A Conceptual Proof of Rosenbrock's Pole Assignment Theorem

by

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(1975)

Submitted in Partial Fulfillment

of the Requirements for the

Degree of

Master of Science

at the

Massachusetts Institute of Technology

July 1977

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Abstract

A new proof is given of Rosenbrock's theorem on pole allocation for linear time-invariant systems by state feedback. The necessary conditions are proven by a geometrical argument. The sufficiency of these conditions is proven by a recursive construction. Questions regarding the multiplicity of feedbacks and module-theoretic interpretations are discussed.

Thesis Supervisor: Sanjoy K. Mitter, Professor of Electrical Engineering

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Notation and Conventions

The following definitions and assumptions are used throughout:

$U \cong R^m$; $X \cong R^n$.

A: $X \to X$ and B: $U \to X$ are linear maps.

$\beta = \mathrm{Im}(B)$.

$k = \dim(\beta)$.

$\{X_j;\ j=1,\ldots,k'\}$ is the set of subspaces in the invariant factor decomposition of A.

$\{p_j(\lambda);\ j=1,\ldots,k'\}$ is the set of invariant factors of A.

$\alpha(j) = \mathrm{degree}(p_j(\lambda))$.

$\{\kappa(j);\ j=1,\ldots,k\}$ is the set of controllability indices of (A,B).

(S) is the linear span of S.

$[S]_A = (S, AS, \ldots, A^{n-1}S)$, the controllability subspace generated by S under A.

Assume $X = [\beta]_A$ ((A,B) is controllable) and $k' = k$ (with no loss of generality, see Appendix 1).

$\delta(j) = \alpha(k-j+1)$.

$\varepsilon(j) = \kappa(k-j+1)$.

$\sigma(j) = \sum_{i=1}^{j} \varepsilon(i)$ for $1 \le j \le k$. $\sigma(0) = 1$.

$\eta(j) = \sum_{i=1}^{j} \delta(i)$ for $1 \le j \le k$. $\eta(0) = 1$.

Shift basis for A -- a special basis for X when $\det(\lambda I - A) = \lambda^n$; see Appendix 1.

Feedback -- a linear map $F: X \to U$.

Elementary feedback -- a feedback with $\dim(\mathrm{Im}(F)) = 1$.

Introduction

One of the important issues in linear systems theory is the possibility of changing the poles of a system by feedback. The most complete result on the allocation of poles of a finite dimensional linear time-invariant dynamical system by linear constant feedback is due to Rosenbrock [9] (p. 190).

Suppose we have a system (either discrete or continuous time) represented in the (Z- or Laplace-) transform domain by $\lambda x(\lambda) = A x(\lambda) + Bu(\lambda)$ with $x(\lambda)$ and $u(\lambda)$ being polynomials with coefficients in $X = R^n$ and $U = R^m$ respectively, where (A, B) is a controllable pair. Let $k = \dim\beta = \dim(\mathrm{Im}(B))$ and let $\{\kappa(i);\ i = 1, \ldots, k\}$ be the controllability indices of (A, B). The main concern of this paper is the following result:

Rosenbrock's theorem on pole assignment. Given (A, B), k, and $\{\kappa(i)\}$ as above. Let $\{p_i'(\lambda);\ i = 1, \ldots, k\}$ be any set of polynomials which satisfies $p_i'(\lambda) | p_{i-1}'(\lambda)$ and $\sum_{i=1}^{k} \text{degree}(p_i'(\lambda)) = n$. Let $\alpha'(i) = \text{degree}(p_i'(\lambda))$. Then the conditions

$$\sum_{i=1}^{j} \alpha'(i) \geq \sum_{i=1}^{j} \kappa(i) \qquad \text{for all } j \text{ such that } 1 \leq j \leq k$$

are necessary and sufficient for the existence of a linear map $F: X \to U$ such that $\{p_i'(\lambda)\}$ is the set of invariant polynomials of $A' = A + BF$.

Rosenbrock's proof is sketched in Appendix 3.

The significance of the invariant polynomials lies in the invariant factor theorem (Appendix 1), which describes the structure of a linear transformation of vectors spaces. This is a consequence of a more general theorem on the decomposition of torsion modules. (See the references in Appendix 1.)

The direct application of these algebraic idea to dynamical systems is due to Kalman [7]. He introduced the notion that a certain input-output map associated with a system may be viewed as a module homomorphism, which gives rise to an algebraic view of realization theory.

The polynomial matrix techniques of Rosenbrock bear a strong resemblance to this view. Unfortunately Rosenbrock's proofs are in terms of matrix manipulations which tend to have no obvious intuitive interpretation. His proof of the above theorem is essentially a sequence of elementary equivalence transformations which result in a certain special matrix. The interpretation of this matrix as a representation of some homomorphism is difficult. Furthermore, his sufficiency arguments rely on assertions about a certain polynomial matrix (F' in Appendix 3) which are not proven.

Kalman [6] claimed to give a proof of this theorem, which is incomplete (Appendix 4). Basically, he tried to prove the sufficiency of Rosenbrock's inequalities with an explicit construction for a particular example. His necessity argument amounted to an unsubstantiated claim of the generality of his construction.

There is also a sufficiency proof by Dickinson [2], which may be of interest. Appendix 5 contains a sketch and discussion of this paper. When coupled with transformation to and from Brunovsky form, the proof shows how to construct a feedback. This is a more concise proof than that presented in the following pages. A necessity proof is mentioned, but not given in the paper.

In this paper we first give a conceptual proof of Rosenbrock's theorem. In Chapter 1, we show a geometrical proof of the necessity of the inequalities in the theorem. In Chapter 2 we prove the sufficiency by means of a construction in which the inequalities clearly appear. This might be viewed as making good the claim of Kalman. However the general situation is significantly more complicated than indicated by Kalman. The construction we give is not the only one possible. This is contrary to Kalman's necessity arguments.

In Chapter 3 we consider the issue of the multiplicity of feedbacks available to obtain given invariant polynomials. In Chapter 4 we discuss connections to the module theory viewpoint and possible interpretations of Rosenbrock's proof.

Chapter 1. The Necessary Conditions

We shall prove the necessary conditions of Rosenbrock's theorem, which are:

Theorem 1. For any linear map $F: X \to U$, let $A' = A - BF$. Then the set $\{\alpha'(i); i=1,\ldots,k\}$ corresponding to the degrees of the invariant polynomials of A' satisfies the inequalities

$$\sum_{i=1}^{j} \alpha'(i) \geq \sum_{i=1}^{j} \kappa(i) \text{ for all } j \text{ such that } 1 \leq j \leq k.$$

The essential view of the proof is to regard $\{\alpha'(i)\}$ as providing upper bounds on the dimensions of controllability subspaces in terms of the generating subspaces' dimensions. The above inequalities then arise as a special case when the generating subspaces are subspaces of β.

Proof: The key geometrical fact is expressed in the following lemma:

Lemma 1. For any subspace $S \subseteq X$, with $\dim(S) \leq k$,

$$\dim([S]_{A'}) \leq \sum_{i=1}^{\dim(S)} \alpha'(i).$$

The intuitive idea of the lemma is that for a fixed value of $s = \dim(S)$, $\dim([S]_{A'})$ will be largest when the projection of S onto each element of X_i' has independent components which are cyclic generators of the s largest X_i'. When this occurs, the lemma is satisfied with equality.

Proof of lemma: By Lemma A1 (Appendix 1) $\{\kappa(i); i=1,\ldots,k\}$ is the set of controllability indices of (A',B).

$\dim([S]_{A'}) = \dim(\{A'^{(i-1)}S;\ i=1,\ldots,j\}) +$

$\dim([S]_{A'}/(A'^{(i-1)}S;\ i=1,\ldots,j))$.

$$[S]_{A'}/(A'^{(i-1)}S;\ i=1,\ldots,\alpha'(j)) \cong [p_j'(A')S]_{A'}.$$

$$[p_i'(A')S]_{A'} \subseteq \bigoplus_{j=1}^{k} [p_i'(A')X_j']_{A'} = \bigoplus_{j=1}^{i-1} [p_i'(A')X_j']_{A'}.$$

Therefore

$$\dim([p_i'(A')S]_{A'}) \leq \dim(\bigoplus_{j=1}^{i-1} [p_i'(A')S]_{A'}) = \sum_{j=1}^{i-1} \dim([p_i'(A')X_j']_{A'}).$$

Since for $j \leq i$

$$\dim([p_i'(A')X_j']_{A'}) = \mathrm{degree}(p_j'(\lambda)/p_i'(\lambda)) = \alpha'(j) - \alpha'(i),$$

we have

$$\dim([p_i'(A')S]_{A'}) \leq \sum_{j=1}^{i-1} (\alpha'(j) - \alpha'(i)) = \sum_{j=1}^{i-1} \alpha'(j) - (i-1)\cdot\alpha'(i).$$

Clearly $\dim((A'^{(j-1)}S;\ j=1,\ldots,\alpha'(i))) \leq \alpha'(i)\cdot\dim(S)$.

Thus $\dim([S]_{A'}) \leq \alpha'(i)\cdot\dim(S) + \sum_{j=1}^{i-1} \alpha'(j) - (i-1)\cdot\alpha'(i)$.

Taking $i = 1 + \dim(S)$ completes the proof of the lemma.

We now complete the proof of the theorem using the geometric significance of the controllability indices. Take $S = \beta_j \subseteq \beta$, where β_j satisfies

$$\dim(\beta_j) = j \quad \text{and} \quad \dim([\beta_j]_{A'}) \geq \sum_{i=1}^{j} \kappa(j)$$

as in Lemma A2 (Appendix 1). Combining this last inequality with Lemma 1, we have the desired result.

Chapter 2. The Sufficiency of Rosenbrock's Conditions

The proof of the sufficiency of Rosenbrock's conditions rests on three steps:

1) Obtaining the Brunovsky canonical form. This form can be obtained by feedback and an appropriate choice of basis. See Appendix 2.

2) Given the Brunovsky canonical form, constructing a feedback F for which the degrees of the invariant factors of $A' = A - BF$ are a specified set subject to Rosenbrock's conditions. At the conclusion of this step the invariant polynomials will be of the form $\lambda^{\alpha'(i)}$, as in the Brunovsky form, but the cyclic generators of the invariant factors will not generally all lie in β.

3) Constructing a feedback to change the invariant polynomials as desired, without changing their degrees.

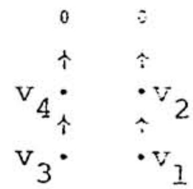
We shall only construct the feedbacks in steps 2 and 3. Appendix 2 gives references on the construction in step 1.

First we give three examples of the construction in step 2. The examples are presented first since the construction is quite involved, and the examples introduce a geometrical picture of the technique. The examples illustrate the increasing complexity of required feedback that can arise, motivating the different cases in the proof.

Examples of Feedbacks to Change Degrees of Invariant Polynomials

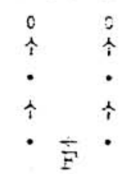
In the following examples, drawings are provided to give a geometrical picture of the vectors and maps under consideration. In these figures, "·" symbols represent basis vectors, arrows indicate the image of a vector under a given map.

Example 1. $\kappa(1) = \kappa(2) = 2$



Thus we are given $v_1 = Bb_1$, $Av_1 = v_2$, $Av_2 = 0$, $v_3 = Bb_2$, $Av_3 = v_4$, $Av_4 = 0$. So the representation of A in this basis is $A = \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix}$ and the representation of B is

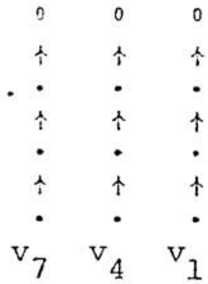
$B = \begin{pmatrix} 1 & \\ 0 & \\ & 1 \\ & 0 \end{pmatrix}$. With feedback F we wish to obtain $\delta(1) = 1$ and $\delta(2) = 3$. We take F as indicated



, which has the representation $F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Then we obtain $A_2 = A + BF$, with representation $A_2 = \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ 1 & & 0 & 0 \\ & & 1 & 0 \end{pmatrix}$. We take as a new basis $v_1^{(2)} = v_2$; $v_2^{(2)} = v_1$; $v_3^{(2)} = v_2 + v_3$; $v_4^{(2)} = v_4$. With this basis we have representations $A_2 = \begin{pmatrix} 0 & & & \\ & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Example 2. $\kappa(1) = \kappa(2) = \kappa(3) = 3$.

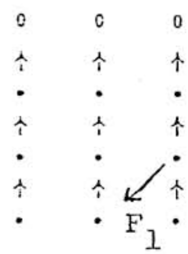
We want $\delta(1) = 1$, $\delta(2) = 4$, $\delta(3) = 4$.



We have the relations $v_1 = Bb_1$, $v_2 = Av_1$, $v_3 = Av_2$, $Av_3 = 0$, $v_4 = Bb_2$, $v_5 = Av_4$, $v_6 = Av_5$, $Av_6 = 0$, $v_7 = Bb_3$, $v_8 = Av_7$, $v_9 = Av_8$, $Av_9 = 0$. In this basis

$$A = \begin{pmatrix} 0&0&0&&&&&& \\ 1&0&0&&&&&& \\ 0&1&0&&&&&& \\ &&&0&0&0&&& \\ &&&1&0&0&&& \\ &&&0&1&0&&& \\ &&&&&&0&0&0 \\ &&&&&&1&0&0 \\ &&&&&&0&1&0 \end{pmatrix} \qquad B = \begin{pmatrix} 1&& \\ 0&& \\ 0&& \\ &1& \\ &0& \\ &0& \\ &&1 \\ &&0 \\ &&0 \end{pmatrix}.$$

Now we take F_1 as indicated



so $A_2 = A + BF_1$. Thus

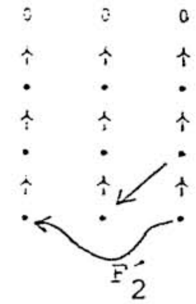
$$F_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0&0&0&&&&&& \\ 1&0&0&&&&&& \\ 0&1&0&&&&&& \\ &1&&0&0&0&&& \\ &&&1&0&0&&& \\ &&&0&1&0&&& \\ &&&&&&0&0&0 \\ &&&&&&1&0&0 \\ &&&&&&0&1&0 \end{pmatrix}.$$

We pick as a new basis $v_1^{(2)} = v_3$; $v_2^{(2)} = v_1$; $v_3^{(2)} = v_2$; $v_4^{(2)} = v_3 + v_4$; $v_5^{(2)} = v_5$; $\ldots$; $v_9^{(2)} = v_9$. In this basis we have

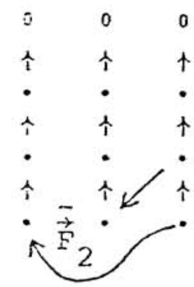
$$A_2 = \begin{pmatrix} 0&&&&&&&& \\ &0&0&0&0&0&&& \\ &1&0&0&0&0&&& \\ &0&1&0&0&0&&& \\ &0&0&1&0&0&&& \\ &0&0&0&1&0&&& \\ &&&&&&0&0&0 \\ &&&&&&1&0&0 \\ &&&&&&0&1&0 \end{pmatrix} \quad B = \begin{pmatrix} &-1& \\ 1&& \\ 0&& \\ 0&1& \\ 0&& \\ 0&& \\ &&1 \\ &&0 \\ &&0 \end{pmatrix}.$$

Now if we pick F_2 as

we did F_1 , as indicated, we find that v_1 has minimal polynomial λ^5 instead of λ^4.



To remedy this we must also take F_2.



In the given basis we then have representations

$$F_2' + F_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A_3 = A_2 + B(F_2 + F_2) = \begin{pmatrix} 0 & & & & & & 1 & & \\ & 0 & 0 & 0 & 0 & 0 & & & \\ & 1 & 0 & 0 & 0 & 0 & & & \\ & 0 & 1 & 0 & 0 & 0 & -1 & & \\ & 0 & 0 & 1 & 0 & 0 & & & \\ & 0 & 0 & 0 & 1 & 0 & & & \\ & 1 & & & & & 0 & 0 & 0 \\ & & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 1 & 0 \end{pmatrix}$

Now we pick the new basis $v_1^{(3)} = v_1^{(2)}$; $v_2^{(3)} = v_3^{(2)}$; $v_3^{(3)} = v_4^{(2)}$; $v_4^{(3)} = v_5^{(2)}$; $v_5^{(3)} = v_6^{(2)}$; $v_6^{(3)} = v_2^{(2)}$; $v_7^{(3)} = v_3^{(2)} + v_7$; $v_8^{(3)} = v_8^{(2)} + v_1^{(2)}$; $v_9^{(3)} = v_9^{(2)}$.

In this basis we have the representations

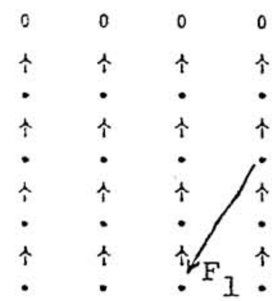
$$A_3 = \begin{pmatrix} 0 & & \\ & \begin{array}{c}0000\\1000\\0100\\0010\end{array} & \\ & & \begin{array}{c}0000\\1000\\0100\\0010\end{array} \end{pmatrix} \qquad B = \begin{pmatrix} & -1 & \\ & & -1 \\ & 1 & 0 \\ & & 0 \\ & & 0 \\ 1 & & \\ 0 & & 1 \\ 0 & & \\ 0 & & \end{pmatrix}.$$

Example 3. $\kappa(1) = \kappa(2) = \kappa(3) = \kappa(4) = 4$ and we want $\delta(1) = 1$, $\delta(2) = \delta(3) = \delta(4) = 5$.

We have the relations $v_1 = Bb_1$, $v_2 = Av_1$, $v_3 = Av_2$, etc.

In this basis we have the representations

$$A = \begin{pmatrix} \begin{array}{c}0000\\1000\\0100\\0010\end{array} & & & \\ & \begin{array}{c}0000\\1000\\0100\\0010\end{array} & & \\ & & \begin{array}{c}0000\\1000\\0100\\0010\end{array} & \\ & & & \begin{array}{c}0000\\1000\\0100\\0010\end{array} \end{pmatrix} \qquad B = \begin{pmatrix} 1 & & & \\ 0 & & & \\ 0 & & & \\ 0 & & & \\ & 1 & & \\ & 0 & & \\ & 0 & & \\ & 0 & & \\ & & 1 & \\ & & 0 & \\ & & 0 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{pmatrix}$$



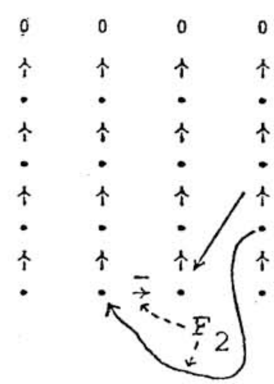
Taking F_1 as indicated we have the representations

$$F_1 = \begin{pmatrix} 0000000000000000 \\ 0010000000000000 \\ 0000000000000000 \\ 0000000000000000 \end{pmatrix}$$ and

$$A_2 = A + BF_1 = \begin{pmatrix} 0000 & & & \\ 1000 & & & \\ 0100 & & & \\ 0010 & & & \\ \quad 1 & 0000 & & \\ & 1000 & & \\ & 0100 & & \\ & 0010 & & \\ & & 0000 & \\ & & 1000 & \\ & & 0100 & \\ & & 0010 & \\ & & & 0000 \\ & & & 1000 \\ & & & 0100 \\ & & & 0010 \end{pmatrix}$$

We pick the new basis $v_1^{(2)} = v_4$; $v_2^{(2)} = v_1$; $v_3^{(2)} = v_2$; $v_4^{(2)} = v_3$; $v_5^{(2)} = v_4 + v_5$; $v_6^{(2)} = v_6$; ...; $v_{16}^{(2)} = v_{16}$. In this basis we have the representations

$$A_2 = \begin{pmatrix} 0 & & & \\ \;0000000 & & & \\ \;1000000 & & & \\ \;0100000 & & & \\ \;0010000 & & & \\ \;0001000 & & & \\ \;0000100 & & & \\ \;0000010 & & & \\ & 0000 & & \\ & 1000 & & \\ & 0100 & & \\ & 0010 & & \\ & & 0000 & \\ & & 1000 & \\ & & 0100 & \\ & & 0010 & \end{pmatrix} \qquad B = \begin{pmatrix} & -1 & & \\ 1 & & & \\ 0 & & & \\ 0 & & & \\ 0 & 1 & & \\ 0 & & & \\ 0 & & & \\ 0 & & & \\ & & 1 & \\ & & 0 & \\ & & 0 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{pmatrix}$$



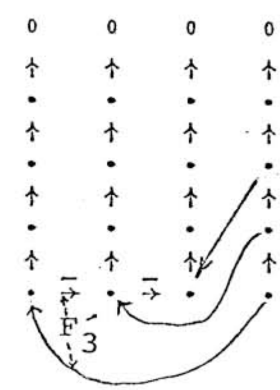
Now we take F_2 as indicated, and we have representations

$$F_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$ and

$$A_3 = A_2 + BF_2 = \begin{pmatrix}
0 & & & & & & & & 1 & & & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & & & & & & & \\
 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & & & & & \\
 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & & & & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & & & & & & & \\
 & & 1 & & & & & & 0 & 0 & 0 & 0 & & & & \\
 & & & & & & & & 1 & 0 & 0 & 0 & & & & \\
 & & & & & & & & 0 & 1 & 0 & 0 & & & & \\
 & & & & & & & & 0 & 0 & 1 & 0 & & & & \\
 & & & & & & & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & 1 & 0 & 0 & 0 \\
 & & & & & & & & & & & & 0 & 1 & 0 & 0 \\
 & & & & & & & & & & & & 0 & 0 & 1 & 0
\end{pmatrix}$$

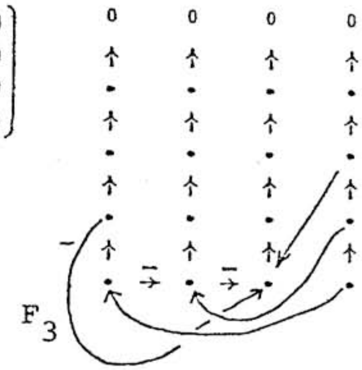
We change basis to $v_1^{(3)} = v_1^{(2)}$; $v_2^{(3)} = v_4^{(2)}$; $v_3^{(3)} = v_5^{(2)}$; $v_4^{(3)} = v_6^{(2)}$; $v_5^{(3)} = v_7^{(2)}$; $v_6^{(3)} = v_8^{(2)}$; $v_7^{(3)} = v_2^{(2)}$; $v_8^{(3)} = v_3^{(2)}$; $v_9^{(3)} = v_4^{(2)} + v_9^{(2)}$; $v_{10}^{(3)} = v_1^{(2)} + v_{10}^{(2)}$; $v_{11}^{(3)} = v_{11}^{(2)}$; ...; $v_{16}^{(3)} = v_{16}^{(2)}$. Our new representations are

$$A_3 = \begin{pmatrix}
0 & & & & & & & & & & & & & & & \\
 & 0&0&0&0&0 & & & & & & & & & & \\
 & 1&0&0&0&0 & & & & & & & & & & \\
 & 0&1&0&0&0 & & & & & & & & & & \\
 & 0&0&1&0&0 & & & & & & & & & & \\
 & 0&0&0&1&0 & & & & & & & & & & \\
 & & & & & & 0&0&0&0&0&0 & & & & \\
 & & & & & & 1&0&0&0&0&0 & & & & \\
 & & & & & & 0&1&0&0&0&0 & & & & \\
 & & & & & & 0&0&1&0&0&0 & & & & \\
 & & & & & & 0&0&0&1&0&0 & & & & \\
 & & & & & & 0&0&0&0&1&0 & & & & \\
 & & & & & & & & & & & & 0&0&0&0 \\
 & & & & & & & & & & & & 1&0&0&0 \\
 & & & & & & & & & & & & 0&1&0&0 \\
 & & & & & & & & & & & & 0&0&1&0
\end{pmatrix} \qquad B = \begin{pmatrix}
 & -1 & & \\
 & & -1 & \\
 & 1 & 0 & \\
 & & 0 & \\
 & & 0 & \\
 & & 0 & \\
1 & & & \\
0 & & & \\
0 & & 1 & \\
0 & & & \\
0 & & & \\
0 & & & \\
 & & & 1 \\
 & & & 0 \\
 & & & 0 \\
 & & & 0
\end{pmatrix}$$



If we take F_3' as indicated in the manner of F_2, we find that v_1 has minimal polynomial λ^7 instead of λ^5 as we would wish. So we take F_3 also, which solves the problem.

$$F_3 + F_3' = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$



$$A_4 = A_3 + B(F_3 + F_3') = \begin{pmatrix}
0 &&&&&&&&&&&&& 1 && \\
& 0 & 0 & 0 & 0 & 0 &&&&&&& 1 &&& \\
& 1 & 0 & 0 & 0 & 0 &&&&&&&& -1 && \\
& 0 & 1 & 0 & 0 & 0 &&&&&&&&&& \\
& 0 & 0 & 1 & 0 & 0 &&&&&&&&&& \\
& 0 & 0 & 0 & 1 & 0 &&&&&&&&&& \\
&&&&&& 0 & 0 & 0 & 0 & 0 & 0 &&&& \\
&&&&&& 1 & 0 & 0 & 0 & 0 & 0 &&&& \\
&&&&&& 0 & 1 & 0 & 0 & 0 & 0 & -1 &&& \\
&&&&&& 0 & 0 & 1 & 0 & 0 & 0 &&&& \\
&&&&&& 0 & 0 & 0 & 1 & 0 & 0 &&&& \\
&&&&&& 0 & 0 & 0 & 0 & 1 & 0 &&&& \\
&&&&&& 1 &&&&&& 0 & 0 & 0 & 0 \\
&&&&&&&&&&&& 1 & 0 & 0 & 0 \\
&&&&&&&&&&&& 0 & 1 & 0 & 0 \\
&&&&&&&&&&&& 0 & 0 & 1 & 0
\end{pmatrix}$$

Our final basis change is $v_1^{(4)} = v_1^{(3)}$; ...; $v_6^{(4)} = v_6^{(3)}$; $v_7^{(4)} = v_8^{(3)}$; ...; $v_{11}^{(4)} = v_{12}^{(3)}$; $v_{12}^{(4)} = v_7^{(3)}$; $v_{13}^{(4)} = v_8^{(3)} + v_{13}^{(3)}$; $v_{14}^{(4)} = v_2^{(3)} + v_{14}^{(3)}$; $v_{15}^{(3)} = v_1^{(3)} + v_{15}^{(3)}$; $v_{16}^{(4)} = v_{16}^{(3)}$. This gives us the following representations:

$$A_4 = \begin{pmatrix}
0 &&&&&&&&&&&&&&& \\
& 0&0&0&0&0 &&&&&&&&&& \\
& 1&0&0&0&0 &&&&&&&&&& \\
& 0&1&0&0&0 &&&&&&&&&& \\
& 0&0&1&0&0 &&&&&&&&&& \\
& 0&0&0&1&0 &&&&&&&&&& \\
&&&&&& 0&0&0&0&0 &&&&& \\
&&&&&& 1&0&0&0&0 &&&&& \\
&&&&&& 0&1&0&0&0 &&&&& \\
&&&&&& 0&0&1&0&0 &&&&& \\
&&&&&& 0&0&0&1&0 &&&&& \\
&&&&&&&&&&& 0&0&0&0&0 \\
&&&&&&&&&&& 1&0&0&0&0 \\
&&&&&&&&&&& 0&1&0&0&0 \\
&&&&&&&&&&& 0&0&1&0&0 \\
&&&&&&&&&&& 0&0&0&1&0
\end{pmatrix} \qquad B = \begin{pmatrix}
& -1 && \\
&& -1 & \\
& 1 & 0 & \\
&& 0 & \\
&& 0 & \\
&& 0 & \\
&&& -1 \\
&& 1 & 0 \\
&&& 0 \\
&&& 0 \\
&&& 0 \\
1 &&& \\
0 &&& 1 \\
0 &&& \\
0 &&& \\
0 &&&
\end{pmatrix}$$

Step 2. Changing the Degrees of the Invariant Factors.

We demonstrate the construction as a concatenation of feedbacks. For the construction of the $(i+1)^{st}$ feedback, we assume that the previous i feedbacks have resulted in a certain cyclic structure with respect to bases chosen in previous steps. In other words, our sequence of feedbacks will change the degrees of the invariant factors one factor at a time. Consequently, the desired feedback F is obtained by means of a recursion giving intermediary maps arising from the sequence of feedbacks.

Suppose (A,B) is in Brunovsky form, and that $V = \{v_1, \ldots, v_{n-1}\}$ is a shift basis for A with $\beta = \{v_{\sigma(j-1)+1};\ j=1, \ldots, k\}$. Let $\{b_i;\ i=1, \ldots, k\}$ be such that $Bb_i = v_{\sigma(i-1)+1}$. Assume $\{\alpha'(i);\ i=1, \ldots, k\}$ satisfies the conditions

$$\sum_{i=1}^{j} \alpha'(i) \geq \sum_{i=1}^{j} \kappa(i) \text{ for } j = 1, \ldots, k \text{ with equality for } j = k.$$

It is easy to see that this is equivalent to $\eta'(i) \leq \sigma(i)$ with equality for $i = k$.

The recursion will have the following hypothesis:

(*) The map A_r: $X \rightarrow X$ and the set $V_r = \{v_1^{(r)}, \ldots, v_n^{(r)}\}$ satisfy (i) to (viii) below.

(i) V_r is a basis for X.

(ii) (A_r, B) has controllability indices $\{\kappa(1), \ldots, \kappa(k)\}$.

(iii) A_r has invariant polynomials $\{\lambda^{\delta'(1)}, \ldots, \lambda^{\delta'(r)}, \lambda^{\sigma(r+1)-\eta'(r)}, \lambda^{\sigma(r+2)}, \ldots, \lambda^{\sigma(k)}$, and V_r is a shift basis for A_r.

(iv) If $j \leq r$ and $\sigma(j) \neq \eta'(j)$, then there exists $x \in \{v^{(r)}_{\eta'(j)+1}, v^{(r)}_{\eta'(j)+2}, \ldots, v^{(r)}_{\sigma(r+1)}\}$ such that $Bb_{j+1} = x - v^{(r)}_{\eta'(j-1)+1}$.

(v) If $j \leq r$ and $\sigma(j) = \eta'(j)$, $Bb_{j+1} = v^{(r)}_{\eta'(s-1)+1}$ where $\ell = \min\{t |\ k \geq t \geq j+1 \text{ and } \sigma(t) = \eta'(t)\}$ and $s = \min(\ell, r+1)$.

(vi) If $k > j > r$, $Bb_{j+1} = v^{(r)}_{\sigma(j)+1}$.

(vii) For all $x \in X$, there exists $b \in (b_1, \ldots, b_{r+1})$ such that $A_r x = Ax + Bb$.

(viii) For all ℓ such that $k \geq \ell \geq 1$, if $j \leq \sigma(\ell)$, then $v^{(r)}_j \in \{v_1, \ldots, v_{\sigma(\rho)}\}$.

We now prove the following proposition:

Proposition 1. There exists a feedback $\tilde{F}$ such that $\tilde{A}' = A + B\tilde{F}$ has invariant polynomials $\{\lambda^{\alpha'(i)};\ i=1,\ldots,k\}$.

Proof: The is proof is a recursive construction of a sequence of maps A_r such that A_{k-1} has the desired invariant polynomials. The desired feedback will be the sum of the feedbacks F_r which yield A_r from A_{r-1}. We proceed as follows: $r = 1$: Let F_1 be the elementary feedback given by $v_{\varepsilon(1)-\delta'(1)} \longmapsto b_2$. Let $A_1 = A + BF_1$. Let $V_1 = \{v^{(1)}_1, \ldots, v^{(1)}_{n-1}\}$ be given by

$$v^{(1)}_i = \begin{cases} v_{\varepsilon(1)-\delta'(1)+i} & \text{for} \quad 1 \leq i \leq \delta'(1) \\ v_{i-\delta'(1)} & \text{for} \quad \delta'(1) < i \leq \sigma(1) \end{cases}$$

$$\begin{cases} v_{i-\delta'(1)} + A^{i-\delta'(1)+1}Bb_2 = v^{(1)}_{i-\delta'(1)-\sigma(1)} + v_i & \text{for } \sigma(1) < i \leq \sigma(1)+\delta'(1) \\ v_i & \text{for } \sigma(1)+\delta'(1) < i \leq n \end{cases}$$

It is easy to directly verify that (*) hold for $r = 1$. See Example 1 for illustration.

Now suppose that (*) holds for $r = i$. There are three cases we must consider: (a) $\eta'(i+1) = \sigma(i+1)$, (b) $\eta'(i+1) < \sigma(i+1)$ and $\varepsilon(i+2) \geq \delta'(i+1)$, and (c) $\eta'(i+1) < \sigma(i+1)$ and $\varepsilon(i+2) < \delta'(i+1)$. We see that case (a) is trivial, case (b) utilizes the same idea as Example 1 and the construction for $r = 1$, and case (c) includes Examples 2 and 3 and requires more effort.

We now demonstrate A_{i+1}, V_{i+1} and F_{i+1}.

Case (a). Take $A_{i+1} = A_i$ and $V_{i+1} = V_i$. (*) still holds for $r = i + 1$.

Case (b). Take F_i to be the elementary feedback given by $v^{(i)}_{\sigma(i+1)-\delta'(i+1)} \longmapsto b_{i+2}$. Let $A_{i+1} = A_i + BF_i$. Define

$$v^{(i+1)}_j = \begin{cases} v^{(i)}_j & \text{for } 1 \leq j \leq \eta'(i) \\ v^{(i)}_{\sigma(i+1)-\delta'(i+1)+j-\eta'(i)} & \text{for } \eta'(i) < j \leq \eta'(i+1) \\ v^{(i)}_{j-\delta'(i+1)} & \text{for } \eta'(i+1) < j \leq \sigma(i+1) \\ v^{(i)}_{j-\delta'(i+1)} + A_i^{j-\sigma(i+1)-1}Bb_{i+2} = v^{(i+1)}_{j-\sigma(i+1)+\eta'(i)} + v^{(i)}_j & \text{for } \sigma(i+1) < j \leq \sigma(i+1)+\delta'(i+1) \end{cases}$$

$$\left| v_j^{(i)} \quad \text{for} \quad \sigma(i+1)+\delta'(i+1) < j \le n. \right.$$

We explicitly check now that (*) holds for $r = i+1$:

(i) By definition of V_{i+1} in terms of V_i it is clear that the two sets are related by an invertible linear transformation.

(ii) Feedback does not change the controllability indices, by Lemma A1.

(iii) Let $\tilde{X}_j = \begin{cases} [v^{(i+1)}_{\eta'(j-1)+1}]_{A_{i+1}} & \text{for } 1 \le j \le i+2 \\ [v^{(i+1)}_{\sigma(j-1)+1}]_{A_{i+1}} & \text{for } i+1 < j \le k. \end{cases}$

But only $\tilde{X}_{i+1}$ and $\tilde{X}_{i+2}$ differ from the cyclic subspaces in the invariant factor decompostion for A_i. Furthermore A_{i+1} restricted to $\{\tilde{X}_j;\ j \ne i+2\}$ is equal to A_i restricted to that subspace. Finally, it is easy to see that $\tilde{X}_{i+1}$ and $\tilde{X}_{i+2}$ have minimal polynomials $\lambda^{\delta'(i+1)}$ and $\lambda^{\sigma(i+2)-\delta'(i+1)}$, respectively.

(iv) The only difference from V_i is that $Bb_{i+2} = v^{(i+1)}_{\sigma(i+1)+1} - v^{(i+1)}_{\eta'(i)+1}$.

(v) For $j \le i$ and $\ell \le i+1$ this is true by hypothesis. For $j \le i$ and $\ell > i+1$, $Bb_{j+1} = v^{(i)}_{\eta'(i)+1}$, and our definition gives $v^{(i+1)}_{\eta'(i+1)+1} = v^{(i)}_{\eta'(i)+1} = Bb_{j+1}$, as desired.

(vi) This is clear by hypothesis and the definition of V_{i+1}.

(vii) Since $A_{i+1} - A_i$ is non-zero only on a subspace of dimension one, we merely note that $A_{i+1}v^{(i+1)}_{\sigma(i+1)} = A_i v^{(i+1)}_{\sigma(i+1)} +$

Bb_{i+2}. The rest follows by hypothesis.

(viii) This is obvious from the definition.

Case (c). In this case, Examples 2 and 3 are our guide. We want to obtain $A_{i+1}^{\sigma(i+2)-\eta'(i+1)} v_{\eta'(i)+1}^{(i)} = 0$, and the assumption that $\varepsilon(i+2) < \delta'(i+1)$ prevents the construction of case (b) from accomplishing this. Now we contruct F_{i+1} recursively from elementary feedbacks. Let G_1 be the elementary feedback given by $v_{\sigma(i+1)-\delta'(i+1)}^{(i)} \longmapsto b_{i+1}$. Let $T_1 = A_i + BG_1$. Since $\varepsilon(i+1) < \delta'(i+1)$, and since by (viii) $v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)} \varepsilon \left(v_1, \ldots, v_{\sigma(i)}\right)$, $A^{\varepsilon(i)} v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)} = 0$. But $T_1^{\varepsilon(i)} v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)} = A_i^{\varepsilon(i)} v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)} = v_{\sigma(i+1)-(\delta'(i+1)-\varepsilon(i))+1}^{(i)} \neq 0$. So by (vii) we conclude there is some smallest s_1 such that $0 < s_1 \leq \varepsilon(i)$ and $A_i^{s_1} v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)} \neq A^{s_1} v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)}$. Thus

$T_1^{s_1} v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)} = A^{s_1} v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)} + Bb'$, where

$b' \varepsilon \left(b_1, \ldots, b_{i+1}\right)$ and $b' \neq 0$. Let G_2 be the elementary feedback given by $v_{\sigma(i+1)+s_1}^{(i)} \longmapsto -b'$. Let $T_2 = T_1 + BG_2$.

Thus $T_2^j(v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)} + Bb_{i+2}) = A^j(v_{\sigma(i+1)-\delta'(i+1)}^{(i)} + Bb_{i+1})$ for $j \leq s_1$ (and $T_2 = A_i$ on V_i except for $v_{\sigma(i+1)-\delta'(i+1)}^{(i)}$ and $v_{\sigma(i+1)+s_1}^{(i)}$). Accordingly

$$T_2^{\sigma(i+2)-\eta'(i+1)} v_{\eta'(i)+1}^{(i)} = T_2^{\varepsilon(i+2)} (v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)} + Bb_{i+2})$$

$$= T_2^{\varepsilon(i+2)-s_1} (A^{s_1} v_{\sigma(i+1)-\delta'(i+1)+1}^{(i)} + Bb') +$$

$$+ T_2^{\varepsilon(i+2)-s_1}(A^{s_1}Bb_{i+2} - Bb')$$

$$= T_2^{\varepsilon(i+2)-s_1}A^{s_1}(v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} + Bb_{i+2})$$

$$= A_i^{\varepsilon(i+2)-s_1}A^{s_1}(v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} + Bb_{i+2}).$$

Now suppose we have integers $0 < s_1 < s_2 < \ldots < s_{\ell-1} \leq \varepsilon(i)$ and a map T_ℓ such that $T_\ell = A_i$ on V_i except for $v^{(i)}_{\sigma(i+1)-\delta'(i+1)}, v^{(i)}_{\sigma(i+1)+s_1}, \ldots, v^{(i)}_{\sigma(i+1)+s_{\ell-1}}$, where we have $T_\ell v^{(i)}_{\sigma(i+1)-\delta'(i+1)} = v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} + Bb_{i+2}$ and

$$T_\ell v^{(i)}_{\sigma(i+1)+s_j} = v^{(i)}_{\sigma(i+1)+s_j+1} + (A-A_i)A^{s_j}v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1}$$

with $1 \leq j \leq \ell-1$. This implies

$$T_\ell^j(v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} + Bb_{i+2}) = A^j(v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} + Bb_{i+2})$$

for $j \leq s$. As above then, we either have $T_\ell^{\varepsilon(i)}(v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} + Bb_{i+2}) = 0$, or there exists a smallest s_ℓ and $b^{(\ell-1)}$ with $\varepsilon(i) \geq s_\ell > s_{\ell-1}$ and $0 \neq b^{(\ell-1)} \in \{b_1, \ldots, b_{i+1}\}$ such that $T_\ell^{s_\ell}v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} = A^{s_\ell}v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} + Bb^{(\ell-1)}$. In this case we define the elementary feedback G_ℓ: $v^{(i)}_{\sigma(i+1)+s_\ell} \mapsto -b^{(\ell-1)}$ and we take $T_{\ell+1} = T_\ell + G_\ell$. In this manner we obtain a strictly increasing sequence of "s_j"'s. Since the sequence is bounded above by $\varepsilon(i)$, after at most $s \leq \varepsilon(i)$ steps we obtain $T_s^{\varepsilon(i)}(v^{(i)}_{\sigma(i+1)-\eta'(i+1)+1} + Bb_{i+2}) = T_s^{\varepsilon(i)}Bb_{i+2} =$

$v^{(i)}_{\sigma(i+1)+\varepsilon(i)+1}$. Then $T_s^{\sigma(i+2)-\eta'(i+1)}v^{(i)}_{\eta'(i)+1} =$

$T_s^{\varepsilon(i+2)+1}v^{(i)}_{\sigma(i+1)-\delta'(i+1)} = T_s^{\varepsilon(i+2)}(v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} + b_{i+2})$

$= 0$. We now take $A_{i+1} = T_s$, and we define V_{i+1} as follows:

$$v_j^{(i+1)} = \begin{cases} v_j^{(i)} & \text{for } 1 \leq j \leq \eta'(i) \\ v^{(i)}_{j+\sigma(i+1)-\eta'(i+1)} & \text{for } \eta'(i) < j \leq \eta'(i+1) \\ A_{i+1}^{j-\eta'(i+1)-1}v^{(i)}_{\eta'(i)+1} & \text{for } \eta'(i+1) < j \leq \sigma(i+2) \\ v_j^{(i)} & \text{for } \sigma(i+2) < j \leq n. \end{cases}$$

Finally we check that (*) holds for $r = i + 1$.

(i) It is sufficient to note that

$$A_{i+1}^{j-\eta'(i+1)-1}v^{(i)}_{\eta'(i)+1} = v^{(i)}_{j-\delta'(i+1)}$$

for $\eta'(i+1)+1 \leq j \leq \sigma(i+1)$ and

$$A_{i+1}^{j-\eta'(i+1)-1}v^{(i)}_{\eta'(i)+1} = A_{i+1}^{j-\sigma(i+1)-1}(v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} + b_{i+2})$$

$$= v_j^{(i)} + x$$

where $x \in \left(v_1^{(i)}, \ldots, v^{(i)}_{\sigma(i+1)}\right)$ for $\sigma(i+1)+1 \leq j \leq \sigma(i+2)$.

(ii) A_{i+1} differs from A_i by feedback.

(iii) Only two invariant factors were changed in this step. The definition of V_{i+1} makes it a shift basis, and the construction of A_{i+1} made $[v^{(i)}_{\eta'(i)+1}]_{A_{i+1}}$ and $[v^{(i)}_{\sigma(i+1)-\eta'(i+1)+1}]_{A_{i+1}}$ independent subspaces with minimal polynomials $\lambda^{\sigma(i+2)-\delta'(i+1)}$ and $\lambda^{\delta'(i+1)}$, respectively.

(iv) We need only note that $Bb_{i+2} = v^{(i)}_{\sigma(i+1)+1} =$

$v^{(i)}_{\sigma(i+1)+1} - v^{(i)}_{\sigma(i+1)-\delta'(i+1)+1} = v^{(i+1)}_{\sigma(i+1)+1} - v^{(i+1)}_{\eta'(i)+1}$

(v) If $\ell > i+1$, $Bb_{j+1} = v^{(i)}_{\eta'(i)+1} = v^{(i+1)}_{\eta'(i+1)+1}$.

If $\ell < i+1$, $v^{(i)}_{\eta'(\ell-1)+1} = v^{(i+1)}_{\eta'(\ell-1)+1}$.

(vi) By definition.

(vii) A_i has been modified only by feedback into $(b_1, \ldots, b_{i+2})$.

(viii) This is by construction.

Using this recursion, we take $\tilde{F} = \sum_{i=1}^{k-1} F_i$ and obtain

$\tilde{A}' = A_{k-1} = A + B\tilde{F}$ having the desired invariant polynomials.

<u>Corollary 1</u>. Let V_{k-1} result from the last step in the proof of Proposition 1. Then by reordering $\{b_1, \ldots, b_k\}$ into $\{\tilde{b}_1, \ldots, \tilde{b}_k\}$ we have

$$v^{(k-1)}_{\eta'(j)+1} = \begin{cases} B\tilde{b}_{j+1} \text{ or} \\ -B\tilde{b}_{j+1} + x \quad \text{where } x \in \{v^{(k-1)}_{\eta'(j+1)+1}, \ldots, \\ \qquad v^{(k)}_{n}\} \end{cases}$$

Proof: This is because the last step of the recursion satisfies (iv) and (v) for $r = k-1$.

Step 3. Changing the Coefficients of the Invariant Factors

As before, we exhibit the desired feedback as the sum of feedbacks . We assume we have a map $\tilde{A}': X \to X$ and a basis W_1 such that $\tilde{A}'$ has invariant polynomials $\{\lambda^{\alpha'(i)}; i=1,\ldots,k\}$. W_1 is a shift basis for $\tilde{A}'$, with $w^{(1)}_{\eta'(i-1)+1}$ being the cyclic generator of the $(k-i)^{th}$ subspace in the invariant factor decomposition for $\tilde{A}'$. Furthermore, $\beta = \left(w^{(1)}_{\eta'(i-1)+1} + x_i;\ i=1,\ldots,k\right)$, where $x_i \in \left(w^{(1)}_{\eta'(i)+1}, \ldots, w^{(1)}_n\right)$ as in the conclusion of Corollary 1.

Now the feedbacks we shall construct will, at the i^{th} step, give the i^{th} cyclic subspace the desired minimal polynomial. It may be noted during the construction, however, that these minimal polynomials of subspaces will not necessarily be the invariant factors at that point. At the conclusion of the construction this will have been remedied.

The idea of the construction is to imitate the scalar feedback "trick" as suggested by Kalman [6]. Unfortunately, this does not work in the obvious manner, because in general the projection of β on the subspaces of the invariant factor decompostion has components which are not generators of the factors. See Appendix 4. Therefore it is necessary to change basis after the application of the "trick" to each cyclic subspace to eliminate interconnections. An example is given in Appendix 4.

We now prove the following:

Proposition 2. Given $\tilde{A}'$ as in Proposition 1, and $W_1 = V_{k-1}$ as in Corollary 1, we can find a feedback $\tilde{F}$ such that $A' = \tilde{A}' + B\tilde{F}$ has invariant polynomials $\{\lambda^{\delta'(i)} + \sum_{j=1}^{\delta'(i)} q_{ij}\lambda^{j-1};\ i=1,\ldots k\}$ where the q_{ij} are arbitrary.

Proof: Let $p'_{k-i}(\lambda) = \lambda^{\delta'(i)} + \sum_{j=1}^{\delta'(i)} q_{ij}\lambda^{j-1}$. Let r_i be the smallest integer such that $(\tilde{A}')^{r_i}x_i = 0$. Let s_i be the largest integer such that $q_{is_i} \neq 0$. Suppose $r_i > \delta'(i)$. Then let $c_{i1}, \ldots, c_{i(r_i-s_i)}$ be defined by the equations

$$
\begin{aligned}
&c_{i1}\, q_{is_i} &&= -1\\
&c_{i1}\, q_{i(s_i-1)} + c_{i2}\, q_{is_i} &&= 0\\
&\vdots && \vdots\\
&c_{i1}\, q_{i1} + c_{i2}\, q_{i2} + \ldots + c_{is_i}\, q_{is_i} &&= 0\\
&\qquad c_{i2}\, q_{i1} + \ldots + c_{is_i}\, q_{i(s_i-1)} + c_{i(s_i+1)}\, q_{is_i} &&= 0\\
&\vdots && \vdots\\
&c_{i(r_i-2s_i+1)}\, q_{i1} + \ldots \qquad + c_{i(r_i-s_i)}\, q_{is_i} &&= 0.
\end{aligned}
$$

Note that $r_i > \delta'(i)$ implies $r_i > s_i$. We can write these equations in matrix form as

$$Q_i\, c_i = e_i$$

where $Q_i = \begin{pmatrix} q_{is_i} & 0 & \cdots & & & 0 \\ q_{i(s_i-1)} & q_{is_i} & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ q_{i1} & & \ddots & \ddots & \ddots & \vdots \\ 0 & q_{i1} & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & q_{i1} & \cdots & q_{is_i} \end{pmatrix}$

is a $(r_i-s_i)\times(r_i-s_i)$-dimensioned matrix. (If $r_i-s_i < s_i$, one takes the first r_i-s_i equations and rows for Q_i.)

$e_i = (-1\ 0\ 0\ \ldots\ 0)^T$, an (r_i-s_i)-vector, and

$c_i = (c_{i1}\ c_{i2}\ \cdots\ c_{i(r_i-s_i)})^T$, also an (r_i-s_i)-vector.

Since Q_i is triangular, and $q_{is_i} \neq 0$, it is easy to see that this system of equations has a unique solution.

We define recursively the maps $\tilde{F}_i\colon\ X \to U$ by $\tilde{F}_i w_j^{(i)} = 0$, except for $w^{(i)}_{\eta'(i-1)+\ell} \mapsto -q_i \cdot \tilde{b}_i$, for $\ell = 1,\ldots,\delta'(i)$. We assume that the $\tilde{b}_i$ are ordered as in the conclusion of Corollary 1. W_{i+1} is defined by

$$w_j^{(i+1)} = w_j^{(i)} \quad \text{for } j \leq \eta'(i-1)$$

$$w^{(i+1)}_{\eta'(i-1)+1} = w^{(i)}_{\eta'(i-1)+1} + x_i + \sum_{j=1}^{s_i-r_i} c_{ij}(\tilde{A}')^{s_i+j-1}x_i$$

$$= \tilde{b}_i + \sum_{j=1}^{s_i-r_i} c_{ij}(\tilde{A}'+\tilde{F}_i)^{s_i+j-1}x_i$$

$$w^{(i+1)}_{\eta'(i-1)+j} = (\tilde{A}'+\tilde{F}_i)w^{(i+1)}_{\eta'(i-1)+j-1} + c_{ij}w^{(i+1)}_{\eta'(i-1)+1}$$

$$\text{for } 1 < j \le \delta'(i)$$

$$w_j^{(i+1)} = w_j^{(i)} = w_j^{(1)} \quad \text{for } \eta'(i) < j \le n.$$

If $r_i \le \delta'(i)$, we simply take $c_{ij} = 0$. Notice that in each basis change we have only changed basis for one cyclic subspace, to keep it as an invariant subspace with the addition of feedback. Let $W' = W_k$.

Now we take $A' = \tilde{A}' + \sum_{i=1}^{k} B\tilde{F}_i$. We have $p_i'(A')w'_{\eta'(i-1)+1} = 0$, which we see as follows: Let $\tilde{A}_i = \tilde{A}' + \sum_{j=1}^{i} B\tilde{F}_j$. By construction, on the subspace $\{w_1', \ldots, w'_{\eta'(i)}\}$ we have $A' = \tilde{A}_i$. So $p_i'(A')w'_{\eta'(i-1)+1} = p_i'(\tilde{A}_i)(w^{(1)}_{\eta'(i-1)+1} + x_i + \sum_{j=1}^{s_i-r_i} c_{ij}(\tilde{A}')^{s_i+j-1}x_i)$. Since

$$p_i'(\tilde{A}_i)w^{(1)}_{\eta'(i-1)+1} = ((\tilde{A}')^{\delta'(i)} - p_i'(\tilde{A}'))x_i \qquad \text{and}$$

$$p_i'(\tilde{A}_i)(\sum_{j=1}^{s_i-r_i} c_{ij}(\tilde{A}')^{s_i+j-1}x_i) = -(\tilde{A}')^{\delta'(i)}x_i,$$ we obtain the desired result. With $F = \sum_{i=1}^{k} \tilde{F}_i$, this completes the proof of Proposition 2.

This immediately gives

<u>Theorem 2</u>. There exists a feedback F such that $A' = A + BF$ has arbitrary invariant polynomials, subject to the necessary conditions stated in Theorem 1.

Proof: Take $F = \tilde{F} + \tilde{F}'$, from Propositions 1 and 2.

Chapter 3. The Multiplicity of Feedbacks

In this chapter we suggest a classification of feedbacks that produce a given set of invariant factors from a given Brunovsky form. The feedbacks will be divided into equivalence classes in which the elements differ by an element of the stabilizer subgroup of the so-called feedback group.

This group has as elements triples of maps (P,K,Q) which act on pairs (A,B). See Brockett [1]. We take $(P,K,Q)\cdot(A,B) = (P^{-1}AP + P^{-1}BK, P^{-1}BQ)$. It is easy to see that this defines a group with identity (I,0,I). We call this group F. This group acts on any pair of matrices of appropriate dimensions. For a fixed pair (A,B) the stabilizer subgroup H of F consists of those elements $h \in F$ such that $h\cdot(A,B) = (A,B)$.

Since the identity is in H, it is easy to see that H is a normal subgroup for (A,B) fixed. Thus we can form equivalence classes of F modulo H.

First we give an example to illustrate the structure of H. Take (A,B) in Brunovsky form with $\kappa(1)=2$ and $\kappa(2)=1$. So $A = \begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & \\ 1 & \\ & 1 \end{pmatrix}$. Brockett gives formulas for the general form of an element of H when (A,B) are in Brunovsky form. In this case we obtain $\left(\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & d \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}\right)$. Thus H is generated by one feedback, one similarity which changes the choice of invariant subspaces in the invariant factor decomposition, and two similarities which are in a sense trivial.

It seems that in general H can be decomposed in this manner. The generating feedbacks go from elements in one invariant subspaces to elements in other invariant subspaces which have lower cyclic order. The similarities replace a basis element with linear co bination   it and other elements of lower cyclic order. Thus it is seen how H does not disturb the geometric structure of (A,B).

Let π: $F \to F/H$ be the canonical map. Let F^* be the subgroup of F consisting of f such that $f\cdot(A,B) = (A',B')$ for a fixed A' and any B'. We wish to characterize $\pi(F^*)$.

One possibility is that some elements of $\pi(F^*)$ differ by pure similarities. In the case $A' = A$ for the previous example, for any nonsingular Q, $\left(\begin{pmatrix}1&0&1\\0&1&0\\0&0&-1\end{pmatrix},0,Q\right)$ lies in a non-zero coset of F/H.

From the construction of Dickinson (Appendix 5), it seems plausible that the cardinality of $\pi(F^*)$ is related to the number of choices of B' such that (A',B') has the correct controllability indices. Note that in the preceding example Im(B') is not necessarily isomorphic under similarity.

An alternative is to take $F_{A'B'}$ to be those f such that $f\cdot(A',B') = (A',B'')$ for some B''. Then $\pi(F^*) \cong F_{A'B'}/H_{A'B'}$.

Chapter 4. Connections to the Module Theory

From the module theory point of view it would be pleasing to give an interpretation of $\begin{pmatrix} \lambda I-A & -B \\ F & I \end{pmatrix}$ as a module homorphism. Rosenbrock's proof would then amount to a choice of bases for the domain and range such that the homorphism has a diagonal representation.

For example, $\lambda I-A$ can be viewed as a module endomorphism on the polynomials $X[\lambda]$. Take $\Phi = X[\lambda]$. Define the evaluation map ε: $\Phi \to X$ by $\sum x_i\lambda^i \to \sum A^i x_i$. Let $\Psi = \ker(\varepsilon)$. Then $(\lambda I - A)$ is a $R[\lambda]$-isomorphism $\Phi \to \Psi$. We obtain a representation of $\lambda I-A$ in Smith canonical form by choosing two bases for $X[\lambda]$ as follows.

Let $\oplus X_i$ be an invariant factor decomposition of X under A. Let x_i be a cyclic generator of X_i, and let $p_i(\lambda)$ be the minimal polynomial of X_i. Then we take as bases
$V_1 = \{(\lambda-A)x_1, \ldots, (\lambda^{\alpha(1)-1}-A^{\alpha(1)-1})x_1, \ldots, (\lambda-A)x_k, \ldots, (\lambda^{\alpha(k)-1}-A^{\alpha(k)-1})x_k, x_1, x_2, \ldots, x_k\}$ for the range and $V_2 = \{(\lambda I-A)^{-1}(\lambda I-A)x_1, \ldots, (\lambda I-A)^{-1}(\lambda^{\alpha(k)-1}-A^{\alpha(k)-1})x_k, (\lambda I-A)^{-1}p_1(\lambda)v_1, \ldots, (\lambda I-A)^{-1}p_k(\lambda)\}$. Represented as a matrix in terms of these bases, $\lambda I-A$ is in Smith canonical form. The interpretation of $\begin{pmatrix} \lambda I-A & -B \\ F & I \end{pmatrix}$ is more complicated. It is difficult to give an interpretation to the range when the domain is any usual space. Taking the domain to be the extended input module $U((\lambda^{-1}))$ direct summed with the space

$X((\lambda^{-1}))$ appears most natural. See Wyman [12]. Then those elements of the domain which map onto elements of $X((\lambda^{-1}))+0$ with only finitely many terms constitute the free response of the system with feedback. Then an appropriate basis for the domain would be 0 direct summed with any basis for U union the basis for the domain in the Smith form of $\lambda I-A+BF$ direct summed with F times this same basis. The basis for the range would be analogous to the case in the Smith form of $\lambda I-A$. Still, this only gives an interpretation when we restrict the domain to the free response, and it gives no help in picking F.

This approach to Rosenbrock's methods leads to similar questions about the interpretation of Rosenbrock's system matrices and the MacMillan form. It seems that the significance of zeros of a system must first be understood, since these are the non-trivial invariant factors of the system matrix. Note via the MacMillan form that zeros give rise to the possibility of finite response to infinite (in terms) input, a situation beyond the scope of the ordinary "Kalman" input-output map. Thus Wyman's input-output map seems to be the appropriate object of study.

Finally we remark that for (A,B) controllable, if the matrix C is chosen appropriately, the basis for $U[\lambda]$ which will diagonalize the transfer function consists of those elements which map to the generators of the invariant subspaces viewed as submodules. For example take

$A = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ & & & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & \\ 0 & 1 \\ 1 & \\ & -1 \end{pmatrix}$. Choose $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then

$C(\lambda I-A)^{-1}B = \begin{pmatrix} \lambda^{-3} & \lambda^{-2} \\ 0 & -\lambda^{-1} \end{pmatrix}$. The generators of the invariant

subspaces are Bb_1 and λBb_1-b_2. This change of basis then

has matrix representation $\begin{pmatrix} 1 & \lambda \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \lambda \\ 0 & -1 \end{pmatrix}$.

$\begin{pmatrix} \lambda^{-3} & \lambda^{-2} \\ 0 & -\lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & -1 \end{pmatrix} \cdot = \begin{pmatrix} \lambda^{-3} & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, which is the MacMillan form.

In the case when A has invariant factors of the form λ^i, such a choice of basis for $U[\lambda]$ seems to give the MacMillan form whenever there are no zeros. When there are zeros, a left equivalence transformation is needed, and the one for the input may not be the same.

Appendix 1. Controllability Indices and Invariant Factors

In this section we present three results needed in the text.

Controllability indices also go by the names of "Kronecker invariants" and "minimal indices." Useful discussions may be found in Gantmacher (V. II, p. 37), Rosenbrock (pp. 90-95), Brunovsky (p. 175), Popov (p. 255) and Wonham (p. 123). Essentially, Gantmacher defines these indices in terms of the kernel of a pencil of matrices $A + \lambda B$, and the other define the indices in terms of a basis for the image of B and linear dependence of the images of these basis elements under application of A. Rosenbrock proves the equivalence of these two definitions (p. 96).

Invariant factors are well-known in algebra from the theory of decomposition under a linear transformation. See, for example Jacobson (pp. 79-86), Newman (p. 28) or Lang (p. 397).

We use the following result (Invariant Factor Theorem):

Theorem: Suppose $X = R^n$. Let $A: X \to X$ be an endomorphism of X. There exist a positive integer k and subspaces $X_i \subset X$ for $i = 1, \ldots, k$ with the following properties:

(i) $X = X_1 \oplus X_2 \oplus \ldots \oplus X_k$

(ii) For $i = 1, \ldots, k$ A restricted to X_i is cyclic.

(iii) If $p_i(\lambda)$ is the minimal polynomial of A restricted to X_i then $p_1(\lambda)$ is the minimal polynomial of A and $p_2(\lambda) \mid p_1(\lambda)$, ..., $p_k(\lambda) \mid p_{k-1}(\lambda)$.

(iv) k and $\{p_i(\lambda)\}$ are unique, for $\chi_k \neq 0$.

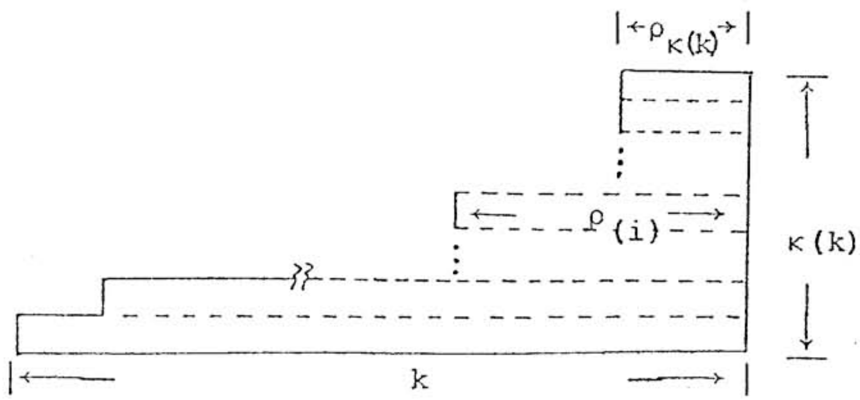
(This phraseology is approximately that of Wonham, p. 16)

This theorem is adequately discussed in the cited references, and we give no proof here. The set $\{p_i(\lambda)\}$ is variously called the invariant factors or the invariant polynomials of A. For each of the polynomials $p_i(\lambda)$ we define $\alpha(i)$ as $\alpha(i) = \text{degree}(p_i(\lambda))$. Thus $\alpha(1) \geq \alpha(2) \geq \ldots \geq \alpha(k)$. It is an easy corollary that

$$\sum_{i=1}^{k} \alpha(i) = n.$$

The second result we need is that controllability indices are invariant under feedback. We shall prove this as an easy consequence of the definition. We first give the definition of controllability indices given by Wonham (p.124).

The geometrical motivation for this definition is as follows. Let S_j be the linear span of β and its images under j applications of the map A. It is clear that the increments in the dimension of S_j with increasing j must be non-increasing. We can represent this in the following manner:



The bottom horizontal line represents β by its length k, the dimension of β. The i^{th} horizontal line up represents the increase in dimension of S_i over S_{i-1} by the length which we call ρ_i. The height of the whole polygon i-1 units from the right edge we call $\kappa(i)$.

It is easily checked that this scheme corresponds to the following definition:

<u>Definition</u>: Let $S_j = \beta + A\beta + \ldots + A^j\beta$ for i = 0, 1, ..., n-1. Let $\rho_0 = k$. $\rho_j = \dim(S_j / S_{j-1})$. The i^{th} controllability index $\kappa(i)$ of (A,B) is the number of integers in $\{\rho_0, \rho_1, \ldots, \rho_{n-1}\}$ which are greater than or equal to i, for i = 1, ..., k.

It is easy to see that the ρ_i are nonnegative and non-increasing, as are the $\kappa(i)$, and that the sums of both these sets of integers are each n, the dimension of X. We then obtain

<u>Lemma A1</u>. $\{\kappa(i)\}$ is invariant under feedback.

proof: For any F: $X \to U$ and $A' = A + BF$, we still have $S_j = \beta + A'\beta + \ldots + A'^j\beta$.

From the preceeding discussion of the geometrical meaning of the definition, or as a corollary to Wonham's Theorem 5.10 (p. 122), we have also

<u>Lemma A2</u>. For any j such that $1 \leq j \leq k$ there exists a subset $\beta_j \subset \beta$ such that $\dim(\beta_j) = j$ and $[\beta_j]_A \geq \sum_{i=1}^{k} \kappa(i)$.

This is most easily seen by picking a basis for β, and then applying the definition of the controllability indices, keeping track of the dependencies of the images of the basis. The lemma would be immediate from such a definition involving an explicit choice of basis.

Note: We have used the integer k both for the dimension of β and for the number of invariant factors of A. A priori these need not be the same number. Suppose the latter is k'. Since we assume (A,B) is controllable, we must have $k \geq k'$. So we remove the restriction of $X_k \neq 0$ in the Invariant Factor Theorem, and take $X_{k'+1} = \dots = X_k = 0$. Accordingly, $p_{k'+1}(\lambda) = \dots = p_k(\lambda) = 1$. Thus we can let $k' = k$.

Appendix 2. Brunovsky Canonical Form

The existence of the following was apparently first proven by Brunovsky in a paper not available in translation (ref. [6] in Brunovsky's paper listed in our bibliography). Rosenbrock (p. 97) proves it, as does Warren and Eckberg (p. 438). The result is as follows:

Given a controllable pair (A',B) there exists a feedback F such that $A'+BF$ has invariant polynomials $\lambda^{\kappa(1)}, \ldots, \lambda^{\kappa(k)}$ where $\{\kappa(i)\}$ is the set of controllability indices of (A,B). Furthermore we can pick bases $\{b_i;\ i=1,\ldots,m\}$ and $\{v_i;\ i=1,\ldots n\}$ for U and X respectively, such that $(A=A'+BF)$

a) $v_{\sigma(i-1)+1} = Bb_i$ for $i = 1, \ldots, k$

$Bb_i = 0$ for $i = k+1, \ldots, m$

and

b) $Av_i = v_{i+1}$ for $i \notin \{\sigma(1), \ldots, \sigma(k)\}$ and $i \leq n$

$Av_i = 0$ for $i \in \{\sigma(1), \ldots, \sigma(k)\}$.

We call $\{v_i\}$ a shift basis for A.

In matrix form, this says that there is a representation of A and B as

$$A = \begin{pmatrix} T_1 & 0 & \cdots & & 0 \\ 0 & T_2 & 0 & \cdots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & T_k \end{pmatrix} \quad B = \begin{pmatrix} r_1 & r_2 & \cdots & r_k & 0 & \cdots & 0 \end{pmatrix}$$

where T_i is of dimensions $\kappa(i)\times\kappa(i)$ and is zero everywhere except for ones on the lower diagonal, and r_i is column of length $\kappa(i)$ with all zeros except for a one in the $\kappa(i-1)+1^{th}$ entry. We call (A,B) in this representation the Brunovsky canonical form for (A ,B).

$$T_i = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \qquad r_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Appendix 3. Rosenbrock's Proof

The following is a brief sketch of Rosenbrock's proof, which appears on pp. 190-192 of his book. Assume A and B have been transformed to Brunovsky canonical form by feedback and change of basis.

We first make note of the facts that the invariant polynomials of any linear map M are given by the non-unit diagonal elements of the Smith form of the matrix $\lambda I-M$. "$\sim$" denotes the equivalence relation of multiplication by unimodular polynomial matrices. It can be shown that equivalent matrices have the same Smith form.

Let F be an arbitrary feedback map. To sketch the proof we indicate the results of a sequence of equivalence transformations which yield a simple form equivalent to $\lambda I-A-BF$. By a right equivalence we have

$$\begin{pmatrix} \lambda I-A & B \\ F & I \end{pmatrix} \sim \begin{pmatrix} \lambda I-A-BF & 0 \\ 0 & I \end{pmatrix}$$

so that the matrix on the left has the same invariant polynomials as $\lambda I-A-BF$, except for additional units.

Now by right equivalence,

$$\begin{pmatrix} \lambda I-A & B \\ F & I \end{pmatrix} \sim \begin{pmatrix} -A & B \\ P & I \end{pmatrix}$$

where P consists of polynomials with coefficients which are the elements of F. Then by left equivalence,

$$\begin{pmatrix} -A & B \\ P & I \end{pmatrix} \sim \begin{pmatrix} -A & B \\ P' & 0 \end{pmatrix}$$

where P' is zero except in columns 1, $\kappa(1)+1$, ..., $\kappa(k-1) + 1$.

Finally, by right equivalence we have

$$\begin{bmatrix} -A & B \\ P' & 0 \end{bmatrix} \sim \begin{bmatrix} I & 0 \\ 0 & F' \end{bmatrix}$$

where $F' = \left(f_1'(\lambda) \quad f_2'(\lambda) \quad \dots \quad f_k'(\lambda)\right)$ and

$$f_i'(\lambda) = \lambda^{\kappa(i)} u_i + \sum_{j=1}^{\kappa(i)} f_{\sigma(i-1)+j} \lambda^{j-1} .$$

u_j is the unit vector with a one in the j^{th} row, and f_j is the j^{th} column of F.

Using the fact that the i^{th} invariant polynomial is given by the ratio of the greatest common divisor of all $i \times i$ sub-determinants to the greatest common divisor of all $(i-1)\times(i-1)$ sub-determinants, the presence of the term $\lambda^{\kappa(i)}$ in the i^{th} diagonal element of F' gives the necessary conditions of Rosenbrock's theorem.

For the sufficient conditions, it is asserted that the choice of F will give any desired invariant polynomials subject to the necessary conditions. A reference is made to the proof of the previous theorem (Theorem 4.1, p. 186), but since an inequality (4.26, p. 188) does not apply, this does not work (4.27 does necessarily hold, and so Lemma 4.1, p. 184, need not apply).

This can probably be corrected, and the idea of the proof of Lemma 4.1 can possibly be modified for the general case.

We demonstrate Rosenbrock's proof on Example 2 from Chapter 2. This is a case in which his Lemma 4.1 does not apply.

$$
\begin{pmatrix} \lambda I-A & -B \\ F & I \end{pmatrix} = \begin{pmatrix}
\lambda & -1 & & & & & & & & 0 & & \\
 & \lambda & -1 & & & & & & & 0 & & \\
 & & \lambda & & & & & & & -1 & & \\
 & & & \lambda & -1 & & & & & & 0 & \\
 & & & & \lambda & -1 & & & & & 0 & \\
 & & & & & \lambda & & & & & -1 & \\
 & & & & & & \lambda & -1 & & & & 0 \\
 & & & & & & & \lambda & -1 & & & 0 \\
 & & & & & & & & \lambda & & & -1 \\
f_{11} & \cdots & & \cdots & & \cdots & & & f_{19} & 1 & & \\
f_{21} & \cdots & & \cdots & & \cdots & & & f_{29} & & 1 & \\
f_{31} & \cdots & & \cdots & & \cdots & & & f_{39} & & & 1
\end{pmatrix}
$$

$$
\sim \begin{pmatrix}
\lambda & -1 & & & & & & & & 0 & & \\
 & \lambda & -1 & & & & & & & 0 & & \\
 & & 0 & & & & & & & -1 & & \\
 & & & \lambda & -1 & & & & & & 0 & \\
 & & & & \lambda & -1 & & & & & 0 & \\
 & & & & & 0 & & & & & -1 & \\
 & & & & & & \lambda & -1 & & & & 0 \\
 & & & & & & & \lambda & -1 & & & 0 \\
 & & & & & & & & 0 & & & -1 \\
f_{11} & \cdots & (f_{13}+\lambda) & \cdots & & f_{15} & \cdots & & f_{19} & 1 & & \\
f_{21} & \cdots & f_{23} & \cdots & & (f_{25}+\lambda) & \cdots & & f_{29} & & 1 & \\
f_{31} & \cdots & f_{33} & \cdots & & f_{36} & \cdots & & (f_{39}+\lambda) & & & 1
\end{pmatrix}
$$

$$
\sim \begin{pmatrix}
 & -1 & & & & & & & & 0 & & \\
 & & -1 & & & & & & & 0 & & \\
 & & & & & & & & & -1 & & \\
 & & & & -1 & & & & & & 0 & \\
 & & & & & -1 & & & & & 0 & \\
 & & & & & & & & & & -1 & \\
 & & & & & & & -1 & & & & 0 \\
 & & & & & & & & -1 & & & 0 \\
 & & & & & & & & & & & -1 \\
 & & & & & & & & & 1 & & \\
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & & 1 & \\
 & & & & & & & & & & & 1
\end{pmatrix}
$$

where the p_i are vector polynomials. We take special note of

$$p_1 = \begin{pmatrix} \lambda^3+f_{13}\lambda^2+f_{12}\lambda+f_{11} \\ f_{23}\lambda^2+f_{22}\lambda+f_{21} \\ f_{33}\lambda^2+f_{32}\lambda+f_{31} \end{pmatrix} \quad p_4 = \begin{pmatrix} f_{15}\lambda^2+f_{15}\lambda+f_{14} \\ \lambda^3+f_{25}\lambda^2+f_{25}\lambda+f_{24} \\ f_{36}\lambda^2+f_{35}\lambda+f_{34} \end{pmatrix}$$

$$p_7 = \begin{pmatrix} f_{19}\lambda^2+f_{13}\lambda+f_{17} \\ f_{29}\lambda^2+f_{28}\lambda+f_{27} \\ \lambda^3+f_{39}\lambda^2+f_{33}\lambda+f_{37} \end{pmatrix} .$$

Continuing with equivalences,

$$\sim \begin{bmatrix} -1 & & & & & & & & & 0 & & \\ & -1 & & & & & & & & 0 & & \\ & & & & & & & & & -1 & & \\ & & & -1 & & & & & & & 0 & \\ & & & & -1 & & & & & & 0 & \\ & & & & & & & & & & -1 & \\ & & & & & & -1 & & & & & 0 \\ & & & & & & & -1 & & & & 0 \\ & & & & & & & & & & & -1 \\ p_1 & 0 & 0 & p_4 & 0 & 0 & p_7 & 0 & 0 & 0 & & \\ & & & & & & & & & & 0 & \\ & & & & & & & & & & & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & & & & & & & & & & & \\ & -1 & & & & & & & & & & \\ & & -1 & & & & & & & & & \\ & & & -1 & & & & & & & & \\ & & & & -1 & & & & & & & \\ & & & & & -1 & & & & & & \\ & & & & & & -1 & & & & & \\ & & & & & & & -1 & & & & \\ & & & & & & & & -1 & & & \\ & & & & & & & & & p_1 & p_4 & p_7 \\ & & & & & & & & & & & \\ & & & & & & & & & & & \end{bmatrix}$$

$\sim \begin{bmatrix} I & 0 \\ 0 & F' \end{bmatrix}$. Now we wish to pick F so as to obtain the the desired Smith form for F'. Rosenbrock's idea is to transform to an equivalent form where the diagonal elements are monic polynomials whose degrees are the degrees of the desired invariant polynomials and are higher than the degree of any other element in the same column. Then by choice of F set the off-diagonal elements to zero and the diagonal elements to the invariant polynomials, obtaining the Smith form directly.

Unfortunately his formula for this transformation is recursive and assumes that the diagonal elements in F' are of unequal degrees. It is not obvious what a generalization would be. Furthermore, it is not shown that after transformation there is sufficient freedom from F to pick all the coefficients as desired.

Appendix 4. Kalman's Paper

We give a brief sketch and discussion of the paper by Kalman mentioned in the introduction (see bibliography).

--Existence of the Brunovsky canonical form. Kalman's proof is essentially the same as Rosenbrock's proof mentioned in Appendix 3, except that Kalman's formulas for the basis in which A is represented in generalized companion form are incorrect.

--Proof of Rosenbrock's theorem.

Sufficiency: Kalman gives a simple example and asserts that "the general case is similar." The example the same as our Example 1 in Chapter 2. This case of $\kappa(1) = \kappa(2) = 2$ is really the simplest non-trivial example possible. He did as we have done, that is first change the degrees of the invariant factors, and then change the coefficients of the invariant polynomials. This latter step he does incorrectly. He asserts that "the scalar pole shifting trick still works." It is easy to see that this is not so, unless one proceeds step-wise and changes basis as we did in our general construction. Specifically, Kalman has obtained by feedback and change of basis the matrices

$$A' = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ & & & 0 \end{pmatrix} \qquad B' = \begin{pmatrix} 0 & \\ 0 & 1 \\ 1 & \\ & 1 \end{pmatrix} .$$

He suggests taking $F = \begin{pmatrix} 0000 \\ 000\alpha \end{pmatrix}$ which gives

$A = A' - B'F = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & -\alpha \\ 0 & 0 & 0 & \\ & & & -\alpha \end{pmatrix}$. He says that the $A_{2,4}$ term

"has no effect" on the characteristic polynomial [true], and hence the invariant polynomials "can be arbitrarily chosen." This latter statement does not follow for the following reasons:

First, as given, A has minimal polynomial $\lambda^3(\lambda + \alpha)$, so there is only one invariant factor. Presumably this was an oversight, and Kalman intended also to apply the scalar feedback trick to what was the larger of the two invariant factors before the above feedback. So suppose we want invariant polynomials $(\lambda - 1)$ and $(\lambda - 1)^2(\lambda + 2) = \lambda^3 - 4\lambda^2 + 5\lambda - 2$. Applying the scalar feedback trick directly we take $F = \begin{pmatrix} -2 & 5 & -4 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ and so

$A' = A - BF = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & 1 \\ 2 & -5 & 4 & \\ & & & 1 \end{pmatrix}$. Unfortunately, A' has minimal

polynomial $(\lambda - 1)^3(\lambda + 2)$. This is easily seen by noting that the vector $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ has cyclic order four, and computing

the determinant of A'.

For comparison with our construction, we first obtain

$A = \begin{pmatrix} 0&1&0& \\ 0&0&1&1 \\ 0&0&0& \\ &&&1 \end{pmatrix} \qquad B = \begin{pmatrix} 0& \\ 0&1 \\ 1& \\ &1 \end{pmatrix}$. Then we change basis to the columns

of $T = \begin{pmatrix} 1&0&0&1 \\ 0&1&0&1 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$. Then $T^{-1}AT = \begin{pmatrix} 0&1&0& \\ 0&0&1& \\ 0&0&0& \\ &&&1 \end{pmatrix}$ $T^{-1}B = \begin{pmatrix} 0&-1 \\ 0&0 \\ 1&0 \\ 0&1 \end{pmatrix}$

Then we take $F' = \begin{pmatrix} -2&5&-4&0 \\ 0&0&0&0 \end{pmatrix}$ and obtain

$A' = T^{-1}AT - T^{-1}BF = \begin{pmatrix} 0&1&0& \\ 0&0&1& \\ 2&-5&4& \\ &&&1 \end{pmatrix}$ as desired. Note that the

correct feedback to be applied to the orinal matrix

$\begin{pmatrix} 0&1&0& \\ 0&0&1& \\ 0&0&0& \\ &&&0 \end{pmatrix}$ would be $\begin{pmatrix} 0&0&0&0 \\ 0&0&0&-1 \end{pmatrix} + FT^{-1} = \begin{pmatrix} -2&5&-4&3 \\ 0&0&0&-1 \end{pmatrix}$, which is

not an obvious consequence of the scalar feedback trick.

Secondly, if we look at our Example 2 in Chapter 2, we see that Example 1 does not show how to change the orders of the invariant factors. Example 3 further compounds the difficulties. Kalman's proof by example, then, is the simplest case, incompletely done.

Necessity: Kalman's argument is as follows: i) The Brunovsky canonical form satisfies the necessary conditions, and ii) every controllable pair (A,B) can be obtained from its Brunovsky canonical form by his construction in the sufficiency part. Therefore the necessary conditions always hold [presumably since the controllability indices are invariant under feedback].

Unfortunately, there is no proof offered of the assertion ii). Indeed, we have seen that the construction given by Kalman does not generally work. The question of uniqueness of the feedback to obtain a given cyclic structure is discussed in Chapter 3.

Appendix 5. Dickinson's Paper

Dickinson asserts that necessity follows directly from the structure of the multivariable controller form of Luenberger. This does not seem to be clear.

The sufficiency proof is based on the following result, for which an explicit construction is given:

Lemma. Let (F,G) have controllability indices $\{\kappa(i); i=1, \ldots, m\}$. If for some i and j $\kappa(i) \geq \kappa(j)+2$, then there is a matrix $\overline{G}$ such that $(F,\overline{G})$ have controllability indices differing only in that $\kappa(i)$ is replaced by $\kappa(i)-1$ and $\kappa(j)$ is replaced by $\kappa(j)+1$. Reordering may be necessary.

To prove sufficiency from this, let F_o be the map we wish to obtain from F by feedback, in rational canonical form (diagonal blocks of companion matrices). If F_o satisfies the necessary conditions, then there is a finite sequence of sets of indices such that the first set is the degrees of the invariant factors of F_o, the last set is the controllability indices of (F,G), only two indices in each set differ from the succeding set, and these indices differ only by a unit increment in one index and a unit decrement in the other.

By repeated application of the lemma, we can then construct a matrix G_o such that (F_o,G_o) has the same controllability indices as (F,G). Of course, then these two pairs have the same Brunovsky form.

The desired feedback is then simply described, given the transformations to Brunovsky form. Suppose (F°, G°) is the Brunovsky form. Assume

$$F^\circ = T_1^{-1}(F_o - G_o K_1)T_1 = T^{-1}(F - GK)T \quad \text{and}$$

$$G^\circ = T_1^{-1} G_o S_1 = T^{-1} GS.$$

Then it is easy to check that

$$F_o = T_1 T^{-1}(F - G(K - SS_1^{-1} K_1 T_1 T^{-1}))TT_1^{-1} .$$

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