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Convexity and Duality in Optimization Theory

Stephen Kinyon Young

Preface

This doctoral thesis was written under my direction in 1977. Chapters VIII and IX of the thesis have been published in 1984 Annali di Matematica pura ed applicata (IV), Vol. CXXXVII, pp. 1-39 . The remainder of the thesis has never been published. I am issuing this as a technical report after fifteen years since I believe that it contains material which might still be new and have relevance to optimization problems arising in control systems design.

Sanjoy K. Mitter
August 1992

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CONVEXITY AND DUALITY IN OPTIMIZATION THEORY

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Stephen Kinyon Young

This report is based on the unaltered thesis of Stephen Kinyon Young submitted to the Department of Mathematics on July 15, 1977 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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CONVEXITY AND DUALITY IN OPTIMIZATION THEORY

by

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B.S., Yale
(1971)

SUBMITTED IN PARTIAL FULFILLMENT

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ABSTRACT

The duality approach to solving convex optimization problems is studied in detail using tools in convex analysis and the theory of conjugate functions. Conditions for the duality formalism to hold are developed which require that the optimal value of the original problem vary continuously with respect to perturbations in the constraints only along feasible directions; this is sufficient to imply existence for the dual problem and no duality gap. These conditions are also posed as certain local compactness requirements on the dual feasibility set, based on a characterization of locally compact convex sets in locally convex spaces in terms of non-empty relative interiors of the corresponding polar sets.

These results are applied to minimum norm and spline problems and improve previous existence results, as well as expressing them in a duality framework. Related results include conditions for the sum of two closed convex sets to be closed, leading to an extended separation principle for closed convex sets.

The continuous linear programming problem is also studied. An extended dual problem is formulated, and a condition sufficient for dual solutions to exist with no duality gap is given which is natural in the context of several examples. Moreover the dual solutions can be taken to be extreme points, which suggests the possibility of a simplex-like algorithm.

Finally, the problem of characterizing optimal quantum detection and estimation is studied using duality techniques. The duality theory for the quantum estimation problem entails studying oeprator-valued measures, developing a generalized Riesz Representation Theorem, and looking at the approximation property for the space of linear operators on a Hilbert space.

Thesis Supervisor: Sanjoy K. Mitter
Title: Professor of Electrical Engineering

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CONTENTS

I. Overview of Thesis

Overview of thesis

The idea of duality theory for solving convex optimization problems is to transform the original problem into a "dual" problem which is easier to solve and which has the same value as the original problem; constructing the dual solution corresponds to formulating extremality conditions which characterize optimality in the original problem. This thesis investigates and extends the duality approach to optimization and applies this approach to several problems of interest.

Chapter II defines basic concepts and develops basic techniques in convex analysis and the theory of conjugate functions which are relevant to studying the duality formalism. It includes an investigation of the relationships between nonempty relative interiors of convex sets and local compactness of the polar sets, which culminates in a characterization of relative continuity points of convex functions in terms of local compactness properties of the conjugate functions.

Chapter III presents a detailed study of the duality approach to optimization using the techniques developed in Chapter II. Conditions for duality to hold are derived which require that the optimal value of the original problem vary "relatively continuously" with respect to perturbations in the constraints only along feasible directions; this is sufficient to imply existence for the dual problem and no duality gap. These conditions are also posed as certain local compactness requirements on the dual feasibility set, based on the work in Chapter II.

Chapter IV applies the duality approach of Chapter III to minimum norm and spline problems, thereby yielding improved existence results as well as expressing them in a duality framework. Related results include conditions for the sum of two closed convex sets in a Banach space to be closed, extending Dieudonne's results and leading to an extended separation principle for disjoint closed convex (possibly unbounded) sets.

Chapter V studies the continuous-time linear programming problem. Previous results in the literature have formulated the dual linear programming problem in too restrictive a space, so that conditions guaranteeing dual solutions are not satisfied in interesting cases. By imbedding the dual problem in a larger space, it is possible to get dual solutions wiht no duality gap under assumptions which are natural in the context of a communications network problem and a dynamic economic model. Moreover, the dual solutions may be taken to be extreme points of the (possibly unbounded, but locally compact) feasibility set; a simple example is presented which shows how this might lead to a "primal-dual" type of algorithm (in analogy to the finite dimensional simplex algorithm) for solving the linear problem. However, much work remains in investigating this approach and in understanding the extreme point structure of the feasibility set.

The remaining chapters consider the problem of characterizing optimal quantum detection and estimation. The quantum nature of these statistical problems requires the use of operator-valued measures; a chapter is devoted to developing general integration theory for operator-valued measure and proving an extended Riesz Representation Theorem for duality purposes. The estimation problem also entails looking at certain somewhat esoteric properties of tensor product spaces, needed to properly formulate the problem; however, the actual duality results then follow without too much difficulty.

II. Convex Analysis

Abstract. Techniques in convex analysis and the theory of conjugate functions are studied. A characterization of locally compact convex sets in locally convex spaces is given in terms of nonempty relative interiors of the corresponding polar sets. This result is extended in a detailed investigation of the relationships between relative continuity points of convex functions and local compactness properties of the level sets of corresponding conjugate functions.

1. Notation and basic definitions

This section assumes a knowledge of topological vector spaces and only serves to recall some concepts in functional analysis which are relevant for optimization theory. The extended real line $[-\infty,+\infty]$ is denoted by $\overline{R}$. Operations in $\overline{R}$ have the usual meaning with the additional convention that

$$(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty.$$

Let X be a set, $f: X \to \overline{R}$ a map from X into $[-\infty,+\infty]$. The epigraph of f is

$$\text{epi}f \overset{\Delta}{=} \{(x,r) \in X \times R: r \geq f(x)\}.$$

The effective domain of f is the set

$$\text{dom}f \overset{\Delta}{=} \{x \in X: f(x) < +\infty\}.$$

The function f is proper iff $f \not\equiv +\infty$ and $f(x) > -\infty$ for every $x \in X$. The indicator function of a set $A \subset X$ is the map $\delta_A: X \to \overline{R}$ defined by

$$\delta_A(x) = \begin{cases} +\infty & \text{if } x \notin A \\ 0 & \text{if } x \in A \end{cases}.$$

Let X be a vector space. A map $f: X \to \overline{R}$ is convex iff epif is a convex subset of $X \times R$, or equivalently iff

$$f(\varepsilon x_1+(1-\varepsilon)x_2) \leq \varepsilon f(x_1)+(1-\varepsilon)f(x_2)$$

for every $x_1, x_2 \in X$ and $\varepsilon \in [0,1]$. The convex hull of f is the largest convex function which is everywhere less than or equal to f; it is given by

$$(\mathrm{co}f)(x) = \sup\{f'(x): f' \text{ is convex } X \to \overline{R},\ f' \leq f\}$$

$$= \sup\{f'(x): f' \text{ is linear } X \to \overline{R},\ f' \leq f\}.$$

Equivalently, the epigraph of cof is given by

$$\mathrm{epi}(\mathrm{co}f) = \{(x,r) \in X\times R: (x,s) \in \mathrm{coepi}f \text{ for every } s > r\},$$

where coepif denotes the convex hull of epif.

Let X be a topological space. A map $f: X \to \overline{R}$ is lower semicontinuous (lsc) iff epif is a closed subset of $X\times R$, or equivalently iff $\{x \in X: f(x) \leq r\}$ is a closed subset of X for every $r \in R$. The map $f:X \to \overline{R}$ is lsc at x_o iff given any $r \in (-\infty, f(x_o))$ there is a neighborhood N of x_o such that $r < f(x)$ for every $x \in N$. The lower semi-continuous hull of f is the largest lower semicontinuous functional on X which everywhere minorizes f, i.e.

$$(\mathrm{lsc}f)(x) = \sup\{f'(x): f' \text{ is lsc } X \to \overline{R},\ f' \leq f\}$$

$$= \liminf_{x'\to x} f(x').$$

Equivalently, $\mathrm{epi}(\mathrm{lsc}f) = \mathrm{cl}(\mathrm{epi}f)$ in $X\times R$.

A *duality* $\langle X,X^*\rangle$ is a pair of vector spaces X,X^* with a bilinear form $\langle\cdot,\cdot\rangle$ on $X\times X^*$ that is separating, i.e. $\langle x,y\rangle = 0\ \forall y \in X^* \Rightarrow x = 0$ and $\langle x,y\rangle = 0\ \forall x \in X \Rightarrow x = 0$. Every duality is equivalent to a Hausdorff locally convex space X paired with its topological dual space X^* under the natural bilinear form $\langle x,y\rangle \stackrel{\Delta}{=} y(x)$ for $x \in X$, $y \in X^*$. We shall also write $xy \equiv \langle x,y\rangle \equiv y(x)$ when no confusion arises.

Let X be a (real) Hausdorff locally convex space (HLCS), which we shall always assume to be real. X^* denotes the topological dual space of X. The *polar* of a set $A \subset X$ and the *(pre-)polar* of a set $B \subset X^*$ are defined by †

$$A^{o} \stackrel{\Delta}{=} \{y \in X^*: \sup_{x\in A} \langle x,y\rangle \leq 1\}$$

$$^{o}B \stackrel{\Delta}{=} \{x \in X: \sup_{y\in B} \langle x,y\rangle \leq 1\}.$$

The *conjugate* of a functional $f: X \to \overline{R}$ and the *(pre-)conjugate* of a functional $g: X^* \to \overline{R}$ are defined by

$$f^*: X^* \to \overline{R}: y \mapsto \sup_{x\in X}[\langle x,y\rangle - f(x)]$$

$$g^*: X \to \overline{R}: x \mapsto \sup_{y\in Y}[\langle x,y\rangle - g(y)].$$

† We use the convention $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$. Hence $\emptyset^{0} = X^*$.

If X is a HLCS there are several topologies on X which are important. By τ we denote the original topology on X; by the definition of equicontinuity, τ is precisely that topology which has a basis of 0-neighborhoods consisting of polars of equicontinuous subsets of X^*. The <u>weak topology</u> $w(X,X^*)$ is the weakest topology compatible with the duality $\langle X,X^*\rangle$, i.e. it is the weakest topology on X for which the linear functionals $x \mapsto \langle x,y\rangle$, $y \in X^*$ are continuous. Equivalently, $w(X,X^*)$ is the locally convex topology on X generated by the seminorms $x \mapsto |\langle x,y\rangle|$ for $y \in X^*$; it has a basis of 0-neighborhoods given by polars of finite subsets of X^*. The <u>Mackey topology</u> $m(X,X^*)$ on X is the strongest topology on X compatible with the duality $\langle X,X^*\rangle$†; it has a 0-neighborhood basis consisting of polars of all $w(X^*,X)$-compact convex†† subsets of X^*. The <u>strong topology</u> $s(X,X^*)$ is the strongest locally convex topology on X that still has a basis consisting of $w(X,X^*)$-closed sets;

† A topology τ_0 on the vector space X is <u>compatible</u> with the duality $\langle X,X^*\rangle$ iff $(X,\tau_0)^* = X^*$, i.e. the space of all continuous linear functionals on X with the τ_0-topology may be identified with X^*.

†† The word "convex" here may not be omitted unless X is a barrelled space. In general there may be $w(X^*,X)$-compact subsets of X^* whose closed convex hulls are not compact for the $w(X^*,X)$ topology.

it has as 0-neighborhood basis all $w(X,X^*)$-closed convex absorbing subsets of X, or equivalently all polars of $w(X^*,X)$-bounded subsets of X^*. We shall often write w,m,s for $w(X,X^*)$, $m(X,X^*)$, $s(X,X^*)$, and also w^* for $w(X^*,X)$. The strong topology need not be compatible with the duality $\langle X,X^*\rangle$. In general we have $w(X,X^*) \subset \tau \subset m(X,X^*) \subset s(X,X^*)$. For a convex set A, however, it follows from the Hahn Banach separation theorem that A is closed iff A is $w(X,X^*)$-closed iff A is $m(X,X^*)$-closed. More generally,

$$w\text{-cl}A = \text{cl}A = m\text{-cl}A \supset s\text{-cl}A$$

when A is convex. Similarly, if a convex function $f: X \to \overline{R}$ is $m(X,X^*)$-lsc then it is lsc and even $w(X,X^*)$-lsc. It is also true that the bounded sets are the same for every compatible topology on X.

Let X be a HLCS and $f: X \to \overline{R}$. The conjugate function $f^*: X^* \to \overline{R}$ is convex and $w(X^*,X)$-lsc since it is the supremum of the $w(X^*,X)$-continuous affine functions $y \mapsto \langle x,y\rangle - f(x)$ over all $x \in \text{dom} f$. Similarly, for $g: X^* \to \overline{R}$ it follows that the preconjugate $^*g: X \to R$ is convex and lsc. The conjugate functions f^*, $^*g$ never take on $-\infty$ values, unless they are identically $-\infty$ or equivalently $f \equiv +\infty$ or $g \equiv +\infty$. Finally, from the Hahn-Banach separation theorem it follows that

$$*(f^*) = \text{lsccof} \qquad (1)$$

whenever f has an affine minorant, or equivalently whenever $f^* \not\equiv +\infty$; otherwise lsccof takes on $-\infty$ values and $f^* \equiv +\infty$, $*(f^*) \equiv -\infty$.

The following lemma is very useful.

1.1 Lemma Let X be a HLCS, $f: X \to \overline{R}$. Then $co(\text{dom}f) = \text{dom}(\text{co}f)$. If $f^* \not\equiv +\infty$, then $\text{clcodom}f = \text{cldom}*(f^*)$.

Proof. Now $\text{co}f \le f$, so $\text{dom}(\text{co}f) \supset \text{dom}f$ and hence (since dom cof is convex) $\text{dom}(\text{co}f) \supset \text{codom}f$. Conversely, cof + $\delta_{\text{codom}f}$ is a convex function everywhere dominated by f, hence by cof, and so $\text{codom}f \supset \text{dom}(\text{co}f)$. Thus $\text{dom}(\text{co}f) = \text{co}(\text{dom}f)$.

Similarly, $*(f^*) \le f$ so $\text{dom}*(f^*) \supset \text{dom}f$ and hence $\text{cldom}*(f^*) \supset \text{clcodom}f$ (since $\text{dom}*(f^*)$ is convex). Conversely, $*(f^*)+\delta_{\text{clcodom}f}$ is a convex lsc function everywhere dominated by f, and since $*(f^*)$ is the largest convex lsc function dominated by f (in the case that $f^* \not\equiv +\infty$, by (1)) we have $*(f^*) + \delta_{\text{clcodom}f} \le *(f^*)$ and $\text{clcodom}f \supset \text{dom}*(f^*)$. Thus $\text{cldom}*(f^*) = \text{clcodom}f$ and the lemma is proved. ⊠

A barrelled space is a HLCS X for which every closed convex absorbing set is a 0-neighborhood; equivalently, the $w(X^*,X)$-bounded sets in X^* are conditionally $w(X^*,X)$-compact. It is then clear that the $m(X,X^*)$ topology is the original topology, and the equicontinuous sets in X^* are the conditionally w*-compact sets. Every Banach space or Frechet space is barrelled, by the Banach-Steinhaus theorem.

We use the following notation. If $A \subset X$ = HLCS, then intA, corA, riA, rcorA, clA, span A, affA, coA denote the interior of A, the algebraic interior or core of A, the relative interior of A, the relative core or algebraic interior of A, the closure of A, the span of A, the affine hull of A, and the convex hull of A. By relative interior of A we mean the interior of A in the relative topology of X on affA; that is $x \in$ riA iff there is a 0-neighborhood N such that $(x+N) \cap \text{aff}A \subset A$. Similarly, $x \in$ rcorA iff $x \in A$ and A-x absorbs affA-x, or equivalently iff $x+[0,\infty)\cdot A \supset A$ and $x \in A$. By affine hull of A we mean the smallest (not necessarily closed) affine subspace containing A; affA = A +span(A-A) = x_0 + span(A-x_0) where x_0 is any element of A.

Let A be a subset of the HLCS X and B a subset of X^*. We have already defined A^o, ${}^oB$. In addition, we make the following useful definitions:

$$A^+ \stackrel{\Delta}{=} \{y \in X^*: \langle x,y\rangle \geq 0 \ \ \forall x \in A\}$$
$$A^- \stackrel{\Delta}{=} -A^+ = \{y \in X^*: \langle x,y\rangle \leq 0 \ \ \forall x \in A\}$$
$$A^\perp \stackrel{\Delta}{=} A^+ \cap A^- = \{y \in X^*: \langle x,y\rangle = 0 \ \ \forall x \in A\}.$$

Similarly, for $B \subset X^*$ the sets ${}^{+}B$, ${}^{-}B$, ${}^{\perp}B$ are defined in X in the same way. Using the Hahn-Banach separation theorem it can be shown that for $A \subset X$, ${}^{o}(A^{o})$ is the smallest closed convex set containing $A \cup \{0\}$; ${}^{+}(A^{+}) = {}^{-}(A^{-})$ is the smallest closed convex cone containing A; and ${}^{\perp}(A^{\perp})$ is the smallest closed subspace containing A. Thus, if A is nonempty[†] then

$${}^{o}(A^{o}) = \mathrm{clco}(A \cup \{0\})$$

$${}^{+}(A^{+}) = \mathrm{cl}[0,\infty) \cdot \mathrm{co}A$$

$${}^{\perp}(A^{\perp}) = \mathrm{clspan}A$$

$$A + {}^{\perp}((A-A)^{\perp}) = \mathrm{claff}A.$$

[†]If $A = \emptyset$, then ${}^{o}(A^{o}) = {}^{+}(A^{+}) = {}^{\perp}(A^{\perp}) = \{0\}$.

2. Recession cones, lineality subspaces, recession functionals

Let A be a nonempty subset of the HLCS X. The <u>recession cone</u> of A is defined to be the set A_∞ of all half-lines contained in clcoA; that is, a vector x is in A_∞ iff for any fixed point $a \in A$ the half-line $a+[0,\infty)\cdot x$ starting at a and passing through x is entirely contained in clcoA. A_∞ is a closed convex cone with vertex at 0; in fact $A_\infty = {}^-(A^o)$. For consistency we define $\emptyset_\infty = \{0\}$. The following proposition (modelled after [R66]) provides a detailed characterization of A_∞.

2.1 <u>Proposition</u>. Let A be a nonempty subset of the HLCS X. Then the following are equivalent:

1) $x \in A_\infty$
2) $A + [0,\infty)\cdot x \subset \text{clco}A$
3) $x \in \bigcap_{t>o} \bigcap_{a\in A} t\cdot(\text{clco}A-a)$
4) $\exists\, a \in A$ st $a + [0,\infty)\cdot x \subset \text{clco}A$
5) $\exists\, a \in A$ st $x \in \bigcap_{t>o} t\cdot(\text{clco}A-a)$
6) $\exists$ nets of scalars $t_i > 0$ and vectors $x_i \in \text{co}A$ st $t_i \to 0$, $t_i x_i \to x$
7) $x \in \bigcap_{\varepsilon>o} [\text{cl}(0,\varepsilon)\cdot\text{co}A]$
8) $x \in {}^-(\text{dom}\delta_A^*)$
9) $x \in {}^-(A^o)$

10) $A + x \subset \mathrm{clco}A$.

Proof. 1) <=> 2) is the definition of A_∞. 2) <=> 3), 2) => 4), 4) <=> 5) are trivial.

4) => 6). Let $\mathcal{B}$ be a basis of 0-neighborhoods in X and consider the directed set $\mathcal{B} \times (0,\infty)$ with the ordering $(B,\varepsilon) \geq (B',\varepsilon')$ iff $B \subset B'$, $\varepsilon \leq \varepsilon'$. For every $B \in \mathcal{B}$, $\varepsilon > 0$ take $t_{B,\varepsilon} = \varepsilon$ and $x_{B,\varepsilon} \in \mathrm{co}A \cap (a+\varepsilon^{-1}x+B)$, where the intersection is nonempty since $a+\varepsilon^{-1}x \in \mathrm{clco}A$ by hypothesis 4). Then $t_{B,\varepsilon} \to 0$ and $t_{B,\varepsilon} \cdot x_{B,\varepsilon} \in x+\varepsilon\cdot a+\varepsilon\cdot B$, so $t_{B,\varepsilon} \cdot x_{B,\varepsilon} \to x$.

6) => 7). By hypothesis $\exists\, t_i \to 0^+$, $x_i \in \mathrm{co}A$, $t_i x_i \to x$. Given any $\varepsilon > 0$, the t_i eventually belong to $(0,\varepsilon)$, so $t_i x_i \in (0,\varepsilon)\cdot \mathrm{co}A$. But then $x = \lim t_i x_i \in \mathrm{cl}(0,\varepsilon)\cdot \mathrm{co}A$.

7) => 6). Again, consider the directed set $\mathcal{B} \times (0,\infty)$. For every 0-neighborhood $B \in \mathcal{B}$, $\varepsilon > 0$ take $t_{B,\varepsilon} \in (0,\varepsilon)$ and $x_{B,\varepsilon} \in \mathrm{co}A$ such that $t_{B,\varepsilon} \cdot x_{B,\varepsilon} \in x+B$; this is possible since $x \in \mathrm{cl}(0,\varepsilon)\cdot \mathrm{co}A$ by hypothesis 7). Then $t_{B,\varepsilon} \to 0$ and $t_{B,\varepsilon} \cdot x_{B,\varepsilon} \to x$.

6) => 8). Suppose $y \in \mathrm{dom}\, \delta_A^*$, i.e. $M = \sup_{a \in A} \langle a,y\rangle$ is finite. Now $\langle x_i,y\rangle \leq M$ since $x_i \in \mathrm{co}A$, so $\langle x,y\rangle = \lim \langle t_i x_i, y\rangle \leq \lim t_i \cdot M = 0$. Thus $\langle x,y\rangle \leq 0$ whenever $y \in \mathrm{dom}\, \delta_A^*$.

8) <=> 9). By definition dom $\delta_A^* = [0,\infty)\cdot A^o$; hence $^-(\text{dom } \delta_A^*) = {}^-(A^o)$.

9) => 10). Suppose $A+x \not\subset$ clcoA; then $\exists\, a \in A$ st $a+tx \notin$ clcoA. By the Hahn-Banach separation theorem there is a separating linear functional $y \in X^*$ for which $\sup_{x \in \text{clcoA}} \langle x,y\rangle < \langle a+x,y\rangle$, i.e. $\delta_A^*(y) < \langle a+x,y\rangle$. Clearly $y \in \text{dom } \delta_A^*$. Also $\langle a,y\rangle < \langle a+x,y\rangle$, so $\langle x,y\rangle > 0$ and $x \notin {}^-[\text{dom}\delta_A^*]$.

10) => 1). Take any $a \in A$. By hypothesis 10), $a+x \in$ clcoA. But then, by repeated application of 10), $a+x+x \in$ clcoA, etc., so $a+nx \in$ clcoA for $n = 1,2,\ldots,$ and by convexity 1) follows. ⊠

<u>Remarks</u>. From 5) it is clear that $A_\infty = (\text{clcoA})_\infty$, since $A_\infty = \bigcap_{t>0} t\cdot(\text{clcoA}-a)$ for any fixed $a \in A$. Similarly, 3) implies that $A_\infty = ({}^o(A^o))_\infty$, since $({}^o(A^o))^o = A^o$ and $A_\infty = {}^-(A^o)$. Thus A, clcoA, ${}^o(A^o) = \text{clco}(A \cup \{0\})$ all have the same recession cone. Applying 10) to clcoA also yields

$$\text{clcoA} + A_\infty = \text{clcoA}.$$

The <u>lineality space</u> of $A \subset X$ is defined to be the set of all lines contained in clcoA, i.e. $\text{linA} \overset{\Delta}{=} A_\infty \cap (-A_\infty) = \bigcap_{t \in R} t\cdot(\text{clcoA}-a)$ where a is any fixed element of A. Lin A is a closed subspace; in fact it is the annihilator ${}^{\perp}(\mathrm{span}A^{o})$ of the smallest subspace containing A^{o}.

2.2 Corollary. Let A be a nonempty subset of the HLCS X. The following are equivalent:

1) $x \in \mathrm{lin}A$
2) $\forall\, a \in A,\ a+(-\infty,+\infty)\cdot x \subset \mathrm{clco}A$
3) $\exists\, a \in A$ st $a+(-\infty,+\infty)\cdot x \subset \mathrm{clco}A$
4) $x \in {}^{\perp}(A^{o}) \equiv {}^{\perp}(\mathrm{dom}\delta_A^{*}) \equiv {}^{\perp}(\mathrm{span}\ A^{o})$
5) $(A+x) \cup (A-x) \subset \mathrm{clco}A$.

Proof. Simply apply Proposition 2.1 to x and -x. ☒

The recession function f_∞ of a function $f\colon X \to \overline{R}$ is defined to be

$$f_\infty(x) = \sup_{y \in \mathrm{dom}f^*} \langle x,y\rangle. \tag{1}$$

This is defined in analogy to the concept of recession cones; $f_\infty(\cdot)$ is that function whose epigraph is the recession cone of epif,

$$\mathrm{epi}(f_\infty) = (\mathrm{epi}f)_\infty. \tag{2}$$

Since $f_\infty(\cdot)$ is the supremum of continuous linear functionals on X, it is convex, positively homogeneous ($f_\infty(tx) = tf_\infty(x)$ for $t > 0$), and lsc. The following proposition provides alternate characterizations of f_∞ when f is convex and lsc. In general $f_\infty = (^*(f^*))_\infty$, since $f^* = (^*(f^*))^*$.

2.3 Proposition. Let $f: X \to \overline{R}$ be a convex lsc proper function on the HLCS X. Then $f_\infty(x)$ is given by each of the following:

1) $\min\{r \in R: (x,r) \in (\text{epi}f)_\infty\}$

2) $\sup\limits_{a \in \text{dom}f} \sup\limits_{t>o} [f(a+tx)-f(a)]/t$

3) $\sup\limits_{t>o} [f(a+tx)-f(a)]/t$ for any fixed $a \in \text{dom}f$

4) $\sup\limits_{a \in \text{dom}f} [f(a+x)-f(a)]$

5) $\sup\limits_{y \in \text{dom}f^*} \langle x,y \rangle$.

In 1) the minimum is always attained (whenever it is not $+\infty$), since $(\text{epi}f)_\infty$ is a closed set.

Proof. It suffices to show that for any $r \in R$, the following are equivalent:

1') $(x,r) \in (\text{epi}f)_\infty$

2') $\forall a \in \text{dom}f$, $\forall t > 0$, $[f(a+tx)-f(a)]/t \leq r$

3') $\exists a \in \text{dom}f$ st $\forall t > 0$, $[f(a+tx)-f(a)/t \leq r$

4') $\forall a \in \text{dom}f$, $f(a+x)-f(a) \leq r$

5') $\sup\limits_{y \in \text{dom}f^*} \langle x,y \rangle \leq r$.

Using the fact that epif contains all points above the graph of f, it is easy to see that 1') through 5') are respectively equivalent to

1") $(x,r) \in (\text{epi}f)_\infty$

2") $\forall (a,s) \in \text{epi}f,\ \forall t > 0,\ (a+tx, s+tr) \in \text{epi}f$

3") $\exists (a,f(a)) \in \text{epi}f$ st $\forall t > 0,\ (a+tx, f(a)+tr) \in \text{epi}f$

4") $\forall (a,s) \in \text{epi}f,\ (a+x, s+r) \in \text{epi}f$

5") $\sup_{y \in \text{dom}f^*} \langle x,y \rangle \le r.$

The equivalence of 1") through 4") now follows directly from Proposition 2.1. If 5") holds, then $\forall a \in \text{dom}f$, $\forall s \geq f(a)$,

$$f(a+x) = {}^*(f^*)(a+x) = \sup_{y \in \text{dom}f^*} [\langle a+x,y \rangle - f^*(y)]$$

$$\le \sup_{y \in \text{dom}f^*} \langle x,y \rangle + \sup_{y \in \text{dom}f^*} [\langle a,y \rangle - f^*(y)]$$

$$\le r + {}^*(f^*)(a) = r + f(a),$$

and hence 4') holds. Conversely, if 4') holds then

$$f^*(y) = \sup_{a \in \text{dom}f} [\langle a,y \rangle - f(a)] \le \sup_{a \in \text{dom}f} [\langle a,y \rangle + r - f(a+x)]$$

$$\le r + \sup_{a \in X} [\langle a,y \rangle - f(a+x)] = r + \sup_{a \in X} [\langle a-x,y \rangle - f(a)]$$

$$= r - \langle x,y \rangle + f^*(y).$$

Hence $\langle x,y \rangle \le r$ whenever $f^*(y) < +\infty$ and 5") holds. ☒

3. Direction derivatives, subgradients

Let X be a HLCS, f a function $X \to \overline{R}$. If $f(x_o)$ is finite, then the <u>directional derivative</u> $f'(x_o;\cdot)$ of f at x_o is defined to be

$$f'(x_o;x) \overset{\Delta}{=} \lim_{t\to o^+} [f(x_o+tx)-f(x_o)]/t,$$

whenever the limit exists (it may be $\pm\infty$). In the case that $f(\cdot)$ is convex, $t \to [f(x_o+tx)-f(x_o)]/t$ is an increasing function for $t > 0$, so that $f'(x_o;\cdot)$ exists whenever $f(x_o) \in R$ and is given by

$$f'(x_o;x) = \inf_{t>o} [f(x_o+tx)-f(x_o)]/t.$$

Convexity of f also implies that $f'(x_o;\cdot)$ is positively homogeneous and convex (equivalently, sublinear), and $f(\cdot)$ is linearly minorized by its directional derivative in the sense that $f(x_o+tx) \geq f(x_o)+tf'(x_o;x)$ for every $x \in X$, $t \geq 0$.

The <u>subgradient set</u> of $f: X \to \overline{R}$ at $x_o \in X$ is defined to be

$$\partial f(x_o) \overset{\Delta}{=} \{y \in X^*: f(x) \geq f(x_o)+\langle x-x_o,y\rangle \ \forall x \in X\}.$$

Note that $\partial f(x_o)$ is always the empty set whenever $f(x_o) = +\infty$ (assuming $f \not\equiv +\infty$). When $f(x_o)$ is finite,

$y \in \partial f(x_o)$ iff the functional $x \to f(x_o)+\langle x-x_o,y\rangle$ is a continuous affine minorant of $f(\cdot)$ exact at the point x_o. Since $*(f^*)$ is the supremum of all continuous affine minorants of f, it is clear that $\partial f(x_o) \neq \emptyset$ implies that $f(x_o) = *(f^*)(x_o)$ and $\partial f(x_o) = \partial *(f^*)(x_o)$; the latter follows since f and $*(f^*)$ have the same affine minorants which are exact at x_o. The subgradient set is always convex and $w(X^*,X)$ closed.

3.1 Proposition. Let $f\colon X \to \overline{R}$ be a function on the HLCS X. The following are equivalent:

1) $y \in \partial f(x_o)$
2) $f(x) \geq f(x_o)+\langle x-x_o,y\rangle \ \forall x \in X$.
3) x_o solves $\inf_x [f(x)-xy]$, i.e. $f(x_o)-\langle x_o,y\rangle = \inf_x [f(x)-\langle x,y\rangle]$
4) $f^*(y) = \langle x_o,y\rangle-f(x_o)$
5) $x_o \in \partial f^*(y)$ and $f(x_o) = *(f^*)(x_o)$.

If $f(\cdot)$ is convex and $f(x_o) \in R$, then each of the above is equivalent to

6) $f'(x_o;x) \geq \langle x,y\rangle \ \forall x \in X$.

Proof. 1) <=> 2). This is the definition of $\partial f(x_o)$.

2) => 3) => 4). Trivial.

4) => 5). Since $*(f^*) \leq f$, 4) implies

$f^*(y) \leq \langle x_o,y\rangle - {}^*(f^*)(x_o)$. But the definition of ${}^*(f^*)(x_o)$ yields $f^*(y) \geq \langle x_o,y\rangle - {}^*(f^*)(x_o)$, so that $f^*(y) = \langle x_o,y\rangle - {}^*(f^*)(x_o)$. Comparison with 4) now yields $f(x_o) = {}^*(f^*)(x_o)$. Also $f^*(y) = \langle x_o,y\rangle - {}^*(f^*)(x_o) = \langle x_o,y\rangle - \sup_{y'}[\langle x_o,y'\rangle - f^*(y')]$ so that $f^*(y) \leq \langle x_o,y\rangle - \langle x_o,y'\rangle + f^*(y')$ for every y' and $x_o \in \partial f^*(y)$.

5) => 2). Since $x_o \in \partial f^*(y)$, the implication 1) => 4) applied to f^* yields ${}^*(f^*)(x_o) = \langle x_o,y\rangle - f^*(y)$, and hence that $f(x_o) = \langle x_o,y\rangle - f^*(y)$ by 5). But then by definition of f^*, $f(x_o) \leq \langle x_o,y\rangle - \langle x,y\rangle + f(x)$ $\forall x$ and 2) follows.

6) <=> 2). Assuming $f(\cdot)$ convex and finite at x_o, the directional derivative is given by $f'(x_o;x) = \inf_{t>o} [f(x_o+tx)-f(x_o)]/t$. Clearly 2) implies that for every $t > 0$, $[f(x_o+tx)-f(x_o)]/t \geq \langle x_o+tx-x_o,y\rangle/t = \langle x,y\rangle$ and hence 6) holds. Conversely, if 6) holds then $[f(x_o+tx)-f(x_o)]/t \geq \langle x,y\rangle$ for every $t > 0$, and setting $t = 1$ yields 2). ☒

<u>Remark</u>. Since it is always true that $f^*(y) \geq \langle x_o,y\rangle - f(x_o)$ we could replace 4) by 4') $f^*(y) \leq \langle x_o,y\rangle - f(x_o)$.

From condition 4) it follows that if $\partial f(x_o) \neq \emptyset$ for a convex function $f: X \to \overline{R}$, then the directional derivative

$f'(x_o;\cdot)$ is bounded below on some 0-neighborhood in X, i.e. the value of f at x does not drop off too sharply as x moves away from the point x_o. The following theorem shows that this property is actually equivalent to the subdifferentiability of f at x_o when f is convex, and also provides other insights into what $\partial f(x_o) \neq \emptyset$ means.

3.2 Theorem. Let $f: X \to \overline{R}$ be a convex function on the HLCS X, with $f(x_o)$ finite. Then the following are equivalent:

1) $\partial f(x_o) \neq \emptyset$

2) $f'(x_o;\cdot)$ is bounded below on a 0-neighborhood in X, i.e. there is a 0-neighborhood N such that

$$\inf_{x\in N} f'(x_o;x) > -\infty$$

3) $\exists$ 0-nbhd N, $\delta > 0$ st $\displaystyle\inf_{\substack{x\in N\\ 0<t<\delta}} \frac{f(x_o+tx)-f(x_o)}{t} > -\infty$

4) $\displaystyle\liminf_{x\to o} f'(x_o;x) > -\infty$

5) $\displaystyle\liminf_{\substack{x\to o\\ t\to o^+}} \frac{f(x_o+tx)-f(x_o)}{t} > -\infty$

6) $\exists\, y \in X^*$ st $f(x_o+x)-f(x_o) \geq \langle x,y\rangle \ \forall x \in X$.

If X is a normed space, then each of the above is equivalent to:

7) $\exists M > 0$ st $f(x_o+x)-f(x_o) \geq -M|x| \quad \forall x \in X$

8) $\exists M > 0, \varepsilon > 0$ st whenever $|x| \leq \varepsilon$, $f(x_o+x)-f(x_o) \geq -M|x|$

9) $\liminf_{|x|\to 0} \dfrac{f(x_o+x)-f(x_o)}{|x|} > -\infty$.

Proof. 1) => 2). This follows directly from Proposition 3.1

1) => 6).

2) => 1). Let N_1 be a convex 0-neighborhood in X such that $\inf_{x \in N_1} f'(x_o;x) > -c$, where c is a sufficiently large positive constant. Let $N = N_1/c$ and define the set E in X×R by

$$E \overset{\Delta}{=} \{(x,-t) \in X \times R \colon t>0,\ x/t \in N\}.$$

Since N is convex it follows that E is convex; for if $x_1 = t_1n_1$ and $x_2 = t_2n_2$ and $\varepsilon \in [0,1]$, where $n_1,n_2 \in N$ and $t_1,t_2 > 0$, then $\varepsilon x_1+(1-\varepsilon)x_2 =$

$$[\varepsilon t_1+(1-\varepsilon)t_2]\cdot\left[\frac{\varepsilon t_1}{\varepsilon t_1+(1-\varepsilon)t_2}n_1 + \frac{(1-\varepsilon)t_2}{\varepsilon t_1+(1-\varepsilon)t_2}n_2\right] \in [\varepsilon t_1+(1-\varepsilon)t_2]\cdot N$$

so $(\varepsilon x_1+(1-\varepsilon)x_2, -\varepsilon t_1-(1-\varepsilon)t_2) \in E$. Since N is a 0-neighborhood, E has nonempty interior; in fact, E contains $N\times[M,\infty)$. Moreover, $E \cap \text{epi} f'(x_o;\cdot)$ is empty; for otherwise it would contain a point $(x,-t)$ satisfying

$-t \geq f'(x_o;x) = \frac{t}{c}f'(x_o;\frac{cx}{t}) > \frac{t}{c}\cdot(-c) = -t$, a contradiction.

Hence it is possible to separate E and epi$f'(x_o;\cdot)$ by a closed hyperplane, i.e. there is a nonzero $(y,r) \in X^* \times R$ such that

$$\inf_{(x,t)\in \operatorname{epi} f'(x_o;\cdot)} \langle x,y\rangle + t\cdot r \geq \sup_{(x,-t)\in E} \langle x,y\rangle + (-t)\cdot r.$$

Since epi$f'(x_o;\cdot)$ is a convex cone ($f'(x_o;\cdot)$ is convex and positively homogeneous), the infimum on the LHS can remain bounded below only if the infimum is 0 and (y,r) is nonpositive on epi$f'(x_o;\cdot)$; in particular $\langle x,y\rangle + f'(x_o;x)\cdot r \geq 0$ for every $x \in$ dom$f'(x_o;\cdot)$. Moreover it must be true that $r \neq 0$; for if $r = 0$ then in particular $0 \geq \langle x,y\rangle$ for every $x \in N$ (taking t sufficiently large in the RHS so that $\frac{x}{t} \in N$ and $(x,-t) \in E$), implying the contradiction that y is also 0 (since N is a 0-neighborhood). Thus $\langle x,\frac{y}{r}\rangle + f'(x_o;x) \geq 0$ for every $x \in$ dom$f'(x_o;x)$, which by Proposition 3.1 6) => 1) yields $-\frac{y}{r} \in \partial f(x_o)$.

2) <=> 3). If $f(\cdot)$ is convex and $f(x_o) \in R$, then $t \to \frac{f(x_o+tx)-f(x_o)}{t}$ is increasing in $t > 0$. Hence, for any $\delta > 0$,

$$\inf_{t>o} \frac{f(x_o+tx)-f(x_o)}{t} = \inf_{o<t<\delta} \frac{f(x_o+tx)-f(x_o)}{t} = \lim_{t\to o^+} \frac{f(x_o+tx)-f(x_o)}{t}.$$

It is now immediate that 2) <=> 3).

2) <=> 4). This follows directly from the definition of lim inf, since

$$\liminf_{x\to o} f'(x_o;x) \equiv \sup_{N=0\text{-nbhd}} \inf_{x\in N} f'(x_o;x)$$

is bounded below iff there is a 0-nbhd N such that 2) holds.

3) <=> 5). This is immediate as in 2) <=> 4), since

$$\liminf_{\substack{x\to o\\ t\to o^+}} \frac{f(x_o+tx)-f(x_o)}{t} \equiv \sup_{\substack{N=0\text{-nbhd}\\ \delta>o}} \inf_{\substack{x\in N\\ t\in(o,\delta)}} \frac{f(x_o+tx)-f(x_o)}{t}.$$

1) <=> 6). This is just the definition of subgradient as in Proposition 3.1, 2).

6) => 7) => 8) <=> 9). Immediate.

8) => 2). Set $\delta = 1$. Then for $t \leq 1$, $|x| \leq \varepsilon$, it follows from the hypothesis 8) that

$$\frac{f(x_o+tx)-f(x_o)}{t} = \frac{f(x_o+tx)-f(x_o)}{|tx|}\cdot|x|$$

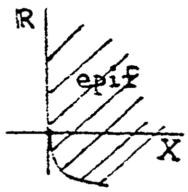
$$\geq -M\cdot|x| \geq -M\varepsilon.$$

Hence 2) holds. ⊠

<u>Remarks.</u> Some parts of Theorem 3.2 are implicit in Rockafellar's formula

$$f'(x_o;\cdot)^* = \delta_{\partial f(x_o)}(\cdot)$$

where $f: X \to \overline{R}$ is convex and $f(x_o) \in R$ [R73, Theorem 11]. In the finite dimensional case $X = R^n$, it is actually true that $\partial f(x_o) = \emptyset$ iff $f'(x_o;x) = -\infty$ for some $x \in X$, assuming $f: X \to \overline{R}$ convex and $f(x_o) \in R$. There is also a closely-related formula $\partial f(x_o) = \partial f'(x_o;\cdot)(0)$ given by [IL72]. Condition 8) is a kind of "local lower Lipschitzness" requirement which is easy to verify in optimization problems in which "state constraints" are absent, as we shall see. The standard example for which the sub-gradient set is empty is $f(x) = \begin{cases} -\sqrt{x}, & x \geq 0 \\ -\infty, & x < 0 \end{cases}$ for $x \in R$, where $\partial f(0) = \emptyset$, $f'(0;x) = -\infty$ whenever $x > 0$, and the supporting hyperplane to epif at $(0,f(0))$ is vertical.



In the finite dimensional case, every convex function has a derivative almost everywhere on its domain. There is also an interesting result in [ET 73] which states that if X is a Banach space, then the set of points where a convex lsc function $f: X \to \overline{R}$ is subdifferentiable is dense in domf.

The following theorem provides the simplest and most widely used condition which guarantees that the subgradient set is nonempty.

3.3 Theorem. Let $f: X \to \overline{R}$ be a convex function on the HLCS X. If $f(\cdot)$ is bounded above on a neighborhood of $x_o \in X$, then $f(\cdot)$ is continuous at x_o, $\partial f(x_o) \neq \emptyset$, and (assuming $f(x_o) > -\infty$) $\partial f(x_o)$ is $w(X^*,X)$-compact.

Proof. This is a corollary of the more general Theorem 5.3 which we prove later, where $\partial f(x_o)$ is the level set

$$\{y \quad X^*: f^*(y) - \langle x,y\rangle \leq -f(x_o)\}. \quad \boxtimes$$

Remark. Convex functions which have $-\infty$ values are very special and are generally excluded from consideration in meaningful situations. In particular, lsc convex functions with $-\infty$ values can have no finite values.

It is also a standard result that under the conditions of Theorem 3.3, there is a sensitivity interpretation of the subgradient set given by

$$f'(x_o;x) = \max_{y \in \partial f(x_o)} \langle x,y\rangle.$$

4. Relative interiors of convex sets and local equicontinuity of polar sets.

The relationship between neighborhoods of 0 in a locally convex space and equicontinuous sets in the dual space is well known: a subset which is a neighborhood of 0 has an equicontinuous polar, and an equicontinuous set in the dual space has a polar which is a neighborhood of 0. Hence, a closed convex set which contains 0 is a 0-neighborhood iff its polar is equicontinuous. We wish to extend this result to show the equivalence between convex sets with nonempty relative interior with respect to a closed affine hull of finite codimension, and local equicontinuity of the corresponding polar sets in an appropriate topology. This will also lead to a characterization of locally compact sets in locally convex spaces.

Throughout this section we shall assume that (X,τ) is a real Hausdorff locally convex topological linear space (HLCS) with topology τ and (continuous) dual space X^*. For $x \in X$, $y \in X^*$ we write $\langle x,y\rangle$ or simply xy to denote $y(x)$. By a τ^*-topology on X^* we mean a Hausdorff locally convex topology τ^* on X^* which is compatible with the duality $\langle X,X^*\rangle$, i.e. $(X^*,\tau^*)^*$ is again X,[†] and which is sufficiently weak so that every equicontinuous set in X^* has τ^*-compact

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[†] More precisely, we mean that $(X^*,\tau^*)^* = JX$, where J is the natural imbedding $x \to \langle x,\cdot\rangle$ of X into the algebraic dual $(X^*)'$ of all linear functionals on X^*.

closure. For example, given any topology τ on X we may always take τ^* to be the $w(X^*,X)$ topology on X^*, since by the Banach-Alaoglu Theorem every τ-equicontinuous set is $w(X^*,X)$-relatively compact. Conversely, a given (compatible) topology τ^* on X^* is a "τ^*-topology if τ is any compatible locally convex topology on X which contains the Arens topology $a(X,X^*)$ given by uniform convergence on τ^*-compact convex sets of X^* (with a basis of 0-neighborhoods being the polars of τ^*-compact convex sets in X^*). This generality allows us to specialize to various interesting cases later.

The polar of a set A in X is defined to be

$$A^{o} = \{y \in X^*: \sup_{x \in A} xy \leq 1\}.^{\dagger}$$

Similarly, the polar of a set B in X^* is

$${}^{o}B = \{x \in X: \sup_{y \in B} xy \leq 1\}.$$

The following properties of polar sets are well known, where $A \subset X$ and $B \subset X^*$:

i). A^{o} and ${}^{o}B$ are closed, convex, and contain 0.

ii). ${}^{o}(A^{o}) = \text{cl co}(A \cup \{0\})$, $({}^{o}B)^{o} = \text{cl co}(B \cup \{0\})$.

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† The supremum over a null set is taken to be $\sup\emptyset \equiv -\infty$. Thus $\emptyset = X^*$, ${}^{o}(\emptyset^{o}) = \{0\}$.

iii). $0 \in \text{int}A \Rightarrow A^o$ is equicontinuous (and hence compact).

iv). B is equicontinuous $\Leftrightarrow 0 \in \text{int}^o B$.

Thus, we see that the closed convex 0-neighborhoods in X are precisely the polars of closed convex equicontinuous sets containing 0 in X*, and vice versus.

It is also knwon that sets with nonempty interior in X have polars which, though not necessarily equicontinuous or even bounded, are nevertheless w(X*,X)-locally compact in X* (cf. [Fan 65]). Recall that a set B in X* is <u>locally compact</u> (resp. <u>locally equicontinuous</u>) at a point $y_o \in B$ iff there is a neighborhood W of y_o in X* such that $B \cap W$ is compact (resp. equicontinuous). We shall characterize local compactness and local equicontinuity in X* by showing its relation to nonempty relative interiors of polar sets in X. To provide some preliminary results (of interest in their own right), and to get a feel for what is going on, we first consider the case of locally equicontinuous convex cones.

4.1 <u>Theorem</u>. Let X be a HLCS, X* its dual with a τ^*-topology, and C a convex cone in X* with $C \cap (-C) = \{0\}$. Then the following are equivalent:

i). C has an equicontinuous base.

ii). $\text{int}^{\circ}C \neq \emptyset$ in X.

iii). C is locally equicontinuous.

iv). 0 has an equicontinuous neighborhood in C.

Proof. We assume $C \neq \{0\}$, since otherwise the theorem is trivial.

i) => ii). Recall that B is a base for C iff there is a closed affine set H such that $B = C \cap H$ and $[0,\infty)\cdot B \supset C$; it is then true that every nonzero $y \in C$ has a unique representation $t\cdot y_o$ where $t > 0$ and $y_o \in B$. Let B be an equicontinuous base for C; then there exists an $x_o \in X$ with $B = C \cap \{y: x_o y = 1\}$ and $[0,\infty)\cdot B \supset C$, and moreover $0 \in \text{int}^{\circ}B$. Now for any $t \geq 0$, $y \in B$, and $x \in {}^{\circ}B$ we have $(-x_o+x)(ty) = t(-1+xy) \leq t(-1+1) \leq 0$; hence $(-x_o+{}^{\circ}B) \subset {}^{-}([0,\infty)\ B) \subset {}^{-}C$. Thus ${}^{\circ}C \equiv {}^{-}C$ contains a neighborhood of $-x_o$, i.e. $-x_o \in \text{int}^{\circ}C$. We remark that x_o is strictly positive on $\text{cl}C \setminus \{0\} = C_\infty \setminus \{0\}$.

ii) => iii). Suppose $-x_o \in \text{int}^{\circ}C$; then $o \in \text{int}(x_o+{}^{\circ}C)$, so $(x_0+{}^{\circ}C)^{\circ}$ is equicontinuous. Given $y_o \in C$, we wish to show that y_o has a τ^* neighborhood W such that $C \cap W$ is equicontinuous. Let $W = \{y: x_o y \leq 1+x_o y_o\}$; W is clearly a neighborhood of y_o. But $C \cap W = \{y: y \in C,\ x_o y \leq 1+x_o y_o\}$ $\subset \{y:\ (x_o+x)y \leq 1+x_o y_o \text{ for all } x \in {}^{-}C\} = r\cdot(x_o+{}^{\circ}C)^{\circ}$, so $C \cap W$ is equicontinuous.

iii) => iv). This trivial.

iv) => i). This is the difficult part of the proof, but the idea is well-known in the literature. Let W be a 0-neighborhood in X* such that $C \cap W$ is equicontinuous. In particular, $\text{clco}(C \cap W)$ is equicontinuous and hence τ^*-compact. Let $D = C \cap (W \setminus \text{int}\frac{W}{2})$; note $0 \notin \text{cl}D$. We claim that $0 \notin \text{clco}D$. For suppose $0 \in \text{clco}D$; then $0 \in \text{ext}D$ since $0 \in \text{ext}C$ and $D \subset C$, and hence $0 \in \text{cl}D$ by the Krein-Milman Theorem on extreme points of compact sets,† which is a contradiction. Since $0 \notin \text{clco}D$ there is a closed affine set H which strongly separates 0 from clcoD. But then $B = C \cap H$ is a base for C (since $[0,\infty)\cdot D \supset C$, so $[0,\infty)\cdot H \supset C$) and $B \subset C \cap W$, so B is equicontinuous. △

Note that in Theorem 1.1 we assumed that C contained no lines, so that $\text{span}^{\circ}C = {}^{-}C - {}^{-}C$ was all of X and ${}^{\circ}C$ had nonempty interior. If however we allow $L = C \cap (-C)$ to be a (finite dimensional) subspace, local equicontinuity of C would no longer imply $\text{int}^{\circ}C \neq \emptyset$, but it would still be true that $\text{ri}^{\circ}C \neq \emptyset$ with respect to $\text{span}^{\circ}C = {}^{\perp}L$, a

†This is the basic tool here, namely that if a set D in a HLCS has compact closed convex hull then $\text{ext}(\text{clco}D) \subset \text{cl}D$.

closed subspace of finite codimension. In fact, these results remain true for the case of an arbitrary convex set in X^*. The basic idea is as follows: if C is a nonempty convex locally equicontinuous set in X^*, then the (finite dimensional) subspace $L = C_\infty \cap (-C_\infty)$ of all lines contained in $\mathrm{cl}C$ is precisely the annihilator of $\mathrm{span}^O C = {}^\perp L$ in X; and those elements of X which are strictly negative on all the remaining half-lines contained in $\mathrm{cl}C$ (that is, on $C_\infty \cap M \setminus \{0\}$ where M is any closed complement of L in X^*) are relative interior points of ${}^O C$ (if there are no such half-lines, i.e. C_∞ is itself a subspace and $C_\infty \cap M = \{0\}$, then $0 \in \mathrm{ri}^O C$).

Before proceeding, we require some lemmas concerning decomposition of finite dimensional subspaces.

4.2 Lemma. Let X be a HLCS. If L is a finite dimensional subspace of X, then there is a closed subspace M of X such that $X = L+M$ and $L \cap M = \{0\}$.

Proof. This is a standard application of the Hahn-Banach Theorem. Let $\{x_1,\ldots,x_n\}$ be a basis for L and define the continuous linear functionals $y_1,\ldots,y_n$ on L by $\langle x_i, y_j\rangle = \delta_{ij}$, $1 \le i,j \le n$. By the Hahn-Banach Theorem we may extend the functionals y_j so that they are elements of X^*. Let $M = {}^\perp\{y_1,\ldots,y_n\}$. Clearly M is a closed subspace of X. Moreover, $L \cap M = \{0\}$; for if $x \in L$, then $x = \sum_j a_j x_j$ for some $a_j \in R$, and if x is also in M then $0 = \langle x, y_j \rangle = a_j$ for every j. Finally, any $x \in X$ can be (uniquely) expressed as

$$x = (\sum_j \langle x, y_j \rangle x_j) + (x - \sum_j \langle x, y_j \rangle x_j) \in L+M. \quad \Delta$$

4.3 Lemma. Let X be a HLCS with $X = L+M$, where L is a finite dimensional subspace, M is a closed subspace, and $L \cap M = \{0\}$. Then $X^* = L^{\perp}+M^{\perp}$, where $L^{\perp} \cap M^{\perp} = \{0\}$ and $M^{\perp}$ is finite dimensional.

Proof. Let $\{x_1,\ldots,x_n\}$ be a base for L. Note that the projection of X onto L is continuous since it has finite dimensional range and its null space M is closed. Hnece, for $i = 1,\ldots,n$ we can define the continuous linear functionals y_i b $\langle m+\sum_j a_j x_j, y_i \rangle = a_i$ whenever $m \in M$ and $a_j \in R$, $1 \leq j \leq n$. Clearly $M \subset {}^{\perp}\{y_1,\ldots,y_n\}$; moreover $L \cap {}^{\perp}\{y_1,\ldots,y_n\} = \{0\}$ so $M \supset {}^{\perp}\{y_1,\ldots,y_n\}$. Hence $M = {}^{\perp}\{y_1,\ldots,y_n\}$ and $M^{\perp} = \text{span}\{y_1,\ldots,y_n\}$. Also $L^{\perp} \cap \text{span}\{y_1,\ldots,y_n\} = \{0\}$, so $L^{\perp} \cap M^{\perp} = \{0\}$. Finally, $X^* = L^{\perp}+M^{\perp}$ since for any $y \in X^*$ we have

$$y = (y - \sum_j \langle x_j, y \rangle y_j) + (\sum_j \langle x_j, y \rangle y_j) \in L^{\perp}+M^{\perp}. \quad \Delta$$

We remark that for a convex subset C of X^*, local equicontinuity at a single point of C is sufficient to imply local equicontinuity of the entire set, in fact of the closure of the set; later we shall see that it also implies local equicontinuity (and hence local compactness) of $({}^{o}C)^{o}$.

4.4 Proposition. Let X be a HLCS, X* its dual with a τ^*-topology. Suppose C is a convex subset of X* and C is locally equicontinuous at a point $y_o \in C$. Then C is locally equicontinuous and clC is locally equicontinuous (hence locally compact).

Proof. We may assume without loss of generality that $y_o = 0$ (otherwise simply replace C by $C-y_o$). Let W be an open τ^* 0-neighborhood such that $C \cap W$ is equicontinuous. Now $C/t \subset C$ for any $t \geq 1$ by convexity, hence $C \cap tW \subset t(C \cap W)$ is equicontinuous. Given any $y \in C$, we simply take t sufficiently large so that $y/t \in W$; then $C \cap tW$ is an equicontinuous relative neighborhood of y in C, so C is locally equicontinuous at every point in C.

To show that clC is locally equicontinuous, we need only show (by what we have just proved, since clC is convex) that 0 has an equicontinuous relative neighborhood in clC. But we claim that $clC \cap W$ is a subset of $cl(C \cap W)$ which is equicontinuous since $C \cap W$ is; hence $clC \cap W$ is an equicontinuous relative neighborhood of 0 in clC and we are done. To show that $\mathrm{cl}C \cap W \subset \mathrm{cl}(C \cap W)$, let $y \in \mathrm{cl}C \cap W$; then $y \in W$ and there is a net $\{y_i\}_{i \in I}$ in C such that $y_i \to y$. But W is open so eventually the y_i are contained in W, i.e. eventually the y_i belong to $C \cap W$. But then $y = \lim y_i \in \mathrm{cl}(C \cap W)$. $\triangle$

We now proceed to the main results. First, a lemma adapted from Dieudonne [D66] to show when a locally equicontinuous set is equicontinuous.

4.5 Lemma. Let X be a HLCS, X* its dual with a τ^*-topology. A nonempty convex locally equicontinuous subset C of X* is equicontinuous iff $C_\infty = \{0\}$.

Proof. If C is equicontinuous, then it is certainly bounded, so $C_\infty = \{0\}$. Suppose C is not equicontinuous. We show that there is a nonzero $x_0 \in C_\infty$. Without loss of generality we may suppose that $0 \in C$. Let W be a 0-neighborhood with $C \cap W$ equicontinuous. Now for $t \geq 1$, $C \cap tW \subset t(C \cap W)$ by convexity of C and hence $C \cap tW$ is equicontinuous; but C itself is not equicontinuous, so we must have $C \setminus tW \neq \emptyset$ for all $t \geq 1$. For $t \geq 1$, define the sets $D_t = ([0,\infty) \cdot (C \setminus tW)) \cap C \cap W \setminus \mathrm{int}(W/2)$; note that $C \cap W \setminus \mathrm{int}(W/2)$ intersects any half-line which intersects C, so that D_t is nonempty since $C \setminus tW$ is nonempty. The D_t are equicontinuous ($D_t \subset C \cap W$), hence relatively compact, and decrease as t increases; thus their closure must have nonempty intersection, i.e. there is an $x_o \in \bigcap_{t \geq 1} \mathrm{cl} D_t$. Clearly $x_o \neq 0$, since $x_o \in W \setminus \mathrm{int}(W/2)$. All that remains is to show $x_o \in C_\infty$, i.e. $r \cdot x_o \in \mathrm{cl} C$ for every $r > 0$. Take any $r > 0$. Now $x_o \in \mathrm{cl}[0,\infty) \cdot (C \setminus tW)$ for $t \geq 1$ and $x_o \in W$; hence $rx_o \in \mathrm{cl}[0,\infty) \cdot (C \setminus tW) \cap tW$ whenever $t \geq r$, i.e. $rx_o \in \mathrm{cl}[0,1] \cdot C \subset \mathrm{cl} C$. Thus x_o is in C_∞. Δ

4.6 Theorem. Let X be a HLCS, X* its dual with a τ^*-topology, C a convex set in X*. Then the following are equivalent:

i). C is locally equicontinuous.

ii). $\mathrm{ri}^o C \neq \emptyset$, where $\mathrm{span}^o C$ is closed and has finite codimension in X.

Moreover if either of the above is true then $\mathrm{span}^o C = {}^\perp(C_\infty \cap (-C_\infty))$, and $0 \in \mathrm{ri}^o C$ iff C_∞ is a subspace, in which case $\mathrm{span}^o C = {}^\perp(C_\infty)$. If C is closed, it is also complete and locally compact.

Proof. i) => ii). Since clC is locally equicontinuous iff C is by Proposition 1.4, and since $(\mathrm{cl} C)_\infty = C_\infty$, we may assume C is closed. Let $L = C_\infty \cap (-C_\infty)$; L is a subspace, and since a translate of L lies in C, L is locally equicontinuous, hence locally totally bounded and finite dimensional. By Lemmas 1.2 and 1.3 applied to L in X*, there is a closed complement M of L in X* with $X^* = L+M$, $L \cap M = \{0\}$, $X = {}^{\perp}L + {}^{\perp}M$, ${}^{\perp}L \cap {}^{\perp}M = \{0\}$, and M finite dimensional. If C is a subspace, i.e. $C \subset L$, then we are done; hence we assume C is not a subspace and $C \cap M \neq \{0\}$. Now $C_{\infty} \cap M$ is a convex cone which contains no lines, and since a translate of it lies in C it is locally equicontinuous. Applying Theorem 1.1, we see that if $C_{\infty} \cap M \neq \{0\}$, there is an $x_o \in X$ such that x_o is strictly negative on $C_{\infty} \cap M \setminus \{0\}$; if $C_{\infty} \cap M = \{0\}$, i.e. in the case that C is a subspace and $L = C_{\infty}$, we simply take $x_o = 0$. We may assume that $x_o \in L$ by taking its projection onto L.

Consider the sets $B_r = \{y \in C \cap M: x_o y \geq r\}$ for $r \in R$. Each B_r is a subset of C, hence locally equicontinuous. Now $(B_r)_{\infty} = C_{\infty} \cap M \cap \{x_o\}^+$ is $\{0\}$ since x_o is strictly negative on $C_{\infty} \cap M \setminus \{0\}$; thus the B_r are actually equicontinuous by Lemma 1.5 and hence compact. Clearly $\bigcap_{r>0} B_r$ is empty, and since the sets B_r are compact and monotone in r there is a finite $r_o > 0$ for which $B_{r_o - 1} = \emptyset$, so that $\sup_{y \in C \cap M} x_o y \leq r_o - 1$.

Take B to be any of the sets B_r which are nonempty; B is equicontinuous so ${}^{o}B$ is a 0-neighborhood. We shall show that $(x_o + {}^{o}B) \cap {}^{\perp}L \subset r_o \cdot {}^{o}C$, i.e. that x_o is in the interior of ${}^{o}C$ relative to the subspace ${}^{\perp}L$; since ${}^{\perp}L$ clearly contains ${}^{o}C$ (a translate of L lies in C), we then see that ${}^{\perp}L = \mathrm{span}\,{}^{o}C$ and $x_o \in \mathrm{ri}\,{}^{o}C$. Moreover, $\mathrm{codim}\,{}^{\perp}L = \dim X/{}^{\perp}L = \dim L$ is finite. So, all that remains is to show $(x_o + {}^{o}B) \cap {}^{\perp}L \subset r_o \cdot {}^{o}C$.

Take $x \in {}^{o}B$ and $y \in C$ with $x_o + x \in {}^{\perp}L$. Now $y = \ell + m$ where $\ell \in L$ and $m \in M$; note m is also in C since $m = y - \ell \in C - L \subset C$ (recall $L \subset C_\infty$), i.e. $m \in C \cap M$. But then $(x_o+x)y = (x_o+x)(\ell+m) = (x_o+x)m \leq (r_o - 1) + xm \leq r_o + 1 - 1 = r_o$. Hence we have shown $x_o + x \in r_o \cdot {}^{o}C$ for every such x, so $(x_o + {}^{o}B) \cap {}^{\perp}L \subset {}^{o}C$.

Concerning the remarks at the end of the theorem, we have already shown that $\mathrm{span}\,{}^{o}C = {}^{\perp}L$ and $0 = x_o \in \mathrm{ri}\,{}^{o}C$ if C_∞ is a subspace. To complete the remarks, we need only show that $0 \in \mathrm{ri}\,{}^{o}C$ implies that C_∞ is a subspace. But if $0 \in \mathrm{ri}\,{}^{o}C$ then ${}^{o}C$ absorbs $\mathrm{span}\,{}^{o}C = {}^{\perp}L$ and hence $C_\infty \equiv ({}^{o}C)^{-} = (\mathrm{span}\,{}^{o}C)^{-} = ({}^{\perp}L)^{-} = L$.

ii) => i). In the next theorem we prove that for $A = {}^{o}C$, ii) implies that $A^{o} = ({}^{o}C)^{o}$ is complete and locally equicontinuous. But C is a subset of $({}^{o}C)^{o}$, so C is locally equicontinuous, also complete and locally compact if it is closed. △

We remark that in Theorem 4.6 we have $C = L + (C \cap M)$, where $L = C_\infty \cap (-C_\infty)$ is finite dimensional and M is a closed complement of L. Moreover $C \cap M$ is equicontinuous iff its asymptotic cone is $\{0\}$, i.e. iff C_∞ is a subspace or equivalently $0 \in ri^o C$.

4.7 Theorem. Let X be a HLCS, X* its dual with a τ^*-topology. Suppose $A \subset X$ has $riA \neq \emptyset$, where affA is closed with finite codimension in X. Then A^o is complete and locally equicontinuous (also convex, closed, and hence locally compact). Moreover, $(A^o)_\infty = A^-$, $(A^o)_\infty \cap (-A^o)_\infty = A^\perp$, and $0 \in ri^o(A^o)$ iff $(A^o)_\infty$ is a subspace.

Proof. Let $x_o \in riA$, or equivalently $0 \in ri(A-x_o)$. Define $M = span(A-x_o) = affA-x_o$, a closed subspace of finite codimension. Let N be any (algebraic) complement of M in X; N is finite dimensional (hence closed) since it is isomorphic to X/M and dim X/M = codim M is finite. Let $\{x_1, \ldots, x_n\}$ be a basis for N. Note $M^\perp = (affA-x_o)^\perp = (A-x_o)^\perp$.

We frist prove that A^o is complete. Let $\{y_i\}_{i \in I}$ be a Cauchy net in A^o, and define the linear functional f on X to be the pointwise limit $f(x) = \lim xy_i$. We will show that f is continuous (i.e. can be taken as an element of X), and hence lies in A^o since A^o is closed. Now the y_i are bounded above by 1 on A, and $(x_o y_i)$ is Cauchy in R so $x_o y_i$ is bounded by some $r > 0$; hence the y_i are bounded above by $1+r$ on $A-x_o$, so f is bounded above by $1+r$ on $A-x_o$. But $A-x_o$ is a 0-neighborhood in M, so f is continuous on M. Since f is certainly continuous on the finite dimensional subspace N, and since the projections from X onto M and N are continuous, f is continuous on $M+N = X$.

We now show that A^o is locally equicontinuous, i.e. that given any $y \in A^o$ there is a τ^* neighborhood W of y_o for which $A^o \cap W$ is equicontinuous. By Proposition 1.4 we may simply take $y_o = 0$. The basic idea is to choose W so as to eliminate all half-lines in $(A^o)_\infty$. Hence, we set $W = \{y: -x_o y \leq 1 \text{ and } \max_{1 \leq i \leq n} |x_i y| \leq 1\} = \{-x_o, \pm x_1, \ldots, \pm x_n\}^o$. Clearly W is a 0-neighborhood in X^*. Now we claim that $U = (A-x_o) + \{\sum_j a_j x_j : |a_j| \leq 1\}$ is a 0-neighborhood in X, and we will show that $A^o \cap W \subset r \cdot U^o$ for r sufficiently large, so that $A^o \cap W$ is equicontinuous; this finishes the proof that A^o is locally equicontinuous.

To show that U is a 0-neighborhood in X, we note that $(A-x_o)$ is a 0-neighborhood in M and $\{\sum_j a_j x_j : |a_j| \leq 1\}$ is a 0-neighborhood in N. But the projections of X onto M and N are continuous, and U is simply the intersection of the inverse images of the two sets under the corresponding projections.

We now show that $A^{\circ} \cap W \subset 2(1+n) \cdot U^{\circ}$. Take any $y \in A^{\circ} \cap W$; then $\sup_{x \in A} xy \leq 1$, $-x_o y \leq 1$, and $\max_{1 \leq i \leq n} |x_i y| \leq 1$, so in particular $\sup_{x \in A} (x-x_o)y \leq 1+1 = 2$ and $\max_{1 \leq i \leq n} |x_i y| \leq 1 < 2$. Hence $y/2 \in (A-x_o)^{\circ} \cap \{\pm x_1, \ldots, \pm x_n\}^{\circ} \subset (1+n) \cdot U^{\circ}$.

All that remains is to verify the concluding remarks in the theorem. To show $(A^{\circ})_{\infty} = A^{-}$, we have
$y \in (A^{\circ})_{\infty} \iff ty \in A^{\circ}\ \forall t > 0 \iff x(ty) \leq 1\ \forall t > 0$,
$x \in A \iff xy \leq 0\ \forall x \in A \iff y \in A^{-}$. Finally, the fact that $(A^{\circ})_{\infty}$ is a subspace iff $0 \in ri^{\circ}(A^{\circ})$ follows from Theorem 1.6 i) => ii) applied to $C = A^{\circ}$. □

We now summarize our results for the $w(X^*,X)$ topology on X^*, in which equicontinuous sets are always relatively compact.

4.8 <u>Corollary</u>. Let (X,τ) be a HLCS with dual space X^*, and suppose $A \subset X$, $B \subset X^*$.

If A has nonempty relative interior, and if affA is closed and has finite codimension in X, then A° is complete and locally equicontinuous (also closed, convex and hence locally compact) in the $w(X^*,X)$ topology on X^*. Moreover $(A^{\circ})_{\infty} = A^{-}$, $(A^{\circ})_{\infty} \cap (-A^{\circ})_{\infty} = A^{\perp}$, and $0 \in ri^{\circ}(A^{\circ}) = rcor^{\circ}(A^{\circ})$ iff $(A^{\circ})_{\infty} = A^{-}$ is a subspace.

Conversely, if B is convex and locally τ-equicontinuous in the $w(X^*,X)$ topology on X^* then ${}^{o}B$ has nonempty relative interior, $\mathrm{span}\,{}^{o}B = {}^{\perp}(B_\infty \cap (-B_\infty))$ is closed with finite codimension, and $0 \in \mathrm{ri}\,{}^{o}B$ iff B_∞ is a subspace. Moreover $\mathrm{cl}B$ and $({}^{o}B)^{o}$ are complete and locally compact in the $w(X^*,X)$ topology.

Proof. This is just a direct consequence of Theorems 4.6 and 4.7, where we take τ to be the original topology on X and τ^* the $w(X^*,X)$ topology on X^*. □

We remark that if X is a barrelled space (i.e. every closed convex absorbing set has nonempty interior, for example any Banach space or Frechet space), then the given topology on X is the $m(X,X^*)$ topology and moreover every bounded set in X^* is relatively sompact in the $w(X^*,X)$ topology. In this case locally equicontinuous simply means $w(X^*,X)$-locally bounded in Corollary 4.8.

In the general case, we can still imbed X^* in the algebraic daul X' to characterize local boundedness in X^*.

4.9 Corollary. Let X be a HLCS with dual space X^*, and suppose $A \subset X$, $B \subset X^*$.

If affA is closed and has finite codimension, and if A has nonempty relative core, then A^o is locally bounded in the $w(X^*,X)$ topology on X^*. Moreover $(A^{\circ})_{\infty} = A^{-}$, $(A^{\circ})_{\infty} \cap (-A^{\circ})_{\infty} = A^{\perp}$, and $0 \in \text{rcor}^{\circ}(A^{\circ})$ iff $(A^{\circ})_{\infty} = A^{-}$ is a subspace. If X is a barrelled space, then A° is closed, convex, complete, and locally compact in the $w(X^*,X)$ topology on X^*.

Conversely, if B is convex and locally bounded in the $w(X^*,X)$ topology on X^*, then ${}^{\circ}B$ has nonempty relative core, $\text{span}\,{}^{\circ}B = {}^{\perp}(B_{\infty} \cap (-B_{\infty}))$ is closed with finite codimension, and $0 \in \text{rcor}\,{}^{\circ}B$ iff B_{∞} is a subspace. If X is a barrelled space, then $\text{ri}\,{}^{\circ}B \neq \emptyset$, and ${}^{\circ}(B^{\circ})$ is complete and locally compact in the $w(X^*,X)$ topology.

Proof. Let X' be the algebraic dual of X, put the "convex core" or strongest locally convex topology on X (i.e. every convex absorbing set is a 0-neighborhood), and let $A^{\oslash}$ denote the polar of A with respect to the duality between X and X'. Of course, $X^* \subset X'$, the $w(X^*,X)$ topology is the restriction of the $w(X',X)$ topology to X^*, and $A^{\circ} = A^{\oslash} \cap X^*$. Moreover we note that X^* is $w(X',X)$-dense in X', since $w(X',X)\text{-cl}(X^*) = ({}^{\oslash}X^*)^{\oslash} = \{0\}^{\oslash} = X'$. Similarly, we have the decomposition $X' = M^{\perp} + w(X',X)\text{-cl}(N\ )$ with $M^{\perp}$ finite dimensional, whenever $X = M + N$ and M is a closed subspace of X, N is a finite dimensional subspace of X, $M \cap N = \{0\}$.

The results then follow by a straightforward application of Corollary 4.8 to X and X'. □

Finally, we characterize local compactness in a HLCS in terms of the Arens topology $a(X^*,X)$ on X^* of uniform convergence on compact convex sets in X (a basis of 0-neighborhoods for $a(X^*,X)$ being the polars of all compact convex sets in X; note this depends on the topology on X, not just on the duality between X and X^*). In particular, we characterize weak local compactness in terms of the Mackey topology $m(X^*,X)$ on X^*, which is the strongest locally convex topology on X^* which still has dual space X.

4.10 Corollary. Let A be a closed convex subset of a HLCS X. Then A is locally compact iff A^o has nonempty relative interior in the $a(X^*,X)$ topology on X^* and $\text{span}(A^o)$ is closed with finite codimension, in which case A is also complete. A is weakly locally compact iff A^o has nonempty relative interior in the $m(X^*,X)$ topology on X^* and $\text{span}(A^o)$ is closed with finite codimension, in which case A is also weakly complete. In either case, $\text{span}(A^o) = (A_\infty \cap (-A_\infty))^\perp$.

Proof. This is a direct consequence of Theorems 4.6 and 4.7 where τ is taken to be the $a(X^*,X)$ topology (resp. the

$m(X^*,X)$ topolog) on X^* and τ^* is the original topology (resp. the weak topology) on X. □

An interesting consequence of this corollary is that if A is a closed convex locally compact subset of a HLCS X, then it is actually weakly locally compact. For, A^o has nonempty relative interior in $a(X^*,X)$ by Corollary 1.10, so A^o certainly has nonempty interior in $m(X^*,X)$, hence A is locally compact and complete in $w(X,X^*)$. Note it is obvious that compactness always implies weak compactness; however it is not so obvious that local compactness implies weak local compactness (for closed convex sets). However the proofs of the theorems show that the compact relative neighborhoods of any x_o in A can be taken to be of the form $A \cap (x_o + {}^o\{y_o, \pm y_1, \ldots, \pm y_n\})$ where, for a complement L of the finite dimensional subspace $A_\infty \cap (-A_\infty)$, y_o is strictly positive on $A_\infty \cap L \setminus \{0\}$ and $\{y_1, \ldots, y_n\}$ forms a basis for $L^\perp$.

5. Continuity of convex functions and equicontinuity of conjugate functions.

We wish to describe here the relationship between continuity of a convex function and equicontinuity of level sets of the conjugate function. Moreau [M64] and Rockafellar [R66] have shown that continuity of a convex function at a given point is equivalent to equicontinuity of certain level sets of the conjugate function. We shall complete this result and also extend it to show the equivalence between relative continuity of a convex function with respect to a closed affine set of finite codimension and local equicontinuity of the level sets of the conjugate function. We then examine relative continuity in a more general context using quotient topologies.

We recall some basic definitions about conjugate functions. Throughout this section we shall again take (X,τ) to be a HLCS with topology τ and (continuous) dual space X^* topologized by a τ^*-topology, i.e. τ^* is compatible with the duality $\langle X,X^*\rangle$ and τ-equicontinuous sets in X^* have τ^*-compact closure. Let $R = [-\infty,+\infty]$; if S is a set and f a function $f:S \to \overline{R}$, we define the effective domain of f to be

$$\text{dom}f = \{s \in S: f(s) < +\infty\}$$

and the <u>epigraph</u> of f to be

$$\text{epi}f = \{(s,r) \in S\times R : f(s) \leq r\}.$$

If $f:X \to \overline{R}$ and $g:X^* \to \overline{R}$, the <u>conjugate</u> functions $f^*:X^* \to \overline{R}$ and ${}^*g:X \to \overline{R}$ are defined by

$$f^*(y) = \sup_{x\in X}(xy - f(x))$$

$${}^*g(x) = \sup_{y\in X^*}(xy - g(y)).$$

The conjugate functions are always convex and lower semi-continuous (in fact, weakly lsc), being the supremum of continuous affine functions (e.g. f^* is the supremum of the functions $y \mapsto xy - r$ over all $(x,r) \in \text{epi}f$), and they never take on $-\infty$ values except in the case they are identically $-\infty$. Note that the conjugate of an indicator function $\delta_A(x) = \begin{cases} +\infty, & x \in A \\ 0, & x \notin A \end{cases}$ for $A \subset X$ is precisely the support function $\delta_A{}^*(y) = \sup_{x\in A} xy$ of A. Finally, it is well known that

$${}^*(f^*) = \text{lsc co } f$$

unless lsc co f takes on $-\infty$ values (or equivalently $f^* \equiv +\infty$), in which case ${}^*(f^*) \equiv -\infty$. By cof we mean the largest convex function dominated by f, and by lscf we mean the largest lower semicontinuous function dominated by f (i.e. $(\text{lscf})(x) = \liminf_{x'\to x} f(x')$), so that epi(lsccof) = clco(epif). And since f^* is convex and lsc, we have $(^*(f^*))^*$ again equal to f^*.

We recall the following important property of convex functions: if $f: X \to \bar{R}$ is convex, then f is continuous relative to affdomf (that is, the restriction of f to affdomf with the induced topology is continuous) at every point of ridomf whenever f is bounded above on any relative neighborhood in affdomf, or equivalently whenever riepif is nonempty. We shall consider the relationship between points of continuity of f and equicontinuity of level sets of f^* of the form

$$\{y \in X^*: f^*(y) - xy \leq r\}, \quad x \in X, \ r \in R.$$

Note that by definition of $^*(f^*)$ the level set is nonempty whenever $r > -{}^*(f^*)(x)$ and empty whenever $r < -{}^*(f^*)(x)$ (the latter entails $x \in \text{dom}^*(f^*)$). We remark that the level set is precisely the ε-subgradient $\partial f_\varepsilon(x)$ of f at x when $r = \varepsilon + f(x)$ and precisely the subgradient set when $r = f(x)$, assuming $f(x) \in R$. In the case that δ_A is the indicator function of a set $A \subset X$, then the level sets of $\delta_A{}^*$ are precisely $r \cdot (A-x)^\circ$ when $r > 0$; thus we have a generalization of the notion of polarity. More generally, the level set for a given $r \in R$ consists of all continuous linear functionals $y \in X^*$ for which $f(\cdot)$ dominates the affine functional $x \mapsto xy-r$, i.e. it is $\{y \in X^*: f(x') \geq (x'-x)y-r \ \forall x' \in X\}$.

We first prove two lemmas which relate polars of level sets of a function with level sets of the conjugate function.

5.1 Lemma. Let X be a HLCS, $f: X \to \overline{R}$. Then

$$\{y \in X^*: f^*(y) \leq s\} \subset (r+s)\cdot\{x \in X: f(x) \leq r\}^{\circ}$$

whenever $r+s > 0$.

Proof. Let A denote the set $\{x \in X: f(x) \leq r\}$. Clearly $f \leq r+\delta_A$, so taking conjugates yields $f^* \geq -r+\delta_A{}^*$. Hence $\{y: f^*(y) \leq s\} \subset \{y: -r+\delta_A{}^*(y) \leq s\} \subset \{y: \sup_{x \in A} xy \leq r+s\} \subset (r+s)\cdot A^{\circ}$. □

5.2 Lemma. Let X be a HLCS with dual X^*, g convex $X^* \to \overline{R}$. Then for any $\varepsilon > 0$,

$$\varepsilon\cdot{}^{\circ}\{y \in X^*: g(y) \leq \varepsilon+g(0)\} \subset \{x \in X: {}^*g(x) \leq \varepsilon+{}^*g(0)\}.$$

Proof. Let $f = {}^*g$, $B = \{y \in X^*: g(y) \leq \varepsilon+g(0)\}$. The trivial cases $g(0) = +\infty$ or $g(0) = -\infty$ are easily checked, so we assume $g(0)$ is finite. In particular, $f(x) > -\infty$ for every x. If $f(0) = +\infty$ the result is also trivial, so we assume $f(0)$ finite.

We shall first show that $g(y) \geq -f(0)-\varepsilon+\varepsilon\delta_{^oB}*(y)$ for every $y \in X^*$. Now if $y \in ({}^oB)^o$, i.e. $\delta_{^oB}*(y) \leq 1$, then $0 \geq -\varepsilon+\varepsilon\delta_{^oB}*(y)$ and so $g(y) \geq -f(0)-\varepsilon+\varepsilon\delta_{^oB}*(y)$, since $f(0) \geq -g(y)$ for every y. On the other hand if $y \notin ({}^oB)^o$, i.e. $\delta_{^oB}*(y) > 1$, then $y/r \notin B$ whenever $1 < r < \delta_{^oB}*(y)$, i.e. $g(y/r)-g(0) > \varepsilon$. Now $g(y)-g(0) \geq r\cdot(g(y/r)-g(0))$ since $r > 1$ and $(g(ty)-g(0))/t$ decreases as $t \downarrow 0$ by convexity, so we have $g(y) - g(0) > \varepsilon\cdot r$. Taking $r \uparrow \delta_{^oB}*(y)$, we get $g(y)-g(0) \geq \varepsilon\delta_{^oB}*(y)$, so $g(y) \geq g(0) + \varepsilon\delta_{^oB}*(y) \geq -f(0)-\varepsilon+\varepsilon\delta_{^oB}*(y)$.

Thus $g \geq -f(0)-\varepsilon+\varepsilon\delta_{^oB}*$; taking conjugates yields $f(x) \leq f(0)+\varepsilon+\delta_{^oB}(x/\varepsilon)$. Hence if $x \in \varepsilon\cdot{}^oB$ we have $\delta_{^oB}(x/\varepsilon) = 0$ and so $f(x) \leq f(0)+\varepsilon$, proving the lemma. □

We are now in a position to use the results of Section 4 on polar sets to show the correspondence between continuity and equicontinuity of level sets.

5.3 <u>Theorem</u>. Let (X,τ) be a HLCS, X^* its dual with τ^*-topology, and let $f:X \to \overline{R}$. If affdomf is closed with finite codimension, and if f is bounded above on some relative neighborhood of affdomf, then cof is continuous on ricodomf and the level sets

$$B = \{y \in X^*: f^*(y)-xy \leq r\},\ x \in X,\ r \in R,$$

are complete and locally equicontinuous (also closed convex and hence locally compact). Moreover if B is nonempty then $B_\infty = (\text{domf}-x)^-$, $B_\infty \cap (-B_\infty) = (\text{domf}-x)^\perp$, $B = (\text{domf}-x)^\perp + (B \cap L^\perp)$ where L is any (finite dimensional) complement of span(domf-x) in X, and the following are equivalent:

i). $x \in$ rcor co domf

ii). cof is finite and continuous at x

iii). B_∞ is a subspace

iv). $B \cap L^\perp$ is compact.

We remark that B is always empty in the degenerate case $f^* \equiv +\infty$ and $*(f^*) \equiv -\infty$. Otherwise $f^* \not\equiv +\infty$ and $*(f^*)$ and cof never take on $-\infty$ values, and $*(f^*) \equiv$ cof except possibly on relative boundary points of codomf.

Proof. We assume $f^* \not\equiv +\infty$, since otherwise B is always empty and $*(f^*) \equiv -\infty$.

Take $x_o \in X$, and let $B = \{y \in X^*: f^*(y)-x_oy \leq r\}$ be nonempty. Define $\tilde{f}(x) = f(x+x_o)$ and $A = \{x: \tilde{f}(x) \leq s\} = \{x: f(x+x_o) \leq s\}$, where s is sufficiently large so that $s+r > 0$ and A contains a point in ridomf. We then have

$\mathrm{ri}A \neq \emptyset$, where $\mathrm{aff}A = \mathrm{affdom}f - x_o$ is closed with finite codimension. By Lemma 5.1 we have

$$B = \{y: f^*(y) - x_o y \leq r\} = \{y: \tilde{f}^*(y) \leq r\} \subset (r+s)\cdot A^o.$$

But then by Theorem 4.7 we know that B is complete and locally equicontinuous, since it is a closed subset of $(r+s)\cdot A^o$ and $\mathrm{ri}A \neq \emptyset$. A straightforward calculation shows that $B_\infty = (\mathrm{dom}f - x_o)^-$ when B is nonempty, and hence that $B_\infty \cap (-B)_\infty = (\mathrm{dom}f - x_o)^\perp$. Now $\mathrm{span}(\mathrm{dom}f - x_o)$ is a closed subspace with finite codimension, since it equals $(\mathrm{affdom}f - x_1) + (-\infty, +\infty)\cdot(x_1 - x_o)$ for any $x_1 \in \mathrm{affdom}f$ and hence is the sum of the closed affine subspace affdomf and the subspace $(-\infty, +\infty)\cdot(x_1 - x_o)$ of dimension at most one (note $\mathrm{span}(\mathrm{dom}f - x) = \mathrm{affdom}f - x_o$ precisely in the case $x_o \in \mathrm{affdom}f$). Thus by Lemma 4.3 we have the decomposition $X^* = (\mathrm{dom}f - x_o)^\perp + L^\perp$ where L is any (finite dimensional) complement of $\mathrm{span}(\mathrm{dom}f - x_o)$ and $L^\perp$ is then a closed complement of $(\mathrm{dom}f - x_o)^\perp$. But then $B = (\mathrm{dom}f - x_o)^\perp + (B \cap L^\perp)$ since $(\mathrm{dom}f - x_o)^\perp \subset B_\infty$. It only remains to show the equivalence of i) through iv).

Note that since f is bounded above on a relative neighborhood in affdomf, cof is also bounded above on the same neighborhood (and of course affdomf = affdom(cof)), so that cof is continuous in ricodomf (note co domf = dom cof by Lemma 1.1) and i) is equivalent to ii) by convexity. Moreover, $B \cap L^{\perp}$ is compact iff $(B \cap L^{\perp})_{\infty} = B_{\infty} \cap L^{\perp}$ is $\{0\}$ by Lemma 4.5; but $B_{\infty} \cap L^{\perp} = \{0\}$ precisely in the case that $B_{\infty} \subset (\text{domf}-x_o)^{\perp} = B_{\infty} \cap (-B_{\infty})$, i.e. B_{∞} is a subspace, so that iii) and iv) are equivalent. Now if $x_o \in$ rcorcodomf, then codomf$-x_o$ absorbs affdomf$-x_o$, so that $(\text{domf}-x_o)^{-} = (\text{codomf}-x_o)^{-}$ is actually $(\text{domf}-x_o)^{\perp}$; thus $B_{\infty} = (\text{domf}-x_o)^{\perp}$ and i) => ii). Conversely, suppose $x_o \notin$ rcorcodomf; since codomf has nonempty relative interior in affdomf, there is a separating $y \in X^*$ such that either $y \equiv 0$ on span(domf$-x_o$) and $\sup_{x \in \text{domf}} xy \leq x_o y$ (in the case $x_o \in$ affdomf), or $y \equiv 0$ on affdomf$-x_1$ and $(x_1-x_o)y < 0$ for some $x_1 \in$ domf (in the case $x_o \notin$ affdomf). But in both cases we then have $y \in (\text{domf}-x_o)^{-} = B_{\infty}$, with $y \notin (\text{domf}-x_o)^{\perp} = B_{\infty} \cap (-B_{\infty})$, so that B_{∞} is not a subspace and iii) => i). □

5.4 <u>Theorem</u>. Let X be a HLCS, X^* its dual with a τ^*-topology, and suppose g is convex $X^* \to \overline{R}$, $^*g \equiv +\infty$. If the level set $B_o = \{y \in X^*: g(y)-x_o y < s_o\}$ is nonempty and locally equicontinuous for some $x_o \in X$, $s_o \in R$, then affdom$^*g$ is closed with finite codimension and $^*g$ is finite and relatively continuous on rcordom$^*g \neq \emptyset$. Moreover all the level sets $B = \{y: g(y)-xy < s\}$, $x \in X$, $s \in R$ are locally equicontinuous, and if nonempty $B_{\infty} = (\text{dom}^*g-x)^{-}$, affdom$^*g = x + {}^{\perp}(B_{\infty} \cap (-B_{\infty}))$ if $x \in$ affdom$^*g$, and $^*g$ is finite and relatively continuous at x iff B_∞ is a subspace.

Proof. First, let us note that if B_o is locally equicontinuous then epig is locally equicontinuous (in the product topologies on $X \times R$ and $X^* \times R$) and hence all the level sets B are equicontinuous. For, if $y_o \in B_o$ and W is a y_o-neighborhood with $B_o \cap W$ equicontinuous, then $g(y_o)-1,x_o \times W$ is a neighborhood of $(g(y_o),y_o)$ whose intersection with epig is contained in $(g(y_o)-1,s_o) \times (B_o \cap W)$ which is equicontinuous. Since epig is convex, we have by Proposition 4.4 that all of epig is locally equicontinuous, and hence all the level sets B are locally equicontinuous. Note also that $^*g$ never has $-\infty$ values, since epig $\neq \emptyset$.

We wish to show that $^*g$ has relative continuity points. Now $^*g \not\equiv +\infty$ by assumption; since all the level sets B are locally equicontinuous we may assume that $x_o \in \mathrm{dom}\,^*g$ in the definition of B_o. Let $y_o \in B_o$ and take some $\varepsilon > 0$ such that $g(y_o)-x_o y_o < s_o-\varepsilon$, and define $B_1 = \{y: g(y)-x_o y \leq \varepsilon+g(y_o)-x_o y_o\}$. B_1 clearly contains y_o and is locally equicontinuous since $B_1 \subset B_o$. Now define $\tilde{g}(y) = g(y_o+y)-x_o y$; then $B_1-y_o = \{y: \tilde{g}(y) \leq \varepsilon+g(0)\}$ and applying Lemma 5.2 yields

$$\varepsilon\cdot{}^{o}(B_1-y_o) \subset \{x \in X: {}^*\tilde{g}(x) \leq \varepsilon+{}^*\tilde{g}(0)\} = \{x \in X: {}^*g(x_o+x)-xy_o \leq \varepsilon+{}^*g(0)\}.$$

But $(B_1 - y_o)$ is convex and locally equicontinuous, so by Theorem 4.6 ${}^{o}(B_1 - y_o)$ has nonempty relative interior with respect to L, where $L = (B_1 - y_o)_\infty \cap (-B_1 + y_o)_\infty$. This means that $x \mapsto {}^*g(x_o + x) - xy_o$ is bounded above on some relative neighborhood of L, so that ${}^*g$ is bounded above on some relative neighborhood of $x_o + L$. We need only show that $x_o + L$ contains affdom${}^*g$. Now since $L \subset (B_1 - y_o)_\infty$, we see that $\tilde{g}(y_o + ty) \leq \varepsilon + \tilde{g}(0)$ for every $t > 0$, $y \in L$ and so

$$^*\tilde{g}(x) \geq \sup_{\substack{y \in L \\ t>0}} (x(y_o + ty) - \tilde{g}(y_o + ty)) \geq xy_o - \varepsilon - \tilde{g}(0) + \sup_{\substack{y \in L \\ t>0}} t \cdot xy$$

is $+\infty$ unless $x \in {}^{\perp}L$. Thus dom${}^*\tilde{g} \subset {}^{\perp}L$, i.e. dom${}^*g$ $x_o + {}^{\perp}L$, so we see that ${}^*g$ is bounded above on some relative neighborhood of $x_o + {}^{\perp}L$ = affdom${}^*g$ and hence is relatively continuous on rcordom${}^*g$.

To prove the remarks at the end of the theorem, we show that the level sets $B = \{y: g(y) - xy < s\}$ have closures which contain and are contained in the level sets of $({}^*g)^*$, and then we simply apply Theorem 2.3 to $f = {}^*g$. Since $({}^*g)^* \leq g$, it is clear that $B \subset \{y: ({}^*g)^*(y) - xy \leq s\}$, hence $B_\infty \subset (\text{dom}{}^*g - x)^-$ by Theorem 2.3. On the other hand, for any $\varepsilon > 0$ we have $\{y: ({}^*g)^*(y) - xy \leq s - \varepsilon\} \subset \text{cl}B$ since $({}^*g)^* = \text{lsc}g$, and hence (taking ε sufficiently small so that the level set of $({}^*g)^*$ is nonempty) $(\text{dom}{}^*g - x)^- \subset B_\infty$ by Theorem 5.3. Thus $B_\infty = (\text{dom}{}^*g - x)^-$, and ${}^*g$ is relatively continuous at x iff B_∞ is a subspace □

We note in particular that for any HLCS X Theorems 5.3 and 5.4 are true for the $w(X^*, X)$ topology on X^*, in which equicontinuous sets are always relatively compact. If X is a barrelled space, then the equicontinuous sets are precisely the $w(X^*,X)$-bounded sets in X^*, so that locally equicontinuous simply means locally bounded in the $w(X^*,X)$ topology. If X is not a barrelled space, we could still characterize $w(X^*,X)$-locally bounded level sets of a convex function $g: X^* \to \overline{R}$ in terms of $\text{rcordom}^*g \neq \emptyset$ and affdom^*g closed with finite codimension, by imbedding X^* in X' just as in Corollary 4.9.

We summarize the results for convex functions with locally compact level sets in a HLCS.

5.5 Corollary. Let X be a HLCS, $f: X \to \overline{R}$ convex and lsc, $g = f^*$. If one of the level sets $B_o = \{x \in X: f(x) - xy_o \leq s_o\}$ is locally compact (resp. weakly locally compact) for some $y_o \in X^*$, $s_o > \inf_x (f(x) - xy_o) \equiv -g(y_o)$, then affdomg is closed with finite codimension and the restriction of g to affdomg is continuous on rcordomg (which is nonempty unless $g \equiv +\infty$) in the $a(X^*,X)$ topology (resp. the $m(X^*,X)$ topology) on X^*. Conversely, if affdomg is closed with finite codimension and g has finite relative continuity points in affdomg in the $a(X^*,X)$ topology (resp. the $m(X^*,X)$ topology), then all the level sets $B = \{x \in X: f(x) - xy \leq s\}$ are closed, convex, complete, and locally compact in X (resp. in the weak topology on X), and if B is nonempty, $B_\infty = {}^{-}(\mathrm{dom}g - y)$, $\mathrm{aff}\,\mathrm{dom}g = y + (B_\infty \cap (-B_\infty))^\perp$ if $y \in \mathrm{aff}\,\mathrm{dom}g$, and g is finite and relatively continuous at y iff $y \in \mathrm{rcor}\,\mathrm{dom}g$ iff B_∞ is a subspace.

Proof. This is a direct consequence of Theorems 2.3 and 2.4 where τ is taken to be the $a(X^*,X)$ topology (resp. the $m(X^*,X)$ topology) on X^*, τ^* is the original topology (resp. the weak topology) on X, and the roles of X and X^* have been reversed. □

6. Closed subspaces with finite codimension.

This section serves only to provide some very basic results about what it means to be a closed subspace with finite codimension; the ideas are simple but it is important to be careful here.

Let X be a HLCS. Let M be an affine subspace of X; the subspace parallel to M is $M-M = M-m_o$ where m_o is any fixed element of M. We have

$$M = \text{aff}M = (M-M)+M = (M-m_o)+m_o.$$

The <u>dimension</u> of M is defined to be the dimension of the subspace $M-M$. More generally, if $C \subset X$ t n the dimension of C is defined to be the dimension of aff$C$:

$$\dim C \overset{\Delta}{=} \dim \text{aff } C = \dim \text{span}(C-C),$$

where of course

$$\text{aff } C = C + \text{span}(C-C) = c_o + \text{span}(C-c_o)$$

$$= \{ \sum_{i=1}^{n} t_i x_i : n \in \mathbb{N},\ t_i \in R,\ x_i \in C,\ \sum_{i=1}^{n} t_i = 1 \}.$$

If N is an affine subspace of M, then we say N has <u>finite codimension</u> in M iff the subspace $N-N$ parallel to N has finite codimension in the subspace $M-M$ parallel to M, i.e. if $\dim (M-M/N-N)$ is finite.

6.1 Proposition. Let X be a HLCS, M an affine subspace of X, N an affine subspace of M. Let M have the topology induced by that of X. Then the following are equivalent:

1) N is closed with finite codimension in M
2) N-N is closed with finite codimension in M-M
3) N is closed in M and M-M/N-N is finite dimensional
4) N is closed in M and M/N-N is a finite dimensional affine subspace of X/N-N
5) N is closed in M and $\exists$ a finite dimensional subspace L such that N+L = M and $(N-N) \cap L = \{0\}$
6) N is closed in M and $\exists$ a finite dimensional subspace L such that $N+L \supset M$
7) $\exists$ finite subset $F \subset X^*$ st $N = (n_o + {}^{\perp}F) \cap M$ for some (and hence every) $n_o \in N$
8) $\exists\, r_1, \ldots, r_n \in R$, $y_1, \ldots, y_n \in X^*$ st $N = M \cap \bigcap_{i=1}^{n} y_i^{-1}\{r_i\}$.

Proof. Throughout the proof we shall assume that n_o is a fixed element of N; in particular, $N-N = N-n_o$ and $M-M = M-n_o$.

1) <=> 2). N is closed in M iff $N-n_o$ is closed in $M-n_o$ by translation invariance of vector topologies. The result now follows from the definition of finite codimension.

2) <=> 3). The codimension of N-N in M-M is precisely dim (M-M/N-N).

3) <=> 4). In 4) we are using the following notation:

if $C \subset X$ and if L is a subspace of X then C/L is the image of C under the canonical quotient map of X into X/L. Now $M/N-N = [n_o] + (M-M)/N-N$ is an affine subspace of X/N-N which is a translation of the subspace M-M/N-N (here $[n_o]$ denotes the equivalence class of n_o in X/N-N); hence dim (M/N-N) = dim (M-M/N-N).

3) => 5). Let L be an algebraic complement of N-N in M-M, i.e. $L+(N-N) = M-M$ and $L \cap (N-N) = \{0\}$. Now L is algebraically isomorphic to M-M/N-N under the quotient map $Q: L \to (M-M/N-N): \ell \to [\ell]$; for Q is linear, one-to-one since $L \cap (N-N) = \{0\}$, and onto since $L+(N-N) = M-M$. Thus by hypothesis 3), dim (L) = dim (M-M/N-N) is finite. Finally, we have

$$M = n_o + (M-M) = n_o + L + (N-N) = L+N.$$

5) <=> 6). Trivially 5) => 6). Suppose 6) holds. Let L' be a complement of (N-N) in (M-M). Then $L' \cap (N-N) = \{0\}$ and $L'+N = M$. But $L' \subset (N-N)+L$; since $L' \cap (N-N) = \{0\}$, $L' \subset L$. Thus L' is finite dimensional and 5) holds for L'.

5) => 7). Define the projection map $P: (M-M) \to L$, where $P \equiv 0$ on (N-N), $P \equiv I$ on L. Let $\{\phi_1, \ldots, \phi_n\}$ be a basis for L*. P is a continuous map $(M-M) \to L$ since P has finite dimensional range and the null space (N-N) of P

is closed in $(M-M)$. Hence each $\phi_i \circ P \in (M-M)^*$. By the Hahn-Banach extension theorem we may extend each $\phi_i \circ P$ to an element y_i of X^*, so that $y_i = \phi_i \circ P$ on $(M-M)$. Let $F = \{y_1,\dots,y_n\}$. Clearly $N-N$, which is the null space of P, is contained in ${}^{\perp}F$. Conversely, $(M-M) \cap {}^{\perp}F \subset N-N$; for if $x \in (M-M)$ then $x = n+\ell$ where $n \in (N-N)$ and $\ell \in L$ and if also $x \in {}^{\perp}F$ then $\ell = 0$ (since $n \in {}^{\perp}F$ and F spans L^*). Thus $(N-N) = (M-M) \cap {}^{\perp}F$. Equivalently, $(N-n_o) = (M-n_o) \cap {}^{\perp}F$ i.e. $N = M \cap (n_o + {}^{\perp}F)$.

7) => 8). Assume 7) holds, i.e. $F = \{y_1,\dots,y_n\} \subset X^*$ and $N = M \cap (n_o + {}^{\perp}F)$. Set $r_i = y_i(n_o)$. Then $n_o + {}^{\perp}F = \{n_o + x\colon y_i(x)=0,\ i=1,\dots,n\} =$ $\{x\colon y_i(x-n_o) = 0\ \forall i = 1,\dots,n\} = \bigcap_{i=1}^{n} y_i^{-1}\{r_i\}$, and 8) follows.

8) => 9). Clearly N is closed in M, since each y_i is continuous on M. Now $y_i(n) = r_i$ for every $n \in N$ and $i = 1,\dots,n$, so $y_i(n-n_o) = 0$ and $N-n_o \subset {}^{\perp}\{y_1,\dots,y_n\}$. But then $\dim (M-M/N-n_o) \le \dim (M-M/{}^{\perp}\{y_1,\dots,y_n\}) \le$ $\dim(X/{}^{\perp}\{y_1,\dots,y_n\}) = \dim ({}^{\perp}\{y_1,\dots,y_n\})^{\perp} =$ $\dim \operatorname{span} \{y_1,\dots,y_n\} \le n$. □

7. Weak dual topologies.

Let (X,τ) be a HLCS, and suppose M is a subspace of X with the induced topology $M \cap \tau$. By the Hahn-Banach theorem we may identify M^* with $X^*/M^\perp$, where $\langle x,[y]\rangle \equiv \langle x,y\rangle$ for $x \in M$ and $[y]$ the equivalence class $y+M^\perp \in X^*/M$ of $y \in X^*$. We shall be concerned with various topologies pertaining to the duality between M and $X^*/M^\perp$. The following notation will be used: if $B \subset X^*$, then $B/M^\perp$ denotes $\{[b]:b \in B\} = \{b+M^\perp : b \in B\}$, a subset of X^*.

We have already defined the $w(X^*,X)$ topology, with 0-neighborhood basis

$$\{F^\circ : F \text{ finite} \subset X\}.$$

A net $\{y_i\}$ converges to 0 in $w(X^*,X)$ iff $\langle x,y_i\rangle \to 0$ for every $x \in X$. A set $B \subset X^*$ is bounded in $w(X^*,X)$ iff for every $x \in X$, $\sup_{y \in B} \langle x,y\rangle < +\infty$. B is $w(X^*,X)$ conditionally compact whenever B is equicontinuous, or equivalently $0 \in \operatorname{int}^\circ B$.

A weaker topology is the $w(X^*,M)$ topology, with 0-neighborhood basis

$$\{F^\circ : F \text{ finite} \subset M\}.$$

A net $\{y_i\}$ converges to 0 in $w(X^*,M)$ iff $\langle x,y_i\rangle \to 0$ for every $x \in M$, or equivalently iff eventually $y_i \in \{x\}^\circ = \{x\}^\circ + M^\perp$ for every $x \in M$. Note that the $w(X^*,M)$ topology need not be Hausdorff; it is Hausdorff iff $M^{\perp} = \{0\}$, iff M is dense in X. Since the closure of $\{0\}$ in $w(X^*,M)$ is $M^{\perp}$, the associated HLCS is X^*/M with the $w(X^*/M^{\perp},M)$ topology; hence $y_i \to 0$ in $w(X^*,M)$ iff $[y_i] \to 0$ in $w(X^*/M\ ,M)$. Similarly, a subset B of X^* is $w(X^*,M)$-bounded iff $\forall x \in M$, $\sup_{y \in B} \langle x,y \rangle < +\infty$, iff B/M is $w(X^*/M\ ,M)$-bounded; B is conditionally compact in $w(X^*,M)$ whenever B is equicontinuous as a subset of M^*, or equivalently $B/M^{\perp}$ is equicontinuous as a subset of M^*. Of course, $(X^*,w(X^*,M))^*$ may be identified with M; for if $z \in (X^*,w(X^*,M))^*$ then there is a finite subset F of M such that $|z(y)| \leq 1$ whenever $y \in F^{\circ}$, hence $\{y : z(y) = 0\} \supset \bigcap_{x \in F} \{y : \langle x,y \rangle = 0\}$ and $z \in \operatorname{span} F \subset M$.

We say that a subset B of X^* is <u>M-equicontinuous</u> iff the restriction of the continuous linear functions in B to the subspace M is equicontinuous for the induced topology $M \cap \tau$ on M.

7.1 <u>Proposition</u>. Let (X,τ) be a HLCS, M a subspace of X with the induced topology $M \cap \tau$, and $B \subset X^*$. Then the following are equivalent:

1) B is M-equicontinuous
2) $B/M^{\perp}$ is equicontinuous as a subset of $M^* \cong X^*/M$

3) $^{\circ}B$ contains a relative 0-nbhd in M, i.e.

$\exists$ 0-nbhd U st $^{\circ}B \supset U \cap M$

4) $\exists$ 0-nbhd U st $\sup_{x \in U \cap M} \sup_{y \in B} \langle x,y \rangle \leq 1$, i.e.

$B \subset (U \cap M)^{\circ}$

5) $\exists$ 0-nbhd U in X st $B \subset U^{\circ} + M$

6) $\exists$ 0-nbhd U in X st $B/M^{\perp} \subset U^{\circ}/M$.

Proof. 1) <=> 2). This is simply the definition of M-equicontinuous.

2) <=> 3). This is what equicontinuity means, for linear functionals.

3) <=> 4). If U is a closed convex 0-nbhd, then $^{\circ}B \supset U \cap M \iff B \subset (U \cap M)^{\circ}$ since $^{\circ}((U \cap M)^{\circ}) = U \cap M$.

4) => 5). This is the only nontrivial part. Suppose $B \subset (U \cap M)^{\circ}$. Let V be a closed convex 0-neighborhood such that $V \subset \text{int}\, U$. Then $\text{cl}(U \cap M) \supset V \cap \text{cl}M$; for if $x \in V$ is the limit of a net $\{x_i\}$ in M, then the $\{x_i\}$ eventually belong to U (since $x \in \text{int}\, U$) and hence $x \in \text{cl}(U \cap M)$. Now V° is $w(X^*,X)$-compact, so $V^{\circ} + M^{\perp}$ is a $w(X^*,X)$-closed convex set containing $V^{\circ} \cup M^{\perp}$; thus $V^{\circ} + M^{\perp} = \text{clco}(V^{\circ} \cup M^{\perp})$. But then $^{\circ}(V^{\circ} + M^{\perp}) = {}^{\circ}(V^{\circ} \cup M^{\perp}) = {}^{\circ}(V^{\circ}) \cap {}^{\circ}(M^{\perp}) = V \cap \text{cl}M$, and so

$$B \subset (U \cap M)^{\circ} = (\text{cl}(U \cap M))^{\circ} \subset (V \cap \text{cl}M)^{\circ} = ({}^{\circ}(V^{\circ} + M^{\perp}))^{\circ} = V^{\circ} + M^{\perp}.$$

Thus 5) holds for the 0-neighborhood V.

5) => 4). Immediate, since $U^o + M^\perp \subset (U \cap M)^o$.

5) <=> 6). Immediate, since $B/M^\perp \subset U^o/M^\perp \iff B \subset U^o + M^\perp$. □

It is also natural to consider the quotient topology of $w(X^*,X)$ on $X^*/M^\perp$, i.e. the strongest topology on $X^*/M^\perp$ for which the canonical quotient map $Q\colon (X^*,w(X^*,X)) \to X^*/M^\perp$ is continuous; we denote this topology by $w(X^*,X)/M^\perp$. A basis of 0-neighborhoods for $w(X^*,X)/M^\perp$ is given by all sets of the form $F^o/M^\perp = (F^o+M^\perp)/M^\perp$, where F is a finite subset of X; $\{[y_i]\} \to 0$ in $w(X^*,X)/M^\perp$ iff eventually $y_i \in \{x\}^o + M^\perp$ for every $x \in X$. We shall also use $w(X^*,X)/M^\perp$ to denote the topology on X^* with 0-neighborhood basis all sets of the form $F^o+M^\perp$, F finite $\subset X$ (it will be clear from context whether the topology is on X^* or on $X^*/M^\perp$), that is $w(X^*,X)/M^\perp = Q^{-1}(w(X^*,X)/M^\perp)$. Of course, $\{y_i\} \to 0$ in $w(X^*,X)/M^\perp$ iff $\{[y_i]\} \to 0$ in $w(X^*,X)/M^\perp$ iff $\forall x \in X$, eventually $y_i \in \{x\}^o+M$. A subset B of X^* is bounded in $w(X^*,X)/M^\perp$ iff for every $x \in X$, $\sup_{y \in B} \inf_{y' \in M^\perp} \langle x, y-y' \rangle < +\infty$.

The $w(X^*,X)/M^\perp$ topology is closely related to the $w(X^*,M)$ topology.

7.2 <u>Proposition</u>. Let (X,τ) be a HLCS, M a subspace of X. Then $w(X^*,X)/M^\perp = w(X^*,\bar{M})$, where $\bar{M}$ denotes the closure of M in X.

Proof. Let F be a finite subset of $\overline{M}$. Since $F^{O} \supset F^{O}+M^{\perp}$, it is clear that F^{O} has nonempty $w(X^*,X)/M^{\perp}$-interior; hence $w(X^*,X)/M^{\perp} \supset w(X^*,\overline{M})$. Conversely, let F be an arbitrary finite subset of X. Since F is finite, it is straightforward to see that

$$\operatorname{clco}(F \cup \{0\}) \cap \overline{M} = \operatorname{clco}((F \cap \overline{M}) \cup \{0\}),$$

or equivalently ${}^{O}(F^{O}) \cap \overline{M} = {}^{O}((F \cap \overline{M})^{O})$. But then

$$(F \cap \overline{M})^{O} = ({}^{O}(F^{O}) \cap \overline{M})^{O} = w^*\text{-}\operatorname{clco}(F^{O} \cup M^{\perp}) \subset w^*\text{-}\operatorname{cl}(F^{O}+M\ ) \subset$$
$$w(X^*,X)/M^{\perp} - \operatorname{cl}(F^{O}+M^{\perp}),$$

where the last step follows since clearly the $w^* = w(X^*,X)$ topology is stronger than the $w(X^*,X)/M^{\perp}$ topology. Hence the closures of sets in the 0-neighborhood base of $w(X^*,X)/M^{\perp}$ have nonempty $w(X^*,\overline{M})$-interior, so $w(X^*,\overline{M}) \supset w(X^*,X)/M$. □

7.3 Corollary. Let X be a HLCS, M a subspace. Then $w(X^*,M) = w(X^*,X)/M$ on X^* iff M is closed. Equivalently $w(X^*/M^{\perp},M) = w(X^*,X)/M^{\perp}$ on X^*/M iff M is closed.

Proof. From Proposition 7.2 we have $w(X^*,X)/M^{\perp} = w(X^*,\overline{M})$. But $w(X^*,\overline{M}) = w(X^*,M)$ iff $\overline{M} = M$, since $(X^*,w(X^*,\overline{M}))^* \cong \overline{M}$ and $(X^*,w(X^*,M)) \cong M$. □

8. Relative continuity points of convex functions

The relationship between continuity points of a functional $f\colon X \to \bar{R}$ and local equicontinuity of the level sets of the conjugate function f^* has been thoroughly investigated in Section 5 for the case that affdomf is closed with finite codimension. We may still ask what happens in the case that affdomf does not necessarily have finite codimension; note that the level sets will contain the (infinite dimensional) subspace $(\text{domf-domf})^\perp$ and we cannot hope for local equicontinuity. However, by characterizing the level sets of f^* modulo their behavior on $(\text{domf-domf})^\perp$, i.e. by considering the duality between affdomf (the natural space determined by f) and $X^*/(\text{domf-domf})^\perp$, we obtain a generalization of the previous results.

For simplicity we consider only the original topology on X and the weak * dual topologies. We consider the following propositions about a function $f\colon X \to \bar{R}$ and an affine subspace M of X which contains domf. Of course, M-M is the subspace parallel to M. We shall often specialize to the case M = affdomf, or M = domf + $(\text{domf-domf})^\perp$ = claffdomf.

1a. $\exists$ open set U, $y_1,\ldots,y_n \in X^*$, $r_1,\ldots,r_n \in R$ st $U \cap M \cap \bigcap_{i=1}^{n} y_i^{-1}\{r_i\} \neq \emptyset$ and $f(\cdot)$ is bounded above on $U \cap M \cap \bigcap_{i=1}^{n} y_i^{-1}(r_i)$.

1b. $f(\cdot)$ is bounded above on a subset C of X, where $riC \neq \emptyset$ and $affC$ is closed with finite codimension in M.

2a. $ricoepif \neq \emptyset$ and $affdomf$ is closed with finite codimension in M.

2b. $rcorcodomf \neq \emptyset$, $cof \restriction rcorcodomf$ is continuous, and $affdomf$ is closed with finite codimension in M.

3a. $f^* \equiv +\infty$, or $\exists\, x_o \in M$, $r_o > -f(x_o)$ st $\{y \in X^*: f^*(y) - x_o y \leq r_o\}$ is $w(X^*, M-x_o)$-locally $(M-x_o)$-equicontinuous.

3b. $f^* \equiv +\infty$, or $\exists\, x_o \in M$, $y_o \in domf^*$, $r_o > f^*(y_o) - x_o y_o$, finite $F \subset M-x_o$, 0-nbhd U st $\{y \in X^*: f^*(y) - x_o y \leq r_o\} \cap (y_o + F^o) \subset U^o + (M-x_o)^{\perp}$

3c. $\forall x_o \in M$ $\exists$ finite $F \subset M-x_o$ st $\forall y_o \in X^*$, $\forall r_o \in R$ $\{y \in X^*: f^*(y) - x_o y \leq r_o\} \cap (y_o + F^o)$ is $(M-x_o)$-equicontinuous, i.e. $\subset U^o + (M-x_o)^{\perp}$ for some 0-nbhd U.

4a. $f^* \equiv +\infty$, or $\exists\, x_o \in M$, $r_o > -f(x_o)$ st $\{y \in X^*: f^*(y) - x_o y \leq r_o\}$ is $w(X^*, M-x_o)$-locally compact.

4b. $f^* \equiv +\infty$, or $\exists\, x_o \in M$, $y_o \in domf^*$, $r_o > f^*(y_o) - x_o y_o$, finite $F \subset M-x_o$ st $\{y \in X^*: f^*(y) - x_o y \leq r_o\} \cap (y_o + F^o)$ is $w(X^*, M-x_o)$-compact.

4c. $\forall x_o \in M \;\; \exists$ finite $F \subset M-x_o$ st $\forall y_o \in X^*$, $\forall r_o \in R$, $\{y \in X^*: f^*(y)-x_o y \leq r_o\} \cap (y_o+F^o)$ is $w(X^*,M-x_o)$-compact.

4d. affdom*(f*) is closed with finite codimension in M, rcordom*(f*) $\neq \emptyset$, and *(f*) $\upharpoonright$ rcordomf is continuous for the topology $M+m(M-M,X^*/(M-M)^{\perp})$.

5a. $f^* \equiv +\infty$, or $\exists x_o \in X$, $r_o > -f(x_o)$ st $\{y \in X^*: f^*(y)-x_o y \leq r_o\}$ is $w(X^*,X)$-locally (M-M)-equicontinuous.

5b. $f^* \equiv +\infty$, or $\exists x_o \in X$, $y_o \in$ domf*, $r_o > f^*(y_o)-x_o y_o$, finite $F \subset X$, 0-nbhd U st $\{y \in X^*: f^*(y)-x_o y \leq r_o\} \cap (y_o+F^o) \subset U^o+(M-M)^{\perp}$.

5c. $\forall x_o \in X \;\; \exists$ finite $F \subset X$ st $\forall y_o \in X^*$, $r_o \in R$, $\{y \in X^*: f^*(y)-x_o y \leq r_o\} \cap (y_o+F^o)$ is $w(X^*,X)$-locally (M-M)-equicontinuous.

5d. $\forall x_o \in X$, $r_o \in R$, $\{y \in X^*: f^*(y)-x_o y \leq r_o\}$ is $w(X^*,X)$-locally (M-M)-equicontinuous.

5e. epif* is $w(X^* \times R, X \times R)$-locally $(M-M) \times R$-equicontinuous.

6a. $f^* \equiv +\infty$, or $\exists x_o \in X$, $r_o > -f(x_o)$ st $\{y \in X^*: f^*(y)-x_o y \leq r_o\}$ is $w(X^*,X)/M^{\perp}$-locally compact.

6b. $f^* \equiv +\infty$, or $\exists x_o \in X$, $y_o \in$ domf*, $r_o > f^*(y_o)-x_o y_o$, finite $F \subset X$ st $\{y \in X^*: f^*(y)-x_o y \leq r_o\} \cap (y_o+F^o)$ is $w(X^*,X)/M^{\perp}$-compact.

6c. $\forall x_o \in X$ $\exists$ finite $F \subset X$ st $\forall y_o \in X^*$, $r_o \in R$, $\{y \in X^*: f^*(y) - x_o y \leq r_o\} \cap (y_o + F^o)$ is $w(X^*,X)/M^{\perp}$-compact.

6d. $\forall x_o \in X$, $r_o \in R$, $\{y \quad X^*: f^*(y) - x_o y \leq r_o\}$ is $w(X^*,X)/M^{\perp}$-compact.

6e. epif^* is locally compact for the $w(X^*,X)/M^{\perp} \times R$ topology.

7a. $f^* \equiv +\infty$, or $\exists x_o \in X$, $r_o > -f(x_o)$ st $\{y \in X^*: f^*(y) - x_o y \leq r_o\}$ is $w(X^*,X)/M^{\perp}$-locally bounded.

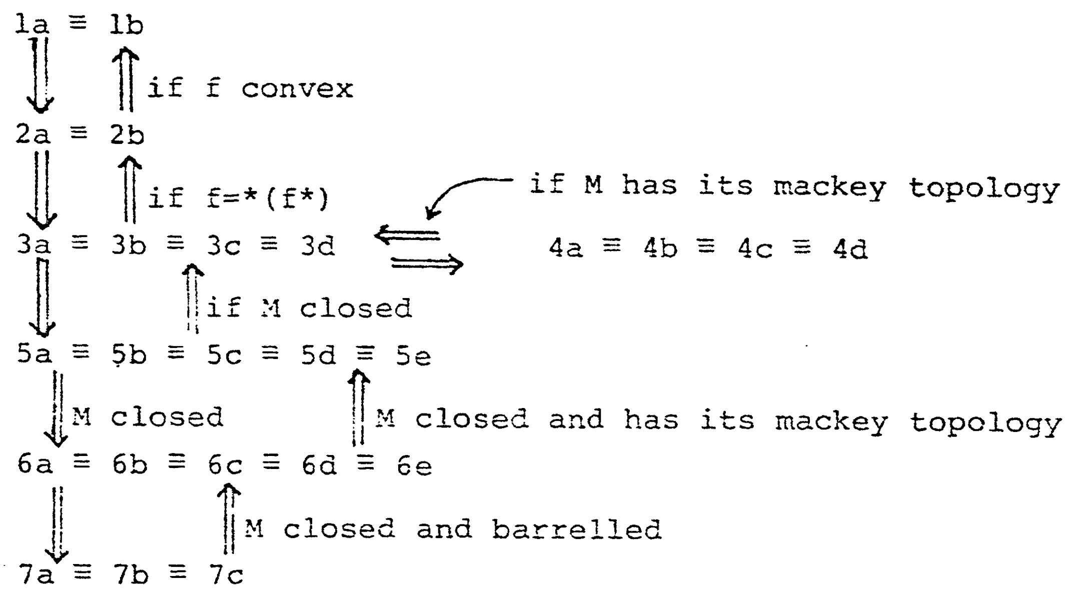
7b. $f^* \equiv +\infty$, or $\exists x_o \in X$, $y_o \in$ domf^*, $r_o > f^*(y_o) - x_o y_o$, finite $F \subset X$ st $\forall x \in X$,

$$\sup\{\inf_{y' \in M^{\perp}} \langle x, y-y' \rangle : y \in y_o + F^o,\ f^*(y) - x_o y \leq r_o\} < +\infty.$$

7c. $\forall x_o \in X$, $\exists$ finite $F \subset X$ st $\forall y_o \in X^*$, $r_o \in R$, $x \in X$,

$$\sup\{\inf_{y' \in M^{\perp}} \langle x, y-y' \rangle : y \in y_o + F^o,\ f^*(y) - x_o y \leq r_o\} < +\infty.$$

8.1 Theorem. Let X be a HLCS, $f: X \to \overline{R}$, M an affine subset of X with the induced topology, $M \supset$ dom$f$. Then we have the following relations:



<u>Remarks</u>. The degenerate case $f^* \equiv +\infty$ is usually excluded in applications. We have $M \supset \text{dom} f$ if (iff, assuming M closed) $M = x_o + {}^{\perp}N$ where $x_o \in \text{dom} f$ and N is a subspace satisfying $N \subset (\text{dom} f - x_o)^{\perp} = \{y \in X^*: y \equiv \text{const on dom} f\} = \{y \in X^*: (f^*)_{\infty}(-y) = -(f^*)_{\infty}(y)\}$. In particular, if $M = {}^{\perp}N$ where $N \subset (\text{dom} f)^{\perp}$, then M is closed and $M \supset \text{dom} f$.

8.2 <u>Corollary</u>. Let X be a metrizable HLCS, $f: X \to \overline{R}$ proper convex lsc, M an affine subset $\supset \text{dom} f$. If M is closed, then all but 6 are equivalent. If M is complete, then all of 1-6 are equivalent.

Proof of Corollary. Since M is metrizable in the induced topology, its parallel subspace $M-M$ has the Mackey topology $m(M-M,(M-M)^*)$. If M is complete, then $M-M$ is also complete, hence barrelled. □

Proof of Theorem.

1a => 1b. Take $C = U \cap M \cap \bigcap_{i=1}^{n} y_i^{-1}\{r_i\}$. Then $\text{aff}C = M \cap \bigcap_{i=1}^{n} y_i^{-1}\{r_i\}$ is closed with finite codimension in M by Proposition 6.1,8). Moreover $U \cap \text{aff}C \subset C$, so $U \cap \text{aff}C \subset \text{ri}C$ and $\text{ri}C \neq \emptyset$.

1b => 1a. Note $C \subset \text{dom}f \subset M$. By Proposition 6.1,8) there are $y_1,\ldots,y_n \in X^*$ and $r_1,\ldots,r_n \in R$ such that $\text{aff}C = M \cap \bigcap_{i=1}^{n} y_i^{-1}\{r_i\}$. Moreover $\text{ri}C \neq \emptyset$, so $\exists$ open set U such that $C \supset U \cap \text{aff}C \neq \emptyset$. But $f(\cdot)$ is bounded above on C, hence on $U \cap \text{aff}C = U \cap M \cap \bigcap_{i=1}^{n} y_i^{-1}\{r_i\}$.

1b => 2a. This is essentially the same argument as that used to prove that every nonempty finite dimensional convex set has nonempty relative interior. We argue by induction on the (finite) dimension of a complementary subspace of $\text{aff}\,\text{dom}f$ in M. Let us first note that $\text{aff}\,\text{dom}f$ is closed with finite codimension in M; for $\text{aff}C \subset \text{aff}\,\text{dom}f \subset M$, so that $\text{aff}\,\text{dom}f$ is the algebraic sum of the closed (in M) flat aff$C$ and an at most finite dimensional subspace of $(M-M)$, hence closed and finite codimensional. Equivalently, affepi$f$ is closed with finite codimension in $M\times R$. Now by hypothesis 1b, epi$f \subset M\times R$ and epi$f$ contains a set B_o with nonempty relative interior and with affB_o closed with finite codimension in $M\times R$; for if f is bounded above by r_o on C, set $B_o = C\times[r_o,\infty)$ and affB_o = aff$C\times R$. If affepi$f$ = affB_o we are done, for then riepi$f \supset$ riepi$B_o \neq \emptyset$. Otherwise $\exists\, z_1 \in$ epi$f \setminus$ affB_o. Now $B_1 \overset{\Delta}{=} \text{co}(\{z_1\}\cup B_o)$ is a subset of coepi$f$, and moreover B_1 has nonempty interior in the flat affB_1 = aff$(\{z_1\}\cup B_o) \subset$ affepi$f$. Proceeding, if affepi$f$ = affB_1 we are done; otherwise $\exists\, z_2 \in$ epi$f\setminus$affB_1 for which $B_2 \overset{\Delta}{=} \text{co}(\{z_2\}\cup B_1)$ is contained in coepi$f$ and has nonempty relative interior in affB_2. Eventually we obtain a linearly independent set $\{z_1,\ldots,z_n\} \subset$ coepi$f$ for which $B_n \overset{\Delta}{=} \text{co}(\{z_1,\ldots,z_n\}\cup B_o)$ is contained in coepi$f$ and has nonempty relative interior in affB_n = aff$(\{z_1,\ldots,z_n\}\cup B_o) \supset$ epi$f$. Hence ricoepi$f \neq \emptyset$.

2a => 1b if f convex. Take any $(x_o,r_o) \in$ riepi$f$; since $(x_o,r_o) \in$ riepi$f$, $\exists$ open set U, $\varepsilon > 0$ such that $(x_o,r_o) \in (U\cap \text{affdom}f) \times (r_o-\varepsilon, r_o+\varepsilon) \subset$ epi$f$. Simply define $C = U\cap\text{affdom}f$; then $f(\cdot)$ is bounded above by r_o on C, and aff$C$ = affdom$f$ is closed with finite codimension in M.

2a => 2b. Epicof $\supset$ coepif and affepicof = affcoepif, so riepicof $\neq \emptyset$. It is now a well-known result in the literature that cof is relatively continuous on rcordomf, since cof is of course convex $X \to \overline{R}$. Note that if cof takes on $-\infty$ values, then cof $\equiv -\infty$ on ricodomf.

2b => 2a. Trivial.

2b => 3a, 3a => 2b when $f = *(f^*)$. Suppose $f^* \not\equiv +\infty$; in particular f cannot take on $-\infty$ values. Take any $x_o \in M$, $r > -f(x_o)$. Let $L = M-x_o$ be the subspace parallel to M, with the induced topology and associated dual space $X^*/L^\perp$. On L define the function $\tilde{f}$: $L \to \overline{R}$: $\ell \to f(x_o+\ell)$. Then dom$\tilde{f}$ = domf$-x_o$, $\tilde{f}^*([y]) = f^*(y)-x_oy$. Clearly affdom$\tilde{f}$ = affdomf$-x_o$ is closed with finite codimension in $L = M-x_o$ iff affdomf is closed with finite codimension in M, and $\tilde{f}$ has relative continuity points in L iff f has in M (using translation invariance of vector topologies). Applying Corollary 5.5 we see that the level set $\{[y] \in X^*/L : \tilde{f}^*([y]) \leq r_o\} = \{[y]: f^*(y)-x_oy \leq r_o\}$ is locally (L-) equicontinuous in the $w(X^*/L, L)$-topology if (iff when $f = *(f^*)$) $\tilde{f}$ has relative continuity points in L and affdom$\tilde{f}$ is closed with finite codimension in L. But the former condition is equivalent to the local L-equicontinuity of $\{y: f^*(y)-x_oy \leq r_o\}$ in the $w(X^*,L)$ topology by Proposition 7.1.

3a <=> 3b. Condition 3b simply states that $\{y \in X^*: f^*(y)-x_o y \leq r_o\}$ is $w(X^*,L)$-locally L-equicontinuous at the point y_o. Since $\{y \in X^*: f^*(y)-x_o y \leq r_o\}$ is convex, it follows that 3b is equivalent to local L-equicontinuity at <u>every</u> point $y \in \{y \in X^*: f^*(y)-x_o y \leq r_o\}$; simply apply Proposition 4.4 to the set $\{[y] \in X^*/L : \tilde{f}^*([y]) \leq r_o\}$. Hence 3a <=> 3b.

3a => 3c. We first note that all of the level sets $\{y: f^*(y)-x_o y \leq r_o\}$ are $w(X^*,L)$-locally L-equicontinuous -- this is just a direct application of Theorem 2.4 to $\tilde{f}^*$ just as in the proof of 2b => 3a, where one of the level sets of $\tilde{f}^*$ being $w(X^*,L)$-locally L-equicontinuous implies that all of them are. Note also that $*(f^*)$ has relative continuity points and affdom$*(f^*)$ is closed with finite codimension from 3a => 2b. Now given $x_o \in M$, let $\{x_1,\ldots,x_n\}$ be a basis for a complement of affdomf in M, let $L = M-x_o$, and let x_{n+1} be an element of L which is strictly positive on the $w(X^*,L)$-locally equicontinuous convex cone $(\text{domf}-x_o)^-/L^\perp$. Take $F = \{\pm x_1,\ldots,\pm x_n,\pm x_{n+1}\}$. Since $\{[y] \in X^*/L : f^*(y)-x_o y \leq r_o\}$ is $w(X^*/L^\perp,L)$-locally L-equicontinuous, its intersection with $(y_o+F^o)/L^\perp$ is for every $y_o \in X^*$. But the recession cone of $\{y \in X^*: f^*(y)-x_o y \leq r_o\} \cap (y_o+F^o)$ is contained in $L^\perp$, hence $\{[y]: f^*(y)-x_o y \leq r_o\} \cap (y_o+F^o)/L^\perp$ has recession cone

$\{[0]\}$ and is actually L-equicontinuous by Lemma 4.5. But this is precisely condition 3c by Proposition 7.1.

3c => 3a. If all the level sets are empty, then $f^* \equiv +\infty$. Otherwise there is a nonempty level set for which 3a is true.

3a => 4a, 3b => 4b, 3c => 4c. This is immediate since (M-M)-equicontinuity implies $w(X^*,M-M)$-compactness by the Banach-Alaoglu theorem applied to $(M-M)^* = X^*/(M-M)^{\perp}$.

4a => 3a, 4b => 3b, 4c => 3c when the induced topology on M-M is the mackey topology $m(M-M, X^*/(M-M)^{\perp})$, since then (M-M)-equicontinuity is equivalent to $w(X^*,M-M)$-compactness.

4a <=> 4d. Put the $m(M-M, X^*/(M-M)^{\perp})$ topology on M-M; this induces a topology on M by translation. But now 4a <=> 4d is equivalent to the result 2b <=> 3a.

3a => 5a, 3b => 5b, 3c => 5c. This is immediate since $w(X^*,X) \supset w(X^*,M-M)$.

5a => 3a, 5b => 3b, 5c => 3c if M is closed. Suppose $\{y \in X^*: f^*(u) - x_o y \leq r_o\}$ is $w(X^*,X)$-locally (M-M)-equicontinuous. Since $M \subset \mathrm{dom} f$, we have $M^{\perp} \subset \{y \in X^*: f^*(y) - x_o y \leq r_o\}_{\infty}$; hence $\{y \in X^*: f^*(y) - x_o y \leq r_o\}/M^{\perp}$ is $w(X^*,X)/(M-M)^{\perp}$-locally (M-M)-equicontinuous. But M is closed, so $w(X^*,X)/(M-M)^{\perp} = w(X^*,M-M)$.

5c => 5d => 5e => 5a. Immediate.

5 => 6 if M closed. Suppose 5 holds. Define $L = \text{span } M = M+(-\infty,\infty)\cdot\{m_o\}$ where $m_o \in M$. Clearly L is closed since it is the sum of the closed flat M and a 1-dimensional subspace; moreover affdomf is closed with finite codimension in L since M is closed with finite codimension in L. Now 5 implies (since M is closed) that 3 and hence 2a holds for *(f*) and M; thus 2a also holds for *(f*) in L. But then 5 holds for L replacing M, that is $\{y \in X^*: f^*(y)-x_o y \leq r_o\}$ is w(X*,X)-locally L-equicontinuous, hence $w(X^*,X)/L^\perp$-locally L-equicontinuous. Since L-equicontinuity implies $w(X^*,L^\perp)$-compactness and $w(X^*,L^\perp) = w(X^*,X)/L^\perp$ by Proposition 7.2 (L is closed), and $L^\perp = M^\perp$, 6 follows.

6 => 5 if M closed and has its mackey topology. As in 5 => 6, define $L = \text{span } M = {}^\perp(M^\perp)$, a closed subspace. If the level sets $\{y \in X^*: f^*(y)-x_o y \leq r_o\}$ are $w(X^*,X)/L^\perp$-locally compact, they are $w(X^*,X)/(M-M)^\perp$-locally compact since $L \supset M$ and hence $w(X^*,L) \supset w(X^*,M)$. But M-M has its mackey topology, so $w(X^*,X)/(M-M)^\perp$-local compactness is equivalent to w(X*,X)-local (M-M)-equicontinuity and 5 follows.

6 => 7. Trivial since local compactness implies local boundedness.

7 => 6 if M closed and barrelled. In this case $w(X^*,X)/M^{\perp} = w(X^*, {}^{\perp}(M^{\perp}))$ since M is closed and $w(X^*/M^{\perp}, {}^{\perp}(M^{\perp}))$-boundedness is equivalent to compactness since M is barrelled. □

9. Determining continuity points

In Theorem 8.1 we have given several conditions which characterize when a convex function $f: X \to \bar{R}$ has relative continuity points, or equivalently when riepif $\neq \emptyset$. In this section we characterize those points at which f is relatively continuous assuming that f has such points.

9.1 Theorem. Let X be a HLCS, $f: X \to \bar{R}$ convex. Assume riepif $\neq \emptyset$. Then $f(\cdot)$ is continuous relative to affdomf on rcordomf, and the following are equivalent for a point $x_o \in X$:

1. $f(\cdot)$ is relatively continuous at $x_o \in$ domf
2. $x_o \in$ rcordomf
3. domf$-x_o$ absorbs x_o-domf
4. $\forall x \in$ domf, $\exists \varepsilon > 0$ st $(1+\varepsilon)x_o - \varepsilon x \in$ domf
5. $[\text{domf}-x_o]^- \subset [\text{domf}-x_o]^\perp \equiv \{y \in X^*: y \equiv \text{constant on domf}\}$
6. $[\text{domf}-x_o]^-$ is a subspace
7. $\{y \in X^*: (f^*)_\infty(y) - x_o y \leq 0\}$ is a subspace
8. $x_o \in$ domf, and $\{y \in X^*: f^*(y) - x_o y \leq r\}_\infty$ is a subspace for some $r \geq -f(x_o)$
9. $\partial f(x_o) \neq \emptyset$ and $(\partial f(x_o))_\infty$ is a subspace
10. $\partial f(x_o)$ is nonempty and $w(X^*, \text{affdomf}-x_o)$-compact.

Proof. 1 <=> 2. Standard in the literature.

$2 \Leftrightarrow 3 \Leftrightarrow 4$. Definition of relative core (relative algebraic interior).

$2 \Leftrightarrow 5$. Let $C = \mathrm{domf} - x_o$; C is convex and has nonempty relative interior. Hence by the Hahn-Banach separation and extension theorems, $0 \notin \mathrm{ri}C$ if and only if $\exists y \in X^*$ such that y is not constant on $\mathrm{aff}C = \mathrm{affdomf} - x_o$ and $\sup_{x \in C} \langle x,y \rangle \leq 0$; equivalently, $y \in C^- = [\mathrm{domf} - x_o]^-$ and $y \notin C^\perp = [\mathrm{domf} - x_o]^\perp$.

$5 \Leftrightarrow 6$. Immediate.

$6 \Leftrightarrow 7$. $\{y \in X^*: (f^*)_\infty(y) - x_o y \leq 0\}$

$$= \{y \in X^*: \sup_{x \in \mathrm{dom}^*(f^*)} \langle x,y \rangle - \langle x_o,y \rangle \leq 0\}$$

$$= [\mathrm{dom}^*(f^*) - x_o]^-$$

Now $\mathrm{dom}^*(f^*) \subset \mathrm{cldomf}$, since $^*(f^*)(\cdot) + \delta_{\mathrm{cldomf}}(\cdot)$ is a convex lsc function dominated by f and hence $^*(f^*) + \delta_{\mathrm{cldomf}} \leq {}^*(f^*)$. Of course, $^*(f^*) \leq f$ so $\mathrm{domf} \subset \mathrm{dom}^*(f^*)$. Thus

$$\mathrm{domf} - x_o \subset \mathrm{dom}^*(f^*) - x_o \subset \mathrm{cldomf} - x_o$$

and so

$$[\mathrm{domf} - x_o]^- \supset [\mathrm{dom}^*(f^*) - x_o]^- \supset [\mathrm{cldomf} - x_o]^-.$$

But $[\mathrm{domf} - x_o]^- = [\mathrm{cldomf} - x_o]^-$, so $[\mathrm{domf} - x_o]^- = [\mathrm{dom}^*(f^*) - x_o]^- = \{y \in X^*: (f^*)_\infty(y) - x_o y \leq 0\}$ and $6 \Leftrightarrow 7$ holds.

7 <=> 8. Suppose $x_o \in \mathrm{dom} f$ and $r \geq -f(x_o)$. Then $\{y \in X^*: f^*(y)-x_o y \leq r\}$ contains an element y_o and has recession cone given by

$$\{y \in X^*: f^*(y)-x_o y \leq r\}_\infty = \{y \in X^*: f^*(y_o+ty)-<x_o,y_o+ty>\leq r \ \forall t>0\}$$

$$= \{y \in X^*: \sup_{t>0}[\frac{f^*(y_o+ty)-f^*(y_o)}{t} + \frac{f^*(y_o)-r-x_o y_o}{t}] \leq x_o y\}$$

$$= \{y \in X^*: \sup_{t>0}[\frac{f^*(y_o+ty)-f^*(y_o)}{t}] \leq x_o y\}$$

$$= \{y \in X^*: (f^*)_\infty(y) \leq x_o y\}.$$

Thus 7 <=> 8 holds.

7 <=> 9. This is a special case of 7 <=> 8, since $\partial f(x_o) = \{y \in X^*: f^*(y)-x_o y \leq -f(x_o)\}$ and $\partial f(x_o) \neq \emptyset \Rightarrow x_o \in \mathrm{dom} f$.

9 => 10. Let $M = \mathrm{affdom} f - x_o$, the subspace parallel to affdomf. By Theorem 8.1, $\partial f(x_o) = \{y \in X^*: f^*(y)-x_o y \leq -f(x_o)\}$ is $w(X^*,M)$-locally compact; equivalently $\partial f(x_o)/M^\perp$ is $w(X^*/M^\perp,M)$ locally-compact. But we have shown in 7 <=> 8 and 6 <=> 7 that

$$\partial f(x_o)_\infty = \{y \quad X^*: (f^*)_\infty(y)-x_o y \leq 0\} = [\mathrm{dom} f - x_o]^-.$$

But then 9 implies $\partial f(x_o)_\infty = [\mathrm{dom} f - x_o]^\perp = M^\perp$, so $(\partial f(x_o)/M^\perp)_\infty = \partial f(x_o)_\infty/M^\perp = \{[0]\}$; hence by Lemma 1.5 $\partial f(x_o)/M^\perp$ is actually $w(X^*/M^\perp,M)$-compact and hence 10 follows.

10 => 9. Immediate. □

III. Duality Approach to Optimization

Abstract. The duality approach to solving convex optimization problems is studied in detail using tools in convex analysis and the theory of conjugate functions. Conditions for the duality formalism to hold are developed which require that the optimal value of the original problem vary continuously with respect to perturbations in the constraints only along feasible directions; this is sufficient to imply existence for the dual problem and no duality gap. These conditions are also posed as certain local compactness requirements on the dual feasibility set, based on a characterization of locally compact convex sets in locally convex spaces in terms of nonempty relative interiors of the corresponding polar sets.

1. Introduction

The idea of duality theory for solving optimization problems is to transform the original problem into a "dual" problem which is easier to solve and which has the same value as the original problem.[+] Constructing the dual solution corresponds to solving a "maximum principle" for the problem. This dual approach is especially useful for solving problems with difficult implicit constraints and costs (e.g. state constraints in optimal control problems), for which the constraints on the dual problem are much simpler (only explicit "control" constraints). Moreover the dual solutions have a valuable sensitivity interpretation: the dual solution set is precisely the subgradient of the change in minimum cost as a function of perturbations in hte "implicit" constraints and costs.

Previous results for establishing the validity of the duality formalism, at least in the infinite-dimensional case, generally require the existence of a feasible interior point ("Kuhn-Tucker" point) for the implicit constraint set. This requirement is restrictive and

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[+]Basic references are [R73], [ET76]. A more elementary reference is [L68, Chapters 7-8].

difficult to verify. Rockafellar [R73] has relaxed this to require only continuity of the optimal value function. In this chapter we investigate the duality approach in detail and develop weaker conditions which require that the optimal value of the minimization problem varies continuously with respect to perturbations in the implicit constraints only along feasible directions (that is, we require relative continuity of the optimal value function); this is sufficient to imply existence for the dual problem and no duality gap. Moreover we pose the conditions in terms of certain local compactness requirements on the dual feasibility set, based on the results of Chapter II characterizing the duality between relative continuity points and local compactness.

To indicate the scope of our results let us consider the Lagrangian formulation of nonlinear programming problems with generalized constraints. Let U, X be normed spaces and consider the problem

$$P_o = \inf\{f(u): u \in C,\ g(u) \leq 0\}$$

where C is a convex subset of U, $f: C \to R$ is convex, and $g: C \to X$ is convex in the sense that

$$g(tu_1+(1-t)u_2) \leq tg(u_1)+(1-t)g(u_2), \quad u_1,u_2 \in C,\ t \in [0,1].$$

We are assuming that X has been given the partial ordering induced by a nonempty closed convex cone Q of "positive vectors"; we write $x_1 \geq x_2$ to mean $x_1 - x_2 \in Q$. The dual problem corresponding to P_0 is well-known to be

$$D_0 = \sup_{y \in Q^+} \inf_{u \in C} [f(u) + \langle g(u), y \rangle];$$

this follows from equation (2.4) below by taking $L \equiv 0$, $x_0 = 0$, and

$$F(u,x) = \begin{cases} f(u) & \text{if } u \in C,\ g(u) \leq x \\ +\infty & \text{otherwise .} \end{cases} \tag{1}$$

We also remark that it is possible to write

$$P_0 = \inf_u \sup_y \ell(u,y)$$

$$D_0 = \sup_y \inf_u \ell(u,y)$$

where we have defined the Lagrangian function by

$$\ell(u,y) = \begin{cases} +\infty & \text{if } u \notin C \\ f(u) - \langle g(u), y \rangle & \text{if } u \in C,\ y \in Q^- \\ -\infty & \text{if } u \in C,\ y \notin Q^- . \end{cases}$$

In analyzing the problem P_0 we imbed it in the family of perturbed problems

$$P(x) = \inf\{f(u): u \in C,\ g(u) \leq x\}.$$

It then follows that the dual problem is precisely the second conjugate of P_o evaluated at 0: $D_o = *(P^*)(0)$. Moreover if there is no duality gap $(P_o = D_o)$ then the dual solution set is the subgradient $\partial P(0)$ of $P(\cdot)$ at 0. The following theorem summarizes the duality results for this problem.

1.1 Theorem. Assume P_o is finite. The following are equivalent:

1) $P_o = D_o$ and D_o has solutions

2) $\partial P(0) \neq \emptyset$

3) $\exists \hat{y} \in Q^+$ st $P_o = \inf_{u \in C}[f(u) + \langle g(u), \hat{y}\rangle]$

4) $\exists \varepsilon > 0,\ M > 0$ st $f(u) \geq P_o - M|x|$ whenever $u \in C,\ |x| \leq \varepsilon,\ g(u) \leq x$.

If 1) is true then $\hat{u}$ is a solution for P_o iff $\hat{u} \in C$, $g(u) \leq 0$, and there is a $\hat{y} \in Q^+$ satisfying

$$f(u) + \langle g(u), \hat{y}\rangle \geq f(\hat{u}) \quad \forall u \in C,$$

in which case complementary slackness holds, i.e. $\langle g(\hat{u}), \hat{y}\rangle = 0$, and $\hat{y}$ solves D_o.

Proof. This follows directly from Theorem 2.4 with F defined by (1). Δ

We remark here that criterion 4) is necessary and sufficient for the duality result 1) to hold, and it is critical in determining how strong a norm to use on the perturbation space X (equivalently, how large a dual space X^* is required in formulating a well-posed dual problem).

The most familiar assumption which is made to insure that the duality results of Theorem 1.1 hold is the existence of a Kuhn Tucker point:

$$\exists \bar{u} \in C \quad \text{st} \quad -g(\bar{u}) \in \text{int } Q$$

(see Corollary 3.2). This is a very strong requirement, and again is often critical in determining what topology to use on the perturbation space X. More generally, we need only require that $P(\cdot)$ is continuous at 0 (Theorem 3.1). Rochafellar has presented the following result [R73]: if U is the normed dual of a Banach space V, if X is a Banach space, if g is lower semicontinuous in the sense that

$$\text{epi}\, g \triangleq \{(u,x) : g(u) \leq x\}$$

is closed in $U \times X$ (e.g. if g is continuous), then the duality results of Theorem 1.1 hold whenever

$$0 \in \operatorname{core}[g(C)+Q].$$

In fact, it then follows that $P(\cdot)$ is continuous at 0. The following theorem relaxes this result to relative continuity and also provides a dual characterization in terms of local compactness requirements which are generally easier to verify.

1.2 <u>Theorem</u>. Assume $P_o < +\infty$; U is the normed dual V^* of a normed space V; X is a Banach space; epig is closed in $U \times X$. Then the following are equivalent:

1) $\operatorname{aff}[g(C)+Q]$ is closed; and $0 \in \operatorname{rcor}[g(C)+Q]$, or equivalently
$\forall u \in C, \forall x \geq g(u)$ $\exists \varepsilon > 0$ and
$u_1 \in C$ st $g(u_1) + \varepsilon x \leq 0$.

2) $Q^+ \cap g(C)^+$ is a subspace M; and
there is an $\varepsilon > 0$, an $x_1 \in X$, an $r_1 \in R$ such that $\{y \in Q^+ : \inf_{|v| \leq \varepsilon} \sup_{u \in C} [f(u)+g(u)y-uv] > r_1\}$ is nonempty and $w(X^*,X)/M$-locally bounded.

If either of the above holds, then $P(\cdot)$ is relatively continuous at 0 and hence Theorem 1.1.1) holds. Moreover the dual solutions have the sensitivity interpretation

$$P'(0;x) = \max\{\langle x,y\rangle : y \text{ solves } D_o\}$$

where the maximum is attained and $P'(0;\cdot)$ denotes the directional derivative of the optimal value function $P(\cdot)$ evaluated at 0.

Proof. This follows directly from Theorem 3.6 where $\text{dom } P = g(C)+Q$ and $(F^*)_\infty(v,y) = \delta_{\leq o}(y) + \sup_{u\in C}[uv+g(x)y]$ $\{y \quad X^*: (F^*)_\infty(0,y) \leq 0\} = Q^- \quad g(C)^-$. Δ

2. Problem formulation

In this section we summarize the duality formulation of optimization problems. Let U be a HLCS of controls; X a HLCS of states; $u \mapsto Lu + x_0$ an affine map representing the system equations, where $x_0 \in X$, and $L: U \to X$ is linear and continuous; $F: U \times X \to \overline{R}$ a cost function. We consider the minimization problem

$$P_0 = \inf_{u \in U} F(u, Lu+x_0), \tag{1}$$

for which feasibility constraints are represented by the requirement that $(u, Lu+x_0) \in \mathrm{dom}F$. Of course, there are many ways of formulating a given optimization problem in the form (1) by choosing different spaces U, X and maps L, F; in general the idea is to put explicit, easily characterized costs and constraints into the "control" costs on U and to put difficult implicit constraints and costs into the "state" part of the cost where a Lagrange multiplier representation can be very useful in transforming implicit constraints to explicit constraints. The dual variables, or multipliers will be in X^*, and the dual problem is an optimization in X^*.

In order to formulate the dual problem we consider a family of perturbed problems

$$P(x) = \inf_{u \in U} F(u, Lu+x) \tag{2}$$

where $x \in X$. Note that if $F: U \times X \to \bar{R}$ is convex then $P: X \to \bar{R}$ is convex; however F lsc does not imply that P is lsc. Of course $P_o = P(x_o)$. We calculate the conjugate function of P:

$$P^*(y) = \sup_x [\langle x,y\rangle - P(x)] = \sup_{u,x} [\langle x,y\rangle - F(u,Lu+x)]$$

$$= F^*(-L^*y, y). \tag{3}$$

The dual problem of $P_o = P(x_o)$ is given by the second conjugate of P evaluated at x_o, i.e.

$$D_o = {}^*(P^*)(x_o) = \sup_{y \in X^*} [\langle x_o, y\rangle - F^*(-L^*y, y)] \tag{4}$$

The feasibility set for the dual problem is just $\text{dom}P^* = \{y \in X^*: (-L^*y, y) \in \text{dom}F^*\}$. We immediately have

$$P_o \equiv P(x_o) \geq D_o \equiv {}^*(P^*)(x_o). \tag{5}$$

Moreover, since the primal problem P_o is an infimum, and the dual problem D_o is a supremum, and $P_o \geq D_o$, we see that if $\hat{u} \in U$, $\hat{y} \in X^*$ satisfy

$$F(\hat{u}, L\hat{u}+x_o) = \langle x_o, \hat{y}\rangle - F^*(-L^*\hat{y}, \hat{y}) \tag{6}$$

then $P_o = D_o = F(\hat{u}, L\hat{u}+x_o)$ and (assuming $P_o \in R$) $\hat{u}$ is optimal for P, $\hat{y}$ is optimal for D. Thus, the existence of a $\hat{y} \in X^*$ satisfying (6) is a sufficient condition for optimality of a control $\hat{u} \in U$; we shall be interested in conditions under which (6) is also necessary. It is also clear that any "dual control" $y \in X^*$ provides a lower bound for the original problem: $P_o \geq \langle x_o, y \rangle - F^*(-L^*y, y)$ for every $y \in X^*$.

The duality approach to optimization problems P_o is essentially to vary the constraints slightly as in the perturbed problem $P(x)$ and see how the minimum cost varies accordingly. In the case that F is convex, $P_o = D_o$ or no "duality gap" means that the perturbed minimum cost function $P(\cdot)$ is lsc at x_o. The stronger requirement that the change in minimum cost does not drop off too sharply with respect to perturbations in the constraints, i.e. that the directional derivative $P'(x_o;\cdot)$ is bounded below on a neighborhood of x_o, corresponds to the situation that $P_o = D_o$ and the dual problem D_o has solutions, so that (6) becomes a necessary and sufficient condition for optimality of a control $\hat{u}$. It turns out that the solutions of D_o when $P_o = D_o$ are precisely the elements of $\partial P(x_o)$, so that the dual solutions have a sensitivity interpretation as the subgradients of the change in minimum cost with respect to the change in constraints.

Before stating the above remarks in a precise way, we define the Hamiltonian and Lagrangian functions associated with the problem P_o. We denote by $F_u(\cdot)$ the functional $F(u,\cdot): x \to F(u,x): X \to \overline{R}$, for $u \in U$. The <u>Hamiltonian</u> function $H: U \times X^* \to \overline{R}$ is defined by

$$H(u,y) = \sup_{x \in X} [\langle x,y\rangle - F(u,x)] = F_u^*(y). \tag{7}$$

2.1 <u>Proposition</u> The Hamiltonian H satisfies:

1) $(^*H_u)(x) = {}^*(F_u{}^*)(x)$

2) $(^*H_u)^*(y) = H_u(y) = F_u{}^*(y)$

3) $F^*(v,y) = \sup_u [\langle u,v\rangle + H(u,y)] = (-H(\cdot,y))^*(v)$.

Moreover $H(u,\cdot)$ is convex and w^*-lsc $X^* \to \overline{R}$; $H(\cdot,y)$ is concave $U \to \overline{R}$ if F is convex; if $F(u,\cdot)$ is convex, proper, and lsc then $H(\cdot,y)$ is concave for every y iff F is convex.

<u>Proof</u>. The equalities are straightforward calculations. $H(u,\cdot)$ is convex and lsc since $(^*H_u)^* = H_u$. It is straightforward to show that $-H(\cdot,y)$ is convex if $F(\cdot)$ is convex. On the other hand if ${}^*(F_u{}^*) = F_u$ and $H(\cdot,y)$ is concave for every $y \in X^*$, then

$F(u,x) = {}^*(F_u{}^*)(x) = {}^*H_u(x) = \sup_y [xy - H(u,y)]$ is the supremum of the convex functionals $(u,x) \mapsto \langle x,y\rangle - H(u,y)$ and hence F is convex. □

The Lagrangian function $\ell: U \times X^* \to \overline{R}$ is defined by

$$\begin{aligned} \ell(u,y) &= \inf_x [F(u,Lu+x_o+x) - \langle x,y\rangle] \\ &= \langle Lu+x_o,y\rangle - F_u{}^*(y) \qquad\qquad (8) \\ &= \langle Lu+x_o,y\rangle - H(u,y). \end{aligned}$$

2.2 Proposition The Lagrangian ℓ satisfies

1) $\inf_u \ell(u,y) = \langle x_o,y\rangle - F^*(-L^*y,y)$

2) $D_o \equiv {}^*(F^*)(x_o) = \sup_y \inf_u \ell(u,y)$

3) ${}^*(-\ell_u)(x) = {}^*(F_u{}^*)(Lu+x_o+x)$

4) $P_o \equiv P(x_o) = \inf_u \sup_y \ell(u,y)$ if $F_u = {}^*(F_u{}^*)$ for every $u \in U$.

Moreover $\ell(u,\cdot)$ is convex and w*-ℓsc $X^* \to \overline{R}$ for every $u \in U$; $\ell(\cdot)$ is convex $U \times X^* \to \overline{R}$ if F is convex; if $F_u = {}^*(F_u{}^*)$ for every $u \in U$ then ℓ is convex iff F is convex.

Proof. The first equality 1) is direct calculation; 2) then follows from 1) and (4). Equaltiy 3) is immediate from (8); 4) then follows from 3) assuming that $*(F_u^*) = F_u$. The final remarks follow from Proposition 2.1 and the fact that $\ell(u,y) = \langle Lu+x_o,y\rangle - H(u,y)$. □

Thus from Proposition 2.2 we see that the duality theory based on conjugate functions includes the Lagrangian formulation of duality for inf-sup problems. For, given a Lagrangian function $\ell: U \times X^* \to \overline{R}$, we can define $F: U \times X \to \overline{R}$ by $F(u,x) = *(-\ell_u)(x) = \sup_y[\langle x,y\rangle+\ell(u,x)]$, so that

$$P_o = \inf_u \sup_y \ell(u,y) = \inf_u F(u,0)$$

$$D_o = \sup_y \inf_u \ell(u,y) = \sup_y - F^*(0,y),$$

which fits into the conjugate duality framework.

For the following we assume as before that U,X are HLCS's; $L: U \to X$ is linear and continuous; $x_o \in X$; $F: U \times X \to \overline{R}$. We define the family of optimization problems $P(x) = \inf_u F(u,Lu+x)$, $P_o = P(x_o)$, $D_o = \sup_y[\langle x,y\rangle-F^*(-L^*y,y)]$ $= *(P^*)(x_o)$. We shall be especially interested in the case that $F(\cdot)$ is convex, and hence $P(\cdot)$ is convex.

2.3 Proposition (no duality gap). It is always true that

$$P_o \equiv P(x_o) \geq \inf_u \sup_y \ell(u,y) \geq D_o$$

$$\equiv \inf_u \sup_y \ell(u,y) \equiv *(P^*)(x_o) \qquad (9)$$

If $P(\cdot)$ is convex and D_o is feasible, then the following are equivalent:

1) $P_o = D_o$

2) $P(\cdot)$ is ℓsc at x_o, i.e. $\liminf_{x \to x_o} P(x) \geq P(x_o)$

3) $\sup_{F \text{ finite} \subset X^*} \inf_{\substack{u \in U \\ x \in Lu + x_o + {}^oF}} F(u,x) \geq P_o$.

These imply, and are equivalent to if $F_u = *(F_u{}^*)$ for every $u \in U$,

4) ℓ has a saddle value, i.e.

$$\inf_u \sup_y \ell(u,y) = \sup_y \inf_u \ell(u,y).$$

Proof. The proof is immediate since $P_o = P(x_o)$ and $D_o = *(P^*)(x_o)$. Statement 4) follows from Proposition 2.2 and (9). □

2.4 Theorem (no duality gap and dual solutions).

Assume P_o is finite. The following are equivalent:

1) $P_o = D_o$ and D_o has solutions

2) $\partial P(x_o) \neq \emptyset$

3) $\exists\, \hat{y} \in Y$ st $P_o = \langle x_o, \hat{y}\rangle - F^*(-L^*\hat{y}, \hat{y})$

4) $\exists\, \hat{y} \in Y$ st $P_o = \inf_u \ell(u, \hat{y})$.

If $P(\cdot)$ is convex, then each of the above is equivalent to

5) $\exists$ 0-nbhd N st $\inf_{x \in N} P'(x_o;x) > -\infty$

6) $\liminf_{x \to o} P'(x_o;x) > -\infty$

7) $$\liminf_{\substack{x \to o \\ t \to o^+}} \frac{P(x_o+tx)-P_o}{t} \equiv$$

$$\sup_{N=0\text{-nbhd}} \; \inf_{t>o} \; \inf_{x \in N} \; \inf_{u \in U} \frac{F(u, Lu+x_o+tx)-P_o}{t} > -\infty.$$

If $P(\cdot)$ is convex and X is a normed space, then the above are equivalent to:

8) $\exists\, \varepsilon > 0,\ M > 0$ st $F(u, Lu+x_o+x)-P_o \geq -M|x| \quad \forall u \in U, |x| \leq \varepsilon$.

9) $\exists\, \varepsilon > 0,\ M > 0$ st $\forall u \in U,\ |x| \leq \varepsilon,\ \delta > 0\ \exists u' \in U$ st

$$F(u, Lu+x_o+x)-F(u', Lu'+x_o) \geq -M|x|-\varepsilon.$$

Moreover, if 1) is true then $\hat{y}$ solves D_o iff $\hat{y} \in \partial P(x_o)$, and $\hat{u}$ is a solution for P_o iff there is a $\hat{y}$ satisfying any of the conditions 1')-3') below. The following statements are equivalent:

1') $\hat{u}$ solves P_o, $\hat{y}$ solves D_o, and $P_o = D_o$

2') $F(\hat{u},L\hat{u}+x_o) = \langle x_o,\hat{y}\rangle - F^*(-L^*\hat{y},\hat{y})$

3') $(-L^*\hat{y},\hat{y}) \in \partial F(\hat{u},L\hat{u}+x_o)$.

These imply, and are equivalent to if $F(u,\cdot)$ is proper convex ℓsc $X \to \overline{R}$ for every $u \in U$, the following equivalent statements:

4') $0 \in \partial \ell(\cdot,\hat{y})(\hat{u})$ and $0 \in \partial(-\ell(\hat{u},\cdot))(\hat{y})$, i.e. $(\hat{u},\hat{y})$ is a saddlepoint of ℓ, that is
$\ell(\hat{u},y) \leq \ell(\hat{u},\hat{y}) \leq \ell(u,\hat{y})$ for every $u \in U$, $y \in X^*$.

5') $L\hat{u}+x_o \in \partial H(\hat{u},\cdot)(\hat{y})$ and $L^*\hat{y} \in \partial(-H(\cdot,\hat{y}))(\hat{u})$, i.e.
$\hat{y}$ solves $\inf_y [H(\hat{u},y)-\langle L\hat{u}+x_o,y\rangle]$ and $\hat{u}$ solves
$\inf_u [H(u,\hat{y})+\langle u,L^*\hat{y}\rangle]$.

Proof. 1) => 2). Let $\hat{y}$ be a solution of $D_o = *(P^*)(x_o)$. Then $P_o = \langle x_o,\hat{y}\rangle - P^*(\hat{y})$. Hence $P^*(\hat{y}) = \langle x_o,\hat{y}\rangle - P(x_o)$ and from Proposition II.3.1, 4) => 1) we have $y \in \partial P(x_o)$.

2) => 3). Immediate by definition of D_o.

3) => 4) => 1). Immediate from (9).

If $P(\cdot)$ is convex and $P(x_o) \in R$, then 1) and 4)-9) are all equivalent by Theorem II.3.2. The equivalence of 1')-5') follows from the definitions and Proposition 2.3. □

Remark. In the case that X is a normed space, condition 8) of Theorem 2.4 provides a necessary and sufficient characterization for when dual solutions exist (with no duality gap) that shows explicitly how their existence depends on what topology is used for the space of perturbations. In general the idea is to take a norm as weak as possible while still satisfying condition 8), so that the dual problem is formulated in as nice a space as possible. For example, in optimal control problems it is well known that when there are no state constraints, perturbations can be taken in e.g. an L_2 norm to get dual solutions y (and costate $-L^*y$) in L_2, whereas the presence of state constraints requires perturbations in a uniform norm, with dual solutions only existing in a space of measures.

It is often useful to consider perturbations on the dual problem; the duality results for optimization can then be applied to the dual family of perturbed problems. Now the dual problem D_o is

$$-D_o = \inf_{y \in X^*} [F^*(-L^*y,y) - \langle x_o, y\rangle].$$

In analogy with (2) we define perturbations on the dual problem by

$$D(v) = \inf_{y \in X^*} [F^*(v-L^*y,y) - \langle x_o, y\rangle], \quad v \in U^*. \tag{10}$$

Thus $D(\cdot)$ is a convex map $U^* \to \overline{R}$, and $-D_o = D(0)$. It is straightforward to calculate

$$\begin{aligned}(^*D)(u) &= \sup_v [\langle u,v\rangle - D(v)] \\ &= {}^*(F^*)(u, Lu+x_o).\end{aligned}$$

Thus the "dual of the dual" is

$$-(^*D)^*(0) = \inf_{u \in U} {}^*(F^*)(u, Lu+x_o). \tag{11}$$

In particular, if $F = {}^*(F^*)$ then the "dual of the dual" is again the primal, i.e. dom$^*D$ is the feasibility set for P_o and $-(^*D)^*(0) = P_o$. More generally, we have

$$P_o \equiv P(x_o) \geq -(^*D)^*(0) \geq D_o \equiv -D(0) \equiv {}^*(P^*)(0). \tag{12}$$

3. Duality theorems for optimization problems

Throughout this section it is assumed that U, X are HLCS's; $L: U \to X$ is linear and continuous; $x_o \in X$; and $F: U \times X \to \overline{R}$. Again, $P(x) = \inf_u F(u, Lu+x_o+x)$, $P_o = P(x_o)$, $D_o = {}^*(P^*)(x_o) = \sup_{y \in X^*} [\langle x_o, y\rangle - F^*(-L^*y, y)]$. We shall be interested in conditions under which $\partial P(x_o) \neq \emptyset$; for then there is no duality gap and there are solutions for D_o. These conditions will be conditions which insure that $P(\cdot)$ is relatively continuous at x_o with respect to affdom P, that is $P \upharpoonright$ affdom P is continuous at x_o for the induced topology on affdom P. We then have

$$\partial P(x_o) \neq \emptyset$$

$$P_o = D_o \tag{1}$$

the solution set for D_o is precisely $\partial P(x_o)$

$$P'(x_0; x) = \max_{y \in \partial P(x_o)} \langle x, y\rangle .$$

This last result provides a very important sensitivity interpretation for the dual solutions, in terms of the rate of change in minimum cost with respect to perturbations in the "state" constraints and costs. Moreover if (1) holds then Theorem 2.4, 1')-5'), gives necessary and sufficient conditions for $\hat{u} \in U$ to solve P_o.

3.1 Theorem. Assume $P(\cdot)$ is convex (e.g. F is convex). If $P(\cdot)$ is bounded above on a subset C of X, where $x_o \in \text{ri}C$ and $\text{aff}C$ is closed with finite codimension in an affine subspace M containing affdom P, then (1) holds.

Proof. From Theorem II.8.1, 1b) => 2b), we know that $P(\cdot)$ is relatively continuous at x_o. □

3.2 Corollary (Kuhn-Tucker point). Assume $P(\cdot)$ is convex (e.g. F is convex). If there exists a $\bar{u} \in U$ such that $F(\bar{u},\cdot)$ is bounded above on a subset C of X, where $L\bar{u}+x_o \in \text{ri}C$ and $\text{aff}C$ is closed with finite codimension in an affine subspace M containing affdom P, then (1) holds. In particular, if there is a $\bar{u} \in U$ such that $F(\bar{u},\cdot)$ is bounded above on a neighborhood of $L\bar{u}+x_o$, then (1) holds.

Proof. Clearly $P(x) = \inf_u F(u,Lu+x) \leq F(\bar{u},L\bar{u}+x)$, so Theorem II.8.1 applies. □

The Kuhn-Tucker condition of Corollary 3.2 is the most widely used assumption for duality [ET76]. The difficulty in applying the more general Theorem 3.1 is that, in cases where $P(\cdot)$ is not actually continuous but only relatively continuous, it is usually difficult to determine affdom P. Of course, $\text{dom } P = \bigcup_{u \in U} [\text{dom}F(u,\cdot)-Lu]$, but this may not be easy to calculate. We shall use Theorem II.8.1 to provide dual compactness conditions which insure that $P(\cdot)$ is relatively continuous at x_o.

Let K be a convex balanced $w(U,U^*)$-compact subset of U; equivalently, we could take $K = {}^oN$ where N is a convex balanced $m(U^*,U)$-0-neighborhood in U^*. Define the function $g: X^* \to \overline{R}$ by

$$g(y) = \inf_{v \in K^o} F^*(v-L^*y,y). \tag{2}$$

Note that g is a kind of "smoothing" of $P^*(y) = F^*(-L^*y,y)$ which is everywhere majorized by P^*. The reason why we need such a g is that $P(\cdot)$ is not necessarily ℓsc, which property is important for applying compactness conditions on the level sets of P^*; however $^*g$ is automatically ℓsc and $^*g$ dominates P, while at the same time $^*g$ approximates P.

3.3 Lemma. Define $g(\cdot)$ as in (2). Then $(^*g)(x) \leq \inf_u [F(u,Lu+x)+\sup_{v \in K^o} \langle u,v \rangle]$. If $F = {}^*(F^*)$, then $P(x) \leq (^*g)(x)$ for every $x \in \text{dom } P$. Moreover $\text{dom } {}^*g \supset \bigcup_{u \in \text{span } K} [\text{dom } F(u,\cdot)-Lu]$.

Proof. By definition of $^*g$, we have $(^*g)(x) = \sup_y \sup_{v \in K^o} [\langle x,y \rangle - F^*(v-L^*y,y)]$. Now for every $u \in U$ and

$y \in Y$, $F^*(v-L^*y,y) \geq \langle u,v-L^*y\rangle + \langle Lu+x,y\rangle - F(u,Lu+x) = \langle u,v\rangle + \langle x,y\rangle - F(u,Lu+x)$ by definition of F^*. Hence for every $u \in U$,

$$(^*g)(x) \leq \sup_{v\in K^o} [F(u,Lu+x)-\langle u,v\rangle]$$

$$= F(u,Lu+x) + \sup_{v\in -K^o} \langle u,v\rangle$$

$$= F(u,Lu+x) + \sup_{v\in K^o} \langle u,v\rangle$$

where the last equality follows since K^o is balanced. Thus we have proved the first inequality of the lemma.

Now suppose $F = {}^*(F^*)$ and $x \in \operatorname{dom} P$. Since K^o is a $m(U^*,U)$-0-neighborhood we have

$$(^*g)(x) = \sup_{v\in K^o} \sup_{y} [\langle x,y\rangle - F^*(v-L^*y,y)]$$

$$\geq \lim_{v\to o} \sup \sup_{y} [\langle x,y\rangle - F^*(v-L^*y,y)]$$

$$= - \lim_{v\to o} \inf \inf_{y} [F^*(v-L^*y,y)-\langle x,y\rangle],$$

where the $\lim_{v\to o} \inf$ is taken in the $m(U^*,U)$-topology.

Define $h(v) = \inf_{y} [F^*(v-L^*y,y)-\langle x,y\rangle]$, so that $(^*g)(x) \geq - \lim_{v\to o}\inf h(v)$. Now $(^*h)(u) = \sup_{v} \sup_{y} [\langle u,v\rangle - F^*(v-L^*y,y)+\langle x,y\rangle] = {}^*(F^*)(u,Lu+x) = F(u,Lu+x)$. Hence $P(x) < +\infty$ means that

$\inf_u F(u,Lu+x) < +\infty$, i.e. $*h \not\equiv +\infty$, so that we can replace the lim inf by the second conjugate:

$$(*g)(x) \geq -\liminf_{v\to o} h(v) = -(*h)*(0)$$

$$= \inf_u F(u,Lu+x) = P(x).$$

The last statement in the lemma follows from the first inequality in the lemma. For

$x \in \bigcup_{u\in \text{span } K} [\text{dom } F(u,\cdot)-Lu]$ iff $\exists u \in [0,\infty)\cdot K$ st $F(u,Lu+x) < +\infty$,

iff $\exists u$ st $\sup_{v\in K^o} \langle u,v\rangle < +\infty$ and $F(u,Lu+x) < +\infty$ (since $K = {}^o(K^o))$, iff $\exists u$ st $F(u,Lu+x) + \sup_{v\in K^o} \langle u,v\rangle < +\infty$, and this implies that $x \in \text{dom } *g$. Hence $\text{dom } *g \supset \bigcup_{u\in\text{span } K} [\text{dom } F(u,\cdot)-Lu]$. Note that dom P is given by $\bigcup_{u\in U} [\text{dom } F(u,\cdot)-Lu]$. □

3.4 <u>Theorem</u>. Assume $F = *(F^*)$, $P_o < +\infty$, and there is a $w(U,U^*)$-compact convex subset K of U such that $\text{span } K \supset \bigcup_{x\in X} \text{dom } F(\cdot,x)$. Suppose

1) $\{y \in X^*: (F^*)_\infty(-L^*y,y)-\langle x_o,y\rangle \leq 0\}$ is a subspace M;

2) $\exists$ $m(U^*,U)$-0-neighborhood N in U^*, an $x_1 \in X$, an $r_1 \in R$ such that

$\{y \in X^*: \inf_{v \in M} F^*(v-L^*y,y) - \langle x,y\rangle < r_i\}$ is nonempty and locally ${}^{\perp}M$-equicontinuous for the $w(X^*,X)$-topology.

Then affdom P is closed, $P(\cdot) \upharpoonright$ affdom P is continuous at x_o for the induced topology on affdom$P$, and (1) holds.

<u>Proof</u>. We may assume that K is balanced and contains N^o by replacing K with co bal $(K \cup N^o) = {}^o(K^o \cap -K^o \cap N \cap -N)$. Define $g(\cdot)$ as in (2). We first show that dom P = dom ${}^*g$. Now dom $P = \bigcup_{u \in U} [\text{dom } F(u,\cdot)-Lu] = \bigcup_{u \in \text{span } K} [\text{dom } F(u,\cdot)-Lu] \subset$ dom ${}^*g$ by Lemma 3.3. But also by Lemma 3.3 we have $P(x) \leq ({}^*g)(x)$ for every $x \in X$ (since dom $P \subset$ dom ${}^*g$), so dom $P \supset$ dom ${}^*g$ and hence dom P = dom ${}^*g$.

This also implies that cl dom $^*(P^*)$ = cl dom $^*g$, since cl dom $^*(P^*)$ = cl dom P by Lemma II.1.1 (note $P^* \not\equiv +\infty$ since P^* has a nonempty level set by hypothesis 2)). Hence by the definition (II.2.1) of recession functions we have $(P^*)_\infty = g_\infty = (({}^*g)^*)_\infty$. A straightforward calculation using Proposition II.2.3 and the fact that $P^*(y) = F^*(-L^*y,y)$ yields

$$g_\infty(y) = (P^*)_\infty(y) = (F^*)_\infty(-L^*y,y).$$

Now $M = \{y \in X^*: g_\infty(y)-\langle x_o,y\rangle \leq 0\} = [\text{dom } g - x_o]^-$ is a subspace, hence $M = [\text{dom } g - x_o]^{\perp}$ and $x_o + {}^{\perp}M$ is a closed affine set containing dom g. But hypothesis 2) then implies that riepi*g $\neq \emptyset$ and affdom g is closed with finite codimension in $x_o + {}^{\perp}M$, by Theorem II.8.1. Moreover by Theorem II.9.1, *g(·) is actually relatively continuous at x_o. Now ${}^{\perp}M = {}^{\perp}([\text{dom } g - x_o]^{\perp}) =$ cl affdom *g − x_0; since affdom *g is a closed subset of $x_o + {}^{\perp}M =$ cl affdom *g, we must have affdom *g = cl affdom *g. Finally, since dom P = dom *g and $P \leq {}^*g$, P(·) is bounded above on a relative neighborhood of x_o and hence is relatively continuous at x_o. □

We shall be interested in two very useful special cases. One is when U is the dual of a normed space V, and we put the $w^* = w(U,V)$ topology as the original topology on U; for then $U^* \cong V$ and the entire space U is the span of a $w(U,V)$-compact convex set (namely the unit ball in U). Hence, if $U = V^*$ where V is a normed space, and if F(·) is convex and $w(U\times X, V\times X^*)$-ℓsc, then conditions 1) and 2) of Theorem 2.4 are automatically sufficient for (1) to hold.

The other case is when X is a barrelled space, so that interior conditions reduce to core conditions for closed sets (equivalently, compactness conditions reduce to boundedness conditions in X^*). For simplicity we consider only Frechet spaces for which it is immediate that all closed subspaces are barrelled.

3.5 <u>Theorem</u>. Assume $F = {}^*(F^*)$; $P_c < +\infty$; X is a Frechet space or Banach space; and there is a $w(U,U^*)$-compact convex set K in U such that $\operatorname{span} K \supset \bigcup_{x \in X} \operatorname{dom} F(\cdot,x)$. Then the following are equivalent:

1) affdom P is closed; and $x_0 \in \operatorname{rcor} \operatorname{dom} P$, or equivalently $F(u_0, Lu_0 + x_0 + x) < +\infty \Rightarrow \exists \varepsilon > 0$ and $u_1 \in U$ st $F(u_1, Lu_1 + x_0 - \varepsilon x) < +\infty$.

2) $\{y \in X^*: (F^*)_\infty(-L^*y, y) - \langle x_0, y\rangle \leq 0\}$ is a subspace M; and there exists a $m(U^*,U)$-0-neighborhood N in U^*, an $x_1 \in X$, an $r_1 \in R$ such that $\{y \in X^*: \inf_{v \in N} F^*(v - L^*y, y) - \langle x_0, y\rangle < r_1\}$ is nonempty and $w(X^*,X)/M$-locally bounded.

If either of the above holds, then $P(\cdot) \upharpoonright \operatorname{affdom} P$ is continuous at x_0 for the induced metric topology on affdom P and (1) holds.

<u>Proof</u>. We first note that since $\operatorname{span} K \supset \bigcup_{x \in X} \operatorname{dom} F(\cdot,x)$ we have as in Theorem 3.4 that $\operatorname{dom} P = \operatorname{dom} {}^*g$ and $g_\infty(y) = (P^*)_\infty(y) = (F^*)_\infty(-L^*y, y)$.

1) => 2). We show that $g(\cdot)$ is relatively continuous at x_o, and then 2) will follow. Now $\text{dom}\, P = \text{dom}\, {}^*g$, so $x_o \in \text{rcordom}\, P$. Let $W = \text{affdom}\, P - x_o$ be the closed subspace parallel to dom P, and define $h: W \to \overline{R}: w \to {}^*g(x_o+w)$. Since ${}^*g$ is ℓsc on X, h is ℓsc on the barrelled space W. But $0 \in \text{core dom}\, h$ (in W), hence h is actually continuous at 0 (since W is barrelled), or equivalently ${}^*g$ is relatively continuous at x_o. Applying Theorem II.9.1 we now see that M is the subspace $W^{\perp}$; the remainder of 2) then follows from Theorem II.8.1, since $g(y) = \inf_{v \in M} F^*(v-L^*y,y) \geq ({}^*g)^*(y)$.

2) => 1). Note that ${}^{\perp}M$ is a Frechet space in the induced topology, so $w(X^*,X)/M$-local boundedness is equivalent to $w(X^*,X)/M$-local compactness. But now we may siply apply Theorem 3.4 to get $P(\cdot)$ relatively continuous at x_o and affdom P closed; of course, 1) follows. □

3.6 <u>Corollary</u>. Assume $P_o < +\infty$; $U = V^*$ where V is a normed space; X is a Frechet space or Banach space; $F(\cdot)$ is convex and $w(U\times X, V\times X^*)$-$\ell sc$. Then the following are equivalent:

1) affdom P is closed; and $x_o \in \text{rcordom}\, P$, or

equivalently $F(u_o, Lu_o+x_o+x) < +\infty \Rightarrow \exists \varepsilon > 0$ and $u_1 \in U$ st $F(u_1, Lu_1+x_o-\varepsilon x) < +\infty$.

2) $\{y \in X^*;\ (F^*)_\infty(-L^*y, y) - \langle x_o, y\rangle \leq 0\}$ is a subspace M; and there is an $\varepsilon > 0$, an $x_1 \in X$, an $r_1 \in R$ such that $\{y \in X^*: \inf_{|v|\leq\varepsilon} F^*(v-L^*y, y) - \langle x_o, y\rangle < r_1\}$ is nonempty and $w(X^*, X)/M$-locally bounded.

If either of the above holds, then $P(\cdot)\restriction$ affdom P is continuous at x_o for the induced metric topology on affdom P and (1) holds.

Proof. Take K to be the closed unit ball in $U = V^*$; then K is $w(U,V)$-compact and span $K = U$. The corollary then follows from Theorem 3.5. □

In the case that affdom P is the entire space X, we have the following useful corollary. Note that conditon 1) considerably generalizes the Kuhn Tucker condition of Corollary 3.2.

3.7 Corollary. Assume $P_o < +\infty$; $U = V^*$ where V is a normed space; X is a Frechet space or Banach space; $F(\cdot)$ is convex and $w(U\times X, V\times X^*)$-$\ell$sc. Then the following are equivalent:

1) $x_o \in$ cordom $P \equiv$ cor $\bigcup_{u\in U} [\text{dom } F(u,\cdot) - Lu]$

2) $\{y \in X^*: (F^*)_\infty(-L^*y,y)-\langle x_o,y\rangle \leq 0\} = \{0\}$;

and there is an $\varepsilon > 0$, an $x_1 \in X$, an $r_1 \in R$ such that $\{y \in X^*: \inf_{|v|\leq\varepsilon} F^*(v-L^*y,y)-\langle x_o,y\rangle < r_1\}$

is nonempty and $w(X^*,X)$-locally bounded.

3) there is an $\varepsilon > 0$, an $r_o \in R$ such that

$\{y \in X^*: \inf_{|v|\leq\varepsilon} F^*(v-L^*y,y)-\langle x_o,y\rangle < r_o\}$ is

nonempty and $w(X^*,X)$-bounded.

If any of the above holds, then $P(\cdot)$ is continuous at x_o and (1) holds.

Proof. Immediate from Corollary 3.6 with affdomP = X. □

We can also apply these theorems to perturbations on the dual problem to get existence of solutions to the original problem P_o and no duality gap $P_o = D_o$. As an example, we give the dual version of Corollary 3.6.

3.8 Corollary. Assume $P_o > -\infty$; $U = V^*$ where V is a Frechet space or Banach space; X is a normed space; $F(\cdot)$ is convex and $w(U\times X, V\times X^*)$-ℓsc. Suppose $\{u \in U: F_\infty(u,Lu+x_o) \leq 0\}$ is a subspace M, and there is an $\varepsilon > 0$, an $x_1 \in X$, an $r_1 \in R$ such that $\{u \in U: \inf_{|x|\leq\varepsilon} F(u,Lu+x_o+x) < r_1\}$ is nonempty and $w(U,U^*)/M$-locally compact. Then $P_o = D_o$ and P_o has solutions.

Proof. Apply Corollary 3.6 to the dual problem (2.10). □

IV. Minimum Norm and Spline Problems and a Separation Theorem

Abstract. Results in duality theory for optimization problems are applied to minimum norm and spline problems and improve previous existence results, as well as expressing them in a duality framework. Related results include conditions for the sum of two closed convex sets to be closed leading to an extended separation principle for closed convex sets.

1. Minimum norm extremals and the spline problem

We apply our results on the relationship between continuity points of convex functionals and locally equicontinuous level sets of conjugate functionals to derive a duality principle for minimum norm problems. It is well known, for example, that in a normed space X the minimum distance from a point x_o to a nonempty convex set C is equal to the maximum of the distances from the point to the closed hyperplanes separating the point and the convex set C. In other words,

$$\inf_{x \in C} |x-x_o| = \max_{y \in B} \inf_{x \in C} (x-x_o)y,$$

where B denotes the closed unit ball in X^* and the maximum on the RHS is attained by some $\hat{y} \in B$. This also characterizes the minimum-norm solution: $\hat{x} \in C$ attains the infimum on the LHS iff $\hat{x}-x_o$ is aligned with some $\hat{y} \in B$, i.e. $|\hat{x}-x_o| = (\hat{x}-x_o)\hat{y}$; and it is easy to see that such solutions exist whenever C is closed and X is either reflexive or the dual of a separable normed space. We generalize these results to include the spline problem and also develop sufficient conditions for a solution to the minimum norm problem to exist.

We consider the following generalized spline problem. Let U, X be normed linear spaces, C a nonempty convex subset of U, T a bounded linear map from U into X; then for $x \in X$, $P(x)$ is the minimum norm problem

$$P(x) = \inf_{u \in C} |Tu+x|.$$

We consider perturbations in x, i.e. calculate the conjugate of $P(\cdot)$, and develop a dual problem $*(P^*)(x)$. We then take perturbations on the dual problem to derive existence conditions for the original problme $P(x)$.

To calculate the dual problem, define $f(u) = \delta_C(u)$ and $g(x) = |x|$; then $P(x) = \inf_u [f(u)+g(Tu+x)]$. Now f^* is just the support function $\delta_C{}^*$ of C and g^* is just the indicator δ_B of the ball $B = \{y \in X^*: |y| \leq 1\}$; hence $P^*(y) = f^*(-T^*y) + g^*(y) = \sup_{u \in C} u(-T^*y) + \delta_B(y)$ $= \delta_B(y) - \inf_{u \in C} (Tu)y$. Thus, the dual problem is $*(P^*)(x) = \sup_y [xy - P^*(y)] = \sup_{y \in B} \inf_{u \in C} (Tu+x)y$. Clearly $P(x) \geq *(P^*)(x)$, with equality iff $P(\cdot)$ is lsc at x.

We now define perturbations on the dual problem. For each $x \in X$, let $D_x(\cdot)$ be the functional on U^* given by

$$D_x(v) = \inf_y [f^*(v-T^*y)+g^*(y)-xy] = \inf_{y \in B} [xy + \sup_{u \in C} u(v+T^*y)].$$

Of course, for $v = 0$ $D_x(v)$ is just the dual problem (with a change in sign to make $D_x(\cdot)$ convex): $D_x(0) = -*(P^*)(x)$. To calculate the conjugate of perturbations on the dual problem, we have

$$(*D_x)(u) = *(f^*)(u) + *(g^*)(Tu+x) = \delta_{clC}(u) + |Tu+x|$$

where the norm $g(\cdot) = |\cdot|$ is weakly lsc so $g = *(g^*)$. Hence, the dual of the dual is

$$(*D_x)^*(0) = -\inf_{u \in clC} |Tu+x|,$$

which is again (minus) the primal problem $P(x)$ if C is closed. In general we have

$$P(x) \geq -(*D_x)^*(0) \geq -D_x(0) \equiv *(P^*)(x).$$

We are now ready to state the main results. We denote the null space of T by $N \equiv T^{-1}(\{0\})$, and for $r > 0$ we write $N^r \equiv \{u \in U: d(u,N) < r\} \equiv N + r\cdot\mathring{B}$ where $\mathring{B}$ is the open unit ball in U.

<u>Theorem 1</u>. Let U,X be normed linear spaces, C a nonempty convex subset of X, T a bounded linear map from U into X. For $x \in X$, let $P(x)$ be the minimum norm problem

$$P(x) = \inf_{u \in C} |Tu+x|$$

and consider the dual problem

$$*(P^*)(x) = \max_{y \in B} \inf_{u \in C} (Tu+x)y.$$

Then we always have $P(x) = *(P^*)(x)$, where the maximization in $*(P^*)$ is attained by some $\hat{y} \in B$. Moreover $\hat{u} \in C$ solves $P(x)$ iff there is some $\hat{y} \in B$ for which $|T\hat{u}+x| = (T\hat{u}+x)\hat{y}$, in which case $\hat{y}$ solves $*(P^*)(x)$. Sufficient conditions for $P(\cdot)$ to have minimizing solutions $\hat{u} \in C$ are:

1) U is reflexive, C is closed, TU is closed.

2) $C_\infty \cap N$ is a subspace M

3) $C \cap N^r/M$ is nonempty and weakly locally bounded in U/M, for some $r > 0$.

Before proving the theorem, we make a few remarks about the existence theorems. First, some authors do not assume that U is reflexive, but that X is reflexive and N is finite dimensional. However this actually implies that U is reflexive, since U/N is topologically isomorphic to TU, a closed subspace of X. In fact, when TU is closed we have U reflexive iff N and U/N are reflexive iff N and TU are reflexive, and the latter is certainly true if N and X are reflexive.

Secondly, we examine the condition 3). By $C \cap N^r/M$ we of course mean $\{u+M \in U/M: u \in C \cap N^r\}$. It is straightforward to show that if $C \cap N^r/M$ is locally bounded for some $r > 0$ sufficiently large so that $C \cap N^r$ is nonempty, then it is actually true that $C \cap N^r/M$ is locally bounded for every $r > 0$ (argue along the lines of Proposition II.1.4). Thus 3) is really equivalent to

3') $C \cap N^r/M$ is weakly locally bounded in U/M for every $r > 0$. By weakly locally bounded in U/M we mean locally bounded in the topology $w(U/M, M^{\perp})$, where U/M is a normed space and $M^{\perp}$ is norm-congruent to $(U/M)^*$ (note that $M = C_\infty \cap N$ is closed since C_∞ and N are closed). Since the weak topology $w(U/M, M^{\perp})$ on the quotient normed space U/M is the same as the quotient $w(U,U^*)/M$ of the weak topology $w(U,U^*)$ on U, we see that $C \cap N^r/M$ is weakly locally bounded in U/M iff $C \cap N^r$ is weakly locally M-equicontinuous in U, that is iff there is a finite subset F of U^* and a $c_o \in C \cap N^r$ such that $\sup_{u \in C \cap N^r \cap (c_o + {}^oF)} d(u,M) < +\infty$,

(we note that $C \cap N^r$ is locally $M^{\perp}$-equicontinuous at every point if it is at a single point c_o, as in Proposition II.1.4). Thus 3) is equivalent to

3") $\exists$ finite $F \subset X^*$, $r > 0$, $c_o \in C \cap N^r$ st

$$\sup_{u \in C \cap N^r \cap (c_o + {}^oF)} d(u,M) < +\infty,$$

and implies the existence of such an F for every $r > 0$, $c_o \in C \cap N^r$. Finally, it can also be shown that 3") is also equivalent to

> 3''') there is a finite subset F of U^* and a $c_o \in C$ such that every norm-convergent sequence $u_i + n_i$, for $u_i \in C \cap (c_o + {}^oF)$ and $n_i \in N$, has $d(n_i, M)$ bounded.

These are certainly true if $C \cap N^r$ is itself weakly locally bounded or N is finite dimensional. And they are certainly true if $C \cap N^r$ is actually $M^\perp$-equicontinuous (not just $w(\mathbf{U},\mathbf{U}^*)$-locally so), e.g. if $C \cap N^r$ is bounded or C is bounded or C is $M^\perp$-equicontinuous (i.e. $\sup_{u \in C} d(u,M) < +\infty$). As in 3'''), we note that $C \cap N^r$ is M-equicontinuous iff every norm-convergent sequence $x_i + n_i$, for $x_i \in C$ and $n_i \in N$ has $d(n_i, M)$ bounded.

<u>Proof of the theorem</u>. We first note that $P(\cdot)$ is a finite, convex, and norm-continuous functional on X. For, it is clearly convex since C is convex, T is

linear, and the norm is convex; and if c_o is any element of C then

$$P(x) \leq |Tc_o| + |x|$$

and $P(\cdot)$ is bounded above by a continuous function. Thus we immediately have $P(x) = *(P^*)(x) \equiv -D_x(0)$, and the subgradient $\partial P(x)$ is nonempty and $w(X^*,X)$-compact. But the elements of $\partial P(x)$ are just those $\hat{y} \in X^*$ which attain the supremum in $\sup_y[xy-P^*(y)] = *(P^*)(x)$, so that $*(P^*)(x) \equiv -D_x(0)$ has solutions $\hat{y} \in B$. This proves the first part of the theorem.

To obtain existence of solutions for $P(x)$, we must show that $\partial D_x(0) \neq \emptyset$, for $\partial D_x(0)$ is precisely the solution set of P. We shall actually show that under the conditions 1) to 3) $D_x(\cdot)$ is norm continuous at 0 on affdom$D(\cdot) \equiv M^{\perp}$ in U^*. We first note that $D_x(\cdot)$ is convex and $w(U^*,U)$-lsc at 0 for every $x \in X$; for, both $D_x(0)$ and $(*D_x)^*(0)$ are pinched between the values $-P(x)$ and $-*(P^*)(x)$, so by the equality of the latter we must have $D_x(0) = (*D_x)^*(0)$. (In fact, more is true. If we define a new primal problem $P_v(x) = \inf_{u \in C}(|Tu+x|-uv)$ we get a dual problem $*(P_v^*)(x) = -D_x(v)$, and the same argument yields $D_x(v) = (*D_x)^*(v)$ for <u>every</u> $v \in U^*$, $x \in X$.)

Thus to show that $\partial D_x(0)$ is nonempty, we must show that $({}^*D_x)^*(\cdot)$ is relatively continuous at 0 in the norm topology (which is the $m(U^*,U)$ topology on U^* when U is reflexive), or equivalently (by Theorem II.3.2) that the level sets of $({}^*D_x)(\cdot)$ have weakly locally bounded (equicontinuous ≡ weakly bounded in a reflexive Banach space) image in the quotient space U/M, where $M = \{u: ({}^*D_x)_\infty(u) \leq 0\}$ is required to be a subspace. Now $({}^*D_x)(u) = |Tu+x| + \delta_C(u)$, and since $({}^*D_x)(\cdot)$ is convex and weakly lsc we have the easy calculation

$$({}^*D_x)_\infty(u) = \sup_{t>0} \frac{{}^*D_x(c_o+tx) - {}^*D(c_o)}{t} = |Tu| + \delta_{C_\infty}(u).$$

Thus we require $M = N \cap C_\infty$ to be a subspace as in 3). The level sets of $({}^*D_x)(\cdot)$ are precisely $\{u: ({}^*D_x)(u) \leq r\} = C \cap T^{-1}(-x+rB)$ (i.e. those $u \in C$ for which Tu is within r of $-x$), for $r > 0$. To insure that we take r sufficiently large so that the level set is nonempty, we take $r > |Tc_o+x|$ for any $c_o \in C$ and for convenience $r > |x|$. Then the level set is contained in $C \cap T^{-1}(2rB)$. Now T has closed range, so there is an ε sufficiently small so that

$$\varepsilon d(u,N) \leq |Tu| \leq \varepsilon^{-1} d(u,N)$$

(this merely states that U/N is topologically isomorphic to TU under the mapping T as taken on U/N). But this means that the set $T^{-1}(2rB)$ is certainly contained in the set $N + \frac{3r}{\varepsilon}\cdot\mathring{B} \equiv N^{3r/\varepsilon}$. Thus, it is sufficient to require that $C \cap N^{3r/\varepsilon}$ have weakly locally bounded image in U/M. Noting that $C \cap N^r/M$ is weakly locally bounded for every $r > 0$ iff it is locally bounded for some $r > \inf\{t: C \cap N^t \neq \emptyset\}$ we have the condition 3) or 3'). □

Remarks. If U is not reflexive, it is still true that $P(x)$ has a solution if the other conditions hold and $(C \cap N^r)/M$ is nonemtpy and weakly locally compact in U/M for some $r > 0$. Of course, weak compactness in a nonreflexive space may be difficult to characterize.

It is also possible to prove similar existence results when U,X are the duals of separable normed spaces, T is $w(U,U^*)-w(X,{}^*X)$-sequentially continuous, and C is $w^* \equiv w(U,{}^*U)$-sequentially closed.

Since the spline existence conditions for $P(x)$ do not depend on the point x, we see that we have actually developed a sufficient condition for TC to be closed in X, or equivalently (when TU is closed) for $C+N$ to be closed in U. The standard approach to such spline problems is to apply Dieudonne's theorem [D66] for the closedness of the sum of two closed convex sets, namely that C be locally compact or N finite dimensional (that is, locally compact) and that $C_\infty \cap N = \{0\}$. Our conditions are much weaker, namely that $C_\infty \cap N$ be a subspace M, and that $C \cap N^r/M$ be weakly locally compact in U/M (local compactness in a HLCS always implies weak local compactness, as noted in the remarks following Corollary II.1.10). In particular, the null space N of T need not be finite-dimensional, and $C_\infty \cap N$ need not reduce to $\{0\}$.

Example, N infinite dimensional.

Let $U = H_p^1 = \{u(\cdot): u(\cdot)$ is abs cont on $[0,1]$ and $\dot{u}(\cdot) \in L_p[0,1]\}$, where $1 < p < \infty$. We shall take the cost to depend on the derivative $\dot{u}$ only over the intervals $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$, with no derivative cost on $[\frac{1}{3},\frac{2}{3}]$. Hence take the linear operator to be $T: H_p^1 \to L_p: u \to Tu$ where

$$(Tu)(t) = \begin{cases} \dot{u}(t) & t \in [0,\frac{1}{3}] \cup [\frac{2}{3},1] \\ 0 & t \in (\frac{1}{3},\frac{2}{3}) \end{cases}.$$

The constraint set is $C = \{u \in H_p^1: u(0) = 1,\ u(\frac{1}{3}) = 0,\ u(\frac{2}{3}) = 1,\ u(1) = 0,$ and $|\dot{u}(t)| \leq 1$ for $t \in [\frac{1}{3},\frac{2}{3}]\}$; note that there are constraints on the derivative for $t \in [\frac{1}{3},\frac{2}{3}]$, so that the null space of T is truely infinite dimensional.

Clearly $U = H_p^1$ is reflexive, C is closed convex, T has closed range on H_p^1 (TU is congruent to $L_p[0,\frac{1}{3}] \times L_p[\frac{2}{3},1]$), $N = \{u \in H_p^1: u \text{ is constant on } [0,\frac{1}{3}]$ and constant on $[\frac{2}{3},1]\}$, $C_\infty = \{u \in H_p^1: u(0)=u(1)=0, u(t) \equiv 0$ for $t \in [\frac{1}{3},\frac{2}{3}]\}$. Thus $N \cap C_\infty = \{0\}$ is a subspace. And

$C \cap N^r = \{u \in H_p^1: u \in C \text{ and } d(u,N) < r\} \subset$

$\{u: u(0) = 1, u(\frac{1}{3}) = 0, u(\frac{2}{3}) = 1, u(1) = 0, |\dot{u}(t)| \leq 1$ on $t \in [\frac{1}{3},\frac{2}{3}]$, and $|u(t)| \leq$ some constant function on $[0,\frac{1}{3}] \cup [\frac{2}{3},1]\}$ which is bounded because of the derivative and endpoint constraints. Thus, the existence conditions of Theorem 1 are satisfied and the minimum norm problem $P(x) = \inf_{u \in C} |Tu+x|$ has solutions.

Example, $C_\infty \cap N$ not necessarily $\{0\}$.

Let U,X be reflexive Banach spaces, $T: U \to X$ bounded linear with closed range, and C a closed affine subset of U. Then C_∞ is the subspace $C-C$ parallel to C, hence condition 2) is always satisfied. If $C_\infty \cap N$ is finite dimensional (e.g. N or C is finite dimensional) then the minimum norm problem $P(x) = \inf_{u \in C} |Tu+x|$ has solutions. Alternatively, if C is a finite-codimensional closed flat $C = \bigcap_{k=1}^{n} v_k^{-1}(r_k)$ for $v_k \in U^*$, $r_k \in R$, then

$C \cap N/C_\infty \cap N$ is a finite-dimensional affine set in

$U/C_\infty \cap N$ and hence $C \cap N^r/C_\infty \cap N$ is weakly locally bounded, so again spline solutions exist.

2. On the separation of closed convex sets

The spline existence conditions developed in Theorem 1 essentially constitute a sufficient condition for the sum of two closed convex sets to be closed, namely the sum of the constraint set C and the null space N. We can use the same techniques to develop a general crtierion for the sum of two closed convex sets to be closed in a reflexive Banach space; this extends Dieudonne's theorem [D66] in this context and leads to a separation principle. In what follows we define $B^{\varepsilon} = \{x \in X: \inf_{b \in B} |x-b| < \varepsilon\} =$ $B + \varepsilon \cdot$(open unit ball), for $\varepsilon > 0$ and $B \subset X$.

Theorem 2 Let X be a reflexive Banach space with A,B closed convex subsets of X satisfying:

1) $A_{\infty} \cap B_{\infty}$ is a subspace M

2) $A \cap B^{\varepsilon}$ is nonempty and $w(X,X^*)/M$-locally bounded, for some $\varepsilon > 0$.

Then A-B is closed. In particular, if A and B are disjoint then they can be strongly separated, i.e. there exists $y \in X^*$ such that $\inf_{a \in A} ay > \sup_{b \in B} by$.

Proof. We may assume that A,B are nonempty. Suppose

$z \in A-B$; we show that $z \notin cl(A-B)$, or equivalently that $\inf_{a\in A} \inf_{b\in B} |a-b-z| > 0$. By translation we may assume that $z = 0$. Define the convex lsc function $f: X \to \bar{R}$ by

$$f(x) = \delta_A(x) + \inf_{b\in B} |x-b|.$$

Then f^* is given by

$$f^*(y) = \sup_{a\in A} \sup_{b\in B} [ay-|a-b|].$$

We show that conditions 1) and 2) are sufficient to prove that $f^*(\cdot)$ is relatively continuous at 0. By Theorems II.9.1, 7) => 1), and II.8.1, 7) => 2), it suffices to show that a level set of f is locally bounded in the topology $w(X,X^*)/M$, where $M = \{x: f_\infty(x) \leq 0\} = A_\infty \cap B_\infty$ is required to be a subspace. But the level sets of f are precisely $\{x: f(x) \leq \varepsilon\} = A \cap B^\varepsilon$ for $\varepsilon > 0$, so that 1) and 2) are the required conditions.

Thus $f^*(\cdot)$ is relatively continuous at 0, and consequently $\partial f^*(0) \neq \emptyset$. This means that there is an $x_o \in \partial f^*(0)$, or equivalently that $0 \in \partial f(x_o)$, i.e. x_o solves $\inf_x f(x) = \inf_{x\in A} \inf_{b\in B} |x-b|$. Hence $\inf_{x\in A} \inf_{b\in B} |x_o-b| = \inf_{b\in B} |x_o-b| > 0$, where the last inequality follows since $x_o \notin B$ (recall $A \cap B = \emptyset$ since $0 \notin A-B$) and B is closed. Note that since $0 \in cl(A-b)$, A and B can be strictly separated. □

If $A_\infty \cap B_\infty$ is a subspace and A is locally bounded, then conditions 1) and 2) follow immediately. In Dieudonne's theorem [D66] $A_\infty \cap B_\infty$ is required to be $\{0\}$, with A locally bounded.

Chapter V
(pages 135 to 164)
was removed from thesis.

VII. Optimal Quantum Detection

Abstract. Duality techniques are applied to the problem of specifying the optimal quantum detector for multiple hypothesis testing. Existence of the optimal detector is established and necessary and sufficient conditions for optimality are derived.

1. Introduction

The mathematical characterization of optimal detection in the Bayesian approach to statistical inference is a well-known result in the classical theory of hypothesis testing. In this paper we consider detection theory for quantum systems.

In the classical formulation of Bayesian hypothesis testing it is desired to decide which of n possible hypotheses $H_1,\ldots,H_n$ is true, based on observation of a random variable whose probability distribution depends on the several hypotheses. The decision entails certain costs that depend on which hypothesis is selected and which hypothesis corresponds to the true state of the system. A decision procedure or strategy prescribes which hypothesis is to be chosen for each possible outcome of the observed data; in general it may be necessary to use a randomized strategy which specifies the probabilities with which each hypothesis should be chosen as a function of the observed data. The detection problem is to determine an optimal decision strategy.

In the quantum formulation of the detection problem, each hypothesis H_j corresponds to a possible state ρ_j of the quantum system under consideration. Unlike the classical situation, however, it is not possible to measure all relevant variables associated with the state of the system and to specify meaningful probability distributions for the resulting values. For the quantum detection problem it is necessary to specify not only the procedure for processing the experimental data, but also what data to measure in the first place. Hence the quantum detection problem involves determining the entire measurement process, or, in mathematical terms, determining the probability operator measure corresponding to the measurement process.

We now formulate the quantum detection problem. Let H be a separable complex Hilbert space corresponding to the physical variables of the system under consideration. There are n hypotheses $H_1,\ldots,H_n$ about the state of the system, each corresponding to a different densit operator ρ_j; every ρ_j is a nonnegative definite selfadjoint trace-class operator on H with trace 1 and is the analog of the distribution functions in the classical problem. Let S denote the set $S = \{1,\ldots,n\}$. A general decision strategy is determined by a probability operator measure (POM) $m: 2^S \to \mathcal{L}_s(H)_+$; in this case the POM effecting the decision needs only n components $m_1,\ldots,m_n$ where each m_j is a positive selfadjoint bound linear operator on H and

$$\sum_{i=1}^{n} m_i = I. \qquad (1)$$

The measurement outcome is an integer $i \in S$; the conditional probability that the hypothesis H_i is chosen when the state of the system is ρ_j is given by

$$\Pr\{i|j\} = \operatorname{tr}(\rho_j m_i) \quad i,j=1,\ldots,m. \tag{2}$$

We remark that it is crucial here to formulate the problem in terms of general probability operator measures rather than resolutions of the identity. For example, an instrument which simply chooses an arbitrary hypothesis with probability 1/n without even interacting with the system corresponds to a measurement process with the POM given by

$$m_j = I/n;$$

these are certainly not projections.

We denote by C_{ij} the cost associated with choosing hypothesis H_i when H_j is true. For a specified decision procedure effected by the POM $\{m_1,\ldots,m_n\}$, the risk function is the conditional expected cost given that the system is in the state ρ_j, i.e.

$$R_m(j) = \operatorname{tr}[\rho_j \sum_{i=1}^{n} C_{ij} m_i].$$

If now μ_j specifies a prior probability for hypothesis H_j, the Bayes cost is the posterior expected cost

$$R_m = \sum_{i=1}^{n} R_m(j)\mu_j = \text{tr} \sum_{i=1}^{n} f_i m_i \qquad (3)$$

where f_i is the selfadjoint trace-class operator

$$f_i = \sum_{j=1}^{n} c_{ij}\mu_j\rho_j \qquad i = 1,\ldots,n. \qquad (4)$$

The quantum detection problem is to find $m_1,\ldots,m_n$ so as to minimize (3) subject to the constraint (1) and subject to the condition that the operators m_j be selfadjoint and nonnegative definite, $m_j \geq 0$.

The minimization problem as formulated above is an abstract linear programming problem, where the positive cone is the set of all selfadjoint nonnegative definite bounded linear operators $(m_1,\ldots,m_n) \in (\mathcal{L}_s(H)_+)^n$. We shall pose this problem in a duality framework, construct a dual problem, and give necessary and sufficient conditions which the solution must satisfy. Moreover we shall show that solutions exist, although they need not be unique.

2. The finite dimensional case

It is interesting to explicitly construct the form of the problem in the finite dimensional case. This will not not only exhibit the primary features of the problem, but also show why the usual linear programming techniques do not apply because of the nature of the positive cone. Moreover the finite dimensional case is of interest because it includes the situation where the quantum states $\rho_1,\ldots,\rho_n$ are pure states.

Hence, for this section only, we shall take H to be C^q where q is a positive integer. The compact, trace-class, and bounded selfadjoint operators are all complex $q\times q$ self-adjoint matrices, which we may identify with the real linear space R^{q^2}. For example, in the case $H = \mathbb{C}^2$ we may identify every self-adjoint operator $f \in \mathcal{L}_s(\mathbb{C}^2)$ with an element of R^4 by

$$f = \begin{bmatrix} f^1 & f^2+if^3 \\ f^2-if^3 & f^4 \end{bmatrix} \Longleftrightarrow f = (f^1,f^2,f^3,f^4) \in R^4. \qquad (5)$$

To save notation, we shall write out the problem explicitly only for $H = \mathbb{C}^2$; the general finite dimensional case is an easy extension.

The quantum detection problem for n hypotheses is, from (3),

$$P = \inf\left\{\sum_{j=1}^{n} \mathrm{tr}(m_j f_j) : m_1,\ldots,m_n \in \mathcal{L}_s(\mathbb{C}^2)_+, \sum_{j=1}^{n} m_j = I\right\}$$

where I is the identity operator on $H = \mathbb{C}$ and each m_j or $f_j \in \mathcal{L}_s(\mathbb{C}^2)$ is identified with an element m_j or $f_j = (f_j^1, f_j^2, f_j^3, f_j^4) \in R^4$ as in (5). The positive cone $\mathcal{L}_s(\mathbb{C}^2)_+$ consists of the nonnegative definite matrices; $f \in \mathcal{L}_s(H)_+$ means that $f^1 \geq 0$, $f^4 \geq 0$, and $f^1 f^4 \geq (f^2)^2 + (f^3)^2$. Hence, if we define the positive cone $K = \mathcal{L}_s(C)_+ \subset R^4$ by

$$K = \{m \in R^4 : m^1 \geq 0,\ m^4 \geq 0,\ m^1 m^4 \geq (m^2)^2 + (m^3)^2\} \tag{6}$$

then the problem becomes

$$P = \inf\left\{\sum_{j=1}^{n} \sum_{i=1}^{4} m_j^i f_j^i : (m_1,\ldots,m_n) \in K^n \text{ and}\right.$$

$$\left.\sum_{j=1}^{n} m_j^1 = 1 = \sum_{j=1}^{n} m_j^4,\ \sum_{j=1}^{n} m_j^2 = 0 = \sum_{j=1}^{n} m_j^3\right\}. \tag{7}$$

Note here that the duality between $\mathcal{L}_s(H)$ and $\mathcal{T}_s(H)$ given by $\langle f,m\rangle = \mathrm{tr}(fm)$ has simply reduced to the usual inner product $\sum_{i=1}^{4} f^i \cdot m^i$ for $f \in \mathcal{T}_s(\mathbb{C}^2) \cong R^4$ and $m \in \mathcal{L}_s(\mathbb{C}^2) \cong R^4$. The problem is in the form of a finite dimensional linear programming problem except that the closed convex cone K of "positive" vectors is no longer <u>polyhedral</u>, that is an intersection of a <u>finite</u> number of closed halfspaces. In the next section we shall define the dual problem by taking perturbations with respect to the constraint $\sum_{j=1}^{n} m_j = I \in R^4$; the dual problem here is thus a minimization problem over R^4. In general, for a linear programming problem of the form $\inf_m\{\langle f,m\rangle: m \in Q,\ Am = g\}$ where Q is a closed convex cone and A is a continuous linear map, the dual problem is given by $\sup_u\{\langle g,y\rangle: f - A^*y \in Q^+\}$. We do not derive this here but simply state that the dual problem for (7) is

$$D = \sup\{y^1+y^4: y \in R^4,\ f_j - y \in K^+ \ \forall j=1,\ldots,n\}$$

where the dual positive cone K^+ is (by straightforward but tedious calculation)

$$K^+ \equiv \{y \in R^4: \inf_{m \in K} \sum_{i=1}^{4} m^i y^i \geq 0\} =$$

$$\{y \in R^4: y^1 \geq 0,\ y^4 \geq 0,\ 4y^1y^4 \geq (y^2)^2+(y^3)^2\}. \quad (8)$$

Hence, the explicit form of the constraints for the dual problem is

$$y^1 \leq f_j^{\,1};\ y^4 \leq f_j^{\,4};\ 4(f_j^{\,1}-y^1)(f_j^{\,4}-y^4) \geq (y^2-f_j^{\,2})^2+(y^3-f_j^{\,3})^2$$

for every $j = 1,\ldots,n$. Clearly, the usual duality theory for finite dimensional linear programming is not applicable.

Because of the explicit nature of the linear constraint $\sum_{j=1}^{n} m_j = I$ in the original problem, we shall see that duality theory does work for this problem. In general, however, it is possible to have a finite duality gap for linear programming problems with positive cone of type K. We construct such an example now.

3. A linear programming problem with non-polyhedral cone which has a duality gap

We consider a linear programming problem of the form (7) with $n = 2$ (that is, a problem in R^8), except that we change the linear equality constraint. Define the closed convex cone K in R^4 by (6); K^+ is given by (8). Let $u = (m_1^{\ 1}, m_1^{\ 2}, m_1^{\ 3}, m_1^{\ 4}, m_2^{\ 1}, m_2^{\ 2}, m_2^{\ 3}, m_2^{\ 4})$ represent a vector in R^8 and define the problem P_1 by

$$P_1 = \inf\{u^{(2)} : u \quad K\times K,\ Au = (0,-1,0,0)\}$$

where A is the linear map

$$Au = (u^1-u^6,\ u^2-u^8,\ -u^5,\ u^3+u^7).$$

If $y \in R^4$ is a dual variable, then A^*y is given by

$$A^*y = (y^1,\ y^2,\ y^4,\ 0,\ -y^3,\ -y^1,\ y^4,\ -y^2).$$

The dual problem is

$$D_1 = \sup\{-y^2 : (0,1,0,0,0,0,0,0) - A^*y \in K^+\times K^+\}.$$

First, let's solve the primal problem. From the constraint $Au = (0,-1,0,0)$ we have $u^5 = 0$; but $(u^5,u^6,u^7,u^8) \in K$ so $u^5 = u^6 = u^7 = 0$. Again from $Au = (0,-1,0,0)$ we now have $u^1 = u^6 = 0$, which since

$(u^1,u^2,u^3,u^4) \in K$ implies $u^2 = 0$. Thus $u^2 = 0$ for every feasible u; in fact every feasible u looks like $u = (0,0,0,u^4,0,0,0,1)$ with $u^4 \geq 0$, and $P_1 = 0$.

Now consider the dual problem. The constraints are $(-y^1, 1-y^2, -y^4, 0) \in K^+$ and $(y^3, y^1, -y^4, +y^2) \in K^+$. The first constraint immediately implies $y^2 = 1$; in fact every feasible y is of the form $y = (y^1, 1, y^3, 0)$ where $y_1 \leq 0$ and $y_3 \geq (y^1)^2/4$. Hence $D_1 = -1$ and there is a finite duality gap $P_1 - D_1 = 1$.

Where does the difficulty arise? If $P = \inf\{cu: u \in Q, Au=b\}$ is an abstract linear program, where Q is a closed convex cone in a Banach space U and A is a bounded linear map from U into a Banach space Z, then P has solutions (assuming P is feasible) and P = D where $D = \sup\{yb: y \in Z^*, c-A^*y \in Q^+\}$ whenever $\begin{bmatrix} c \\ A \end{bmatrix}(Q)$ is closed in R×Z, or equivalently (in the case that A has closed range) whenever $Q + \mathcal{N}\begin{bmatrix} c \\ A \end{bmatrix}$ is closed[+]. But consider the cone K; if we fix m^2 and m^3 in (6) with m^2 and m^3 not both zero, then the cross section of K in m^1-m^4 space looks like

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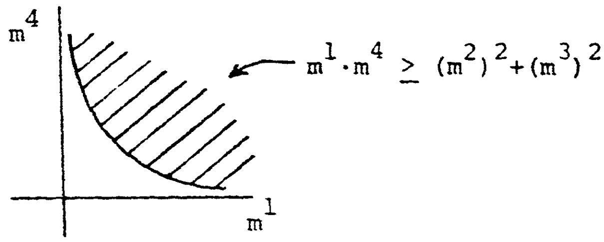
[+] $\mathcal{N}$ denotes null space.



This is precisely the infamous example of a closed convex set whose sum with a closed subspace (e.g. the m^4 axis) need not be closed or equivalently whose image under a closed-range bounded linear map (e.g. the projection onto the m^1 axis) need not be closed.

4. The quantum detection problem and its dual

We formulate the quantum detection problem in a daultiy framework and calculate the associated dual problem. First we summarize some well-known duality relationships between various spaces of operators (cf. [Sch60]).

Let H be a complex Hilbert space. The real linear space of compact self-adjoint operators $\mathcal{K}_s(H)$ with the operator norm is a Banach space whose dual is isometrically isomorphic to the real Banach space $\mathcal{T}_s(H)$ of self-adjoint trace-class operators with the trace norm, i.e. $\mathcal{K}_s(H)^* = \mathcal{T}_s(H)$ under the duality

$$\langle A,B\rangle = tr(AB) \leq |A|_{tr}\cdot|B| \qquad A \in \mathcal{T}_s(H),\ B \in \mathcal{K}_s(H).$$

Here $|B| = \sup\{|B\phi|: \phi \in H, |\phi| \leq 1\} = \sup\{trAB: A \in \mathcal{T}_s(H), |A|_{tr} \leq 1\}$ and $|A|_{tr}$ is the trace norm $\sum_i |\lambda_i| < +\infty$ where $A \in \mathcal{T}_s(H)$ and $\{\lambda_i\}$ are the eigenvalues of A repeated according to multiplicity. The dual of $\mathcal{T}_s(H)$ with the trace norm is isometrically isomorphic to the space of all linear bounded self-adjoint operators, i.e. $\mathcal{T}_s(H)^* = \mathcal{L}_s(H)$ under the duality

$$\langle A,B\rangle = tr(AB) \qquad A \in \mathcal{T}_s(H),\ B \in \mathcal{L}_s(H).$$

Moreover the orderings are compatible in the following sense. If $\mathcal{K}_s(H)_+$, $\mathcal{T}_s(H)_+$, and $\mathcal{L}_s(H)_+$ denote the closed convex cones of nennegative definite operators in $\mathcal{K}_s(H)$, $\mathcal{T}_s(H)$, and $\mathcal{L}_s(H)$ respectively, then

$$[\mathcal{K}_s(H)_+]^+ = \mathcal{T}_s(H)_+ \quad \text{and} \quad [\mathcal{T}_s(H)_+]^+ = \mathcal{L}_s(H)_+$$

where the associated dual spaces are to be understood in the sense defined above.

Let f_j be given elements of $\mathcal{T}_s(H)$ (as defined in (4)), $j = 1,\dots,n$. Define the functionals F_j: $\mathcal{L}_s(H) \to \overline{R}$ by

$$F_j(A) = \delta_{\geq 0}(A) + tr(f_j A) \qquad A \in \mathcal{L}_s(H),\ j = 1,\dots,n, \quad (8)$$

where $\delta_{\geq 0}(\cdot)$ denotes the indicator function for the set $\mathcal{L}_s(H)_+$ of nonnegative definite operators, i.e. $\delta_{\geq 0}(A)$ is 0 if $A \geq 0$ and $+\infty$ otherwise. Each F_j is proper convex and w*-lowersemicontinuous on $\mathcal{L}_s(H)$, since $\mathcal{L}_s(H)_+$ is a w*-closed convex cone and $A \mapsto tr(f_j A)$ is a continuous (in fact w*-continuous) linear functional on $\mathcal{L}_s(H)$. Define the function G: $\mathcal{L}_s(H) \to \overline{R}$ by

$$G(A) = \delta_{\{0\}}(A), \quad A \in \mathcal{L}_s(H), \quad (9)$$

that is $G(A)$ is 0 if $A = 0$ and $G(A)$ is $+\infty$ if $A \neq 0$; G is trivially convex and lower semicontinuous. Let $m = (m_1,\dots,m_n)$ denote an element of $\mathcal{L}_s(H)^n$, the

Cartesian product of n copies of $\mathcal{L}_s(H)$. Then the quantum detection problem (3) may be written

$$P = \inf \left\{ \sum_{j=1}^{n} F_j(m_j) + G(I-Lm) : m = (m_1,\ldots,m_n) \in \mathcal{L}_s(H)^n \right\} \quad (10)$$

where $L: \mathcal{L}_s(H)^n \to \mathcal{L}_s(H)$ is the continuous linear operator

$$L(m) = \sum_{j=1}^{n} m_j , \quad m \in \mathcal{L}_s(H)^n. \quad (11)$$

We consider a family of perturbed problems defined by

$$P(A) = \inf \left\{ \sum_{j=1}^{n} F_j(m_j) + G(A-Lm) : m \in \mathcal{L}_s(H)^n \right\},$$

$$A \in \mathcal{L}_s(H). \quad (12)$$

$P(\cdot)$ is a convex function $\mathcal{L}_s(H) \to \overline{R}$ and $P = P(I)$. Note that we are taking perturbations in the equality constraint, i.e. the problem $P(A)$ requires that every feasible m satisfy $Lm = A$. We remark that $G(\cdot)$ is nowhere continuous, so that there is certainly no Kuhn-Tucker point $\overline{m}$ such that $G(\cdot)$ is continuous at $L\overline{m}$ as required by the duality theorem in [ET76,III 4.1].

In order to construct the dual problem corresponding to the family of perturbed problems (12) we must calculate the conjugate functions of F_j and G. We would like to pose the dual problem in the space $\mathcal{T}_s(H)$, so we consider

$\mathcal{L}_s(H) = \mathcal{T}_s(H)^*$ and compute the pre-conjugates of F_j, G. Clearly $^*G \equiv 0$. By a straightforward calculation we have, for $y \in \mathcal{T}_s(H)$,

$$(^*F_j)(y) = \sup\{\operatorname{tr} yx - F_j(x) : x \in \mathcal{L}_s(H)\}$$

$$= \sup\{\operatorname{tr}(y-f_j)x : x \in \mathcal{L}_s(H)_+\}$$

$$= \begin{cases} 0 & \text{if } f_j - y \in \mathcal{T}_s(H)_+ \\ +\infty & \text{otherwise} \end{cases}$$

$$= \delta_{\leq f_j}(y).$$

Now $L: \mathcal{L}_s(H)^n \to \mathcal{L}_s(H)$ is continuous for the $w^* = w(\mathcal{L}_s(H), \mathcal{T}_s(H))$ topology on $\mathcal{L}_s(H)$, so we can calculate the pre-adjoint (where we identify $\mathcal{L}_s(H)^n = (\mathcal{T}_s(H)^n)^*$) as

$$^*L: \mathcal{T}_s(H) \to \mathcal{T}_s(H)^n : y \mapsto (y, y, \ldots, y).$$

Hence $(^*P)(y) = \sum_{j=1}^{n} (^*F_j)((L^*y)_j) + (^*G)(y) = \sum_{j=1}^{n} \delta_{\leq f_j}(y)$.

Thus the dual problem is $(^*P)^*(I) = \sup_y[\operatorname{tr} yI - {}^*P(y)]$ is given by

$$(^*P)^*(I) = \sup\{\operatorname{tr}(y) : y \in \mathcal{T}_s(H),\ f_j - y \geq 0 \quad j = 1, \ldots, n\}. \tag{12}$$

We have immediately $P(I) \geq (^*P)^*(I)$ with equality iff $P(\cdot)$

is w*-ℓsc at I.

We now define perturbations on the daul problem. Let D(·) be the functional on $\mathcal{T}_s(H)^n$ defined by

$$D(v) = \inf\{-\mathrm{tr}y: y \in \mathcal{T}_s(H),\ y \leq v_j \quad j=1,\dots,n\} \tag{13}$$

where $v = (v_1,\dots,v_n) \in \mathcal{T}_s(H)^n$. Of course, D(f) is just the dual problem (with a change in sign to make D(·) convex) for $f = (f_1,\dots,f_n)$: $D(f) = -(*P)*(I)$. Moreover the dual of the dual problem is again the primal, since F_j and G are w*-ℓsc:

$$\begin{aligned}
*(D^*)(f) &= \sup\{\langle f,m\rangle - D^*(m): m \in \mathcal{L}_s(H)^n\} \\
&= \sup\{\sum_{j=1}^n \mathrm{tr}(f_j m_j) - \sum_{j=1}^n (*F_j)^*(m_j) - (*G)^*(-Lm-I): m \in \mathcal{L}_s(H)^n\} \\
&= \sup\{\sum_{j=1}^n \mathrm{tr}(f_j m_j): -m_j \in \mathcal{L}_s(H)_+\ \forall j=1,\dots,n, -Lm = I\} \\
&= -\inf\{\sum_{j=1}^n \mathrm{tr}(f_j m_j): m_j \in \mathcal{L}_s(H)_+,\ j=1,\dots,n \text{ and } Lm=I\} \\
&= -P(I).
\end{aligned}$$

In general we have $P(I) \equiv -*(D^*)(f) \geq -D(f) \equiv *(P^*)(I)$. We shall show that D(·) is continuous for the norm topology on $\mathcal{T}_s(H)^n$, and hence that $D(f) = *(D^*)(f)$ and $P(I) = *(D^*)(f)$ has solutions. Equavalently, we could show that the level sets

$$\{m \in \mathcal{L}_s(H)^n: D^*(m) - \langle f,m\rangle \leq r\} =$$

$$\{m \in \mathcal{L}_s(H)^n_+: \sum_{j=1}^{n} m_j = I \text{ and } \sum_{j=1}^{n} \mathrm{tr} f_j m_j \leq r\},\ r \in R$$

are bounded and hence $w^* = w(\mathcal{L}_s(H)^n, \mathcal{T}_s(H)^n)$ compact, and then apply Theorem III.11.5 to show that $D(\cdot)$ is continuous at f. In fact, in this case the feasibility set for the primal problem,

$$\mathrm{dom} D^* = \{m \in \mathcal{L}_s(H)^n_+: \sum_{j=1}^{n} m_j = I\},$$

is itself w^* compact and hence it is easy to see that P has solutions.

<u>Proposition 1</u>. $D(\cdot)$ is continuous on $\mathcal{T}_s(H)^n$. Hence $D(f) = {}^*(D^*)(f)$ and ${}^*(D^*)(f) = P$ has solutions in $_s(H)^n$.

<u>Proof</u>. By Theorem III.11.5 applied to the dual problem we need only show that $\mathrm{dom} D = \mathcal{T}_s(H)^n$. Given $v = (v_1,\ldots,v_n) \in \mathcal{T}_s(H)^n$, set $y = -\sum_{j=1}^{n}(v_j^* v_j)^{1/2}$ where $(v_j^* v_i)^{1/2}$ is the unique positive square root of the positive operator $v_j^* v_j \geq 0$. Since $(v_j^* v_j)^{1/2} - v_j \geq 0$ for every j, then $y \leq v_j\ \forall_j$ and hence y is feasible for D, i.e. $D(v) \leq -\mathrm{tr} y < +\infty$. Hence $\mathrm{dom} D = \mathcal{T}_s(H)^n$. □

Proposition 1 shows that there is an optimal solution for the quantum detection problem and that there is no duality gap. The difficult part is to show that the dual problem (*P)*(I) has solutions. It turns out that the level sets of the dual cost function are bounded in $\mathcal{T}_s(H)$ but not weakly compact; equivalently, $P(\cdot)$ is norm-continuous at I but not $m(\mathcal{L}_s(H), \mathcal{T}_s(H))$-continuous. This suggests that we imbed $\mathcal{T}_s(H)$ in its bidual $\mathcal{T}_s(H)^{**} = \mathcal{L}_s(H)^*$ and extend the dual problem to the larger space; it will then turn out that there are solutions in $\mathcal{T}_s(H)$. This approach works because $\mathcal{T}_s(H)$ has a natural topological complement as a subset of $\mathcal{L}_s(H)^*$.

<u>Proposition 2.</u> $\mathcal{L}_s(H)^* = \mathcal{T}_s(H) \oplus_1 (J\,\mathcal{K}_s(H))^{\perp}$ where J is the canonical imbedding of $\mathcal{K}_s(H)$ in $\mathcal{L}_s(H)$. In other words, every bounded linear functional y on $\mathcal{L}_s(H)$ may be uniquely represented in the form $y = y_{ac} \oplus y_{sg}$ where $y_{ac} \in \mathcal{T}_s(H)$ and $y_{sg} \in \mathcal{K}_s(H)^{\perp}$, and

$$y(A) = \mathrm{tr}(y_{ac}A) + y_{sg}(A),\ A \in \mathcal{L}_s(H)$$

$$|y| = |y_{ac}|_{tr} + |y_{sg}|.$$

<u>Proof.</u> From [Sch50,IV.3.5] we have the identification $\mathcal{L}(H)^* = \mathcal{T}(H) \oplus_1 \mathcal{K}(H)^{\perp}$; it is only necessary to show that the same result holds for the <u>real</u> linear space $\mathcal{L}_s(H)$. But every (real-linear) $y \in \mathcal{L}_s(H)^*$ corresponds to a unique (complex-linear) $\Lambda \in \mathcal{L}(H)^*$ satisfying $\Lambda(A^*) = \overline{\Lambda(A)}$, and conversely; this correspondence is given

by

$$y(A) = \frac{1}{2}[\Lambda(A)+\overline{\Lambda(A)}], \quad A \in \mathcal{L}_s(H);$$

$$\Lambda(A) = y(\frac{A+A^*}{2}) + iy(\frac{A-A^*}{2i}), \quad A \in \mathcal{L}(H).$$

Hence, the theorem follows. □

Before calculating the dual problem, it is necessary to determine what the positive linear functions look like in terms of the decomposition provided by Proposition 2.

Proposition 3. Let $y \in \mathcal{L}_s(H)^*$. Then $y \in [\mathcal{L}_s(H)_+]^+$ iff $y_{ac} \in \mathcal{T}_s(H)_+$ and $y_{sg} \in [\mathcal{L}_s(H)_+]^+$.

Proof. It is immediate that $y \in [\mathcal{L}_s(H)_+]^+$ if $y_{ac} \in \mathcal{T}_s(H)_+$ and $y_{sg} \in [\mathcal{L}_s(H)_+]^+$. Conversely, suppose $y \in [\mathcal{L}_s(H)_+]^+$. Then clearly for every compact operator $C \in \mathcal{K}_s(H)_+ \subset \mathcal{L}_s(H)_+$ we have

$$0 \leq y(C) = \mathrm{tr}\, y_{ac}C.$$

Hence $y_{ac} \in [\mathcal{K}_s(H)_+]^+ = \mathcal{T}_s(H)_+$. Now let $A \in \mathcal{L}_s(H)_+$ be an arbitrary positive operator. Take $\{P_i\}$ to be a norm-bounded net of projections with finite rank such that $P_i \uparrow I$ in the sense that $P_i \geq P_{i'}$ for $i \geq i'$ and $P_i \to I$ in the strong operator topology. Then $A^{1/2}P_iA^{1/2}$ has finite rank and $A^{1/2}P_iA^{1/2} \uparrow A$ in the strong operator topology. Hence

$$0 \le y(A-A^{1/2}P_iA^{1/2}) = y_{sg}(A)+\mathrm{tr}[y_{ac}(A-A^{1/2}P_iA^{1/2})] \to y_{sg}(A)$$

where the limit in the last step is valid since $A-A^{1/2}P_iA^{1/2} \to 0$ in the $w^* = w(\mathcal{L}_s(H), \mathcal{T}_s(H))$ topology on $\mathcal{L}_s(H)$ (this is weaker than the strong operator topology). Thus $y_{sg} \in [\mathcal{L}_s(H)_+]^+$. □

With the aid of this last proposition it is now possible to calculate the extended dual problem in $\mathcal{L}_s(H)^*$. The conjugate function of G is $G^* \equiv 0$. The conjugate of F_j is

$$F_j^*(y) = \sup\{\mathrm{tr}[(y_{ac}-f)x]+y_{sg}(x) : x \in \mathcal{L}_s(H)_+\}$$

$$= \begin{cases} 0 & \text{if } f_j-y_{ac} \in \mathcal{T}_s(H)_+ \text{ and } -y_{sg} \in [\mathcal{L}_s(H)_+]^+ \\ +\infty & \text{otherwise} \end{cases}$$

$$= \delta_{\le f_j}(y_{ac}) + \delta_{\le 0}(y_{sg})$$

where by $y_{sg} \le 0$ we mean $-y_{sg} \in [\mathcal{L}_s(H)_+]^+$. The adjoint of $L: \mathcal{L}_s(H)^n \to \mathcal{L}_s(H): m \to \sum_{j=1}^{n} m_j$ is

$$L^*: \mathcal{L}_s(H)^* \to \mathcal{L}_s(H)^{*n}: y \to (y,\ldots,y).$$ Hence

$$P^*(y) = \sum_{j=1}^{n} F_j^*(-(L^*y)_j)+G^*(y) = \sum_{j=1}^{n} \delta_{\le f_j}(y_{ac}) + \delta_{\le 0}(y_{sg}).$$

Thus the dual problem $*(P^*)(I) = \sup_y [y(I) - P^*(y)]$ is given by

$$*(P^*)(I) = \sup\{tr(y_{ac}) + y_{sg}(I) : y \in \mathcal{L}_s(H)^*,\ y_{sg} \leq 0,\ y_{ac} \leq f_j \quad j=1,\dots,n\}.$$

Note that this is consistent with the more restricted dual problem $(*P)*(I)$ given by (12). We prove that $P(\cdot)$ is norm-continuous at I, and hence $P(I) = *(P^*)(I)$, $*(P^*)(I)$ has solutions.

Lemma 4. If $A \in \mathcal{L}_s(H)$ and $|A| \leq 1$, then $I+A \geq 0$. In particular, $I \in \text{int}\, \mathcal{L}_s(H)_+$ and $y(I) > 0$ for every nonzero $y \in [\mathcal{L}_s(H)_+]^+$.

Proof. Suppose $|A| \leq 1$. For every $\phi \in H$,

$$<(I+A)\phi|\phi> = |\phi|^2 + <A\phi|\phi> \geq |\phi|^2 - |A| \cdot |\phi|^2 = (1-|A|)|\phi|^2 \geq 0.$$

Hence $I+A \geq 0$ and $I \in \text{int}\, \mathcal{L}_s(H)_+$. Now suppose $y \in [\mathcal{L}_s(H)_+]^+$, $y \neq 0$. Then there is an $A \in \mathcal{L}_s(H)$ such that $|A| \leq 1$ and $y(A) < 0$. Hence $y(I) > y(I+A) \geq 0$. □

Proposition 5. $P(\cdot)$ is continuous at I, and hence $\partial P(I) \neq \emptyset$. In particular, $*(P^*)(I) = P(I)$ and the dual problem $*(P^*)(I)$ has solutions.

Proof. By Theorem III.11.5 it suffices to show that $I \in \text{int}\, \text{dom}P$. But if $A \in \mathcal{L}_s(H)$ and $|A| \leq 1$, then by

Lemma 4 $I+A \geq 0$ and $m = (I+A,0,0,\ldots,0) \in \mathcal{L}_s(H)^n$ is feasible for $P(I+A)$, i.e. $I+A \in \text{dom}P$. Hence $I \in \text{int dom}P$ and $\partial P(I) \neq \emptyset$. □

It is now an easy matter to show that the dual problem actually has solutions in $\mathcal{T}_s(H)$, that is solutions in $\mathcal{L}_s(H)^*$ with 0 singular part.

Proposition 6. Every solution $y \in \mathcal{L}_s(H)^*$ of the extended dual problem $*(P^*)(I)$ satisfies $y_{sg} = 0$, i.e. y belongs to the canonical image of $\mathcal{T}_s(H)$ in $\mathcal{T}_s(H)^{**}$.

Proof. Suppose $y \in \mathcal{L}_s(H)^*$ is feasible for the dual problem, i.e. $y_{ac} \leq f_j$ for $j = 1,\ldots,n$ and $y_{sg} \leq 0$. If $y_{sg} \neq 0$, then $tr(y_{ac})+y_{sg}(I) < tr(y_{ac})$ by Lemma 4. Hence the value of the objective function is improved by setting $y_{sg} = 0$, while the constraints are not violated. Thus if y is optimal, then $y_{sg} = 0$. □

To summarize the results, we have shown that if we define

$$P = \inf\left\{\sum_{j=1}^{n} tr(f_j m_j) : (m_1,m_2,\ldots,m_n) \in \mathcal{L}_s(H)^n;\right.$$

$$\left. m_j \geq 0 \text{ for } j = 1,2,\ldots,n;\ \sum_{j=1}^{n} m_j = I\right\} \tag{14}$$

$$-D = \sup\{tr(y) : y \in \mathcal{T}_s,\ y \leq f_j \text{ for } j = 1,2,\ldots,n\} \tag{15}$$

then $P = -D$ and both P and $-D$ have optimal solutions. Since P is an infimum and $-D$ is a supremum we immediately get an extremality condition: m solves P and y solves D if and only if m is feasible for P, y is feasible for -D, and

$$\sum_{j=1}^{n} tr(f_j m_j) = try.$$

This leads to the following characterization of the solution to the quantum detection problem.

Theorem 7. Let H be a complex Hilbert space and suppose $(f_1,\ldots,f_n) \in \mathcal{T}_s(H)^n$. Then the quantum detection problem P defined by (1.1) has solutions. Moreover, the following statements are equivalent for $m = (m_1,\ldots,m_n) \in \mathcal{L}_s(H)^n$:

1) m solves P

2) $\sum_{j=1}^{n} m_j = I;\ m_i \geq 0$ for $i=1,\ldots,n;$

$\sum_{j=1}^{n} f_j m_j \leq f_i$ for $i = 1,\ldots,n$

3) $\sum_{j=1}^{n} m_j = I;\ m_i \geq 0$ for $i=1,\ldots,n;$

$\sum_{j=1}^{n} m_j f_j \leq f_i$ for $i = 1,\ldots,n.$

Under any of the above conditions it follows that

$y = \sum_{j=1}^{n} f_j m_j = \sum_{j=1}^{n} m_j f_j$ is self-adjoint and is the unique solution of the dual problem -D given by (15); moreover

$$P = -D = tr(y).$$

Proof We must show that the conditions 2) and 3) are necessary and sufficient for $m \in \mathcal{L}_s(H)^n$ to solve P. Note that the first part of each condition 2), 3) is simply a feasibility requirement.

Suppose u solves P. Then there is a $y \in \mathcal{T}_s(H)$ which solves -D such that $y \leq f_i$ for $i = 1,\ldots,n$ and

$$\sum_{j=1}^{n} tr(f_j m_j) = tr(y).$$

Equivalently, $0 = \sum_{j=1}^{n} tr(f_j m_j) - tr(yI) = \sum_{j=1}^{n} tr(f_j - y)m_j$ since $\sum_{j=1}^{n} m_j = I$. Since $f_j - y \geq 0$ and $m_j \geq 0$ we conclude from Lemma 8 which follows, that $(f_j - y)m_j = 0$ for $j = 1,\ldots,n$. But then $0 = \sum_{j=1}^{n} (f_j - y)m_j = \sum_{j=1}^{n} f_j m_j - y$ and 2) follows. This also shows that y is unique.

Conversely, suppose 2), i.e. m is feasible for P and $\sum_{j=1}^{n} f_j m_j \leq f_i$, $i = 1,\ldots,n$. Then $y = \sum_{j=1}^{n} f_j m_j$ is feasible for -D, and $\sum_{j=1}^{n} tr(f_j m_j) = tr(y)$. Hence m solves P and y solves -D.

Thus 1) <=> 2) is proved. Since $tr(f_j m_j) = tr(m_j f_j)$ the proof for 1) <=> 3) is identical, and $y = \sum_{j=1}^{n} f_j m_j = \sum_{j=1}^{n} m_j f_j$ is the solution of -D. □

We have made use of the following easy lemma.

Lemma 8. Let $A \in \mathcal{T}_s(H)_+$, $B \in \mathcal{L}_s(H)_+$. Then $AB \geq 0$, and $trAB = 0$ iff $AB = 0$.

Proof. If $\phi \in H$, then $\langle AB\phi|\phi\rangle = \langle A^{1/2}B^{1/2}B^{1/2}A^{1/2}\phi|\phi\rangle = \langle B^{1/2}A^{1/2}\phi|B^{1/2}A^{1/2}\phi\rangle = |B^{1/2}A^{1/2}\phi|^2 \geq 0$. Since $AB \geq 0$, $trAB = \sum_i \langle AB\phi_i|\phi_i\rangle$ is 0 iff $AB = 0$, where $\{\phi_i\}$ is a complete orthonormal set. □

Remarks on the literature. [YKL75] claims the necessary and sufficient conditions 2) with the additional constraint that $\sum_{j=1}^{n} f_j m_j = \sum_{j=1}^{n} m_j f_j$, but the proof of these conditions is not correct. [H73] states that the conditions 2) are sufficient, but of course this is the easy part. It is interesting to note that in the commuting case where $[\rho_i - \rho_j, \rho_k - \rho_\ell] = 0$ for i,j,k,ℓ $\{1,\ldots,n\}$, the problem reduces to the classical case, i.e. the optimal quantum detector $m = (m_1,\ldots,m_n)$ corresponds to a finite resolution of the identity and the decision is made in the usual way by maximizing the posterior probability.

Added Remark. Prcfessor Mitter has brought to my attention Holevo's paper [H76] in which the detection results given here are proved using a somewhat different argument. However he does not appear to have extended these results to the more general estimation problem considered in Chapter IX.

VIII. Operator-Valued Measures

Abstract. Let S be a locally compact Hausdorff space and X,Z Banach spaces. A theory is developed whcih represents all bounded linear operators L: $C_o(S,X) \to Z^*$ (without requiring L to be weakly compact) by Borel measures m which have values in $L(X,Z^*)$ and are countably additive in a certain operator topology. Moreover this approach affords a natural characterization of various subspaces of $L(X,Z^*)$ in terms of boundedness conditions on the corresponding representing measures. The uaual results for representing bounded linear maps can then be obtained by considering $L(C_o(S,X),Y)$ as a subspace of $L(C_o(S,X),Y^{**})$, for Y a Banach space. These results have applicaticns in the theory of quantum estimation.

Operator-valued Measures

It is clear that the formulation of quantum estimation problems requires some techniques in the theory of operator-valued measures. While proving the necessary properties of such measures I noticed that the approach I had taken, while natural for $\mathbf{L}_s(H)$-valued measures, was somewhat different from the general theory of operator-valued measures developed in the literature, as we shall see. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let X,Y be Banach spaces with normed duals X*,Y*. $C_o(S,X)$ denotes the Banach space of continuous X-valued functions $f: S \to X$ which vanish at infinity (for every $\varepsilon > 0$, there is a compact set $K \subset S$ such that $|f(s)| < \varepsilon$ for all $s \in S \setminus K$), with the supremum norm $|f|_\infty = \sup_{s \in S}|f(s)|$. It is possible to identify every bounded linear map $L: C_o(S,X) \to Y$ with a representing measure m such that

$$Lf = \int_S m(ds)f(s) \tag{1}$$

for every $f \in C_o(S,X)$. Here m is a finitely additive map $m: B \to L(X,Y^{**})$ with finite semivariation which satisfies:

1. for every $z \in Y^*$, $m_z: \mathcal{B} \to X^*$ is a regular X*-valued Borel measure, where m_z is defined by

$$m_z(E)x = \langle z,m(E)x\rangle \qquad E \in \mathcal{B},\ x \in X; \tag{2}$$

2. the map $z \mapsto m_z$ is continuous for the w* topologies on $z \in Y^*$ and $m_z \in C_o(S,X)^*$.

The latter condition assures that the integral (1) has values in Y even though the measure has values in $L(X,Y^{**})$ rather than $L(X,Y)$ (we identify Y as a subspace of Y^{**}). Under the above representation of maps $L \in \mathbf{L}(C_o(S,X),Y)$, the maps for which L_x: $C_o(S) \to Y$: $g(\cdot) \mapsto L(g(\cdot)x)$ is weakly compact for every $x \in X$ are precisely the maps whose representing measures have values in $L(X,Y)$, not just in $L(X,Y^{**})$. In particular, if Y is reflexive or if Y is weakly complete or more generally if Y has no subspace isomorphic to c_o, then every map in $L(C_o(S,X),Y)$ is weakly compact and hence every $L \in L(C_o(S,X),Y)$ has a representing measure with values in $L(X,Y)$.

In the contex of quantum mechanical measures with values in $L_s(H)$, however, I identified every continuous linear map L: $C_o(S) \to L_s(H)$ (here $X=R$, $Y=L_s(H)$) with a representing measure with values in $L_s(H)$ rather than in $L_s(H)^{**}$, using fairly elementary arguments. Since $Y = L_s(H)$ is neither reflexive nor devoid of subspaces isomorphic to c_o (think of a subspace of compact operators on H having a fixed countable set of eigenvectors), I thought at first I had made an error. Fortunately for my sanity, however, I soon detected the crucial difference: whereas in the usual approach it is assumed that the real-valued set function $zm(\cdot)x$ is countably additive for $x \in X$ and <u>every</u> $z \in Y^*$, I require that it be countably additive only for $x \in X$ and $z \in Z=T_s(H)$, where $Z=T_s(H)$ is a <u>predual</u> of $Y=L_s(H)$, and hence can represent all linear bounded maps $L: C_o(S,X) \to Y$ by measures with values in $L(X,Y)$. In other words, by assuming that the measures $m: B \to L_s(H)$ are countably additive in the weak* topology rather than the weak topology (these are equivalent only when m has bounded variation), it is possible to represent every bounded linear map $L: C_o(S) \to L_s(H)$ and not just the weakly compact maps. This approach is generally applicable whenever Y is a dual space, and in fact yields the usual results by imbedding Y in Y**; moreover it clearly shows the relationships between various boundedness conditions on the representing measures and the corresponding spaces of linear maps. But first we must define what is meant by integration with respect to operator-valued measures. We shall always take the underlying field of scalars to be the reals, although the results extend immediately to the complex case.

Throughout this section we assume that $\mathcal{B}$ is the σ-algebra of Borel sets of a locally compact Hausdorff space S, and X,Y are Banach spaces. Let $M: \mathcal{B} \to L(X,Y)$ be an additive set function, i.e. $m(E_1 \cup E_2) = m(E_1)+m(E_2)$

whenever E_1, E_2 are disjoint sets in B. The <u>semivariation</u> of m is the map $\bar{m}: \mathcal{B} \to \bar{R}_+$ defined by

$$\bar{m}(E) = \sup \left| \sum_{i=1}^{n} m(E_i) x_i \right|,$$

where the supremum is taken over all finite collections of disjoint sets $E_1, \ldots, E_n$ belonging to $\mathcal{B} \cap E$ and $x_1, \ldots, x_n$ belonging to X_1. By $\mathcal{B} \cap E$ we mean the sub-σ-algebra $\{E' \in \mathcal{B}: E' \subset E\} = \{E' \cap E: E \in \mathcal{B}\}$ and by X_1 we denote the closed unit ball in X. The <u>variation</u> of m is the map $|m|: \mathcal{B} \to \bar{R}_+$ defined by

$$|m|(E) = \sup \sum_{i=1}^{n} |m(E_i)|$$

where again the supremum is taken over all finite collections of disjoint sets in $\mathcal{B} \cap E$. The <u>scalar semivariation</u> of m is the map $\bar{\bar{m}}: \mathcal{B} \to \bar{R}_+$ defined by

$$\bar{\bar{m}}(E) = \sup \left| \sum_{i=1}^{n} a_i m(E_i) \right|$$

where the supremum is taken over all finite collections of disjoint sets $E_1, \ldots, E_n$ belonging to $\mathcal{B} \cap E$ and $a_1, \ldots, a_n \in R$ with $|a_i| \leq 1$. It should be noted that the notion of semivariation depends on the spaces X and Y; in fact, if $m: \mathcal{B} \to L(X,Y)$ is taken to have values in $L(R, L(X,Y))$, $L(X,Y)$, $L(X,Y)^{**} = L(L(X,Y),R)$ respectively then

$$\bar{\bar{m}} = \bar{m}_{L(R,L(X,U))} \leq \bar{m} = \bar{m}_{L(X,Y)} \leq |m| = \bar{m}_{L(L(X,U)^*,R)}. \tag{3}$$

When necessary, we shall subscript the semivariation accordingly. By $fa(\mathcal{B},W)$ we denote the space of all finitely additive maps $m: \mathcal{B} \to W$ where W is a vector space.

<u>Proposition 1</u>. If $m \in fa(\mathcal{B},X^*)$ then $\bar{m} = |m|$. More generally, if $m \in fa(\mathcal{B},L(X,Y))$ then for every $z \in Y^*$ the finitely additive map $zm: \mathcal{B} \to X^*$ satisfies $\overline{zm} = |zm|$.

<u>Proof</u>. It is sufficient to consider the case Y = R, i.e. $m \in fa(\mathcal{B},X^*)$. Clearly $\bar{m} \leq |m|$. Let $E \in \mathcal{B}$ and let $E_1,\ldots,E_n$ be disjoint sets in $\mathcal{B} \cap E$. Then $\sum_i |m(E_i)| = \sup_{x_i \in X_1} \Sigma m(E_i)x_i = \sup_{x_i \in X_1} |\Sigma m(E_i)x_i| \leq m(E)$. Taking the supremum over all disjoint $E_i \in \mathcal{B} \cap E$ yields $|m|(E) \leq \bar{m}(E)$. □

We shall need some basic facts about variation and semivariation. Let X,Y be normed spaces. A subset Z of Y^* is a <u>norming subset</u> of Y^* if $\sup\{zy: z \in Z, |z| \leq 1\} = |y|$ for every $y \in Y$.

<u>Proposition 2</u>. Let X,Y be normed spaces, $m \in fa(\mathcal{B}, L(X,Y))$. If Z is a norming subset of Y^*, then

$$\tilde{m}(E) = \sup_{z\in Z, |z|\leq 1} |zm|(E) \qquad , E \in \mathcal{B}$$

$$\bar{m}(E) = \sup_{z\in Z, |z|\leq 1} \sup_{x\in X, |x|\leq 1} |zm(\cdot)x|(E) \quad , E \in \mathcal{B}$$

Moreover $|y^*m(\cdot)x|(E) \leq |x|\cdot|y^*m|(E) \leq |x|\cdot|y^*|\cdot|m|(E)$ for every $x \in X$, $y^* \in Y^*$, $E \in \mathcal{B}$.

Proof. Let $\{E_1,\ldots,E_n\}$ be disjoint sets in $\mathcal{B} \cap E$ and $x_1,\ldots,x_n \in X_1$. Then

$$\left|\sum_{i=1}^{n} m(E_i)x_i\right| = \sup_{z\in Z_1} \langle z, \sum_{i=1}^{n} m(E_i)x_i\rangle = \sum_{i=1}^{n} zm(E_i)x_i .$$

Taking the supremum over $\{E_i\}$ and $\{x_i\}$ yields $\tilde{m}(E) = |zm|(E)$. Similarly,

$$\sup_{|a_i|\leq 1} \left|\sum_{i=1}^{n} a_i m(E_i)\right| = \sup_{|a_i|\leq 1} \sup_{x\in X_1} \sup_{z\in Z_1} \langle z, \sum_{i=1}^{n} a_i m(E_i)x\rangle$$

$$= \sup_{\substack{x\in X_1 \\ z\in Z_1}} \sum_{i=1}^{n} |zm(E_i)x|$$

and taking the supremum over finite disjoint collections $\{E_i\} \subset \mathcal{B}\cap E$ yields $\bar{\bar{m}}(E) = \sup_{|x|\leq 1} \sup_{|z|\leq 1} |zm(\cdot)x|(E)$.

It is straightforward to check the final statement of the theorem. □

Proposition 3. Let $m \in fa(\mathcal{D}, L(X,Y))$. Then $\bar{\bar{m}}$, $\bar{m}$, and $|m|$ are monotone and finitely subadditive; $|m|$ is finitely additive.

Proof. It is immediate that $\bar{m}$, $\bar{\bar{m}}$, $|m|$ are monotone. Suppose $E_1, E_2 \in \mathcal{D}$ and $E_1 \cap E_2 = \emptyset$, and let $F_1,\ldots,F_n$ be a finite collection of disjoint sets in $\mathcal{D} \cap (E_1 \cup E_2)$. Then if $|x_i| \leq 1$, $i = 1,\ldots,n$, we have

$$\left|\sum_{i=1}^{n} m(F_i)x_i\right| = \left|\sum_{i=1}^{n} (m(F_i \cap E_1)+m(F_i \cap E_2))x_i\right|$$

$$\leq \left|\sum_i m(F_i \cap E_1)x_i\right| + \left|\sum_i m(F_i \cap E_2)x_i\right|$$

$$\leq \bar{m}(E_1)+\bar{m}(E_2).$$

Taking the supremum over all disjoint $F_1,\ldots,F_n \in \mathcal{D} \cap (E_1 \cup E_2)$ yields $\bar{m}(E_1 \cup E_2) \leq \bar{m}(E_1)+\bar{m}_2(E_2)$. Using (3) we immediately have $\bar{\bar{m}}$, $|m|$ finitely subadditive. Since $|m|$ is always superadditive by its definition, $|m|$ is finitely additive. □

We now define integration with respect to additive set functions $m: \mathcal{D} \to L(X,Y)$. Let $\mathcal{D} \otimes X$ denote the vector space of all X-valued measurable simple functions on S, that is all functions of the form $f(s) = \sum_{i=1}^{n} 1_{E_i}(s)x_i$ where $\{E_1,\ldots,E_n\}$ is a finite disjoint measurable partition of S, i.e. $E_i \in \mathcal{D}$ $\forall i$, $E_i \cap E_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n} E_i = S$. Then the integral $\int_S m(ds)f(s)$ is defined unambiguously (by finite additivity) as

$$\int_S m(ds)f(s) = \sum_{i=1}^{n} m(E_i)x_i. \tag{4}$$

We make $\mathcal{B}\otimes X$ into a normed space under the uniform norm, defined for bounded maps $f: S \to X$ by

$$|f|_\infty = \sup_{s\in S}|f(s)|.$$

Suppose now that m has finite semivariation, i.e. $\bar{m}(s) < +\infty$. From the definitions it is clear that

$$\left|\int_S m(ds)f(s)\right| \leq \bar{m}(S)\cdot|f|_\infty , \tag{5}$$

so that $f \mapsto \int_S m(ds)f(s)$ is a bounded linear functional on $(\mathcal{B}\otimes X, |\cdot|_\infty)$; in fact, $\bar{m}(S) = \sup\{|\int m(ds)f(s)|: |f|_\infty \leq 1, f\in\mathcal{B}\otimes X\}$ is the bound. Thus, if $\bar{m}(S) < +\infty$ it is possible to extend the definition of the integral to the completion $M(S,X)$ of $\mathcal{B}\otimes X$ in the $|\cdot|_\infty$ norm. $M(S,X)$ is called the space of totally $\mathcal{B}$-measurable X-valued functions on S; every such function is the uniform limit of $\mathcal{B}$-measurable simple functions. For $f\in M(S,X)$ define

$$\int_S m(ds)f(s) = \lim_{n\to\infty} \int_S m(ds)f_n(s) \tag{6}$$

where $f_n \in \mathcal{B}\otimes X$ is an arbitrary sequence of simple functions which converge uniformly to f. The integral is well-defined since if $\{f_n\}$ is a Cauchy sequence in $\mathcal{B}\otimes X$ then $\{\int_S m(ds)f_n(s)\}$ is Cauchy in Y by (5) and hence converges. Moreover if two sequences $\{f_n\}$, $\{g_n\}$ in $\mathcal{B}\otimes X$ satisfy $|g_n - f|_\infty \to 0$ and $|f_n - f|_\infty \to 0$ then $|\int m(ds)f_n(s) - \int m(ds)g_n(s)| \leq \bar{m}(S)|f_n - g_n|_\infty \to 0$ so $\lim_{n\to\infty} \int_S m(ds)f_n(s) = \lim_{n\to\infty} \int_S m(ds)g_n(s)$. Similarly, it is clear that (5) remains true for every $f \in M(S,X)$. More generally it is straightforward to verify that

$$\bar{m}(E) = \sup\{\int_S m(ds)f(s): f \in M(S,X),\ |f|_\infty \leq 1,\ \mathrm{supp} f \subset E\}. \tag{7}$$

<u>Proposition 4</u>. $C_o(S,X) \subset M(S,X)$.

Proof. Every $g(\cdot) \in C_o(S)$ is the uniform limit of simple real-valued Borel-measurable functions, hence every function of the form $f(s) = \sum_{i=1}^{n} g_i(s)x_i = \sum_{i=1}^{n} g_i \otimes x_i$ belongs to $M(S,X)$, for $g_i \in C_o(S)$ and $x_i \in X$. These functions may be identified with $C_o(S)\otimes X$, which is dense in $C_o(S,X)$ for the supremum norm [T67p448]. Hence $C_o(S,X) = \mathrm{cl}C_o(S)\otimes X \subset M(S,X)$. □

To summarize, if $m \in fa(\mathcal{B}, L(X,Y))$ has finite semivariation $\bar{m}(S) < +\infty$ then $\int_S m(ds)f(s)$ is well-defined for

$f \in M(S,X) \supset C_o(S,X)$, and in fact $f \mapsto \int_S m(ds)f(s)$ is a bounded linear map from $C_o(S,X)$ or $M(S,X)$ into Y.

Now let Z be a Banach space and L a bounded linear map from Y to Z. If $m: \mathcal{B} \to \mathbf{L}(X,Y)$ is finitely additive and has finite semivariation then $Lm: \mathcal{B} \to \mathbf{L}(X,Z)$ is also finitely additive and has finite semivariation $\overline{Lm}(S) \leq |L| \cdot \bar{m}(S)$. For every simple function $f \in \mathcal{B} \otimes X$ it is easy to check that $L\int_S m(ds)f(s) = \int_S Lm(ds)f(s)$. By taking limits of uniformly convergent simple functions we have proved

Proposition 5. Let $m \in fa(\mathcal{B}, \mathbf{L}(X,Y))$ and $\bar{m}(S) < +\infty$. Then $Lm \in fa(\mathcal{B}, \mathbf{L}(X,Z))$ for every bounded linear $L: Y \to Z$, with $\overline{Lm}(S) < +\infty$ and

$$L\int_S m(ds)f(s) = \int_S Lm(ds)f(s). \tag{8}$$

Since we will be considering measure representations of bounded linear operators on $C_o(S,X)$, we shall require some notions of countable additivity and regularity. Recall that a set function $m: \mathcal{B} \to W$ with values in a locally convex Hausdorff space W is countably additive iff $m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$ for every countable disjoint sequence $\{E_i\}$ in $\mathcal{B}$. By the Pettis Theorem (DSiv.10.1) countable additivity is equivalent to weak countable additivity, i.e. $m\colon \mathcal{B} \to W$ is countable additive iff it is countably additive for the weak topology on W, that is iff $w^*m\colon \mathcal{B} \to R$ is countably additive for every $w^* \in W^*$. If W is a Banach space, we denote by $ca(\mathcal{B}, W)$ the space of all countably additive maps $m\colon \mathcal{B} \to W$; $fabv(\mathcal{B}, W)$ and $cabv(\mathcal{B}, W)$ denote the spaces of finitely additive and countably additive maps $m\colon \mathcal{B} \to W$ which have bounded variation $|m|(S) < +\infty$.

If W is a Banach space, a measure $m \in fa(\mathcal{B}, W)$ is <u>regular</u> iff for every $\varepsilon > 0$ and every Borel set E there is a compact set $K \subset E$ and an open set $G \supset E$ such that $|m(F)| < \varepsilon$ whenever $F \in \mathcal{B} \cap (G \setminus K)$. The following theorem shows among other things that regularity actually implies countable additivity when m has bounded variation $|m|(S) < +\infty$ (this latter condition is crucial). By $rcabv(\mathcal{B}, W)$ we denote the space of all countably additive regular Borel measures $m\colon \mathcal{B} \to W$ which have bounded variation.

Let X, Z be Banach spaces. We shall be mainly concerned with a special class of $L(X, Z^*)$-valued measures which we now define. Let $\mathcal{M}(\mathcal{B}, L(X, Z^*))$ be the space of all $m \in fa(\mathcal{B}, L(X, Z^*))$ such that $\langle z, m(\cdot)x\rangle \in rcabv(\mathcal{B})$ for every $x \in X$, $z \in Z$. Note that such measures $m \in \mathcal{M}(\mathcal{B}, L(X, Z^*))$ need not be countably additive for the weak operator (equivalently, the strong operator) topology on $L(X,Z^*)$, since $z^{**}m(\cdot)x$ need not belong to $ca(\mathcal{B})$ for every $x \in X$, $z^{**} \in Z^{**}$.

The following theorem is very important in relating various countable additivity and regularity conditions.

Theorem 1. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let X,Y be normed spaces, Z_1 a norming subset of Y^*, $m \in fa(\mathcal{B}, L(X,Y))$. If $zm(\cdot)x: \mathcal{B} \to R$ is countably additive for every $z \in Z_1$, $x \in X$ then $|m|(\cdot)$ is countably additive $\mathcal{B} \to \bar{R}_+$. If $zm(\cdot)x: \mathcal{B} \to R$ is regular for every $z \in Z_1$, $x \in X$, and if $|m|(S) < +\infty$, then $|m|(\cdot)$ $rcabv(\mathcal{B}, R_+)$. If $|m|(S) < +\infty$, then $m(\cdot)$ is countably additive iff $|m|$ is and $m(\cdot)$ is regular iff $|m|$ is.

Proof. Suppose $zm(\cdot)x \in ca(\mathcal{B},R)$ for every $z \in Z_1$, $x \in X$. Let $\{A_i\}$ be a disjoint sequence in $\mathcal{B}$. Let $\{B_1,\ldots,B_n\}$ be a finite collection of disjoint Borel subsets of $\bigcup_{i=1}^{\infty} A_i$. Then

$$\sum_{j=1}^{n} |m(B_j)| = \sum_{j=1}^{n} |m(\bigcup_{i=1}^{\infty} A_i \cap B_j)| = \sum_{j=1}^{n} \sup_{\substack{x_j \in X_1 \\ z_j \in Z_1}} |z_j m(\bigcup_{i=1}^{\infty} A_i \cap B_j) x_j| .$$

Since each $z_j m(\cdot) x_j$ is countably additive, we may continue with

$$= \sum_{j=1}^{n} \sup_{\substack{x_j \in X_1 \\ z_j \in Z_1}} \left| \sum_{i=1}^{\infty} a_j m(A_i \cap B_j) x_j \right| \leq \sum_{j=1}^{n} \sup_{\substack{x_j \in X_1 \\ z_j \in Z_1}} \sum_{i=1}^{\infty} |z_j m(A_i \cap B_j) x_j|$$

$$\leq \sum_{j=1}^{n} \sum_{i=1}^{\infty} |m(A_i \cap B_j)| = \sum_{i=1}^{\infty} \sum_{j=1}^{n} |m(A_i \cap B_j)| \leq \sum_{i=1}^{\infty} |m|(A_i).$$

Hence, taking the supremum over all disjoint $\{B_j\} \subset \bigcup_{i=1}^{\infty} A_i$, we have $|m|(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} |m|(A_i)$. Since $|m|$ is always countably superadditive, $|m|$ is countably additive.

Now assume $zm(\cdot)x$ is regular for every $z \in Z_1$, $x \in X$, and $|m|(S) < +\infty$. Obviously each $zm(\cdot)x$ has bounded variation since $|m|(S) < +\infty$, hence $zm(\cdot)x \in ca(\mathcal{B})$ by [DS III.5.13] and $zm(\cdot)x \in rcabv(\mathcal{B})$. We wish to show that $|m|$ is regular; we already know $|m| \in cabv(\mathcal{B})$. Let $E \in \mathcal{B}$, $\varepsilon > 0$. By definition of $|m|(E)$ there is a finite disjoint Borel partition $\{E_1, \ldots, E_n\}$ of E such that $|m|(E) < \sum_{i=1}^{n} |m(E_i)| + \varepsilon/2$. Hence there are

$z_1, \ldots, z_n \in Z_1$ and $x_1, \ldots, x_n \in X$, $|x_i| \leq 1$, such that

$$|m|(E) < \sum_{i=1}^{n} z_i m(E_i) x_i + \varepsilon/2.$$

Now each $z_i m(\cdot) x_i$ is regular, so there are compact $K_i \subset E_i$

for which $|z_i m(E_i \setminus K_i)x_i| < \varepsilon/2n$, $i = 1,\ldots,n$. Hence

$$|m|(E\setminus K) = |m|(E) - |m|(K)$$

$$< \sum_{i=1}^{n} z_i m(E_i)x_i + \frac{\varepsilon}{2} - \sum_{i=1}^{n} z_i m(E_i \cap K_i)x_i$$

$$= \sum_{i=1}^{n} z_i m(E_i \setminus K_i)x_i + \varepsilon/2$$

$$< \varepsilon,$$

and we have shown that $|m|$ is inner regular. Since $|m|(s) < +\infty$, it is straightforward to show that $|m|$ is outer regular. For if $E \in \mathcal{B}$, $\varepsilon > 0$ then there is a compact $K \subset S\setminus E$ for which $|m|(S\setminus E) < |m|(K) + \varepsilon$ and so for the open set $G = S\setminus K \supset E$ we have

$$|m|(G\setminus E) = |m|(S\setminus E) - |m|(K) < \varepsilon.$$

Finally, let us prove the last statement of the theorem. We assume $m \in fa(\mathcal{B}, L(X,Y))$ and $|m|(S) < +\infty$. First suppose $m(\cdot)$ is countably additive. Then for every disjoint sequence $\{A_i\}$ in $\mathcal{B}$,

$|m(\bigcup_{i=1}^{\infty} A_i) - \sum_{i=1}^{n} m(A_i)| \to 0$, so certainly

$y^*m(\bigcup_{i=1}^{\infty} A_i)x - \sum_{i=1}^{n} y^*m(A_i)x_i \to 0$ for every $y^* \in Y^*$, $x \in X$

and by what we just proved $|m|$ is countably additive. Conversely, if $|m|$ is countably additive then for every disjoint sequence $\{A_i\}$ we have $|m(\bigcup_{i=1}^{\infty} A_i) - \sum_{i=1}^{n} m(A_i)| =$

$|m(\bigcup_{i=1}^{\infty} A_i)| \leq |m|(\bigcup_{i=1}^{\infty} A_i) = |m|(\bigcup_{i=1}^{\infty} A_i) - \sum_{i=1}^{n} |m|(A_i) \to 0.$

Similarly, if m is regular then every $y^*m(\cdot)x$ is regular and by what we proved already $|m|$ is regular. Conversely, if $|m|$ is regular it is easy to show that m is regular. □

Theorem 2. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let X, Z be Banach spaces. There is an isometric isomorphism $L \leftrightarrow m$ between the bounded linear maps $L: C_0(S) \to L(X,Z^*)$ and the finitely additive measures $m: \mathcal{B} \to L(X,Z^*)$ for which $zm(\cdot)x \in rcabv(\mathcal{B})$ for every $x \in X$, $z \in Z$. The correspondence $L \leftrightarrow m$ is given by

$$Lg = \int_S g(s)m(ds), \quad g \in C_0(S) \tag{10}$$

where $|L| = \bar{\bar{m}}(S)$; moreover, $zL(g)x = \int_S g(s)\ zm(ds)x$ and $|zL(\cdot)x| = |zm(\cdot)x|(S)$ for $x \in X$, $z \in Z$.

Remarks. The measure $m \in fa(\mathcal{B}, L(X,Z^*))$ need have neither finite semivariation $\bar{m}(s)$ nor bounded variation $|m|(S)$. It is also clear that $L(g)x = \int_S g(s)m(ds)x$ and

$zL(g) = \int_S g(s)zm(ds)$, by Proposition 5.

<u>Proof</u>. Suppose $L \in \mathbf{L}(C_o(S), \mathbf{L}(X,Z^*))$ is given. Then for every $x \in X$, $z \in Z$ the map $g \mapsto zL(g)x$ is a bounded linear functional on $C_o(S)$, so there is a unique real-valued regular Borel measure $m_{x,z}: \mathcal{B} \to R$ such that

$$zL(g)x = \int_S f(s)m_{x,z}(ds). \tag{11}$$

For each Borel set $E \in \mathcal{B}$, define the map $m(E): X \to Z^*$ by $\langle z, m(E)x \rangle = m_{xz}(E)$. It is easy to see that $m(E): X \to Z^*$ is linear from (11); moreover it is continuous since

$$|m(E)| \leq \bar{\bar{m}}(S) = \sup_{\substack{|x|\leq 1 \\ |z|\leq 1}} |zm(\cdot)x|(S) = \sup_{\substack{|x|\leq 1 \\ |z|\leq 1}} |m_{xz}|(S) =$$

$$\sup_{\substack{|x|\leq 1 \\ |z|\leq 1}} |zL(\cdot)x| = |L|.$$

Thus $m(E) \in \mathbf{L}(X,Z^*)$ for $E \in \mathcal{B}$ and $m \in fa(\mathcal{B}, \mathbf{L}(X,Z^*))$ has finite scalar semivariation $\bar{\bar{m}}(S) = |L|$. Since $\bar{\bar{m}} = \bar{m}_{L(R,L(X,Z^*))}$ is finite the integral in (10) is well-defined for $g \in C_o(S) \subset M(S,R)$ and is a continuous linear map $g \mapsto \int_S m(ds)g(s)$. Now (11) and Proposition 5 imply that

$$zL(g)x = \int_S zm(ds)xg(s) = \langle z, \int_S m(ds)g(s) \cdot x \rangle$$

for every $x \in X$, $z \in Z$. Thus (10) follows.

Conversely suppose $m \in fa(\mathcal{B}, \mathcal{L}(X,Z^*))$ satisfies $zm(\cdot)x \in rcabv(\mathcal{B})$ for every $x \in X$, $z \in Z$. First we must show that m has finite scalar semivariation $\bar{\bar{m}}(S) < +\infty$. Now $\sup_{E \in \mathcal{B}} |zm(E)x| \leq |zm(\cdot)x|(S) < +\infty$ for every $x \in X$, $z \in Z$. Hence successive applications of the uniform boundedness theorem yields $\sup_{E \in \mathcal{B}} |m(E)x| < +\infty$ for every $x \in X$ and $\sup_{E \in \mathcal{B}} |m(E)| < +\infty$, i.e. m is bounded. But then by Proposition 2

$$\bar{\bar{m}}(S) = \sup_{\substack{|x| \leq 1 \\ |z| \leq 1}} |zm(\cdot)x|(S) = \sup_{\substack{|x| \leq 1 \\ |z| \leq 1}} \sup_{E_i \text{ disjoint}} \sum_{i=1}^{n} |zm(E_i)x|$$

$$= \sup_{\substack{|x| \leq 1 \\ |z| \leq 1}} \sup_{E_i \text{ disj}} \Sigma^+ zm(E_i)x - \Sigma^- zm(E_i)x$$

$$= \sup_{\substack{|x| \leq 1 \\ |z| \leq 1}} \sup_{E_i \text{disj}} zm(\cup^+ E_i)x - zm(\cup^- E_i)x$$

$$\leq \sup_{\substack{|x| \leq 1 \\ |z| \leq 1}} 2 \sup_{E \in \mathcal{B}} |zm(E)x| = 2 \sup_{E \in \mathcal{B}} |m(E)| < +\infty,$$

where Σ^+ and $\cup^+$ (Σ^- and $\cup^-$) are taken over those i for which $zm(E_i)x \geq 0$ ($zm(E_i)x < 0$). Thus $\bar{\bar{m}}(s)$ is finite so (10) defines a bounded linear map $L: C_o(S) \to \mathcal{L}(X,Z^*)$. □

We now investigate a more restrictive class of bounded linear maps. For $L \in \mathbb{L}(C_o(S), \mathbb{L}(X,Z^*))$ define the (not necessarily finite) norm

$$||L|| = \sup \left| \sum_{i=1}^{n} L(g_i)x_i \right|$$

where the supremum is over all finite collections $g_1,\ldots,g_n \in C_o(S)_1$ and $x_1,\ldots,x_n \in X_1$ such that the g_i have disjoint support.

Theorem 3. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let X,Z be Banach spaces. There is an isometric isomorphism $L_1 \leftrightarrow m \leftrightarrow L_2$ between the linear maps $L_1\colon C_o(S) \to \mathbb{L}(X,Z^*)$ with $||L_1|| < +\infty$; the measures $m \in fa(\mathcal{B}, \mathbb{L}(X,Z^*))$ with finite semivariation $\bar{m}(S) < +\infty$ for which $zm(\cdot)x \in rcabv(\mathcal{B})$ for every $z \in Z$, $x \in X$; and the bounded linear maps $L_2\colon C_o(S,X) \to Z^*$. The correspondence $L_1 \leftrightarrow m \leftrightarrow L_2$ is given by

$$L_1 g = \int_S m(ds)g(s) \quad , \quad g \in C_o(S) \tag{12}$$

$$L_2 f = \int_S m(ds)f(s) \quad , \quad f \in C_o(S,X) \tag{13}$$

$$L_2(g(\cdot)x) = (L_1 g)x \ , \ g \in C_o(S),\ x \in X. \tag{14}$$

Moreover under this correspondence $||L_1|| = \bar{m}(S) = |L_2|$;

and $zL_2 \in C_o(S,X)^*$ is given by $zL_2 f = \int_S zm(ds)f(s)$

where $zm \in rcabv(\mathcal{B},X^*)$ for every $z \in Z$.

Proof. From Theorem 2 we already have an isomorphism $L_1 \leftrightarrow m$; we must show that $||L_1|| = \bar{m}(S)$ under this correspondence. We first show that $||L_1|| \leq \bar{m}(S)$. Suppose $g_1,\ldots,g_n \in C_o(s)$ have disjoint support with $|g_i|_\infty \leq 1$; $x_1,\ldots,x_n \in X$ with $|x_i| \leq 1$; and $z \in Z$ with $|z| \leq 1$. Then

$$<z, \sum_{i=1}^{n} L_1(g_i)x_i> = \sum_{i=1}^{n} \int_S zm(ds)x_i \cdot g_i(s)$$

$$\leq \sum_{i=1}^{n} |zm(\cdot)x_i|(\text{supp}g_i)$$

$$\leq \sum_{i=1}^{n} |zm|(\text{supp}g_i)$$

where the last step follows from Proposition 2 and $|x_i| \leq 1$. Since $|zm|$ is subadditive by Proposition 3, we have

$$<z, \sum_{i=1}^{n} L_1(g_i)x_i> \leq |zm|(\bigcup_{i=1}^{n} \text{supp}g_i) \leq |zm|(S).$$

Taking the supremum over $|z| \leq 1$, we have, again by Proposition 2,

$$|\sum_{i=1}^{n} L_1(g_i)x_i| \leq \sup_{|z|\leq 1} |zm|(S) = \bar{m}(S).$$

Since this is true for all such collections $\{g_i\}$ and $\{x_i\}$, $||L|| \leq \overline{m}(S)$. We now show $\overline{m}(S) \leq ||L||$. Let $\varepsilon > 0$ be arbitrary, and suppose $E_1,\dots,E_n \in \mathcal{B}$ are disjoint, $|z| \leq 1$, $|x_i| \leq 1$, $i = 1,\dots,n$. By regularity of $zm(\cdot)x_i$, there is a compact $K_i \subset E_i$ such that $|zm(\cdot)x_i|(E_i) < \frac{\varepsilon}{n} + |zm(\cdot)x_i|(K_i)$, $i = 1,\dots,n$. Since the K_i are disjoint, there are disjoint open sets $G_i \supset K_i$. By Urysohn's Lemma there are continuous functions g_i with compact support such that $1_{K_i} \leq g_i \leq 1_{G_i}$. Then

$$\sum_{i=1}^{n} zm(E_i)x_i = \sum_{i=1}^{n} zL(g_i)x_i + \sum_{i=1}^{n} \int (1_{E_i}-g_i)(s)\, zm(ds)x_i$$

$$\leq \sum_{i=1}^{n} zL(g_i)x_i + \sum_{i=1}^{n} \int (1_{E_i}-1_{K_i})(s)\, zm(ds)x_i$$

$$\leq \sum_{i=1}^{n} zL(g_i)x_i + \sum_{i=1}^{n} |zm(\cdot)x|(E_i \setminus K_i) \leq \sum_{i=1}^{n} zL(g_i)x_i+\varepsilon$$

$$\leq \left|\sum_{i=1}^{n} L(g_i)x_i\right| + \varepsilon$$

$$\leq ||L|| + \varepsilon.$$

Taking the supremum over $|z| \leq 1$, finite disjoint collections $\{E_i\}$, $|x_i| \leq 1$ we get $\overline{m}(S) \leq ||L|| + \varepsilon$. Since $\varepsilon > 0$

was arbitrary $\bar{m}(S) \leq \|L\|$ and so $\bar{m}(S) = \|L\|$.

It remains to show how the maps $L_2 \in L(C_o(S,X),Z^*)$ are related to L_1 and m. Now given L_1 or equivalently m, it is immediate from the definition of the integral (6) that (13) defines an $L_2 \in L(C_o(S,X),Z^*)$ with $|L_2| = \bar{m}(S) < +\infty$. Conversely, suppose $L_2 \in L(C_o(S,X),Z^*)$ is given. Then (14) defines a bounded linear map $L_1: C_o(S) \to L(X,Z^*)$, with $|L_1| \leq |L_2|$; moreover it is easy to see that $\|L_1\| \leq |L_2|$. Of course, L_1 uniquely determines a measure $m \in \mathcal{M}(\mathcal{B}, L(X,Z^*))$ with $\bar{m}(S) = \|L_1\| \leq |L_2|$ such that (12) holds. Now suppose $f(\cdot) = \sum_{i=1}^{n} g_i(\cdot)x_i \in C_o(S) \otimes X$; then

$$\int m(ds)f(s) = \sum_{i=1}^{n} L_1(g_i)x_i = \sum_{i=1}^{n} L_2(g_i(\cdot)x_i) = L_2(f).$$

Hence (14) holds for $f(\cdot) \in C_o(S)\otimes X$, and since $C_o(S)\otimes X$ is dense in $C_o(S,X)$ we have

$$|L_2| = \sup_{\substack{f \in C_o(S)\otimes X \\ |f|_\infty \leq 1}} |L_2 f| = \sup_{\substack{f \in C_o(S)\otimes X \\ |f|_\infty \leq 1}} \left|\int m(ds)f(s)\right|$$

$$\leq \sup_{\substack{f \in M(S,X) \\ |f|_\infty \leq 1}} \left|\int m(ds)f(s)\right| = \bar{m}(S).$$

Thus $\bar{m}(S) = |L_2|$.

Finally, it is immediate from Proposition 5 that $zL_2 f = \int_S zm(ds)f(s)$ for $f \in C_o(S,X)$, $z \in Z$. We show that $zm \in rcabv(\mathcal{B},X^*)$ for $z \in Z$. Since $|zm|(S) \leq |z|\cdot\bar{m}(S)$ by Proposition 2, zm has bounded variation. Since for each $x \in X$, $zm(\cdot)x \in rcabv(\mathcal{B})$ we may apply Theorem 1 (with $Y = R$) to get $|zm| \in rcabv(\mathcal{B})$ and $zm \in rcabv(\mathcal{B},X^*)$. □

The following interesting corollary is immediate from $||L_1|| = |L_2|$ in Theorem 3.

<u>Corollary</u>. Let $L_2: C_o(S,X) \to Y$ be linear and bounded, where X,Y are Banach spaces and S is a locally compact Hausdorff space. Then

$$|L_2| = \sup|L_2(\sum_{i=1}^{n} g_i(\cdot)x_i)|,$$

where the supremum is over all finite collections $\{g_1,\ldots,g_n\} \subset C_o(S)$ and all $\{x_1,\ldots,x_n\} \in X$ such that $\{\text{supp}\, g_i\}$ are disjoint and $|g_i|_\infty \leq 1$, $|x_i| \leq 1$.

<u>Proof</u>. Take $Z = Y^*$ and imbed Y in $Z^* = Y^{**}$. Then $L_2 \in L(C_o(S,X),Z^*)$ and the result follows from $||L_1|| = |L_2|$ in Theorem 3. □

We now consider a subspace of linear operators $L_2 \in L(C_o(S,X),Y)$ with even stronger continuity properties, namely those which correspond to bounded linear functionals on $C_o(S, X \otimes_\pi Z)$; equivalently, we shall see that these maps correspond to representing measures $m \in \mathcal{M}(\mathcal{B}, L(X,Z^*))$ which have finite total variation $|m|(S) < +\infty$, so that $m \in rcabv(\mathcal{B}, L(X,Z^*))$. For $L_2 \in L(C_o(S,X),Y)$ we define the (not necessarily finite) norm

$$|||L_2||| = \sup_{\{f_i\}} \sum_{i=1}^{n} |L_2(f_i)|$$

where the supremum is over all finite collections $\{f_1,\ldots,f_n\}$ of functions in $C_o(S,X)$ having disjoint support and $|f_i|_\infty \leq 1$. In applying the definition to $L_1 \in L(C_o(S), L(X,Z^*)) = L(C_o(S,R),Y)$ with $Y = L(X,Z^*)$ we get

$$|||L_1||| = \sup_{\{g_i\}} \sum_{i=1}^{n} |L_1(g_i)|$$

where the supremum is over all finite collections $\{g_1,\ldots,g_n\}$ of functions in $C_o(S)$ having disjoint support and $|g_i|_\infty \leq 1$.

Before proceeding, we should make a few remarks about tensor product spaces. By $X \otimes Z$ we denote a tensor product space of X and Z, which is the vector space of all finite linear combinations $\sum_{i=1}^{n} a_i x_i \otimes z_i$ where

$a_i \in R$, $x_i \in X$, $z_i \in Z$ (of course, a_i, x_i, z_i are not uniquely determined). There is a natural duality between $X \otimes Z$ and $\mathbf{L}(X,Z^*)$ given by

$$\langle \sum_{i=1}^{n} a_i x_i \otimes z_i, L \rangle = \sum_{i=1}^{n} a_i \langle z_i, Lx_i \rangle .$$

Moreover the norm of $L \in \mathbf{L}(X,Z^*)$ as a linear functional on $X \otimes Z$ is precisely its usual operator norm $|L| = \sup_{\substack{|x|\leq 1 \\ |z|\leq 1}} \langle z, Lx \rangle$ when $X \otimes Z$ is made into a normed space $X \otimes_\pi Z$ under the <u>tensor product norm</u> π defined by

$$\pi(u) = \inf\{ \sum_{i=1}^{n} |x_i| \cdot |z_i| : u = \sum_{i=1}^{n} x_i \otimes z_i \}, \ u \in X \otimes Z .$$

It is easy to see that $\pi(x \otimes z) = |x| \cdot |z|$ for $x \in X$, $z \in Z$ (the canonical injection $X \times Z \to X \otimes Z$ is continuous) and in fact π is the strongest norm on $X \otimes Z$ with this property. By $X \hat{\otimes}_\pi Z$ we denote the completion of $X \otimes_\pi Z$ for the π norm. Every $L \in \mathbf{L}(X,Z^*)$ extends to a unique bounded linear functional on $X \hat{\otimes}_\pi Z$ with the same norm. $X \hat{\otimes}_\pi Z$ may be identified more concretely as infinite sums $\sum_{i=1}^{\infty} a_i x_i \otimes z_i$ where $x_i \to 0$ in X, $z_i \to 0$ in Z, and $\sum_{i=1}^{\infty} |a_i| < \infty$ [S71, III.6.4] and we identify $(X \hat{\otimes}_\pi Z)^*$ with $\mathbb{L}(X, Z^*)$ by

$$< \sum_{i=1}^{\infty} a_i x_i \otimes z_i, L> = \sum_{i=1}^{\infty} a_i <z_i, Lx_i>.$$

The following theorem provides an integral representation of $C_o(S, X \hat{\otimes}_\pi Z)^*$.

<u>Theorem 4</u>. Let S be a Hausdorff locally compact space with Borel sets $\mathcal{B}$. Let X, Z be Banach spaces. There is an isometric isomorphism $L_1 \leftrightarrow m \leftrightarrow L_2 \leftrightarrow L_3$ between the linear maps $L_1: C_o(S) \to \mathbb{L}(X, Z^*)$ with $|||L_1||| < +\infty$; the finitely additive measures $m: \mathcal{B} \to \mathbb{L}(X, Z^*)$ with finite variation $|m|(S) < +\infty$ for which $zm(\cdot)x \in rcabv(\mathcal{B})$ for every $z \in Z$, $x \in X$; the linear maps $L_2: C_o(S, X) \to Z^*$ with $|||L_2||| < +\infty$; and the bounded linear functionals $L_3: C_o(S, X \hat{\otimes}_\pi Z) \to R$. The correspondence $L_1 \leftrightarrow m \leftrightarrow L_2 \leftrightarrow L_3$ is given by

$$L_1 g = \int_S m(ds) g(s) \quad , \quad g \in C_o(S) \tag{15}$$

$$L_2 f = \int_S m(ds) f(s) \quad , \quad f \in C_o(S, X) \tag{16}$$

$$L_3 u = \int_S <u(s), m(ds)> \quad , \quad u \in C_o(S, X \hat{\otimes}_\pi Z) \tag{17}$$

$$<z, (L_1 g) x> = <z, L_2(g(\cdot) x)> = L_3(g(\cdot) x \otimes z),$$

$$g \in C_o(S), \ x \in X, \ z \in Z. \tag{18}$$

Under this correspondence $|||L_1||| = |m|(S) = |||L_2||| = |L_3|$, and $m \in rcabv(\mathcal{B}, L(X,Z^*))$.

Proof. From Theorem 3 we already have an isomorphism $L_1 \leftrightarrow m \leftrightarrow L_2$; we must show that the norms are carried over under this correspondence. As in Theorem 2, we assume that $L_1 \leftrightarrow m \leftrightarrow L_2$ with $||L_1|| = \bar{m}(S) = |L_2| < +\infty$.

We first show $|||L_1||| \leq |||L_2|||$. Now if $\{g_1,\ldots,g_n\} \subset C_o(S)_1$ have disjoint support and $|x_i| \leq 1$, then $g_i(\cdot)x_i \in C_o(S,X)$ have disjoint support with $|g_i(\cdot)x_i|_\infty \leq 1$, so

$$\sum_{i=1}^{n} |L_1(g_i)x_i| = \sum_{i=1}^{n} |L_2(g_i(\cdot)x_i)| \leq |||L_2|||.$$

Taking the supremum over $|x_i| \leq 1$ yields $\sum_{i=1}^{n} |L_1(g_i)| \leq |||L_2|||$, and hence $|||L_1||| \leq |||L_2|||$.

Next we show $|||L_2||| \leq |m|(S)$. Let $f_1,\ldots,f_n \in C_o(S,X)$ have disjoint support and $z_1,\ldots,z_n \in Z$ with $|z_i| \leq 1$. Then

$$\sum_{i=1}^{n} z_i L_2(f_i) = \sum_{i=1}^{n} \int_S z_i m(ds) f_i(s) \leq \sum_{i=1}^{n} |z_i m|(\operatorname{supp} f_i)$$

where the last inequality follows from (7) applied to $z \cdot m \in fa(\mathcal{B}, X^*)$. By Propositions 2 and 3 we now have

$$\sum_{i=1}^{n} z_i L_2(f_i) \leq \sum_{i=1}^{n} |m|(\text{supp} f_i) = |m|(\bigcup_{i=1}^{n} \text{supp} f_i) \leq |m|(S).$$

Taking the supremum over $|z_i| \leq 1$ yields $\sum_{i=1}^{n} |L_2 f_i| \leq |m|(S)$, and over $\{f_i\}$ yields $|||L_2||| \leq |m|(S)$.

Now we show $|m|(S) \leq |||L_1|||$. Let $\varepsilon > 0$ be arbitrary, and suppose $E_1,\ldots,E_n \in \mathcal{B}$ are disjoint and $|x_i| \leq 1$, $|z_i| \leq 1$, $i = 1,\ldots,n$. By regularity of $z_i m(\cdot) x_i$, there is a compact $K_i \subset E_i$ such that $|z_i m(\cdot) x_i|(E_i) < \frac{\varepsilon}{n} + |z_i m(\cdot) x_i|(K_i)$, $i = 1,\ldots,n$. Since the K_i are disjoint, there are disjoint open sets $G_i \supset K_i$. Urysohn's Lemma then guarantees the existence of continuous functions g_i with compact support wuch that $1_{K_i} \leq g_i \leq 1_{G_i}$. We have

$$\sum_{i=1}^{n} z_i m(E_i) x_i = \sum_{i=1}^{n} z_i L_1(g_i) x_i + \sum_{i=1}^{n} \int (1_{E_i} - g_i)(s) z_i m(ds) x_i$$

$$\leq \sum_{i=1}^{n} z_i L_1(g_i) x_i + \sum_{i=1}^{n} \int (1_{E_i} - 1_{K_i})(s) z_i m(ds) x_i$$

$$\leq \sum_{i=1}^{n} z_i L_1(g_i) x_i + \sum_{i=1}^{n} |z_i m(\cdot) x_i|(E_i \setminus K_i)$$

$$< \sum_{i=1}^{n} |L_1 g_i| + \varepsilon \leq |||L_1||| + \varepsilon$$

Taking the supremum over $|x_i| \leq 1$ and $|z_i| \leq 1$ yields

$\sum_{i=1}^{n} |m(E_i)| \leq |||L_1||| + \varepsilon$, and the supremum over all disjoint $\{E_1,\ldots,E_n\}$ yields $|m|(S) \leq |||L_1||| + \varepsilon$. Since ε was arbitrary, $|m|(S) \leq |||L_1|||$. We also note that if $|m|(S) < +\infty$, then $m \in rcabv(\mathcal{B}, \mathbb{L}(X,Z^*))$ by Theorem 1.

It remains to show how the maps $L_3 \in C_o(S, X \otimes_\pi Z)^*$ are related to L_1, m, and L_2. Suppose $L_3 \in C_o(S, X \otimes_\pi Z)^*$ is given. Define $L_1: C_o(S) \to \mathbb{L}(X,Z^*)$ by

$\langle z, L_1(g)x\rangle = L_3(g(\cdot)x \otimes z)$, $g \in C_o(S)$, $x \in X$, $z \in Z$. If $g_1,\ldots,g_n \in C_o(S)$ have disjoint support with $|g_i|_\infty \leq 1$, and if $|x_i| \leq 1$, $|z_i| \leq 1$ then $|\sum_{i=1}^{n} g_i(\cdot)x_i \otimes z_i|_\infty \leq 1$ and so

$$\sum_{i=1}^{n} z_i L_1(g_i)x_i = L_3\left(\sum_{i=1}^{n} g_i(\cdot)x_i \otimes z_i\right) \leq |L_3|.$$

Hence $\sum_{i=1}^{n} |L_1 g_i| \leq |L_3|$ and $|||L_1||| \leq |L_3|$. Conversely, let m correspond to L_1; since $|m|(S) = |||L_1||| \leq |L_3| < +\infty$ we know that $m \in rcabv(\mathcal{B}, \mathbb{L}(X,Z^*)) = rcabv(\mathcal{B}, (X \otimes_\pi Z)^*)$. Let us define $W = X \hat{\otimes}_\pi Z$. By Theorem 2 there is an isometric isomorphism between maps $L_3 \in C_o(S,W)^* = \mathbb{L}(C_o(S,W),\mathbb{R})$ and measures $m \in rcabv(\mathcal{B}, \mathbb{L}(W,\mathbb{R})) = rcabv(\mathcal{B}, W^*) = rcabv(\mathcal{B}, \mathbb{L}(X,Z^*))$; under this correspondence

$L_3 u = \int_S \langle u(s), m(ds) \rangle$ and $|L_3| = |m|(S)$. Thus (17) holds and the theorem is proved. □

Thus, to summarize, we have shown that there is a continuous canonical injection

$$C_o(S, X \otimes_\pi Z)^* \to \mathbb{L}(C_o(S,X), Z^*) \to \mathbb{L}(C_o(S), \mathbb{L}(X,Z^*));$$

each of these spaces corresponds to operator-valued measures $m \in \mathcal{M}(\mathcal{B}, \mathbb{L}(X,Z^*))$ which have finite variation $|m|(S)$, finite semivariation $\tilde{m}(S)$, and finite scalar semivariation $\tilde{\tilde{m}}(S)$, respectively. By posing the theory in terms of measures with values in an $\mathbb{L}(X,Z^*)$ space rather than an $\mathbb{L}(X,Y)$ space, we have developed a natural and complete representation of linear operators on $C_o(S,X)$ spaces. Moreover in the case that Y is a dual space (without necessarily being reflexive), it is possible to represent all bounded linear operators $L \in \mathbb{L}(C_o(S,X),Y)$ by operator-valued measures $m \in \mathcal{M}(\mathcal{B}, \mathbb{L}(X,Y))$ with values in $\mathbb{L}(X,Y)$ rather than in $\mathbb{L}(X,Y^{**})$; this is important for the quantum applications we have in mind, where we would like to represent $\mathbb{L}(C_o(S), \mathbb{L}_s(H))$ operators by $\mathbb{L}_s(H)$-valued operator measures rather than $\mathbb{L}_s(H)^{**}$-valued measures. We now give two examples to show how the usual representation theorems follow as corollaries by considering Y as a subspace of Y^{**}.

Corollary [D67, III.19.5]. Let S be a locally compact Hausdorff space and X,Y Banach spaces. There is an isometric isomorphism between bounded linear maps $L: C_o(S,X) \to Y$ and finitely additive maps $m: \mathcal{B} \to \mathbb{L}(X,Y^{**})$ with finite semivariation $\tilde{m}(s) < +\infty$ for which

1) $y^*m(\cdot) \in rcabv(\mathcal{B},X^*)$ for every $y^* \in Y^*$

2) $y^* \mapsto y^*m$ is continuous for the weak * topologies on Y^*, $rcabv(\mathcal{B},X^*) \cong C_o(S,X)^*$. This correspondence $L \leftrightarrow m$ is given by $Lf = \int m(ds)f(s)$ for $f \in C_o(S,X)$, and $|L| = \tilde{m}(S)$.

Proof. Set $Z = Y^*$ and consider Y as a norm-closed subspace of Z^*. An element y^{**} of Y^{**} belongs to Y iff the linear functional $y^* \mapsto y^{**}(y^*)$ is continuous for the w* topology on Y^*. Hence the maps $L \in \mathbb{L}(C_o(S,X),Y^{**})$ which correspond to maps $L \in \mathbb{L}(C_o(S,X),Y)$ are precisely the maps for which $z \mapsto \langle z,Lf\rangle$ are continuous in the w*-topology on $Z = Y^*$ for every $f \in C_o(S,X)$, or equivalently those maps L for which $z \mapsto L^*z$ is continuous for the w* topologies on $Z = Y^*$ and $C_o(S,X)^*$. The results then follow directly from Theorem 3, where we note that when $L \leftrightarrow m$,

$$\langle f,L^*z\rangle = \langle z,Lf\rangle = \int_S zm(ds)f(s). \quad \square$$

Corollary 2 [Do71, 2.2]. [illegible]

$L: C_o(S,X) \to Y$ can be uniquely represented as

$$Lf = \int_S m(ds)f(s) \ , \quad f \in C_o(S,X)$$

where $m \in fa(\mathcal{B}, \mathbf{L}(X,Y))$ has finite semivariation $\overline{m}(s) < +\infty$ and satisfies $y^*m(\cdot)x \in rcabv(\mathcal{B})$ for every $x \in X$, $y^* \in Y$, if and only if for every $x \in X$ the bounded linear operator $L_x: C_o(S) \to Y: g(\cdot) \mapsto L(g(\ )x)$ is weakly compact. In that case $|L| = \overline{m}(s)$ and L^*y^* is given by $(L^*y^*)f = \int_S y^*m(ds)f(s)$ where $y^*m \in rcabv(\mathcal{B},X^*)$ for every $y^* \in Y^*$.

Remark. Suppose $Y = Z^*$ is a dual space. Then by Theorem 2 every $L \in \mathbf{L}(C_o(S,X),Y)$ has a representing measure $m \in \mathcal{M}(\mathcal{B}, \mathbf{L}(X,Y))$. What Corollary 2 says is that the representing measure m actually satisfies $y^*m(\cdot)x \in rcabv(\mathcal{B})$ for every $y^* \in Y^*$ (and not just for every y^* belonging to the canonical image of Z in $Z^{**} = Y^*$), if and only if L_x is weakly compact $C_o(S) \to Y$ for every $x \in X$; i.e. in this case we have (in our notation) $m \in \mathcal{M}(\mathcal{B}, \mathbf{L}(X,Y^{**}))$ where Y is injected into its bidual Y^{**}.

Proof. Again, let $Z = Y^*$ and define $J: Y \to Y^{**}$ to be the canonical injection of Y into $Y^{**} = Z^*$. The bounded linear operator $L_x: C_o(S) \to Y$ is weakly compact iff

$L_x^{**}: C_o(S)^{**} \to Y^{**}$ has image $L_x^{**}C_o(X)^{**}$ which is a subset of JY [DS, VI.4.2]. First, suppose L_x is weakly compact, so that $L_x^{**}: C_o(S)^{**} \to JY$ for every x. Now the map $\lambda \mapsto \lambda(E)$ is an element of $C_o(S)^{**}$ (where we have identified $\lambda \in rcabv(\mathcal{B}) \cong C_o(S)^*$) for $E \in \mathcal{B}$, and

$$L_x^{**}(\lambda \mapsto \lambda(E)) = (z \mapsto \langle z, m(E)x\rangle) \in Y^{**}$$

where $m \in \mathcal{M}(\mathcal{B}, \mathbb{L}(X,Z^*))$ is the representing measure of $JL: C_o(S,X) \to Y^{**}$. Since L_x is weakly compact, $z \mapsto \langle z, m(E)x\rangle$ must actually belong to $JY \subset Y^{**}$, that is $z \mapsto \langle z, m(E)x\rangle$ is w* continuous and $m(E)x \in JY$. Hence m has values in $\mathbb{L}(X,JY)$ rather than just $\mathbb{L}(X,Y^{**})$.

Conversely if $m \in \mathcal{M}(\mathcal{B}, \mathbb{L}(X,JY))$ represents an operator $L \in \mathbb{L}(C_o(S,X),Y)$ by

$$JLf = \int m(ds)f(s),$$

then the map $L_x^*: Y^* \to C_o(S)^* \cong rcabv(\mathcal{B}): z \mapsto \langle z, m(\cdot)x\rangle$ is continuous for the weak topology on $Z = Y^*$ and the weak * topology on $C_o(S)^* \cong rcabv(\mathcal{B})$ since $m(E)x \in JY$ for every $E \in \mathcal{B}$, $x \in X$. Hence by [DS, VI.4.7], L_x is weakly compact. □

IX. Optimal Quantum Estimation

Abstract. Duality techniques are applied to the problem of specifying the optimal estimator for quantum estimation. Existence of the optimal estimator is established and necessary and sufficient conditions for optimality are derived.

1. Introduction

The mathematical characterization of optimal estimation in the Bayesian approach to statistical inference is a well-known result in classical estimation theory. In this paper we consider estimation theory for quantum systems.

In the classical formulation of Bayesian estimation theory it is desired to estimate the unknown value of a random parameter $s \in S$ based on observation of a random variable whose probability distribution depends on the value s. The procedure for determining an estimated parameter value s, as a function of the experimental observation, represents a decision strategy; the problem is to find the optimal decision strategy.

In the quantum formulation of the estimation problem, each parameter $s \in S$ corresponds to a state $\rho(s)$ of the quantum system. The aim is to estimate the value of s by performing a measurement on the quantum system. However, the quantum situation precludes exhaustive measurements of the system. This contrasts with the classical situation, where it is possible in principle to measure all relevant variables determining the state of the system and to specify meaningful probability density functions for the resulting values. For the quantum estimation problem it is necessary to specify not only the best procedure for processing experimental data, but also what to measure in the first place. Hence the quantum decision problem is to determine an optimal measurement procedure, or, in mathematical terms, to determine the optimal probability operator measure corresponding to a measurement procedure.

We now formulate the quantum estimation problem. Let H be a separable complex Hilbert space corresponding to the physical variables of the system under consideration. Let S be a parameter space, with measurable sets $\mathcal{B}$. Each $s \in S$ specifies a state $\rho(s)$ of the quantum system, i.e. every $\rho(s)$ is a nonnegative-definite selfadjoint trace-class operator on H with trace 1. A general decision strategy is determined by a measurement process $m(\cdot)$, where $m: \mathcal{B} \to \mathcal{L}_s(H)$ is a positive operator-valued measure (POM) on the measurable space $(S, \mathcal{B})$ -- $m(E) \in \mathcal{L}_s(H)_+$ is a positive selfadjoint bounded linear operator on H for every $E \in \mathcal{B}$, $m(S) = I$, and $m(\cdot)$ is countably additive for the weak operator topology on $\mathcal{L}_s(H)$. The measurement process yields an estimate of the unknown parameter; for a given value s of the parameter and a given measurable set $E \in \mathcal{B}$, the probability that the estimated value $\hat{s}$ lies in E is given by

$$\Pr\{\hat{s} \in E | s\} = \mathrm{tr}[\rho(s)m(E)]. \tag{1}$$

Finally, we assume that there is a cost function $c(s,\hat{s})$ which specifies the relative cost of an estimate $\hat{s}$ when the true value of the parameter is s.

For a specified decision procedure corresponding to the POM $m(\cdot)$, the risk function is the conditional expected cost given the parameter value s, i.e.

$$R_m(s) = \operatorname{tr}[\rho(s)\int_S c(s,t)m(dt)]. \qquad (2)$$

If now μ is a probability measure on $(S,\mathcal{B})$ which specifies a prior distribution for the parameter value s, the Bayes cost is the posterior expected cost

$$R_m = \int_S R_m(s)\mu(ds). \qquad (3)$$

The quantum estimation problem is to find a POM $m(\cdot)$ for which the Bayes expected cost R_m is minimum.

A formal interchange of the order of integration yields

$$R_m = \operatorname{tr}\int_S f(s)m(ds) \qquad (4)$$

where $f(s) = \int_S c(t,s)\rho(t)\mu(dt)$. Thus, formally at least, the problem is to minimize the linear functional (4) over all POM's $m(\cdot)$ on $(S,\mathcal{B})$. We shall apply duality theory for optimization problems to prove existence of a solution and to determine necessary and sufficient conditions for a decision strategy to be optimal, much as in the detection problem with a finite number of hypotheses (a special case of the estimation problem where S is a finite set). Of course we must first rigorously define what is meant by an integral of the form (4); note that both the integrand and the measure are operator-valued. We must then show the equivalence of (3) and (4); this entails proving a Fubini-type theorem for operator-valued measures. Finally, we must identify an appropriate dual space for POM's consistent with the linear functional (4), so that a dual problem can be formulated.

Before proceeding, we summarize the results in an informal way to be made precise later. Essentially, we shall see that there is always an optimal solution, and that necessary and sufficient conditions for a POM m to be optimal are

$$\int_S f(s)m(ds) \leq f(t) \quad \text{for every} \quad t \in S.$$

It then turns out that $\int_S f(s)m(ds)$ belongs to $\mathcal{T}_s(H)$ (that is, selfadjoint) and the minimum Bayes posterior expected cost is

$$R_m = \operatorname{tr}\int_S f(s)m(ds).$$

2. Integration of real-valued functions with respect to operator-valued measures

In quantum mechanical measurement theory, it is nearly always the case that physical quantities have values in a locally compact Hausdorff space S, e.g. a subset of R^n. The integration theory may be extended to more general measurable spaces; but since for duality purposes we wish to interpret operator-valued measures on S as continuous linear maps, we shall always assume that the parameter space S is a locally compact space with the induced σ-algebra of Borel sets, and that the operator-valued measure is regular. In particular, if S is second countable then S is countable at infinity (the one-point compactification $S \cup \{\infty\}$ has a countable neighborhood basis at ∞) and every complex Borel measure on S is regular; also S is a complete separable metric space, so that the Baire sets and Borel sets coincide.

Let H be a complex Hilbert space. A (self-adjoint) operator-valued regular Borel measure on S is a map $m: \mathcal{B} \to \mathcal{L}_s(H)$ such that $\langle m(\cdot)\phi|\psi\rangle$ is a regular Borel measure on S for every $\phi, \psi \in H$. In particular, since for a vector-valued measure countable additivity is equivalent to weak countable additivity [DS, IV.10.1],

$m(\cdot)\phi$ is a (norm-) countably additive H-valued measure for every $\phi \in H$; hence whenever $\{E_n\}$ is a countable collection of disjoint subsets in $\mathcal{B}$ then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n),$$

where the sum is convergent in the strong operator topology. We denote by $\mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ the real linear space of all operator-valued regular Borel measures on S. We define <u>scalar semivariation</u> of $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ to be the norm

$$\bar{\bar{m}}(S) = \sup_{|\phi| \leq 1} |\langle m(\cdot)\phi|\phi\rangle|(s) \qquad (5)$$

where $|\langle m(\cdot)\phi|\phi\rangle|$ denotes the total variation measure of the real-valued Borel measure $E \mapsto \langle m(E)\phi|\phi\rangle$. The scalar semivariation is always finite, as proved in Theorem VIII.2 by the uniform boundedness theorem (see "Operator-Valued Measures" for alternative definitions of $\bar{\bar{m}}(s)$; note that when $m(\cdot)$ is self-adjoint valued the identity $\bar{\bar{m}}(s) = \sup_{|\phi| \leq 1} \sup_{|\psi| \leq 1} |\langle m(\cdot)\phi|\psi\rangle|(s)$ reduces to (5)).

A <u>positive</u> operator-valued regular Borel measure is a measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ which satisfies

$$m(E) \geq 0 \qquad \forall E \in \mathcal{B},$$

where by $m(E) \geq 0$ we mean $m(E)$ belongs to the positive cone $\mathcal{L}_s(H)_+$ of all nonnegative-definite operators. A <u>probability operator measure</u> (POM) is a positive operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ which satisfies

$$m(S) = I.$$

If m is a POM then every $\langle m(\cdot)\phi|\phi\rangle$ is a probability measure on S and $\bar{\bar{m}}(S) = 1$. In particular, a <u>resolution of the identity</u> is an $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ which satisfies $m(S) = I$ and $m(E)m(F) = 0$ whenever $E \cap F = \emptyset$; it is then true that $m(\cdot)$ is projection-valued and satisfies

$$m(E \cap F) = m(E)m(F), \qquad E,F \in \mathcal{B}.^{+}$$

We now consider integration of real-valued functions with respect to operator-valued measures. Basically, we identify the regular Borel operator-valued measures

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$^{+}$<u>Proof</u>. First, $m(\cdot)$ is projection valued since by finite additivity

$$m(E) = m(E)m(S) = m(E)[m(E)+m(S\setminus E)] = m(E)^2+m(E)m(S\setminus E),$$

and the last term is 0 since $E \cap (S\setminus E) = \emptyset$. Moreover we have by finite additivity

$$m(E)m(F) = [m(E\cap F)+m(E\setminus F)]\cdot[m(E\cap F)+m(F\setminus E)]$$
$$= m(E\cap F)^2+m(E\cap F)m(F\setminus E)+m(E\setminus F)m(E\cap F)+m(E\setminus F)m(F\setminus E),$$

where the last three terms are 0 since they have pairwise disjoint sets.

$m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ with the bounded linear operators $L: C_o(S) \to \mathcal{L}_s(H)$, using the integration theory of Chapter VIII to get a generalization of the Riesz Representation Theorem.

1. Theorem. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let H be a Hilbert space. There is an isometric isomorphism $m \leftrightarrow L$ between the operator-valued regular Borel measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ and the bounded linear maps $L \in \mathbf{L}(C_o(S), \mathcal{L}_s(H))$. The correspondnece $m \leftrightarrow L$ is given by

$$L(g) = \int_S g(s)m(ds), \quad g \in C_o(S) \tag{6}$$

where the integral is well-defined for $g(\cdot) \in M(S)$ (bounded and totally measurable maps $g: S \to R$) and is convergent for the supremum norm on $M(S)$. If $m \leftrightarrow L$, then $\bar{\bar{m}}(S) = |L|$ and $\langle L(g)\phi|\psi\rangle = \int_S g(s)\langle m(\cdot)\phi|\psi\rangle(ds)$ for every $\phi, \psi \in H$. Moreover L is positive (maps $C_o(S)_+$ into $\mathcal{L}_S(H)_+$) iff m is a positive measure; L is positive and $L(1) = I$ iff m is a POM; and L is an algebra homomorphism with $L(1) = I$ iff m is a resolution of the identity, in which case L is actually an isometric algebra homomorphism of $C_o(S)$ onto a norm-closed subalgebra of $\mathcal{L}_s(H)$.

Proof. The correspondence $L \leftrightarrow m$ is immediate from Theorem VIII.2. If m is a positive measure, then $\langle m(E)\phi|\phi\rangle \geq 0$ for every $E \in \mathcal{B}$ and $\phi \in H$, so $\langle L(g)\phi|\phi\rangle = \int_S g(s)\langle m(\cdot)\phi|\phi\rangle(ds) \geq 0$ whenever $g \geq 0$, $\phi \in H$ and L is positive. Conversely, if L is positive then $\langle m(\cdot)\phi|\phi\rangle$ is a positive real-valued measure for every $\phi \in H$, so $m(\cdot)$ is positive. Similarly, L is positive and $L(1) = I$ iff m is a POM. It only remains to verify the final statement of the theorem.

Suppose $m(\cdot)$ is a resolution of the identity. If $g_1(s) = \sum_{j=1}^{n} a_j 1_{E_j}(s)$ and $g_2(s) = \sum_{j=1}^{m} b_j 1_{F_j}(s)$ are simple functions, where $\{E_1,\ldots,E_n\}$ and $\{F_1,\ldots,F_n\}$ are each finite disjoint subcollections of $\mathcal{B}$, then

$$\int g_1(s)m(ds)\cdot\int g_2(s)m(ds) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j b_k m(E_j)m(F_k)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m} a_j b_k m(E_1 \cap F_k)$$

$$= \int g_1(s)g_2(s)m(ds).$$

Hence $g \mapsto \int g(s)m(ds)$ is an algebra homomorphism from the algebra of simple functions on S into $\mathcal{L}_s(H)$. Moreover we show tha. the homomorphism is isometric on simple functions. Clearly

$$|\int g(s)m(ds)| \leq \bar{\bar{m}}(s)|g|_\infty = |g|_\infty.$$

Conversely, for $g = \sum_{j=1}^{n} a_j 1_{E_j}$ we may choose ϕ_j to be in the range of the projection $m(E_j)$, with $|\phi_j| = 1$, to get

$$|\int g(s)m(ds)| \geq \max_{j=1,\ldots,n} \langle \int g(s)m(ds)\cdot\phi_j|\phi_j\rangle$$

$$= \max_{j=1,\ldots,n} |a_j|\langle m(E_i)\phi_j|\phi_j\rangle$$

$$= \max_{j=1,\ldots,n} |a_j| = |g|_\infty.$$

Thus $g \mapsto \int g(s)m(ds)$ is isometric on simple functions. Since simple functions are uniformly dense in $M(S)$, it follows by taking limits of simple functions that $\int g_1(s)m(ds)\cdot\int g_2(s)m(ds) = \int g_1(s)g_2(s)m(ds)$ and $|\int g_1(s)m(ds)| = |g_1|_\infty$ for every $g_1, g_2 \in M(S)$. Of course, the same is then true for $g_1, g_2 \in C_o(S) \subset M(S)$. Since $C_o(S)$ is complete, it follows that L is an isometric isomorphism of $C_o(S)$ onto a closed subalgebra of $\mathcal{L}_s(H)$.

Now assume that L is an algebra homomorphism and $L(1) = I$. Clearly $m(S) = L(1) = I$. Since $L(g^2) = L(g)^2 \geq 0$ for every $g \in C_o(S)$, L and hence m are positive. Let

$M_1 = \{g \in M(S): \int g(s)m(ds) \cdot \int h(s)m(ds) = \int g(s)h(s)m(ds)$
for every $h \in C_o(S)\}$.

Then M_1 contains $C_o(S)$. Now if $g_n \in M(S)$ is a uniformly bounded sequence which converges pointwise to g_o then $\int g_n(s)m(ds)$ converges in the weak operator topology to $\int g_o(s)m(ds)$ by the dominated convergence theorem applied to each of the regular Borel measures $\langle m(\cdot)\phi|\psi\rangle$, $\phi,\psi \in H$ (the integrals actually converge for the norm topology on $\mathcal{L}_s(H)$ whenever $|g_n - g_o|_\infty \to 0$). Hence M_1 is closed under pointwise convergence of uniformly bounded sequences, and so equals all of M(S) by regularity. Similarly, let

$M_2 = \{h \in M(S): \int g(s)m(ds) \cdot \int h(s)m(ds) = \int g(s)h(s)m(ds)$
for every $g \in M(S)\}$.

Then M_2 contains $C_o(S)$ and must therefore equal all of M(S). It is now immediate that whenever E,F are disjoint sets in $\mathcal{B}$ then

$$m(E)m(F) = \int 1_E dm \cdot \int 1_F dm = \int 1_{E\cap F}(s)m(ds) = 0.$$

Thus m is a resolution of the identity. □

Remark. Since every real-linear map from a real-linear subspace of a complex space into another real-linear subspace of a complex space corresponds to a unique "Hermitian" complex-linear map on the complex linear spaces, we could just as easily identify the (self-adjoint) operator-valued regular measures $\mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ with the complex-linear maps $L: C_o(S,\mathbb{C}) \to \mathcal{L}(H)$ which satisfy

$$L(g) = L(\bar{g})^*, \quad g \in C_o(S,\mathbb{C}).$$

3. Integration of $\mathcal{T}_s(H)$-valued functions

We now consider $\mathcal{L}(H)$ as a subspace of the "operations" $\mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$, that is, bounded linear maps from $\mathcal{T}(H)$ into $\mathcal{T}(H)$. This is possible because if $A \in \mathcal{T}(H)$ and $B \in \mathcal{L}(H)$ then AB and BA belong to $\mathcal{T}(H)$ and

$$|AB|_{tr} \leq |A|_{tr}|B|$$

$$|BA|_{tr} \leq |A|_{tr}|B| \qquad 97)$$

$$tr(AB) = tr(BA).$$

Then every $B \in \mathcal{L}(H)$ defines a bounded linear function $L_B: \mathcal{T}(H) \to \mathcal{T}(H)$ by

$$L_B(A) = AB, \quad A \in \mathcal{T}(H)$$

with $|B| = |L_B|$.[+] In particular, $A \mapsto trAB$ defines a continuous (complex-) linear functional on $A \in \mathcal{T}(H)$, and in fact every linear functional in $\mathcal{T}(H)^*$ is of this form for some $B \in \mathcal{L}(H)$ (cf Section VII.4). We note that if A and B are selfadjoint then trAB is real

[+]From (7), $|L_B| \leq |B|$. Conversely, if $\phi, \psi \in H$ and $|\phi| \leq 1$, $|\psi| \leq 1$ then $|L_B| \geq |(\phi \otimes \psi)B|_{tr} = |\phi \otimes B^*\psi|_{tr} = |\phi| \cdot |B^*\psi| \geq |\langle B\phi|\psi\rangle|$; hence $|L_B| \geq |B|$.

(although it is not necessarily true that AB is self-adjoint unless $AB = BA$). Thus, it is possible to identify the space $\mathcal{T}_s(H)^*$ of real-linear continuous functionals on $\mathcal{T}_s(H)$ with $\mathcal{L}_s(H)$, again under the pairing $\langle A,B\rangle = \mathrm{tr}AB$, $A \in \mathcal{T}_s(H)$, $B \in \mathcal{L}_s(H)$. For our purposes we shall be especially interested in this latter duality between the spaces $\mathcal{T}_s(H)$ and $\mathcal{L}_s(H)$, which we shall use to formulate a dual problem for the quantum estimation situation. However, we will also need to consider $\mathcal{L}_s(H)$ as a subspace of $\mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$ so that we may integrate $\mathcal{T}_s(H)$-valued functions on S with respect to $\mathcal{L}_s(H)$-valued operator measures to get an element of $\mathcal{T}(H)$.

Suppose $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ is an operator-valued regular Borel measure, and $f: S \to \mathcal{T}_s(H)$ is a simple function with finite range of the form

$$f(s) = \sum_{j=1}^{n} 1_{E_j}(s)\rho_j$$

where $\rho_j \in \mathcal{T}_s(H)$ and E_j are disjoint sets in $\mathcal{B}$, that is $f \in \mathcal{B} \otimes \mathcal{T}_s(H)$.[+] Then we may unambiguously (by finite additivity of m) define the integral

$$\int_S f(s)m(ds) = \sum_{j=1}^{n} m(E_j)\rho_j .$$

[+]See Chapter VIII.

The question, of course, is to what class of functions can we properly extend the definition of the integral? Now *if* m has finite total variation $|m|(s)$, then the map $f \mapsto \int_S f(s)m(ds)$ is continuous for the supremum norm $|f|_\infty = \sup_s |f(s)|_{tr}$ on $\mathcal{B} \otimes \mathcal{T}_s(H)$, so that by continuity the integral map extends to a continuous linear map from the closure $M(S, \mathcal{T}_s(H))$ of $\mathcal{B} \otimes \mathcal{T}_s(H)$ with the $|\cdot|_\infty$ norm into $\mathcal{T}(H)$. In particular, the integral $\int_S f(s)m(ds)$ is well-defined (as the limit of the integrals of uniformly convergent simple functions) for every bounded and continuous function $f: S \to \mathcal{T}_s(H)$. Unfortunately, it is *not* the case that an arbitrary POM m has finite total variation. Since we wish to consider general quantum measurement processes as represented by POM's m (in particular, resolutions of the identity), we can only assume that m has finite scalar semivariation $\bar{\bar{m}}(S) < +\infty$. Hence we must put stronger restrictions on the class of functions which we integrate.

We may consider every $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ as an element of $\mathcal{M}(\mathcal{B}, \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$ in the obvious way: for $E \in \mathcal{B}$, $\rho \in \mathcal{T}(H)$ we put

$$m(E)(\rho) = \rho m(E).$$

Moreover, the scalar semivariation of m as an element of $\mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ is the same as the scalar semivariation of m as an element of $\mathcal{M}(\mathcal{B}, \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$, since the norm of $B \in \mathcal{L}_s(H)$ is the same as the norm of B as the map $\rho \mapsto \rho B$ in $\mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$. By the representation Theorem VIII.2 we may uniquely identify $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)) \subset \mathcal{M}(\mathcal{B}, \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$ with a linear operator $L \in \mathcal{L}(C_o(S), \mathcal{L}_s(H)) \subset \mathcal{L}(C_o(S), \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$. Now it is well-known that for Banach spaces X, Y, Z we may identify [T67, III.43.12]

$$\mathcal{L}(X \hat{\otimes}_\pi Y, Z) \cong \beta(X, Y\cdot, Z) \cong \mathcal{L}(X, \mathcal{L}(Y, Z))$$

where $X \hat{\otimes}_\pi Y$ denotes the completion of the tensor product space $X \otimes Y$ for the projective tensor product norm

$$|f|_\pi = \inf\{\sum_{j=1}^n |x_j| \cdot |y_j| : f = \sum_{j=1}^n x_j \otimes y_j\}, \quad f \in X \otimes Y;$$

$\beta(X, Y:Z)$ denotes the space of continuous bilinear forms $B: X \times Y \to Z$ with norm

$$|B|_{\beta(X,Y;Z)} = \sup_{|x| \leq 1} \sup_{|y| \leq 1} |B(x,y)|;$$

and $\mathcal{L}(X, \mathcal{L}(Y,Z))$ of course denotes the space of continuous linear maps $L_2: X \to \mathcal{L}(X,Z)$ with norm

$$|L_2|_{(X, \mathcal{L}(Y,Z))} = \sup_{|x| \leq 1} |L_2 x|_{\mathcal{L}(Y,Z)} .$$

The identification $L_1 \leftrightarrow B \leftrightarrow L_2$ is given by

$$L_1(x \otimes y) = B(x,y) = L_2(x)y.$$

In our case we take $X = M(S)$, $Y = Z = \mathcal{T}(H)$ to identify

$$\mathcal{L}(M(S) \hat{\otimes}_\pi \mathcal{T}(H), \mathcal{T}(H)) \cong \mathcal{L}(M(S), \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))). \quad (8)$$

Since the map $g \mapsto \int g(s)m(ds)$ is continuous from $M(S)$ into $\mathcal{L}_\pi(H) \subset \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$ for every $m \in \mathcal{M}(\mathcal{S}, \mathcal{L}_s(H))$, we see that we may identify m with a continuous linear map $f \mapsto \int f dm$ for $f \in M(S) \hat{\otimes}_\pi \mathcal{T}(H)$. Clearly if $f \in M(S) \otimes \mathcal{T}(H)$, that is if

$$f(s) = \sum_{j=1}^{n} g_j(s)\rho_j$$

for $g_j \in M(S)$ and $\rho_j \in \mathcal{T}(H)$, then

$$\int_S f(s)m(ds) = \sum_{j=1}^{n} \rho_j \int g_j(s)m(ds).$$

Moreover the map $f \mapsto \int_S f(s)m(ds)$ is continuous and linear for the $\|\cdot\|_\pi$-norm on $M(S) \otimes \mathcal{T}(H)$, so we may extend the definition of the integral to elements of the completion $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ by setting

$$\int f m(ds) = \lim_{n \to \infty} \int f_n(s)m(ds)$$

where $f_n \in M(S) \otimes \mathcal{T}(H)$ and $f_n \to f$ in the $\|\cdot\|_\pi$-norm. In the section which follows we prove that the completions $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ and $C_o(S) \hat{\otimes}_\pi \mathcal{T}(H)$ may be identified with subspaces of $M(S, \mathcal{T}(H))$ and $C_o(S, \mathcal{T}(H))$ respectively, i.e. we can treat elements f of $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ as totally measurable functions $f: S \to \mathcal{T}(H)$. We shall show that under suitable conditions the maps $f: S \to \mathcal{T}(H)$ we are interested in for quantum estimation problems do belong to $C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$, and hence are integrable against arbitrary operator-valued measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{T}_s(H))$.

2. Theorem. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let H be a Hilbert space. There is an isometric isomorphism $L_1 \leftrightarrow m \leftrightarrow L_2$ between the bounded linear maps $L_1: C_o(S) \hat{\otimes}_\pi \mathcal{T}(H) \to \mathcal{T}(H)$, the operator-valued regular Borel measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$, and the bounded linear maps $L_2: C_o(S) \to \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$. The correspondence $L_1 \leftrightarrow m \leftrightarrow L_2$ is given by the relations

$$L_1(f) = \int_S f(s)m(ds), \quad f \in C_o(S) \hat{\otimes}_\pi \mathcal{T}(H)$$

$$L_2(g)\rho = L_1(g(\cdot)\rho) = \rho\int g(s)m(ds), \quad g \in C_o(S), \ \rho \in \mathcal{T}(H)$$

and under this correspondence $\|L_1\| = \bar{\bar{m}}(S) = \|L_2\|$. Moreover the integral $\int_S f(s)m(ds)$ is well-defined for every $f \in M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ [illegible] and linear from $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ into $\mathcal{T}(H)$.

Proof. From Theorem 4 of the section which follows we may identify $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$, and hence $C_o(S) \hat{\otimes}_\pi \mathcal{T}(H)$, as a subspace of the totally measurable (that is, uniform limits of simple functions) functions $f: S \to \mathcal{T}(H)$. The results then follow from Theorem VIII.2 and the isometric isomorphism

$$\mathcal{L}(C_o(S) \hat{\otimes}_\pi \mathcal{T}(H), \mathcal{T}(H)) \cong \mathcal{L}(C_o(S), \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$$

as in (8). We note that by a $\mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$-valued regular Borel measure we mean a map $m: \mathcal{B} \to \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$ for which $\mathrm{tr}Cm(\cdot)\rho$ is a complex regular Borel measure for every $\rho \in \mathcal{T}(H)$, $C \in \mathcal{K}(H)$, where in the application of Theorem VIII.2 we have taken $X = \mathcal{T}(H)$, $Z = \mathcal{K}(H)$, $Z^* = \mathcal{T}(H)$. In particular this is satisfied for every $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$. □

3. Corollary. If $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ then the integral $\int_S f(s)m(ds)$ is well-defined for every $f \in M(S) \hat{\otimes}_\pi \mathcal{T}(H)$.

Remark. It should be emphasized that the $|\cdot|_\pi$ norm is strictly stronger than the supremum norm $|f|_\infty = \sup_s |f(s)|_{tr}$. Hence, if $f_n, f \in M(S) \otimes_\pi \mathcal{T}(H)$ satisfy $f_n(s) \to f(s)$ uniformly, it is not necessarily true that $|f_n - f|_\pi \to 0$ or that $\int_S f_n(s)m(ds) \to \int_S f(s)m(ds)$.

4. $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ is a subspace of $M(S, \mathcal{T}(H))$

The purpose of this section is to show that we may identify the tensor product space $M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ with a subspace of the totally measurable functions $f: S \to \mathcal{T}_s(H)$ in a well-defined way. The reason why this is important is that the functions $f \in M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ are those for which we may legitimately define an integral $\int_S f(s)m(ds)$ for arbitrary operator-valued measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$, since $f \mapsto \int_S f(s)m(ds)$ is a continuous linear map from $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ into $\mathcal{T}(H)$. In particular, it is obvious that $C_o(S) \otimes \mathcal{T}_s(H)$ may be identified with a subspace of continuous functions $f: S \to \mathcal{T}_s(H)$ in a well-defined way, just as it is obvious how to define the integral $\int_S f(s)m(ds)$ for finite linear combinations $f(s) = \sum_{j=1}^n g_j(s)\rho_j \in C_o(S) \otimes \mathcal{T}_s(H)$. What is <u>not</u> obvious is that the <u>completion</u> of $C_o(S) \hat{\otimes} \mathcal{T}_s(H)$ in the tensor product norm π may be identified with a subspace of continuous functions $f: S \to \mathcal{T}_s(H)$.

Before proceeding, we review some basic facts about tensor product spaces. Let X, Z be normed spaces. By $X \otimes Z$ we denote a tensor product space of X and Z, which is the vector space of all linear finite combinations

$\sum_{j=1}^{n} a_j x_j \otimes z_j$ where $a_j \in R$, $x_j \in X$, $z_j \in Z$ (of course, a_j, x_j, z_j are not uniquely determined). There is a natural duality between $X \otimes Z$ and $\mathcal{L}(X, Z^*)$ given by

$$< \sum_{j=1}^{n} a_j x_j \otimes z_j, L> = \sum_{j=1}^{n} a_j <z_j, Lx_j>.$$

Moreover the norm of $L \in \mathcal{L}(X,Z^*)$ as a linear functional on $X \otimes Z$ is precisely its usual operator norm $|L| = \sup_{|z| \leq 1} \sup_{|x| \leq 1} <z,Lx>$ when $X \otimes Z$ is made into a normed space $X \otimes_\pi Z$ under the <u>tensor product norm</u> $|\cdot|_\pi$ defined by

$$|f|_\pi = \inf\{\sum_{j=1}^{n} |x_j| \cdot |z_j| : f = \sum_{j=1}^{n} x_j \otimes z_j\}, \ f \in X \otimes Z.$$

It is easy to see that $|x \otimes z|_\pi = |x| \cdot |z|$ for $x \in X$, $z \in Z$ (the canonical injection $X \times Z \to X \otimes Z$ is continuous with norm 1) and in fact $|\cdot|_\pi$ is the strongest norm on $X \otimes Z$ with this property. By $X \hat{\otimes}_\pi Z$ we denote the completion of $X \otimes_\pi Z$ for the $|\cdot|_\pi$ norm. Every $L \in \mathcal{L}(X,Z^*)$ extends to a unique bounded linear functional on $X \hat{\otimes}_\pi Z$ with the same norm as its operator norm, so that we identify $(X \hat{\otimes}_\pi Z)^* \cong \mathcal{L}(X,Z^*)$. The space $X \hat{\otimes}_\pi Z$ may be identified more concretely as all infinite sums

$\sum_{j=1}^{\infty} a_j x_j \otimes z_j$ where $x_j \to 0$ in X, $z_j \to 0$ in Z, and $\sum_{j=1}^{\infty} |a_j| < +\infty$ [S71, III.6.4], and the pairing between $X \hat{\otimes}_\pi Z$ and $\mathcal{L}(X,Z^*)$ by

$$< \sum_{j=1}^{\infty} a_j x_j \otimes z_j, L> = \sum_{i=1}^{\infty} a_j <z_i, Lx_i>.$$

A second important topology on $X \otimes Z$ is the ε-topology, with norm

$$|\sum_{i=1}^{n} a_i x_i \otimes z_i|_\varepsilon = \max_{|x^*|\le 1} \max_{|z^*|\le 1} |\sum_{i=1}^{n} a_i <x_i, x^*><z_i, z^*>|$$

It is easy to see that $|\cdot|_\varepsilon$ is a cross-norm, i.e. $|x \otimes z|_\varepsilon = |x|\cdot|z|$, and that $|\cdot|_\varepsilon \le |\cdot|_\pi$, i.e. the π-topology is finer than the ε-topology. We denote by $X \otimes_\varepsilon Z$ the tensor product space $X \otimes Z$ with the ε-norm, and by $X \hat{\otimes}_\varepsilon Z$ the completion of $X \otimes Z$ in the ε-norm. Now the canonical injection of $X \otimes_\pi Z$ into $X \hat{\otimes}_\varepsilon Z$ is continuous (with norm 1 and dense image); this induces a canonical continuous map $X \hat{\otimes}_\pi Z \to X \hat{\otimes}_\varepsilon Z$. It is not known, in general, whether this map is one-to-one. In the case that X, Z are Hilbert spaces we may identify $X \hat{\otimes}_\pi Z$ with the nuclear or trace-class maps $\mathcal{T}(X^*,Z)$ and $X \hat{\otimes}_\varepsilon Z$ with the compact operators $\mathcal{K}(X^*,Z)$, and it is well known that the canonical map

$X \hat{\otimes}_\pi Z \to X \hat{\otimes}_\varepsilon Z$ is one-to-one [cf T67, III.38.4]. We are interested in the case that $X = C_o(S)$ and $Z = \mathcal{T}_s(H)$; we may then identify $C_o(S) \hat{\otimes}_\varepsilon \mathcal{T}_s(H)$ with $C_o(S, \mathcal{T}_s(H))$ (since the $|\cdot|_\varepsilon$ is precisely the $|\cdot|_\infty$ norm when $C_o(S) \otimes \mathcal{T}_s(H)$ is identified with a subspace of $C_o(S, \mathcal{T}_s(H))$, and $C_o(S) \otimes \mathcal{T}_s(H)$ is dense in $C_o(S, \mathcal{T}_s(H))$) and we would like to be able to consider $C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ as a subspace of $C_o(S, \mathcal{T}_s(H))$. Similarly we want to consider $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ as a subspace of $M(S, \mathcal{T}(H))$.

4. Theorem. Let X be a Banach space and H a Hilbert space. Then the canonical mapping of $X \hat{\otimes}_\pi \mathcal{T}(H)$ into $X \hat{\otimes}_\varepsilon \mathcal{T}(H)$ is one-to-one.

Proof. It suffices to show that the adjoint of the mapping in question has weak * dense image in $(X \hat{\otimes}_\pi \mathcal{T}(H))^* \cong \mathcal{L}(X, \mathcal{L}(H))$, where we have identified $\mathcal{T}(H)^*$ with $\mathcal{L}(H)$. Note that the adjoint is one-to-one, since the image of the canonical mapping is clearly dense. What we must show is that the imbedding of $(X \hat{\otimes}_\varepsilon \mathcal{T}(H))^*$, the so-called integral mappings $X \to \mathcal{L}(H) \cong \mathcal{T}(H)^*$, into $\mathcal{L}(X, \mathcal{L}(H))$ has weak * dense image. Of course, the set of linear continuous maps $L_o: X \to \mathcal{L}(H)$ with finite dimensional image belongs to the integral mappings

$(X \hat{\otimes}_\varepsilon \mathcal{T}(H))^*$; we shall actually show that these finite-rank operators are weak* dense in $\mathcal{L}(X, \mathcal{L}(H))$. We therefore need to prove that for every $f \in (X \hat{\otimes}_\pi \mathcal{T}(H))$, $L \in \mathcal{L}(X, \mathcal{L}(H))$, $\varepsilon > 0$ there is an L_o in $\mathcal{L}(X, \mathcal{L}(H))$ with finite rank such that $|\langle f, L-L_o \rangle| < \varepsilon$. Now f has the representation

$$f = \sum_{j=1}^{\infty} a_j x_j \otimes z_j \tag{10}$$

with $\sum_{j=1}^{\infty} |a_j| < +\infty$, $x_j \to 0$ in X, and $z_j \to 0$ in $\mathcal{T}(H)$ [S71, III.6.4], and

$$\langle f, L-L_o \rangle = \sum_{j=1}^{\infty} a_j \langle z_j, (L-L_o) x_j \rangle . \tag{11}$$

The lemma which follows proves the following fact: to every compact subset K of X and every 0-neighborhood V of $\mathcal{L}(H)$, there is a continuous linear map L_o: $X \to \mathcal{L}(H)$ with finite rank such that $(L-L_o)(K) \subset V$. Using the representation (10), we take $K = \{x_j\}_{j=1}^{\infty} \cup \{0\}$ and $V = \{y_1, y_2, \ldots\}^o \cdot \varepsilon / \sum_{j=1}^{\infty} |a_j|$. We then have $|\langle f, L-L_o \rangle| < \varepsilon$ as desired. □

The lemma required for the above proof, which we give below, basically amounts to showing that $\mathcal{T}^* = \mathcal{L}(H)$ satisfies the approximation property, that is for every

Banach space X the finite rank operators are dense in $\mathcal{L}(X,Z^*)$ for the topology of uniform convergence on compact subsets of X. It is not known whether every locally convex space satisfies the approximation property; this question (as in the present situation) is closely related to when the canonical mapping $X \hat{\otimes}_\pi Z \to X \hat{\otimes}_\varepsilon Z$ is one-to-one.

5. Lemma. Let X be a Banach space, H a Hilbert space. For every $L \in \mathcal{L}(X, \mathcal{L}(H))$, every compact subset K of X, and every 0-neighborhood V in $\mathcal{L}(H)$ there is a continuous linear map $L_o: X \to \mathcal{L}(H)$ with finite rank such that

$$(L-L_o)(K) \subset V.$$

Proof. Let P_n be projections in H with $P_n \uparrow I$, where I is the identity operator on H (e.g. take any complete orthonormal basis $\{\phi_j, j \in J\}$ for H; let N be the family of all finite subsets of J, directed by set inclusion; and for $n \in N$ define P_n to be the projection operator $P_n(\phi) = \sum_{j \in n} \langle\phi|\phi_j\rangle\phi_j$ for $\phi \in H$). Suppose $L \in \mathcal{L}(X, \mathcal{L}(H))$. Then $P_n L \in \mathcal{L}(X, \mathcal{L}(H))$ has finite rank and converges pointwise to L, since $(P_n L)(x) = P_n(Lx) \to Lx$. Moreover $\{P_n L\}$ is uniformly bounded, since $|P_n L| \le |P_n| \cdot |L| = |L|$. Thus, by the Banach-Steinhaus Theorem [S, III.4.6] or by the

Arzela-Ascoli Theorem the convergence $P_n L \to L$ is uniform on compact sets. This means that for every 0-neighborhood V in $\mathcal{L}(H)$ and every compact subset K of X, it is true that for n sufficiently large

$$(L-P_n L)(K) \subset V. \quad \square$$

6. Corollary. Let S be a locally compact Hausdorff space, H a Hilbert space. The canonical mapping $C_o(S) \hat{\otimes}_\pi \mathcal{T}(H) \to C_o(S, \mathcal{T}(H))$ is one-to-one, and the canonical mapping $M(S) \hat{\otimes}_\pi \mathcal{T}(H) \to M(S, \mathcal{T}(H))$ is one-to-one.

Proof. This follows from the previous theorem and the fact that $C_o(S) \hat{\otimes}_\varepsilon Z$ may be identified with $C_o(S,Z)$ with the supremum norm, for Z a Banach space. Similarly $M(S) \hat{\otimes}_\varepsilon Z = M(S,Z)$ with the supremum norm. $\square$

Remark. In Theorem VIII.4 we explicitly identified $(C_o(S) \hat{\otimes}_\pi \mathcal{T}(H))^* = \mathcal{L}(C_o(S), \mathcal{L}(H))$ and $(C_o(S) \hat{\otimes}_\varepsilon \mathcal{T}(H))^* = C_o(S, \mathcal{T}(H))^*$ with the measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}(H))$ having finite semivariation and finite total variation, respectively.

5. A Fubini theorem for the Bayes posterior expected cost

In the quantum estimation probkem, a decision strategy corresponds to a probability operator measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ with posterior expected cost

$$R_m = \int_S tr[\rho(s) \int_S C(t,s)m(dt)]\mu(dt)$$

where for each s $\rho(s)$ specifies a state of the quantum system, $C(t,s)$ is a cost function, and μ is a prior probability measure on S. We would like to show that the order of integration can be interchanged to yield

$$R_m = tr \int_S f(s)m(ds)$$

where

$$f(s) = \int_S C(t,s)\rho(t)\mu(dt)$$

is a map $f: S \to \mathcal{T}_s(H)$ that belongs to the space $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ of functions integrable against operator-valued measures.

Let $(S, \mathcal{B}, \mu)$ be a finite nonnegative measure space, X a Banach space. A function $f: S \to X$ is <u>measurable</u> iff there is a sequence $\{f_n\}$ of simple measurable functions converging pointwise to f, i.e. $f_n(s) \to f(s)$ for every $s \in S$. A useful criterion for measurability is the following [DS III.6.9]: f is measurable iff it is separably-valued and for every open subset V of X, $f^{-1}(V) \in \mathcal{B}$. In particular, every $f \in C_0(S,X)$ is measurable, when S is a locally compact Hausdorff space with Borel sets $\mathcal{B}$. A function $f: S \to X$ is <u>integrable</u> iff it is measurable and $\int_S |f(s)| \cdot \mu(ds) < +\infty$, in which case the integral $\int_S f(s)\mu(ds)$ is well-defined as Bochner's integral: we denote by $L_1(S,\mathcal{B},\mu;X)$ the space of all integrable functions $f: S \to X$, a normed space under the L_1 norm $|f|_1 = \int_S |f(s)|\mu(ds)$. The uniform norm $|\cdot|_\infty$ on f8nctions $f: S \to X$ is defined by $|f|_\infty = \sup_{s \in S} |f(s)|$; $M(S,X)$ denotes the Banach space of all uniform limits of simple X-valued functions, with norm $|\cdot|_\infty$, i.e. $M(S,X)$ is the closure of the simple X-valued functions with the uniform norm. We abbreviate $M(S,R)$ to $M(S)$.

7. <u>Proposition</u>. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$, μ a probability measure on S, and H a Hilbert space. Suppose $\sigma: S \to \mathcal{T}_s(H)$ belongs to $M(S, \mathcal{T}_s(H))$, and $C: S \times S \to R$ is a real-valued map satisfying

$$t \mapsto C(t,\cdot) \in L_1(S,\mathcal{B},\mu;M(S)).$$

Then for every $s \in S$, $f(s)$ is well-defined as an element of $\mathcal{T}_s(H)$ by the [illegible]

$$f(s) = \int_S C(t,s)\rho(t)\mu(dt); \tag{12}$$

moreover $f \in M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ and for every operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$, we have

$$\int_S f(s)m(ds) = \int_S \rho(t)\left[\int_S C(t,s)m(ds)\right]\mu(dt) \tag{13}$$

Moreover if $t \mapsto C(t,\cdot)$ in fact belongs to $L_1(S,\mathcal{B},\mu;C_o(S))$ then $f \in C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$.

Proof. Since $t \mapsto C(t,\cdot) \in L_1(S,\mathcal{B},\mu;M(S))$, for each n there is a simple function $C_n \in L_1(S,\mathcal{B},\mu;M(S))$ such that

$$\int_S |C(t,\cdot)-C_n(t,\cdot)|_\infty \mu(dt) < \frac{1}{n^{2n}} . \tag{14}$$

Each simple function C_n is of the form

$$C_n(t,s) = \sum_{k=1}^{k_n} g_{nk}(s) 1_{E_{nk}}(t)$$

where $E_{n,1},\ldots,E_{nk_n}$ are disjoint subsets of $\mathcal{B}$ and $g_{n1},\ldots,g_{nk_n}$ belong to $M(S)$ (in the case that $t \mapsto C(t,\cdot)$ $L_1(S,\mathcal{B},\mu;C_o(S))$ we take $g_{n1},\ldots,g_{nk_n}$ in $C_o(S)$). Since $\rho \in M(S,\mathcal{T}_s(H))$, for each n there is a simple measurable function $\rho_n: S \to \mathcal{T}_s(H)$ such that

$$\sup_t |\rho(t)-\rho_n(t)| < \frac{1}{n^{2n}} . \tag{15}$$

We may assume, by replacing each set E_{nk} with a disjoint subpartition corresponding to the finite number of values taken on by ρ_n, that each ρ_n is in fact of the form

$$\rho_n(t) = \sum_{k=1}^{k_n} \rho_{nk} 1_{E_{nk}}(t).$$

Define $f_n \colon S \to \mathcal{T}_s(H)$ by

$$f_n(s) = \int_S C_n(t,s)\rho_n(t)\mu(dt)$$

$$= \sum_{k=1}^{k_n} g_{nk}(s)\rho_{nk}\mu(E_{nk}).$$

Of course, each f_n belongs to $M(S) \otimes \mathcal{T}_s(H)$. We shall show that $\{f_n\}$ is a Cauchy sequence for the $|\cdot|_\pi$ norm on $M(S) \otimes \mathcal{T}_s(H)$, and that $f_n(s) \to f(s)$ for every $s \in S$; since the $|\cdot|_\pi$-limit of the sequence f_n is a unique function by Theorem 4, we see that f is the $|\cdot|_\pi$-limit of $\{f_n\}$ and hence f belongs to the completion $M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$.

We calculate an upper bound for $|f_{n+1}-f_n|_\pi$. Now

$$f_{n+1}(s)-f_n(s) =$$

$$\sum_{j=1}^{k_{n+1}} \sum_{k=1}^{k_n} \{g_{n+1,j}(s)[\rho_{n+1,j}-\rho_{n,k}]+[g_{n+1,j}(s)-g_{n,k}(s)]\rho_{n,k}\}\mu(E_{n+1,j}\cap E_{n,k})$$

and hence

$$|f_{n+1}-f_n|_\pi \leq \tag{16}$$

$$\sum_{j=1}^{k_{n+1}} \sum_{k=1}^{k_n} \{|g_{n+1,j}|_\infty \cdot |\rho_{n+1,j}-\rho_{n,k}|_{tr}+|g_{n+1,j}-g_{n,k}|_\infty \cdot |\rho_{nk}|_{tr}\}\mu(E_{n+1,j}\cap E_{n,k})$$

Suppose $E_{n+1,j}\cap E_{n,k} \neq \emptyset$, i.e. there exists a $t_o \in E_{n+1,j}\cap E_{n,k}$.

Then from (15) we have

$$|\rho_{n+1,j}-\rho_{n,k}|_{tr} \leq |\rho_{n+1,j}-\rho(t_o)|_{tr} + |\rho_{n,k}-\rho(t_o)|_{tr}$$

$$\leq \frac{1}{(n+1)2^{n+1}} + \frac{1}{n^{2n}} < \frac{1}{n2^{n+1}} .$$

Thus, the first half of the summation in (16) is bounded above by

$$\frac{1}{n2^{n-1}} \sum_{j=1}^{k_{n+1}} \sum_{k=1}^{k_n} |g_{n+1,j}|_\infty \mu(E_{n+1,j}\cap E_{n,k}) = \frac{1}{n2^{n-1}} \int_S |C_{n+1}(t,\cdot)|_\infty \mu(dt)$$

$$= \frac{1}{n2^{n-1}} ||C_{n+1}||_1$$

$$\leq \frac{1}{n2^{n-1}} (1+||C||_1)$$

where by $||C||_1$ we mean the norm of $t \mapsto C(t,\cdot)$ as a element of $L_1(S,\mathcal{B},\mu;M(S))$, and the last inequality follows from (14). Similarly the second half of the summation is bounded above by

$$(|\rho|_\infty+1)\cdot \sum_{j=1}^{k_{n+1}} \sum_{k=1}^{k_n} |g_{n+1,j}-g_{n,k}|_\infty \cdot \mu(E_{n+1,j}\cap E_{n,k})$$

$$= (|\rho|_\infty+1)\cdot ||C_{n+1}-C_n||_1$$

$$< (|\rho|_\infty+1)\cdot \frac{1}{n2^{n-1}}$$

where again the last inequality follows since $||C_n-C||_1 < \frac{1}{n2^n}$ by (14). Let a be a constant larger than $1 + ||C||_1$ and $1 + |\rho|_\infty$; adding the last two inequalities from (16) we have

$$|f_{n+1}-f_n|_\pi < \frac{a}{n2^{n-2}} .$$

Hence for every $m > n \geq 1$ it follows that

$$|f_m-f_n|_\pi \leq \sum_{j=n}^{m-1} |f_{j+1}-f_j|_\pi < \sum_{j=n}^{\infty} \frac{a}{n2^{n-2}} < \frac{1}{n} \sum_{j=1}^{\infty} \frac{a}{2^{n-2}} = \frac{3a}{n} .$$

Thus $\{f_n\}$ is a Cauchy sequence for the $|\cdot|_\pi$ norm on $M(S) \otimes \mathcal{T}_s(H)$, and hence has a limit $f_o \in M(S) \otimes_\pi \mathcal{T}_s(H)$. Since it certainly follows that $f_n \to f_o$ pointwise (in fact in the uniform norm since $|\cdot|_\infty \leq |\cdot|_\pi$), and since it is straightforward to show that $f_n(s) \to f(s)$ for every $s \in S$, $f_o = f$. Moreover in the case that $t \mapsto C(t,\cdot) \in L_1(S,\mathcal{E},\mu;C_o(S))$, we have $f_n \in C_o(S) \otimes \mathcal{T}_s(H)$

and hence $f = |\cdot|_\pi\text{-lim } f_n$ belongs to $C_0(S) \otimes_\pi \mathcal{T}_s(H)$.

It only remains to show that (13) holds. Essentially this follows from the approximations we have already made with simple functions. Now clearly

$$\int f_n(s)m(ds) = \sum_{k=1}^{k_n} \rho_{nk}\mu(E_{nk})\int_S g_{nk}(s)m(ds)$$

$$= \int_S \rho_n(t)[\int C_n(t,s)m(ds)]\mu(dt), \qquad (16)$$

so that (13) is satisfied for the simple approximations. We have already shown that $f_n \to f$ in $M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$, so that $|\int f_n(s)m(ds) - \int f(s)m(ds)|_{tr} \le |f_n - f|_\pi \cdot \bar{\bar{m}}(S) \to 0$ and the LHS of (16) converges to $\int f(s)m(ds)$. We need only show that the RHS of (16) converges to the RHS of (13). But applying the triangle inequality to (16) yields

$$|\int\rho_n(t)[\int C_n(t,s)m(ds)]\mu(dt) - \int\rho(t)[\int C(t,s)m(ds)]\mu(dt)|_{tr}$$

$$\le \int|\rho_n(t)\int[C_n(t,s) - C(t,s)]m(ds)|_{tr}\mu(dt)$$

$$+ \int|(\rho_n(t) - \rho(t))\cdot\int C(t,s)m(ds)|_{tr}\mu(dt)$$

$$\le |\rho_n|_\infty\cdot\int|C_n(t,\cdot) - C(t,\cdot)|_\infty\cdot\bar{\bar{m}}(S)\mu(dt)$$

$$+ |\rho_n - \rho|_\infty\cdot\int|C(t,\cdot)|_\infty\bar{\bar{m}}(S)\mu(dt)$$

$$\le (|\rho|_\infty + 1)\cdot\bar{\bar{m}}(S)\cdot||C_n - C||_1 + |\rho_n - \rho|_\infty\bar{\bar{m}}(S)||C||_1$$

$$\le (|\rho|_\infty + 1)\cdot\bar{\bar{m}}(S)\cdot\frac{1}{n2^n} + \frac{1}{n2^n}\bar{\bar{m}}(S)\cdot||C||_1 \to 0$$

where the last inequality follows from (14) and (15) and again $||C||_1 = \int_S |C(t,\cdot)|_\infty \mu(dt)$ denotes the norm of C as an element of $L_1(S, \mathcal{B}, \mu; M(S))$. □

6. The quantum estimation problem and its dual

We are now prepared to formulate the quantum detection problem in a duality framework and calculate the associated dual problem. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let H be a Hilbert space associated with the physical variables of the system under consideration. For each parameter value $s \in S$ let $\rho(s)$ be a state or density operator for the quantum system, i.e. every $\rho(s)$ is a nonnegative-definite selfadjoint trace-class operator on H with trace 1; we assume $\rho \in M(S, \mathcal{T}_s(H))$. We assume that there is a cost function $C: S \times S \to R$, where $C(s,t)$ specifies the relative cost of an estimate t when the true parameter value is s. If the operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ corresponds to a given measurement and decision strategy, then the posterior expected cost is

$$R_m = \operatorname{tr} \int_S \rho(t) \left[\int_S C(t,s) m(ds) \right] \mu(dt),$$

where μ is a prior probability measure on $(S, \mathcal{B})$. By Proposition 7 this is well-defined whenever the map $t \mapsto C(t,\cdot)$ belongs to $L_1(S, \mathcal{B}, \mu; M(S))$, in which case we may interchange the order of integration to get

$$R_m = \operatorname{tr} \int_S f(s) m(ds) \tag{17}$$

where $f \in M(S) \hat{\otimes}_\lambda \mathcal{T}_s(H)$ is defined by

$$f(s) = \int_S \rho(t)C(t,s)\mu(ds).$$

The quantum estimation problem is to minimize (17) over all operator-valued measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ which are POM's, i.e. the constraints are that $m(E) \geq 0$ for every $E \in \mathcal{B}$ and $m(S) = I$.

We formulate the estimation problem in a duality framework. As in the quantum detection problem, we take perturbations on the equality constraint $m(S) = I$. Define the convex function $F: \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)) \to \overline{R}$ by

$$F(m) = \delta_{\geq 0}(m) + \mathrm{tr}\int_S f(s)m(ds), \quad m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)),$$

where $\delta_{\geq 0}$ denotes the indicator function for the positive operator-valued measures, i.e. $\delta_{\geq 0}(m)$ is 0 if $m(\mathcal{B}) \subset \mathcal{L}_s(H)_+$ and $+\infty$ otherwise. Define the convex function $G: \mathcal{L}_s(H) \to \overline{R}$ by

$$G(x) = \delta_{\{0\}}(x), \quad x \in \mathcal{L}_s(H)$$

i.e. $G(x)$ is 0 if $x = 0$ and $G(x) = +\infty$ if $x \neq 0$. Then the quantum detection problem may be written

$$P_0 = \inf\{F(m)+G(I-Lm) : m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))\}$$

where $L: \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)) \to \mathcal{L}_s(H)$ is the continuous linear operator

$$L(m) = m(S).$$

We consider a family of perturbed problems defined by

$$P(x) = \inf\{F(m)+G(x-Lm) : m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))\}, \quad x \in \mathcal{L}_s(H).$$

Thus we are taking perturbations in the equality constraint, i.e. the problem $P(x)$ requires that every feasible m be nonnegative and satisfy $m(S) = x$; of course, $P_o = P(I)$. Since F and G are convex, $P(\cdot)$ is convex $\mathcal{L}_s(H) \to \overline{R}$.

In order to construct the dual problem corresponding to the family of perturbed problems $P(x)$, we must calculate the conjugate functions of F and G. We shall work in the norm topology of the constraint space $\mathcal{L}_s(H)$, so that the dual problem is posed in $\mathcal{L}_s(H)^*$. Clearly $G^* \equiv 0$. The adjoint of the operator L is given by

$$L^*: \mathcal{L}_s(H)^* \to \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))^*: y \mapsto (m \mapsto y \cdot m(S)).$$

To calculate $F^*(L^*y)$, we have the following lemma.

8. Lemma. Suppose $y \in \mathcal{L}_s(H)^*$ and $f \in M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ satisfy

$$y \cdot m(S) \le \operatorname{tr} \int_S f(s)m(ds) \tag{18}$$

for every positive operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)_+)$. Then $y_{sg} \le 0$ and $y_{ac} \le f(s)$ for every $s \in S$, where $y = y_{ac}+y_{sg}$ is the unique decomposition of y into

$y_{ac} \in \mathcal{T}_s(H)$ and $y_{sg} \in \mathcal{K}_s(H)^{\perp}$.

Proof. Fix any $s_o \in S$. Let x be an arbitrary element of $\mathcal{L}_s(H)_+$, and define the positive operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)_+)$ by

$$m(E) = \begin{cases} x & \text{if } s_o \in E \\ 0 & \text{if } s_o \notin E \end{cases}, \quad E \in \mathcal{B}.$$

Then $y \circ m(S) = y(x) = \operatorname{tr}(y_{ac}x) + y_{sg}(x)$, and $\operatorname{tr}\int f(s)m(ds) = \operatorname{tr} f(s_o)x$. Thus by (18) $\operatorname{tr}[y_{ac} - f(s_o)]x + y_{sg}(x) \leq 0$; since $x \in \mathcal{L}_s(H)_+$ was arbitrary, it follows from Proposition III.3 that $y_{ac} \leq f(s_o)$ (i.e. $f(s_o) - y_{ac} \in \mathcal{T}_s(H)_+$) and $y_{sg} \leq 0$ (i.e. $-y_{sg} \in [\mathcal{L}_s(H)_-]^+ \cap \mathcal{K}_s(H)^{\perp}$). □

With the aid of this lemma it is now easy to verify that

$$F^*(L^*y) = \begin{cases} 0 & \text{if } y_{ac} \leq f(s) \quad s \in S, \text{ and } y_{sg} \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$= \delta_{\leq f}(y_{ac}) + \delta_{\leq 0}(y_{sg}).$$

It now follows that $P^*(y) = F^*(L^*y) + G^*(y)$ is 0 if $y_{sg} \leq 0$ and $y_{ac} \leq f(s)$ for every $s \in S$, and $P^*(y) = +\infty$ otherwise. The dual problem $D_o = *(P^*)(I) = \sup_y [y(I) - P^*(y)]$ is thus given by

$D_o = *(P*)(I)$

$$= \sup\{\text{tr}\, y_{ac} + y_{sg}(I) : y \in \mathcal{L}_s(H)^*,\ y_{sg} \leq 0,\ y_{ac} \leq f(s)\ \forall s \in S\}.$$

We show that $P(\cdot)$ is norm continuous at I, and hence there is no duality gap $(P_o = D_o)$ and D_o has solutions. Moreover we expect, as in the detection case, that the optimal solutions for D_o will always have 0 singular part, i.e. will be in $\mathcal{T}_s(H)$.

9. Proposition. The perturbation function $P(\cdot)$ is continuous at I, and hence $\partial P(I) \neq \emptyset$. In particular, $P_o = D_o$ and the dual problem D_o has optimal solutions. Moreover every solution $\hat{y} \in \mathcal{L}_s(H)^*$ of the dual problem D_o has 0 singular part, i.e. $\hat{y}_{sg} = 0$ and $\hat{y} = \hat{y}_{ac}$ belongs to the canonical image of $\mathcal{T}_s(H)$ in $\mathcal{T}_s(H)^{**}$.

Proof. We show that $P(\cdot)$ is bounded above on a unit ball centered at I. Suppose $x \in \mathcal{L}_s(H)$ and $|x| \leq 1$. By Lemma VII.4, $I+X \geq 0$. Let s_o be an arbitrary element of S and define the positive operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)_+)$ by

$$m(E) = \begin{cases} I+x & \text{if } s_o \in E \\ 0 & \text{if } s_o \notin E \end{cases}, \quad E \in \mathcal{B}.$$

Then m is feasible for $P(x)$ and has cost

$$\mathrm{tr}\int f(s)m(ds) = \mathrm{tr} f(s_o)(I+x) \leq 2|f(s_o)|_{tr}.$$

Thus $P(I+x) \leq 2|f(s_o)|_{tr}$ whenever $|x| \leq 1$, so $P(\cdot)$ is bounded above on a neighborhood of I and so by convexity is continuous at I. By Theorem I.11.1 it follows that $\partial P(x_o) \neq \emptyset$, hence $P_o = D_o$ and D_o has solutions. Suppose now that $\hat{y} \in \mathcal{L}_s(H)^*$ is an optimal solution for D_o. If $\hat{y}_{sg} \neq 0$, then since $\hat{y}_{sg} \leq 0$ and $I \in \mathrm{int}\,\mathcal{L}_s(H)_+$ it follows from Lemma VII.4 that $\mathrm{tr}(\hat{y}_{ac}) + \hat{y}_{sg}(I) < \mathrm{tr}(\hat{y}_{ac})$. Hence the value of the dual objective function is strictly improved by setting $\hat{y}_{sg} = 0$, while the constraints remain satisfied, so that if $\hat{y}$ is optimal it must be true that $\hat{y}_{sg} = 0$. □

In order to show that the problem P_o has solutions, we could define a family of dual perturbed problems $D(v)$ for $v \in C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ and show that $D(\cdot)$ is continuous. Or we could take the alternative method of showing that the set of feasible POM's m is weak* compact and the cost function is weak*-lsc when $\mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)) \cong \mathcal{L}(C_o(S), \mathcal{L}_s(H))$ is identified as the normed dual of the space $C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ under the pairing

$$\langle f, m \rangle = \mathrm{tr}\int f(s)m(ds).$$

Note that both methods require that f belong to the predual $C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ of $\mathcal{M}(\mathcal{B}, \mathcal{T}_s(H))$; by Proposition 7 it suffices to assume that $t \mapsto C(t,\cdot)$ belongs to $L_1(S,\mathcal{B},\mu;C_o(S))$.

10. <u>Proposition</u>. The set of POM's is compact for the weak* $\equiv w(\mathcal{M}(\mathcal{B},\mathcal{L}_s(H)), C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H))$ topology. If $t \mapsto C(t,\cdot) \in L_1(S,\mathcal{B},\mu;C_o(S))$ then P_o has optimal solutions $\hat{m}$.

<u>Proof</u>. Since $\mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$ is the normed dual of $C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ it suffices to show that the set of POM's is bounded; in fact, we show that $\bar{\bar{m}}(S) = 1$ for every POM m. If $\phi \in H$ and $|\phi| = 1$, then $\langle\phi m(\cdot)|\phi\rangle$ is a regular Borel probability measure on S whenever m is a POM, so that the total variation of $\langle\phi m(\cdot)|\phi\rangle$ is precisely 1. Hence

$$\bar{\bar{m}}(S) = \sup_{\substack{\phi \in H \\ |\phi| \le 1}} |\langle\phi m(\cdot)|\phi\rangle|(S) = \sup_{\substack{\phi \in H \\ |\phi| = 1}} |\langle\phi m(\cdot)|\phi\rangle|(S) = 1.$$

Thus the set of POM's is a weak*-closed subset of the unit ball in $\mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$, hence weak*-compact. If now $t \mapsto C(t,\cdot)$ belongs to $L_1(S,\mathcal{B},\mu;C_o(S))$ then $f \in C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ by Proposition 7, so $m \mapsto \mathrm{tr}\int f(s)m(ds)$ is a weak*-continuous linear function and hence attains its infimum on the set of POM's. Thus P_o has solutions. □

The following theorem summarizes the results we have obtained so far, as well as providing a necessary and sufficient characterization of the optimal solution.

11. <u>Theorem</u>. Let H be a Hilbert space, S a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let $\rho \in M(S, \mathcal{T}_s(H))$, $C: S \times S \to R$ a map satisfying $t \mapsto C(t,\cdot) \in L_1(S,\mathcal{B},\mu;C_o(S))$, and μ a probability measure on $(S,\mathcal{B})$. Then for every $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$,

$$\mathrm{tr}\int_S \rho(t)\left[\int_S C(t,s)m(ds)\right]\mu(dt) = \mathrm{tr}\int_S f(s)m(ds)$$

where $f \in C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ is defined by

$$f(s) = \int_S \rho(t)C(t,s)\mu(ds).$$

Define the optimization problems

$$P_o = \inf\{\mathrm{tr}\int_S f(s)m(ds): m\in\mathcal{M}(\mathcal{B},\mathcal{L}_s(H)), m(S)=I, m(E)\geq 0 \text{ for every } E\in\mathcal{B}\}$$

$$D_o = \sup\{\mathrm{tr}y: y\in \mathcal{T}_s(H),\ y \leq f(s) \text{ for every } s\in S\}.$$

Then $P_o = D_o$, and both P_o and D_o have optimal solutions. Moreover the following statements are equivalent for $m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$, assuming $m(S) = I$ and $m(E) \geq 0$ for every $E \in \mathcal{B}$:

1) m solves P_o

2) $\int_S f(s)m(ds) \leq f(t)$ for every $t \in S$

3) $\int_S m(ds)f(s) \leq f(t)$ for every $t \in S$.

Under any of the above conditions it follows that $y = \int_S f(s)m(ds) = \int_S m(ds)f(s)$ is selfadjoint and is the unique solution of D_o, with

$$P_o = D_o = \mathrm{tr}\, y.$$

<u>Proof</u>. We need only verify the equivalence of 1)-3); the rest follows from Propositions 9 and 10. Suppose m solves P_o. Then there is a $y \in \mathcal{T}_s(H)$ which solves D, so that $y \leq f(t)$ for every t and

$$\mathrm{tr}\int_S f(s)m(ds) = \mathrm{tr}\, y.$$

Equivalently $0 = \mathrm{tr}\int_S f(s)m(ds) - \mathrm{tr}\, y = \mathrm{tr}\int_S (f(s)-y)m(ds)$. Since $f(s)-y \geq 0$ for every $s \in S$ and $m \geq 0$ it follows that $0 = \int_S (f(s)-y)m(ds) = \int_S f(s)m(ds) - y$ and hence 2) holds. This last equality also shows that y is unique.

Conversely, suppose 2) holds. Then $y = \int_S f(s)m(ds)$ is feasible for D_o, and moreover $\mathrm{tr}\int f(s)m(ds) = \mathrm{tr}\, y$. Since $P_o \geq D_o$, it follows that m solves P_o and y solves D_o, so that 1) holds.

Thus 1) <=> 2) is proved. The proof of 1) <=> is identical, assuming that $\mathrm{tr}\int f(s)m(ds) = \mathrm{tr}\int m(ds)f(s)$ for every $f \in C_0(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$. But the latter is true since $\mathrm{tr}AB = \mathrm{tr}BA$ for every $A \in \mathcal{T}_s(H)$, $B \in \mathcal{L}_s(H)$ and hence it is true for every $f \in C_0(S) \otimes \mathcal{T}_s(H)$. □

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