, in general, the schematic structure as shown in Figure 5.2.

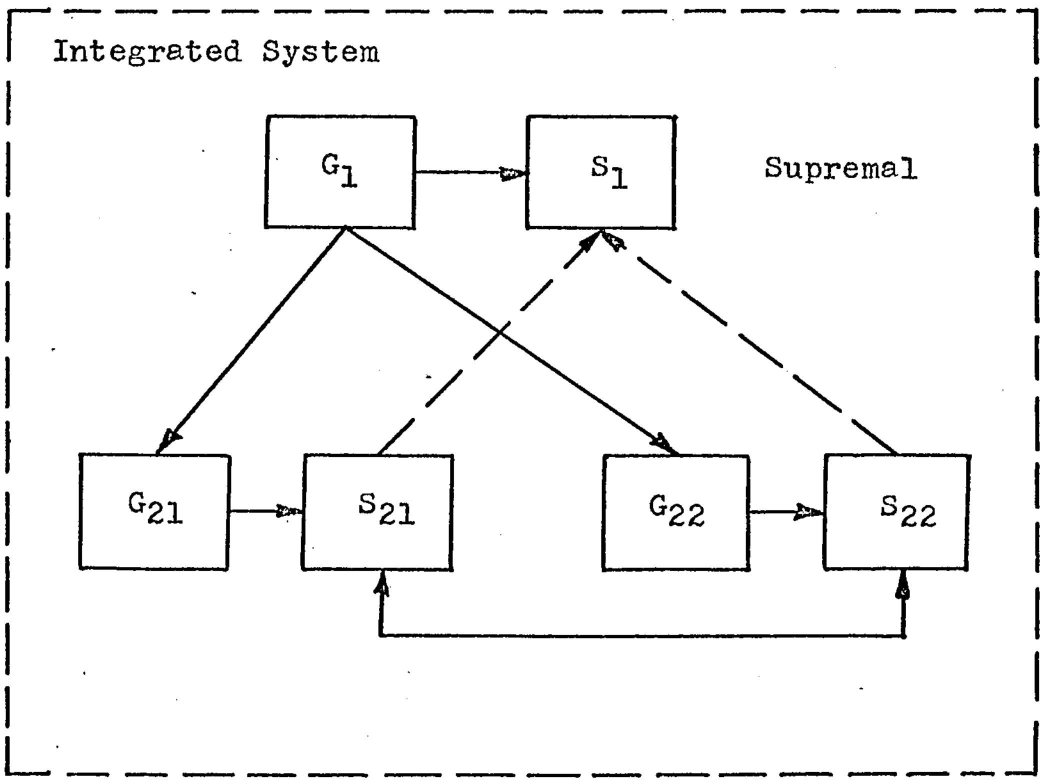


Figure 5.2

In Figure 5.2, the blocks G_1, G_{21}, G_{22} represent the controllers, the blocks S_1, S_{21}, S_{22} represent the controlled systems, the arrows indicate the direction of interactions. The difference from the system described in Figure 5.1 is obvious because the controller G_1 in Figure 5.2 has its own directly controlled subsystems in addition to its duty of coordinating the activities of controller G_{21} and G_{22}. The fact that many complex physical systems can be constructed as two-level linear dynamical system will be demonstrated in next chapter on applications.

5.3 Satisfaction Approach

For any true multi-level control systems, whether it be a Type II, Type III, or Type IV systems, the most significant characteristic for its own identification is the existence of interactions. A direct consequence of this characteristic is the arising of internal disturbance [34], which may otherwise not arise for ordinary systems. The solution of multi-level system control problems can only follow after the successful treatment of internal disturbances.

One of the approaches in dealing with the problem of internal uncertainty was the <u>satisfaction approach</u> [34], which may be defined in following way:

<u>Given</u>:

(i) A system $x = \psi(m, f)$

where x, m, f are state variable, control variable, and disturbance respectively, and ψ is a mapping of the input space M x F into the state space X, where m, f, and x are elements of M, F, and X.

(ii) A performance functional $Q = Q(x, m)$

(iii) Aset M of admissible control functions

(iv) A set F of uncertainties

(v) A functional V(f) defined on F

(vi) A relation R(V(f), Q) between V(f) and Q.

<u>Find</u>:

An element of M which will satisfy R(V(f), Q) for $f \in F$.

Let us now write down the mathematical model of the two-level linear dynamical systems considered in this report:

The supremal:

$$\dot{x} = A(t)x(t) + C(t)m(t) + \sum_i B_i(t)x_i(t) \qquad (5.3\text{-}1)$$

$$M = \{m(t) : \|m\|_{2 \text{ or } \infty} \leq k, \quad t \in J\}$$

The infimals:

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) + \sum_{j\neq i} D_{ij}(t)x_j(t) \tag{5.3-2}$$

$$M_i = \{ m_i(t) : \|m_i\|_{2 \text{ or } \infty} \leq k_i, \quad t \in J\}$$

Since we have defined that the over-all goal of the two-level linear system (5.3-1) (5.3-2) is to coincide with the goal of the supremal, we may use the following interpretation to show how satisfaction approach might be adapted to the study carried out in the present research.

Given:

(i) The system is supremal (5.3-1).

(ii) Let $\varepsilon > 0$ be given. Let $x(t)$ be solution of (5.3-2). Then, the performance functional is defined to be $Q = \|x(T) - x_d\|^2$ for some $T < \infty$, where x_d is the desired state.

(iii) $M = \{m(t) : \|m\|_{2 \text{ or } \infty} \leq k, \ t \in J\}$

(iv) F is the set $\{f(t) = \Sigma B_i(t)x_i(t)\}$, where $x_i(t)$ are generated by the infimals.

(v) $V(f) = \varepsilon^2$ for all $f \in F$.

(vi) $R(V(f), Q)$ is defined as $Q \leq V(f)$ for all $f \in F$.

Based on this satisfaction approach, the control problem of the two-level linear dynamical system may be solved by selecting appropriate constraints on the uncertainty set F. Takahara did obtain some general results related to the above mentioned problem. The basic heuristic in his approach of utilizing the satisfaction approach was the following: By selecting any $f \in F$ to be imposed on the system, he then tried to find a control function $m \in M$ which extremizes the performance functional Q under the influence of the particular f. If such m exists, then, for certain classes of systems, this m will satisfy the relation $R(V(f), Q)$. In other words, this control function m will be the solution to the control problem of the system with uncertainty. However, he also established that, for certain other class of systems, the failure of the above approach, i.e., the finding of m via optimization, does not necessarily imply that a solution to the satisfaction approach does not exist. The systems under consideration in the present research belong in general to the latter case.

In addition to the above observation, there are two major differences in the problems considered by Takahara and the present study. First of all, there is the difference in systems structure. As it can be seen in Figure 5.1, there exist no direct influence from the lower level subsystems to the higher level subsystems in Takahara's model of multi-level systems. On the other hand, as it can be seen from equations (5.3-1) (5.3-2), the infimals are directly influencing the supremal via the state functions $x_i(t)$ as represented by the term $\Sigma B_i(t)x_i(t)$. Secondly, the success in solving the coordination problem of the present case depends heavily on establishing an a priori bound as a constraint on the uncertainty set F. On the contrary, such constraint was not established a priori in Takahara's approach. However, for certain class of systems, a limit on F would emerge iteratively in his work. By comparing these two studies, it appears that unsurmontable computational difficulties might be encountered in trying to find the a priori constraint on F if his approach is followed strictly.

5.4 A Scheme of On-line Coordination

As we had pointed out, internal disturbance is the most important characteristic which distinguishes a multi-level system from other types of systems. Hence, coordination under the influence of internal disturbance becomes the primary concern in the study of multi-level systems.

Iteration techniques were adopted as one method of coordination in most of the previous studies on multi-level systems. But this approach can be used only for a design problem, and thus off-line, from which a fixed system structure would be the result. In this report, since we are primarily concerned with certain behavioral properties of multi-level systems, a scheme of coordination which brings about a prescribed behavior pattern becomes essential. Apparently, such precribed behavior pattern will change from time to time, often, as practical requirements dictate. Consequently, the iterative techniques are no longer satisfactory. An on-line coordination scheme in the sense that adjustments be made from time to time without altering basic systems structure seems to be necessary.

Such a scheme was proposed by Takahara[34], we shall see how his scheme can be adapted to solve the on-line coordination problem in the present case.

Let us introduce two index sets $J = [0, T]$ and $I = \{1, 2, \ldots, N\}$. The index set J represents the set of real time and I the adaptation stages of the supremal.

A. The two-level system is given by the equations (5.4-1) and (5.4-2).

The supremal:

$$\dot{x} = A(t)x(t) + C(t)m(t) + \sum_i B_i(t)x_i(t) \qquad (5.4\text{-}1)$$

$$\dot{x} = A(t)x(t) + C(t)m(t) \qquad (5.4\text{-}1')$$

The infimals:

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) + \sum_{j\neq i} D_{ij}(t)x_j(t),$$

$$i = 1, 2, \ldots, p. \qquad (5.4\text{-}2)$$

$$\dot{x}_i = A_i(t)x_i(t) + C_i(t)m_i(t) \qquad (5.4\text{-}2')$$

The over-all performance criterion is given by

$$Q = \| x(T) - x_d \|^2 \qquad (5.4\text{-}3)$$

where $x(t)$ is the solution of (5.4-1) and x_d is a desired state. And $V(f) = \varepsilon^2$.

The solution of (5.4-1) is given by

$$x(t) = \Phi(t)x_o + \int_o^t \Phi(t, s)C(s)m(s)ds$$

$$+ \int_o^t \Phi(t, s)\sum_i B_i(s)x_i(s)ds \qquad (5.4\text{-}4)$$

Similarly, for the infimals:

$$x_i(t) = \Phi_i(t)x_{io} + \int_o^t \Phi_i(t, s)C_i(s)m_i(s)ds$$

$$+ \int_o^t \Phi_i(t, s) \sum_{j\neq i} D_{ij}(s)x_j(s)ds$$

$$(5.4\text{-}5)$$

Let us attach a superscript i on each variable to denote the i-th adaptation stage. For instance, the desired state x_d at i-th adaptation stage is x_d^i. Let us assume that the adaptation stages are sequential and continuous by defining the following:

$$J^1 = [0,\ T^1],\quad J^2 = (T^1,\ T^2],\ \ldots,\quad J^i = (\sum_{k=1}^{i-1} T^k,\ T^i],\ \ldots,$$

$$J^N = (\sum_{k=1}^{N-1} T^k,\ T^N] \qquad (5.4\text{-}6)$$

Then

$$x^i(t) = x(t) \qquad \text{for } t \in J^i = (\sum_{k=1}^{i-1} T^k,\ T^i]$$

$$x_o^i = x(T^{i-1}) = x(\sum_{k=1}^{i-1} T^k) \qquad (5.4\text{-}7)$$

Similarly

$$x_i^i(t) = x_i(t) \qquad \text{for } t \in J^i$$

$$x_{io}^i = x_i(T^{i-1}) = x_i(\sum_{k=1}^{i-1} T^k) \qquad (5.4\text{-}8)$$

Using previous notations, we have also

$$M^i = \{m^i(t) : \|m^i\|_{2 \text{ or } \infty} \le k^i,\ t \in J^i\}$$

$$M_i^i = \{ m_i^i(t) : \|m_i^i\|_{2 \text{ or } \infty} \leq k_i^i, \quad t \in J^i \}$$

$$X_{io}^i = \{ x_{io}^i \in X : \|x_{io}^i\| \quad k_{oi}^i \}$$

B. The control problem for the i-th imfimal at the j-th adaptation stage is the following: Given the system (5.4-2'), the space M_i^j of admissible controls, the target state x_{id}^j, and the set X_{io}^j of admissible initial conditions . Find a control function $m_i^j(t)$ for $t \in J^j$ such that the system (5.4-2') is transferred from a selected initial state x_{io}^j to the target states x_{id}^j for some $t \in J^j$.

C. The control problem for the supremal during the j-th adaptation stage is the following:

<u>Given</u>:

(i) The system equation (5.4-1).

(ii) The space M^j of admissible controls.

(iii) The initial state $x_o^j = x(T^{j-1})$.

(iv) The target state x_d^j.

<u>Find</u>:

(i) A control function $m^j(t) \in M^i$ which transfer the

System (5.4-1') form x_o^j to x_d^j in some time $t \in J'$

(ii) Choosing an appropriate M_i^j for each infimal according to the results obtained in previous chapters.

(iii) Choosing an appropriate x_{id}^j for each infimal according to previous results.

(iv) Choosing appropriate amount of interactions among the infimals by determining the matrices

$D_{ij}^j(t)$ for $t \in J^j$ according to previous results.

D. On-line coordination scheme: Suppose we are at j-th stage:

(i) Previous informations concerning M_i^{j-1} , x_{id}^{j-1}, x_{io}^{j-1}

$D_{ij}^{j-1}(t)$ are sent to the controller of the supremal.

(ii) The supremal solves its control problem as defined in C.

(iii) The supremal uses its own control function $m^1(t) \in M^1$ and, at the same time, sends orders regarding to

M_i^j, x_{id}^j, $D_{ij}^j(t)$ to the appropriate infimals.

(iv) The infimals solve their control problem as defined in B by choosing an appropriate $m_i^j(t) \in M_i^j$.

(v) The resulting informations for the period J^j are stored and ready to be sent to supremal's controller at (j+1)-th adaptation stage.

When the above coordination scheme is followed, the overall performance criterion will be satisfied, i.e.,

$$\| x^j(T^j) - x_d^j \| \leq \varepsilon^j \quad \text{for all } j \in I$$

CHAPTER VI

APPLICATIONS

Almost invariably, complex industrial systems are constructed using many smaller components or subsystems. For instance, in a power-generation plant there are steam-generation units, turbine-generator units, and electricity dispatching subsystems; in a refinary there are reactors, distillation columns, etc; in an integrated steel processing plant there are blast furnaces, soaking pits, rolling mills, etc. More or less, the operating characteristics of these components or subsystems are known. For a successful operation, one of the problems to be answered is how the integrated system behaves when the subsystems are put togather or interconnected. At the present advanced stage of technology in computer control, there are propositions of using on-line multi-computer systems to control such complicated processes. One way of doing this is to use separate computers to conduct the behaviors of individual subsystems or group of components while another computer situated on a different level of hierarchy is employed to coordinate the control strategies of the other computers. In such cases, the understanding of the operating characteristics of the over-all systems as a function of the operating characteristics of the interconnected subsystems is extremely important. We feel that the results developed in the present study will partially answer the questions posed above.

In the following, we shall work out one numerical example intended to illustrate how the results developed so far may be used to practical situations. It is to be understood, however, that the system described in the following exmple is highly idealized and thus do not necessarily represent the mathematical model of any particular system in real world. At the same time, since we do not consider a quantitative theory, such as the case of optimal control theory, the justification on whether an actual system should be built in the way described in the examples can not be provided here.

6-1 Example

Let us consider the application of the two-level linear system model to the control of a four-stand rolling mill as described in the following schematic diagram.

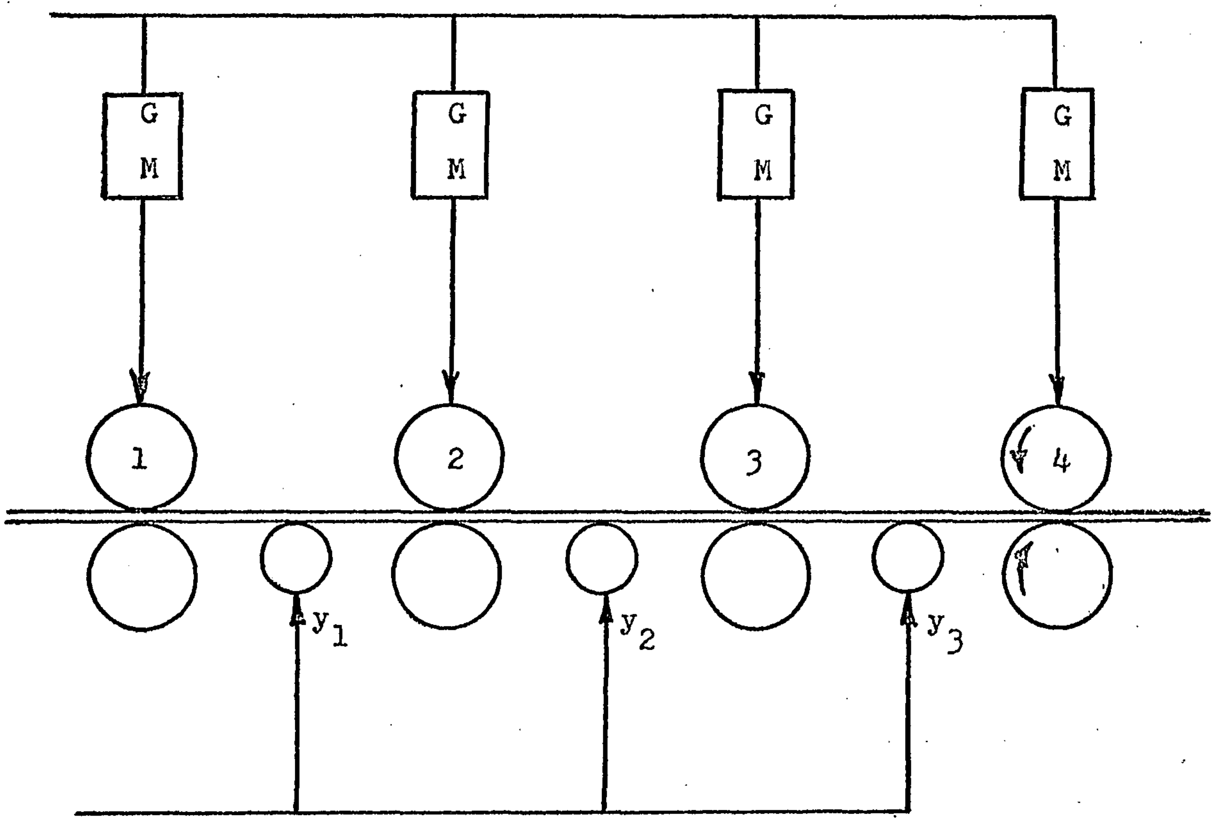


Figure 6.1

Let us assume that the rolling mill is a high speed hot rolling mill for the production of sheet-steels from steel slabs. One of the most important factors which controls the quality of the final products is the tension and its variation on steel strip between stands. In practice, the tension in steel strip can be controlled by adjusting the slack control devices (rollers y_1, y_2, y_3) and the angular speed of the rollers 1, 2, 3, 4. In order to produce a product of uniform quality, it is necessary to control the roller speeds at all rolling stands so that the sensitivity of slack variation due to extroneous disturbances could be minimized.

One common practice in solving such control problem is to select a stand as the reference, while the rolling speeds of other stands are adjusted and controlled with reference to this selected stand. At the same time, the slack control devices are some fixed mechanism, e.g., spring loaded rollers with a predetermined spring rate. However, no dynamical control action, e.g., continuous adjustment of positions or spring rates, is realized on the slack control devices during the operation of the system. The fact that the above control scheme has not been always successful is witnessed by the sometimes looping of the steel strip during operation in many actual cases. Consequently, it is perhaps worthwhile to suggest a new control scheme which would yield better control qualitatively in the above case. This, we feel, might be accomplished by introducing the two-level linear system model developed in the present thesis.

As we find from the dynamical property of the steel rolling mill, the two controlling factors of strip tension between stands are the position of the slack control device and the relative rolling speeds of the adjacent stands. In the previous practices, the only control variable which has been exploited in developing the control acheme is actually the relative rolling speeds. Unfortunately, due to factors such as dynamic interaction between stands, time lag, and sensitivity to external disturbances, people do not always get satisfactory performance. Suppose now that we start to exploit the second controlling variable, the slack control device, via the introduction of the two-level linear system model, would this at least qualitatively improve system performance? Would the introduction of a second control action simplify the modeling for control ore reduce system sensitivity to external disturbance or improve system performance in any other way? These questions can not be answered quantitatively as in the simple and idealized example presented in the following. Nevertheless, we shall show that the task could be tackled via the theoretical analysis developed inthe main body of the present thesis. We shall demonstrate such possibility ysing a modified model given in Cooperman 10 .

Let us assume that y_1, y_2, y_3 represent the controllable positions of the slack-control rollers. Let us also denote by x_i the deviation of rolling speeds of rolling stands i from their nominal values. Let us assume that the rolling speed of each stand is controlled by independent control mechanisms so that, for each stand, its system may be described by equation (6.1-1)

$$\dot{x}_i = -\frac{b_i}{I_i} x_i + \frac{1}{I_i} m_i \;, \; i = 1,2,3,4 \qquad (6.1\text{-}1)$$

where b_i are rolling frictions of the rollers; I_i are the moments of inertia of the rollers; and m_i are the control function for each stand. We shall assume that each rolling-stand will have its own power source, thus no dynamic interaction is directly coupling the rolling-stands. In this way, the x_j term will not appear in the i-th equation.

at the same time, let us assume that the positions

y_1, y_2, y_3 of the slack control rollers are controlled by a linear model described by the follwwing equation

$$\dot{y} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = Ay(t) + Cm(t) \qquad (6.1\text{-}2)$$

where A, C are constant matrices.

In order to form an integrated control system, which will simultaneously adjust the variables x_i and y_j, a two-level linear dynamical system may be formulated in the follwing way:

Supremal

$$\dot{y} = Ay(t) + Cm(t) + \sum_i B_i x_i(t) \qquad (6.1\text{-}3)$$

Infimals

$$\dot{x}_i = A_i x_i(t) + C_i m_i(t) \ , \ i \mp 1,2,3,4 \qquad (6.1\text{-}4)$$

where $A_i = -b_i/I_i$, $C_i = 1/I_i$.

In equation (6.-1-2), the last term indicates the dynamic interaction between the control mecanisms of the slack-control device and of the rolling-stands.

Physically, the above model indicates a form of dynamical interaction among the sybsystems (6.1-1) and

(6.1-2), which may be interpreted in this way: The deviations of rolling speed x_i are sensed by the contol mechanism of y_j such that the supremal may not only adjust its own contpol functions accordingly but also impose constraints on the roller speed control mechanisms .

According to the theoretical analysis developed in the thesis, such a control scheme is possible. Therefore, if certain constraints on x_i and y_j could lead to a product of uniform quality, the control function developed in the two-level linear dynamical system model will fulfill the requirement.

For convenience of illustration, let us make the following assumptions:

(i) $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(ii) $I_i = 1, \quad b_i = 1, \quad C_i = (1), \quad i = 1,2,3,4$

According to Cooperman, we may have the matrices B_i as

$$B_1 = r\begin{pmatrix} b_{11} \\ 0 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_{21} \\ b_{22} \\ 0 \end{pmatrix},$$

$$B_3 = r\begin{pmatrix} 0 \\ b_{32} \\ b_{33} \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 \\ 0 \\ b_{43} \end{pmatrix}$$

where r is the radius for all the rollers.

(iii) $r = 1, \quad b_{ij} = 1, \quad i = 1,2,3,4, \quad j = 1,2,3$

Then, equations (6.1-3) and (6.1-4) may be written respectively as

$$\dot{y} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} m(t)$$

$$+ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x_2(t) + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} x_3(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_4(t)$$

..... (6.1-5)

$$\dot{y} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} m(t) \qquad (6.1\text{-}5')$$

$$\dot{x}_i = -x_i + m_i(t), \quad i = 1,2,3,4 \qquad (6.1\text{-}6)$$

The control problem of the two-level linear dynamical system can thus be defined as follows:

(i) Supremal is described by (6.1-5).

(ii) Infimals are described by (6.1-6).

(iii) Over-all Goal of the System: Let y_d be some desired position of the slack control devices which gives product of desired quality. Let the supremal be started from some initial position y_o. Let the desired position y_d be reacheable at T = 1.0 second under the condition that the rolling-stand speed deviations x_i are zero, with respect to a given control space $M = \{ m: \|m\|_2 \leq 100, t \gtrless 0 \}$, The over-all goal of the two-level linear control system is defined to be the conservation of ε-controllability, for some $\varepsilon > 0$, of system (6.1-5) at T with respect to M and (y_o, y_d) when x_i are identically zero.

(iv) Control Problem of the Supremal: To achieve the over-all goal as defined by the use of coordinative interventions.

For convenience of illustration, let $y_d = (0,0,0)$, $y_o = (1,0,0)$. The fact that system (6.1-5') and (6.1-6) are completely controllable can be easily demonstrated by standard test. Whether system (6.1-5') is controllable with respect to y_o, y_d, and M as given can be tested by direct computation as follows. From formula (2.2-6), we know that

$$m_o(t) = -C'\Phi'(t)W^{-1}(T)y_o$$

is a control function which will transfer system (6.1-5') from y_o to the desired state y_d, where

$$W^{-1}(T) = \int_o^T \Phi(t)CC'\Phi'(t)dt$$

In the present, $\Phi(t) = \exp(At)$. Therefore, we have

$$W(T) = W(1.0) = \int_o^{1.0} \exp(At)\begin{pmatrix}1\\1\\1\end{pmatrix}(1 \quad 1 \quad 1)\exp(At)dt$$

$$= \begin{pmatrix} .48 & .33 & .25 \\ .33 & .25 & .20 \\ .25 & .20 & .17 \end{pmatrix}$$

From which we have

$$W^{-1}(1.0) = \begin{pmatrix} 40.4 & -98.4 & 56.4 \\ -98.4 & 308 & -217.5 \\ 56.4 & -217.5 & 179 \end{pmatrix}$$

Consequently,

$$m_o(t) = -40.4\, e^{-t} + 98.4e^{-2t} - 56.4e^{-3t} \quad (6.1\text{-}7)$$

Now, we should check whether $m_o(t)$ is admissible, which is done by direct computation. Since

$$\|m_o\|_2 = \int_0^1 \|m_o(t)\|^2 dt = 27.4 < 100 \qquad (6.1\text{-}8)$$

we know that m_o is an admissible control function.

Now, the task left to be finished is to show how the supremal could use the four types of coordinative interventions to coordinate the behavior of the infimals, i.e., the control of deviations of roller speeds. Interaction intervention is will not be useful because interaction in the present case is assumed to be absent. As has been developred in the main body of the thesis, the use of image intervention is to require the infimals to be b.i.b.o. stable. By examining equation (6.1-6) we see that this requirement has already been satisfied. This leaves constraint intervention and goal intervention for the use by the supremal. Since x_i represent the deviations of roller speed from some nominal speed settings determined beforehand, we may assume that the desired state for each infimal is $x_i = \theta$. Consequently, we must show that the supremal will achieve the over-all goal as defined by the use of constraint intervention only.

In the present case, the supremal has evidently a large amount of redundant control energy for his use as it has been demonstrated by formula (6.1-8). Then, according to Proposition 3.7-1, the supremal may even achieve the over-all goal when $\varepsilon = 0$. In order to show this, let us assume that the constraint intervention will take the following form:

(i) The supremal selects the set of admissible initial states for each infimal by defining

$$X_{io} = \{ x_{io} \in R : \|x_{io}\| \leq 1 \}, \quad i = 1,2,3,4$$

(ii) The supremal commands the infimals to use only feedback type controls so that

$$m_i(t) = h_i x_i(t) \qquad (6.1\text{-}9)$$

with $h_i < 0$.

When these constraints are imposed, the deviation of roller speed generated at the rolling-stands will take the form

$$x_{io}(t) = x_{io} \exp(-1 + h_i)t \qquad (6.1\text{-}10)$$

Since $-1 + h_i < 0$ under the given constraint, we know that

$$|x_i(t)| \leq 1 \qquad \text{for all} \quad t \geq 0$$

Then, according to Proposition 3.7-5, the over-all goal is indeed achievable when $\varepsilon = 0$. This can be effectively demonstrated by actual computation as follows:

Suppose that the infimals pick any feedback constants, say $h_i = -1$, and some initial state $x_{io} = 1$. Then

$$x_i(t) = e^{-2t} \quad \text{for all } t \geq 0 \qquad (6.1\text{-}11)$$

Under this circumtances, the supremal may put in some extra effort by adding the control function

$$m^*(t) = -2e^{-2t}$$

to the original $m_o(t)$ so that the control function is now $m(t) = m_o(t) + m^*(t)$. The fact that the presence of (6.1-11) in (6.1-5) will not afffect the achievement of over-all goal is simply demonstrated by the fact that $\sum_i B_i x_i(t) = (2 \;\; 2 \;\; 2)' e^{-2t}$. Thus

$$\begin{aligned} \dot{y} &= Ay(t) + C\cdot(m_o(t) + m^*(t)) + \sum_i B_i x_i(t) \\ &= Ay(t) + Cm_o(t) \end{aligned}$$

which is controllable at T = 1.0 second. Since

$$\|m_o + m^*\|_2 = \int_0^1 \|m_o(t) + m^*\|^2 dt = 28.6 < 100$$

we know that $m(t) = m_o(t) + m^*(t)$ is again an admissible control function in the control space M.

Although the computation done for a single set of values (x_{io}, h_i) only, it should be quite obvious that the result will be valid for any chosen x_{io} and h_i so long as they satisfy the given constraints imposed by the supremal.

In general, the steel-rolling mills are batch processis. The conditions and settings may be changed from time to time. In this case, the on-line coordination scheme as outlined in section 5.4 can be employed to solve the control problem of the supremal.

In the above example, we have delt with a highly simplified and idealized problem. In practice, we would expect much more complicated situation and thus much complex modeling proble to take care factors such as external disturbances, dynamic interaction between stands due to strip tension, etc. Nevertheless, the purpose of this example is simply to propose a new way to look at the control problems of complex physical systems. We can claim with considerable confidence that many complex processes might be modeled via the two-level linear dynamical system which we advocate in the present thesis.

CHAPTER VII

CONCLUSIONS AND EXTENSIONS

The two-level linear dynamical systems studied in the present research represents only a small portion of the entire class of multi-level systems. At the same time, controllability is but only one of the many qualitative properties important to any control system. Consequently, the present research can only be considered as a limited exploration in pursuing knowledge relating to multi-level systems theory, and extensions to the present knowledge can be, and have to be done.

One of the possible extensions to the present study stems from the main weakness in the theory developed so far, namely: the stringent constraints in many cases on the state functions $x_i(t)$ of the infimals. This limitation is the result in the way "set of admissible disturbance functions" are defined. Nevertheless, this limitation does not seem to be easily lifted if the above definition is not more carefully explored. In other words, studies will have to be conduted in order to determine the relation between more restrictive classes of admissible disturbance functions and system dynamics.

As we can see from the previous studies and the mathematical structure of the two-level linear dynamical systems, so long as the state functions $x_1(t)$ satisfy certain properties the results obtained will not be affected whether the infimals are linear or nonlinear systems. Thus, another possible extension is to study the case when some nonlinearities present in the subsystems. We should note, however, that this would become a very difficult problem when decomposition is the main concern, because the presence of nonlinearities will impair the effort of reticulation.

In the present research, we explored only the dynamical relationships between the sets of admissible control functions, system dynamics, etc. Further studies on the computational problems of actually finding appropriate control functions, constants, etc. will be highly desirable.

REFERENCES

1. Antosiewicz, H. A., "Linear Control Systems," Archive Rational Mechanics and Analysis, 12(1963), 313-324.

2. Antosiewicz, H. A., "Boundedness and Stability," Nonlinear Differential Equations and Nonlinear Mechanics, J. P. LaSalle and S. Lefschetz, editors, Academic Press, New York, 1963.

3. Bailey, F. N., "The Application of Lyapunov's Second Method to Interconnected Systems," SIAM J. on Control, 3(1965), 443-462.

4. Ben-Israel, A. and Charnes, A., "Contributions to the Theory of Generalized Inverses," SIAM J., 11(1963), 667-669.

5. Bellman, R., Stability of Differential equations, McGraw-Hill, New York, 1953.

6. Bernstein, B. and Truesdell, C., "The Solution of Linear Differential Systems by Quadratures," Journal of Reine and Angewandt Mathematik, 197(1957).

7. Chen, C. T. and Desoer, A. C., "Controllability and Observability of Composite Systems," IEEE Trans. on Automatic Control, AC-12(1967), 402-409.

8. Coddington, E. A. and Levinson, N., "Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

9. Connors, M., "Controllability of Discrete Linear Random Dynamical Systems," SIAM J. on Control, 5(1967), 183-210.

10. Cooperman, S., Coordination of a Multi-stand Steel Rolling Mill Using Decomposition Techniques, systems Research Center Report SRC 104-A-67-46, Case Institute of Technology, Cleveland, Ohio, 1967.

11. Gautschi, W., "The Asymptotic Behavior of Powers of Matrices," Duke Mathematic J., 20(1953),P.127.

12. Gilbert, E. G., "Controllability and Observability of Multivariable Control Systems," SIAM J. on Control, 1(1963), 128-151.

13. Hartman, P., Ordinary Differential Equations, Wiley, New York, 1964.

14. Hermes, H., "Controllability and the Singular Problems," SIAM J. on Control, 2(1964), 241-260.

15. Kalman, R.E., "Contributions to the Theory of Optimal Control," Proc. of Sympsium on Ordinary Differential Equations, Mexico City, 1959.

16. Kalman, R. E., "On the General Theory of Control Systems," Proc. of First International Federation of Automatic Control Congress, Moscow, 1960.

17. Kalman, R. E., New Methods and Results in Linear Prediction and Filtering Theory, RIAS Tech. Rept. 61-1, Baltimore, Md., 1961.

18. Kalman, R. E., Ho, H. C. and Narendra, K. S., "Controllability of Linear Dynamical Systems," Contributions to Differential Equations, 1(1963), 189-213.

19. Kalman, R. E., "Mathematical Description of Linear Dynamical Systems," SIAM J. on Control, 1(1963), 152-192.

20. Kirchmayer, L. K., Economic Control of Interconnected Systems, Wiley, New York, 1959.

21. Kirchmayer, L. K., Economic Operations of Power Systems, Wiley, New York, 1958.

22. Kulikowsky,. R., "Optimization of Aggregated Dynamic Systems," Archiwum Automatyki i Telemechaniki, 11(1966), 227-255.

23. Marcus, L., "Controllability of Nonlinear Processed," SIAM J. on Control, 3(1965), 78-90.

24. Marcus, M. and Minc, H., A Survey of Matrix Theory and Matrix Inequalities, Allyn and Bacon, Boston, 1964.

25. Mesarovic, M. D., "Foundations for a General Systems Theory," Views on General Systems Theory, M. D. Mesarovic, editor, Wiley, New York, 1964.

26. Mesarovic, M. D., A Conceptual Framework for the Studies of Multi-level Multi-goal Systems, Systems Research Center Report SRC 101-A-66-43, Case Institute of Technology, Cleveland, Ohio 1966.

27. Mesarovic, M. D., Pearson, J. D., Macko, D. and Takahara, Y., "On the Synthesis of Dynamic Multi-level Systems," 3rd International Federation of Automatic Control Congress, London, 1966.

28. Mitter, S. K., "An Abstract Theory of Controllability," Conference on the Mathematical Theory of Control, Los Angeles, Calif., 1967.

29. Penrose, R., "A Generalized Inverse for Matrices," Proc. of Cambridge Philosophical Society, 51(1955), 406-413.

30. Rosen, J. R., "Controllable Stability and Equivalent Nonlinear Programming Problem," Nonlinear Defferential Equations and Nonlinear Mechanics, J. P. LaSalle and S. Lefschetz, editors, Wiley, New York, 1963.

31. Roxin, E., "Axiomatic Foundation of the Theory of Control Systems," 2nd International Federation of Automatic Control Congress, Basel, 1963.

32. Simmons, G. F., Introduction to Topology and Modern Analysis, McGraw-Hill, New York, 1963.

33. Sprague, C. F., "On the Reticulation Problem in Multivariable Control Systems," IEEE Trans. on Automatic Control, AC-9 (1964) 398-405.

34. Takahara, Y., Multi-level Systems and Uncertainties, Systems Research Center Report SRC 99-A-66-42, Case Institute of Technology, Cleveland, Ohio, 1966.

35. Taylor, A. E., Introduction to Functional Analysis, Wiley, New York, 1958.

36. Todd, J., A Survey of Numerical Analysis, McGraw-Hill, New York, 1962.

37. Tou, J. T., Modern Control Theory, McGraw-Hill, New York, 1964.

38. Zadeh, L. A. and Desoer, A. C., Linear Systems Theory, McGraw-Hill, New York, 1963.