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FILTERING FOR BILINEAR SYSTEMS

by

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Lawrence Charles Vallot

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ABSTRACT

The problem of filtering for bilinear systems is considered. Because the optimal filter cannot be realized, suboptimal filters are considered. The optimal linear filter is derived, and asymptotic behavior of this filter is discussed.

Unconditional moments of the state of a bilinear system can be calculated by solving a finite set of ordinary differential equations. This property makes these systems suitable for the application of various tensor methods, among them the creation of various pseudo-observables to improve linear filter performance, and nonlinear post-processing of optimal linear estimates. These methods result in suboptimal nonlinear filters that can outperform the optimal linear filter. The nonlinear filters considered in this research share a highly desirable property with the linear filter: the gains do not depend on the observation and thus may be calculated off-line, which greatly reduces the computational burden associated with filter implementation.

The performance of a nonlinear post-processor type filter is evaluated for a phase tracking problem and the cubic sensor problem. Performance for the phase tracking problem is poor. However, in a problem of estimation for rotational processes inherent in the phase tracking problem, the nonlinear post-processor provides up to nine percent reduction in mean square error (relative to the optimal linear filter). A reduction in filtering mean square error is also obtained by nonlinear post-processing in the cubic sensor problem.

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CHAPTER 1

INTRODUCTION

A significant feature of the well-known Kalman filtering problem is that the conditional probability density of the state x_t given the related observation

$$Y_t = \{y_s, t_0 \leq s \leq t\} \tag{1.1}$$

is a Gaussian density. Thus, in order to determine the minimum variance state estimate and its performance, the filtering algorithm needs only to propagate the conditional mean and conditional covariance. These two parameters completely characterize the conditional probability density. However, in the general nonlinear filtering problem

$$\begin{aligned} dx_t &= f(x_t,t)dt + G(x_t,t)d\beta_t \\ dy_t &= h(x_t,t)dt + M(x_t,t)d\eta_t \end{aligned} \tag{1.2}$$

this fortuitous situation does not exist. The conditional density $p(x_t,t|Y_t)$ cannot, in general, be characterized by a finite parameter set. Thus, in most nonlinear problems of practical interest, one is forced to consider suboptimal filters.

The particular problem to be considered in this thesis is filtering for bilinear systems. The term "bilinear" refers to the fact that systems of this form are linear in the state, and linear in the driving term, but not jointly linear. Bilinear systems are sometimes referred to as linear systems with state dependent (or state multiplicative) noise. The exact model to be considered can be found in Chapter 2.

Due to the presence of state-multiplicative noise in the bilinear estimation problem, the conditional density of interest is not Gaussian. Thus the optimal nonlinear filter is infinite dimensional, which leads to the study of suboptimal filters. An approach that can be used is to obtain the minimum variance filter within some limited class. For example, one might try to determine the optimal linear filter. This filter is derived in Chapter 2. Another approach might be to obtain the optimal filter within some particular class of nonlinear filters. Because the unconditional moments of bilinear systems are easily calculated, this approach is particularly useful for these systems. Several nonlinear filters that make use of both the above noted property and the linear minimum variance filter are considered in Chapter 3.

One might ask "Why are we so interested in filtering for bilinear systems?" The answer is that models for many real world problems are naturally bilinear, among them being models for lossless electrical networks, population dynamics, nuclear reactors, and some biological processes. Furthermore, recent work has indicated that arbitrarily good bilinear approximations to deterministic nonlinear systems can be obtained. A problem of current research interest is the extension of these ideas to stochastic nonlinear systems. Significant results in this area could lead to increased interest in the performance of nonlinear filters for bilinear systems.

Two applications of nonlinear filters for bilinear systems are considered in this thesis. In Chapter 4, the methods developed here are applied to the well-known phase tracking problem. Another nonlinear estimation problem, the cubic sensor, is considered in Chapter 5. Quantitative results are given for both of the applications. Finally, comments and conclusions motivated by this research are presented in Chapter 6.

CHAPTER 2

OPTIMAL LINEAR FILTERING FOR BILINEAR SYSTEMS

In this chapter, the optimal linear filter for bilinear stochastic systems is derived and certain properties of the filter are discussed. The linear filter discussed here will play an essential role in the nonlinear filters to be discussed in Chapter 3.

Section 2.1 reviews elements of stochastic calculus, specifically mean square convergence and the definition of the Ito stochastic integral. Following this brief review, the optimal linear filter is derived. The derivation is based on the orthogonal projections theorem, and follows Kailath's innovations derivation of the Kalman filter [1]. It should be noted that this filter has been derived previously by several others, among them Gustafson [2] who obtained the continuous filter from the discrete filter by a careful limiting argument. Section 2.3 discusses asymptotic behavior and stability of the optimal linear filter.

2.1 Properties of the Ito Integral

In order to discuss certain properties of the Ito integral which are needed in the linear filter derivation, a brief review will be given. A more detailed exposition of various elements of stochastic calculus can be found in Jazwinski [3].

Before a stochastic integral can be defined, some type of convergence concept is necessary. The following definition will be used.

Given a sequence of approximations x_n to a random variable x, with

$$E[x_n^2] < \infty \text{ for all } n \tag{2.1}$$

and

$$E[x^2] < \infty \tag{2.2}$$

we say that x_n converges to x in the mean square sense if

$$\lim_{n\to\infty} E[(x - x_n)^2] = 0 \tag{2.3}$$

In this case we call x the mean square limit of $\{x_n\}$, and write

$$\text{l.i.m. } x_n = x \tag{2.4}$$

where l.i.m. stands for "limit in the mean."

The Ito stochastic integral is one way to define integrals of the form

$$\int_a^b g_t(\omega)d\beta_t \tag{2.5}$$

where β_t is a Brownian motion process and the function $g_t(\omega)$ is random. Suppose that $\{g_t^n(\omega)\}$ is a sequence of step functions that converges to $g_t(\omega)$ in the sense that

$$\int_a^b E[(g_t(\omega) - g_t^n(\omega))^2]dt \to 0 \qquad \text{as } n\to\infty \tag{2.6}$$

Then the Ito integral is defined by

$$\int_a^b g_t(\omega)d\beta_t = \underset{n\to\infty}{\text{l.i.m.}} \int_a^b g_t^{\,n}(\omega)d\beta_t$$
$$= \underset{\rho\to 0}{\text{l.i.m.}} \sum_{i=0}^{n-1} g_{t_i}^{\,n}(\omega)(\beta_{t_{i+1}} - \beta_{t_i}) \tag{2.7}$$

where $\rho = \max_i (t_{i+1} - t_i)$. It is shown in Doob [4] that the class of functions $g_t(\omega)$ which satisfy

$$1)\quad g_t(\omega) \text{ independent of } \{\beta_{t_k} - \beta_{t_\ell} : t \leqslant t_\ell \leqslant t_k\} \tag{2.8}$$

$$2)\quad \int_a^b E[(g_t(\omega))^2]dt < \infty \tag{2.9}$$

can be approximated in the sense of (2.6).

With the definition of the Ito integral as given in (2.7), it is easy to show that

$$E[\int_a^b g_t(\omega)d\beta_t] = 0 \tag{2.10}$$

and

$$E[\int_a^b g_t(\omega)d\beta_t \cdot \int_a^b f_t(\omega)d\beta_t] = \sigma^2 \int_a^b E[g_t(\omega)f_t(\omega)]dt \tag{2.11}$$

with σ^2 being the variance parameter of the Brownian motion β_t, and $g_t(\omega)$ and $f_t(\omega)$ being random functions which satisfy (2.8) and (2.9).

These properties will prove useful in the linear filter derivation to follow in Section 2.2.

Stochastic Differential Equations

At this stage we will briefly discuss stochastic differential equations and the Ito differential rule. By a stochastic differential equation we mean equations of the form

$$dx_t = f(x_t,t)dt + G(x_t,t)d\beta_t \tag{2.12}$$

where β_t is a Brownian motion process independent of x_{t_0}. By (2.12) we really mean that

$$x_t = x_{t_0} + \int_{t_0}^{t} f(x_s,s)ds + \int_{t_0}^{t} G(x_s,s)d\beta_s \tag{2.13}$$

with the first integral being a mean square Riemann integral and the second an Ito stochastic integral. In [5] Wong discusses the existence and uniqueness of solutions to (2.12). Wong also discusses the modeling problem, that is, the question of how well (2.12) models stochastic dynamical systems driven by white noise, such as

$$\frac{d}{dt} x_t = f(x_t,t) + G(x_t,t)\dot{w}_t \tag{2.14}$$

where $\dot{w}_t$ can be considered to be the formal derivative of the Brownian motion β_t. Further discussion of the modeling problem can be found in Clark [6].

We are now in a position to state one of the more useful results of stochastic calculus, that being the Ito differential rule. The rule is stated here as given in Jazwinski [3].

Let the random process x_t be the unique solution of the vector Ito equation

$$dx_t = f(x_t,t)dt + G(x_t,t)d\beta_t, \qquad t \geq t_0 \tag{2.15}$$

where x_t and $f(x_t,t)$ are n-vectors, $G(x_t,t)$ is $n \times m$, and β_t is an m-vector Brownian motion process with $E[d\beta_t d\beta_t'] = Q(t)dt$. Let $\phi(x_t,t)$ be a scalar valued real function which is continuously differentiable in t and having continuous second mixed partial derivatives with respect to the elements of x. Then ϕ satisfies the stochastic differential equation

$$d\phi = \phi_t dt + \phi_x' dx_t + \tfrac{1}{2} \text{ tr } GQG'\phi_{xx} dt \tag{2.16}$$

where

$$\phi_t = \frac{\partial\phi}{\partial t}, \qquad \phi_x' = \left[\frac{\partial\phi}{\partial x_1}, \ldots, \frac{\partial\phi}{\partial x_n}\right] \tag{2.17}$$

and

$$\phi_{xx} = \begin{bmatrix} \frac{\partial^2\phi}{\partial x_1^2} & \cdots & \frac{\partial^2\phi}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2\phi}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2\phi}{\partial x_n^2} \end{bmatrix} \tag{2.18}$$

This rule provides a stochastic analog of the chain rule of ordinary calculus. It will prove useful in the following linear filter derivation and in all of the nonlinear filters to be discussed in Chapter 3.

2.2 Derivation of the Optimal Linear Filter

The bilinear system to be considered here is

$$dx_t = F(t)x_t dt + \sum_{i=1}^{q} G_i(t)x_t dw_{i_t} \qquad t \geq 0 \tag{2.19}$$

with the observation

$$dy_t = H(t)x_t dt + \sum_{i=1}^{r} M_i(t)x_t dv_{i_t} \qquad t \geq 0 \tag{2.20}$$

We assume that the w_i's and v_i's are all standard Brownian motions independent of x_0, and that the process noises w_i, $i = 1,\ldots,q$ are independent of the measurement noises v_j, $j = 1,\ldots,r$.

As an aside, note that there is no loss of generality here in terms of the lack of additive noise in the model; we can always augment the original state vector with the constant state "one" and thus account for additive noise in the process, the observation, or both. For example, consider the scalar process

$$dx_t = x_t dt + x_t d\eta_t + dw_t$$

By defining the new state vector

$$z_t = \begin{bmatrix} x_t \\ 1 \end{bmatrix}$$

we can write

$$dz_t = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} z_t dt + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} z_t dn_t + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} z_t dw_t$$

which is in the form of equation (2.19).

Since a linear filter is desired, we write the estimate $\hat{x}_t$ as

$$\hat{x}_t = \hat{x}_0 + \int_0^t K(t,s)d\nu_s \tag{2.21}$$

where ν_s is the linear innovations process given by

$$\begin{aligned} d\nu_s &= dy_s - H(s)\hat{x}_s ds \\ &= H(s)\tilde{x}_s ds + \sum_{i=1}^{r} M_i(s)x_s dv_{i_s} \end{aligned} \tag{2.22}$$

with

$$\tilde{x}_s = x_s - \hat{x}_s \tag{2.23}$$

and the kernel $K(t,s)$ is to be determined.

Following Kailath's derivation of the Kalman filter [1], we invoke the orthogonal projections theorem. In this application the theorem basically states that the error in the minimum variance estimate of x_t must be orthogonal to the entire innovations process up to time t. Thus

$$x_t - \hat{x}_t \perp \nu_\tau, \qquad \tau \leqslant t \tag{2.24}$$

or

$$E[x_t\nu_\tau'] = E[\hat{x}_t\nu_\tau'] = E[(\hat{x}_0 + \int_0^t K(t,s)d\nu_s)\cdot\int_0^\tau d\nu_s'] \qquad (2.25)$$

Utilizing the properties of the Ito integral given in (2.10) - (2.11) and differentiating with respect to τ, we obtain

$$K(t,s) = \frac{\partial}{\partial s} E[x_t\nu_s']\Sigma_\nu^{-1}(s) \qquad (2.26)$$

where

$$E[d\nu_s d\nu_s'] = \Sigma_\nu(s)ds \qquad (2.27)$$

An explicit expression for $\Sigma_\nu(s)$ will be given shortly.

We now evaluate the expression for K(t,s) given in (2.26).

$$K(t,s) = \frac{\partial}{\partial s} E[x_t\nu_s']\Sigma_\nu^{-1}(s)$$

$$= \frac{\partial}{\partial s} E[x_t\cdot(\int_0^s \tilde{x}_u'H'(u)du + \int_0^s x_u' \sum_{i=1}^r M_i'(u)dv_{i_u})]\Sigma_\nu^{-1}(s)$$

$$= E[x_t\tilde{x}_s']H'(s)\Sigma_\nu^{-1}(s) \qquad (2.28)$$

With K(t,s) known, we can write a stochastic differential equation for $\hat{x}_t$ using (2.21) to obtain

$$d\hat{x}_t = K(t,t)d\nu_t + \int_0^t d(K(t,s))_t d\nu_s$$

$$= P(t)H'(t)\Sigma_\nu^{-1}(t)d\nu_t + \int_0^t d(E[x_t\tilde{x}_s'])_t H'(s)\Sigma_\nu^{-1}(s)d\nu_s \qquad (2.29)$$

with $P(t) = E[\tilde{x}_t\tilde{x}_t']$. Expanding the second term on the right hand side of (2.29) gives

$$\int_{0^-}^{t} d(K(t,s))_t d\nu_s = \int_0^t d(E[x_s\tilde{x}_s' + \int_s^t F(u)x_u\tilde{x}_u' du$$

$$+ \int_s^t \sum_{i=1}^{q} G_i(u)x_u\tilde{x}_s' dw_{i_u}])_t H'(s)\Sigma_\nu^{-1}(s)d\nu_s$$

$$= (\int_0^t F(t)E[x_t\tilde{x}_s']H'(s)\Sigma_\nu^{-1}(s)d\nu_s)dt$$

$$= F(t)\hat{x}_t dt \qquad (2.30)$$

Thus we can write

$$d\hat{x}_t = F(t)\hat{x}_t dt + K(t)d\nu_t \qquad (2.31)$$

$$K(t) = P(t)H'(t)\Sigma_\nu^{-1}(t) \qquad (2.32)$$

Variance Calculations

In order to determine a differential equation for $P(t)$, note that

$$P(t) = E[\tilde{x}_t\tilde{x}_t'] = E[x_t x_t' - \hat{x}_t\hat{x}_t']$$

Thus

$$P(t) = X(t) - \Sigma(t) \qquad (2.33)$$

where

$$X(t) = E[x_t x_t'] \quad \text{and} \quad \Sigma(t) = E[\hat{x}_t\hat{x}_t'] \qquad (2.34)$$

First a differential equation for $X(t)$ will be determined. We have (from (2.19)) that the $j^{\underline{th}}$ element of x_t satisfies

$$dx_j = F_j x_t dt + \sum_{i=1}^{q} G_{ij} x_t dw_i \tag{2.35}$$

In the above expression F_j and G_{ij} denote the $j^{\underline{th}}$ rows of $F(t)$ and $G_i(t)$ respectively. Using the Ito differential rule, and taking expectation, we find that the product $x_j x_k$ satisfies

$$\frac{d}{dt}(E[x_j x_k]) = E(x_k F_j x_t + x_j F_k x_t) + \sum_{i=1}^{q} G_{ij} X(t) G_{ik}' \tag{2.36}$$

Generalizing the above result to the matrix case we have

$$\dot{X}(t) = F(t)X(t) + X(t)F'(t) + \sum_{i=1}^{q} G_i(t)X(t)G_i'(t) \tag{2.37}$$

$$X(0) = E[x_0 x_0'] \tag{2.38}$$

The next step is to obtain a differential equation for $\Sigma(t) \triangleq E[\hat{x}_t \hat{x}_t']$. Note that because $\hat{x}_t$ satisfies the linear equation (2.31) we can write that

$$\hat{x}_t = \Phi(t,0)\hat{x}_0 + \int_0^t \Phi(t,\tau)K(\tau)d\nu_\tau \tag{2.39}$$

where $\Phi(t,\tau)$ is the unique solution of

$$\frac{d}{dt}\Phi(t,\tau) = F(t)\Phi(t,\tau), \qquad \Phi(t,t) = I \tag{2.40}$$

Thus

$$\Sigma(t) = E[\hat{x}_t \hat{x}_t'] = \Phi(t,0)\Sigma(0)\Phi'(t,0) + \int_0^t \Phi(t,\tau)K(\tau)\Sigma_\nu(\tau)K'(\tau)\Phi'(t,\tau)d\tau \qquad (2.41)$$

Differentiating with respect to t, we obtain

$$\dot{\Sigma}(t) = F(t)\Sigma(t) + \Sigma(t)F'(t) + K(t)\Sigma_\nu(t)K'(t) \qquad (2.42)$$

$$\Sigma(0) = E[\hat{x}_0 \hat{x}_0'] \qquad (2.42a)$$

Now $\dot{P}(t) = \dot{X}(t) - \dot{\Sigma}(t)$, thus

$$\dot{P}(t) = F(t)P(t) + P(t)F'(t) + \sum_{i=1}^{q} G_i(t)X(t)G_i'(t) - K(t)\Sigma_\nu(t)K'(t) \qquad (2.43)$$

$$P(0) = X(0) - \Sigma(0) = E[x_0 x_0'] - E[x_0]E[x_0'] \qquad (2.44)$$

Finally, an expression must be given for $\Sigma_\nu(s)$. Recall that $\Sigma_\nu(s)ds = E[d\nu_s d\nu_s']$. Expanding this expression gives

$$\begin{aligned}\Sigma_\nu(s)ds &= E[(H(s)\tilde{x}_s ds + \sum_{i=1}^{r} M_i(s)x_s dv_{i_s})(\tilde{x}_s'H'(s)ds + \sum_{j=1}^{r} dv_{j_s} x_s'M_j'(s))] \\ &= E[H(s)\tilde{x}_s\tilde{x}_s'H'(s)(ds)^2 + \sum_{i=1}^{r} M_i(s)x_s\tilde{x}_s'H'(s)dv_{i_s} ds \\ &\quad + \sum_{j=1}^{r} H(s)\tilde{x}_s x_s'M_j(s)dv_{j_s} ds + \sum_{i=1}^{r}\sum_{j=1}^{r} M_i(s)x_s x_s'M_j'(s)dv_{i_s} dv_{j_s}]\end{aligned} \qquad (2.45)$$

Noting that $E[(d\nu_s)^2] = ds$, that the ν_i's are mutually independent, and keeping only first order terms in ds (as in [3], p. 113), we obtain the result

$$\Sigma_\nu(s)ds = \sum_{i=1}^{r} M_i(s)X(s)M_i'(s)ds \tag{2.46}$$

or

$$\Sigma_\nu(s) = \sum_{i=1}^{r} M_i(s)X(s)M_i'(s) \tag{2.47}$$

In summary, the linear minimum variance filter for the system (2.19), (2.20) is given by

$$d\hat{x}_t = F(t)\hat{x}_t dt + K(t)d\nu_t$$

$$K(t) = P(t)H'(t)\Sigma_\nu^{-1}(t)$$

$$\Sigma_\nu(t) = \sum_{i=1}^{r} M_i(t)X(t)M_i'(t)$$

$$\dot{X}(t) = F(t)X(t) + X(t)F'(t) + \sum_{i=1}^{q} G_i(t)X(t)G_i'(t)$$

$$\dot{P}(t) = F(t)P(t) + P(t)F'(t) + \sum_{i=1}^{q} G_i(t)X(t)G_i'(t) - K(t)\Sigma_\nu(t)K'(t)$$

with initial conditions

$$\hat{x}_0 = E[x_0]$$

$$X(0) = E[x_0 x_0']$$

$$P(0) = E[x_0 x_0'] - E[x_0]E[x_0']$$

2.3 Asymptotic Behavior

In this section the asymptotic behavior of the optimal linear filter is discussed. Specifically, the questions of the existence of a finite steady state error covariance and the stability of the filter are considered.

Wonham [7] and Haussmann [8] have considered the existence, uniqueness, and asymptotic behavior of the solutions to a Riccati equation similar to (2.43) that arise in connection with the problem of stochastic control for linear systems with state-dependent noise. As a result of the dual nature of filtering and control, these results are easily adapted to the problem of linear filtering for systems with state-multiplicative noise.

For the purposes of this section, the presence of additive noise will be explicitly included in our process model. Only additive noise will be considered in the measurement, although the results to be presented here are easily applicable to the model used in Section 2.2. We consider the system

$$dx_t = Fx_t dt + \sum_{i=1}^{q} G_i x_t dw_{i_t} + G dw_t \tag{2.48}$$

and the measurement

$$dy_t = Hx_t dt + dv_t \tag{2.49}$$

with $E[dv_t dv_t'] = Rdt$. The standard Brownian motions w_t and w_{i_t}, $i = 1,\ldots,q$, are mutually independent, and independent of x_0 and the measurement noise v_t. The matrices F, G, G_i, $i = 1,\ldots,q$, H, and R are constant. The evolution of the error covariance for this problem is given by

$$\dot{P}(t) = FP(t) + P(t)F' + GG' + \Pi(t) - P(t)H'R^{-1}HP(t) \tag{2.50}$$

with

$$\Pi(t) \triangleq \sum_{i=1}^{q} G_i(t)X(t)G_i'(t) \tag{2.51}$$

The initial condition P(0) is assumed to be positive semidefinite. X(t) denotes the state covariance $E[x_t x_t']$. Wonham's basic result is given in the following theorem.

If (F,H) is detectable, and $\Pi(t)$ has a finite positive definite steady state value Π_∞ which satisfies

$$\inf_K \left\| \int_0^\infty e^{(F-KH)'t}\Pi_\infty e^{(F-KH)t}dt \right\| < 1 \tag{2.52}$$

then

$$P_\infty = \lim_{t\to\infty} P(t) \tag{2.53}$$

is bounded. If, in addition the pair (F,G) is controllable, then P_∞ exists and is positive definite. In this case, P_∞ is the unique positive definite solution of

$$0 = FP + PF' + \Pi_\infty + GG' - PH'R^{-1}HP \tag{2.54}$$

and the matrix

$$F - P_\infty H'R^{-1}H$$

is stable.

Equation (2.52) expresses the requirement that there must not be "too much" state multiplicative noise present. Although this condition does establish limits on the amount of multiplicative noise that can be tolerated (and still allow a finite P_∞), it appears rather difficult to evaluate in practice. For filtering problems of very low dimension a much simpler approach can be taken. Consider for example the following scalar constant coefficient problem:

$$dx_t = ax_t dt + bx_t d\eta_t + cdw_t \tag{2.55}$$

$$dy_t = hx_t dt + dv_t \tag{2.56}$$

where η_t, w_t, and v_t are independent standard Brownian motions independent of x_0. Then the variance equations associated with the optimal linear filter are given by

$$\frac{d}{dt} p(t) = 2ap(t) + b^2\overline{x^2(t)} + c^2 - \frac{p^2(t)h^2}{r} \tag{2.57}$$

$$\frac{d}{dt} \overline{x^2(t)} = 2a\overline{x^2(t)} + b^2\overline{x^2(t)} + c^2 \tag{2.58}$$

It is clear from an examination of these equations that the following conditions must be met in order to have a finite steady state error variance P_∞.

1) The state variance $\overline{x^2(t)}$ must have a finite steady state value, thus $2a + b^2 < 0$.

2) The system (2.55), (2.56) must be detectable.

In addition, for P_∞ to be positive definite, we must have $c \neq 0$. Thus we see that for systems of low order we can deal with simpler conditions on the allowable magnitude of state dependent noise than that given by (2.52).

The final statement of Wonham's theorem is quite important in applications because it provides conditions which insure the stability of the filter. The practical significance of this result is that errors in the initial conditions $\hat{x}_0$ and P_0 or errors in the numerical calculations associated with the filter are eventually forgotten. Given the fact that $\hat{x}_0$ and P_0 are often rather arbitrarily chosen, the design of a stable filter becomes essential.

CHAPTER 3

TENSOR METHODS IN SUBOPTIMAL NONLINEAR FILTERING

The solution of the linear filtering problem is sometimes characterized in the following manner. The optimal linear filtered estimate $\hat{x}_{\ell_t}$ of a random variable x_t is simply the projection of x_t onto the Hilbert space spanned by the related observed process y_s, $0 \leqslant s \leqslant t$. Thus the optimal linear filtered estimate can be written as

$$\hat{x}_{\ell_t} = P[x_t | \mathscr{H}_t^y] \tag{3.1}$$

where P denotes the orthogonal projection, and the Hilbert space $\mathscr{H}_t^y$ denotes the set of all linear combinations of the form $\sum_i \alpha_i y_{t_i}$, $t_i \leqslant t$, and mean square limits of such combinations.

As discussed in Chapter 1, in general the nonlinear filtering problem does not have a finite dimensional solution due to the coupling of conditional moments. Thus one is forced to turn to suboptimal filters. One approach to suboptimal nonlinear filtering is to try to obtain the best filter within a particular class of nonlinear filters. For example, given a scalar observable y_s, $0 \leqslant s \leqslant t$, one might want to obtain the estimate

$$\hat{x}_t = P[x_t | \mathscr{H}_t^y \oplus \mathscr{H}_t^{y^3}] \tag{3.2}$$

Thus the desired estimate $\hat{x}_t$ is the orthogonal projection of x_t onto the direct sum of the Hilbert spaces spanned by y_t and y_t^3. In general, it can be difficult if not impossible to obtain the best filter of the above form for a particular nonlinear problem. As a matter of fact, it is usually not even clear how to solve for the optimal linear filter.

The following definitions will be used in the description of the application of tensor methods in suboptimal filtering for bilinear systems. Let $x^{[p]}$ denote the $p^{\underline{th}}$ order tensor product. That is, $x^{[p]}$ is a vector made up of all possible (suitably scaled) $p^{\underline{th}}$ order monomials in the elements of x. For example, if

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.3}$$

then

$$x^{[3]} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^3 \\ \alpha_1 x_1^2 x_2 \\ \alpha_2 x_1 x_2^2 \\ x_2^3 \end{bmatrix} \tag{3.4}$$

where $\otimes$ denotes the tensor product operation. The weights α_i are chosen so that for the Euclidean norm we have $\|x^{[p]}\| = \|x\|^p$. However, for the research described in this report these weights are not essential and will hereafter be neglected. Higher order tensor products and tensor products for higher dimensional vectors can be defined in a manner similar to (3.4).

3.1 Moment Calculations for Bilinear Systems

As previously stated, conditional moments for the general nonlinear filtering problem cannot be calculated using a finite set of differential equations due to the coupling of the moments. Moreover, for the general nonlinear dynamical system

$$dx_t = f(x_t,t)dt + G(x_t,t)d\beta_t \tag{3.5}$$

even the unconditional moments (that is, not conditioned on the observation of a related process) of the state x_t cannot be calculated due to moment coupling. However, the bilinear model of the form

$$dx_t = F(t)x_t dt + \sum_{i=1}^{q} G_i(t)x_t dw_{i_t} \tag{3.6}$$

has a rather special property; the evolution of the unconditional moments is governed by a <u>finite</u> set of linear ordinary differential equations. For example, if x_t satisfies the scalar stochastic differential equation

$$dx_t = ax_t dt + bx_t dw_t \tag{3.7}$$

then, by the Ito differential rule, we have

$$dx_t^p = apx_t^p dt + bpx_t^p dw_t + \frac{b^2}{2} p(p-1)x_t^p dt \tag{3.8}$$

Using the fact that the expected value of the Ito integral is zero we obtain

$$\frac{d}{dt} (E[x_t^p]) = (a + \frac{b^2}{2} (p-1))pE[x_t^p]dt \tag{3.9}$$

Similar calculations can be made for the moments of vector processes described by bilinear stochastic differential equations. This property of the unconditional moments of bilinear processes will be essential for the nonlinear filters to be discussed in the remainder of this chapter.

3.2 Tensoring of Observables

Consider the stochastic bilinear system ((3.6), repeated here) and observation

$$dx_t = F(t)x_t dt + \sum_{i=1}^{q} G_i(t)x_t dw_{i_t} \tag{3.6}$$

$$dy_t = H(t)x_t dt + \sum_{i=1}^{r} M_i(t)x_t dv_{i_t} \tag{3.10}$$

where the w_{i_t} and v_{i_t} are Brownian motions independent of x_0. If we consider the augmented state vector

$$\tilde{x} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{3.11}$$

and form

$$\tilde{x}_{[p]} = \begin{bmatrix} \tilde{x} \\ \tilde{x}^{[2]} \\ \vdots \\ \tilde{x}^{[p]} \end{bmatrix} \tag{3.12}$$

then for each p, $\tilde{x}_{[p]}$ satisfies a related bilinear stochastic differential equation

$$d\tilde{x}_{[p]_t} = A(t)\tilde{x}_{[p]_t} dt + \sum_{i=1}^{n} B_i(t)\tilde{x}_{[p]_t} d\tilde{w}_t \tag{3.13}$$

Note that if we observe y_t we can also regard $y_t^{[2]},\ldots,y_t^{[p]}$ as being observed as well. These additional "pseudo-observables" can provide for improved performance from the optimal linear filter. For example, by using y_t and $y_t^{[3]}$ in our estimate of x_t, we then have

$$\hat{x}_t = P[x_t | \mathscr{H}_t^y \oplus \mathscr{H}_t^{y^3}] \tag{3.14}$$

This estimate is generally superior to the estimate

$$\hat{x}_{\ell_t} = P[x_t | \mathscr{H}_t^y] \tag{3.15}$$

because $\hat{x}_t$ is obtained by projecting onto a "larger" vector space than that used to obtain $\hat{x}_{\ell_t}$. Notice also that because of the non-Gaussian nature of the x_t process, $P[x_t | \mathscr{H}_t^{y^3}]$ is typically nonzero, which improves the performance of (3.14) relative to (3.15). Since the x_t process is the solution of a bilinear stochastic differential equation, the observed process $\tilde{y}_{[p]}$ satisfies

$$d\tilde{y}_{[p]_t} = C(t)\tilde{x}_{[p]_t}dt + \sum_{i=1}^{k} D_i(t)\tilde{x}_{[p]_t}d\tilde{v}_{i_t} \tag{3.16}$$

where

$$\tilde{y}_{[p]_t} \triangleq \begin{bmatrix} y_t \\ y_t^{[2]} \\ \vdots \\ y_t^{[p]} \end{bmatrix} \tag{3.17}$$

Since both $\tilde{x}_{[p]}$ and $\tilde{y}_{[p]}$ are solutions of bilinear stochastic differential equations, the linear minimum variance filter described in Chapter 2 may be applied.

In general, the optimal linear estimate of the state vector for the system (3.13) satisfied by $\tilde{x}_{[p]}$ using the observations y_t, $y_t^{[2]}, \ldots, y_t^{[p]}$ provides a better estimate of x_t than the optimal linear estimate based on the system satisfied by $\tilde{x}_{[p-1]}$ using the observations y_t, $y_t^{[2]}, \ldots, y_t^{[p-1]}$ [9]. However, it should be noted that the higher moments introduced in these $x_{[p]}$ setups may not always exist.

Another value of the $x_{[p]}$ setup is the following. Consider the situation when we have a polynomial observation, for example, the scalar observation

$$dy_t = x_t^3 dt + dv_t \qquad (3.18)$$

Clearly, by using an $x_{[p]}$ setup we can convert this nonlinear observation to a linear observation. Of course this requires that the x_t process is the solution of a linear or bilinear stochastic differential equation. An example of this approach is discussed in Chapter 5.

Although the idea of improved filter performance through tensoring of the observable seems useful in theory, in practice several difficulties were encountered. One of these difficulties was the fact that the covariance matrix of the innovations process associated with the tensored observation is singular at $t=0$. This can be seen in the following example. Let the original scalar observable be given by

$$dy_t = x_t dt + dv_t \qquad (3.19)$$

Then assume that we want to use a linear filter to calculate $\hat{x}_t = P[x_t | \mathscr{H}_t^y \oplus \mathscr{H}_t^{y^3}]$. We see that

$$dy_t^3 = (3x_t y_t^3 + 3y_t)dt + 3y_t^2 dv_t \qquad (3.20)$$

Now the question arises "$E[y_0^2] = ?$" The statistics of y_0 are not typically specified in a filtering problem as is the case with x_0. If one assumes that y_0 is known to be zero, then the y_t^3 measurement contains no noise at $t=0$, and thus the innovations covariance matrix (which needs to be inverted in the filtering algorithm) is singular. When this occurs, standard techniques for dealing with singular measurement noise covariance matrices can be used. (See for example, Liptser and Shiryayev [10], p. 375.) On the other hand, one could assume that y_0 is not precisely known, and choose $E[y_0^2] \neq 0$. An arbitrary choice of $E[y_0^2]$ does not seem reasonable; however, it is not clear what the proper value should be.

The second difficulty encountered with the tensored observables scheme was more significant than that described above. From equation (2.46) we can see that the variance of the innovation associated with the y_t^3 measurement is equal to $9E[y_t^4]$. Now the y_t process evolves as

$$y_t = y_0 + \int_0^t x_s ds + v_t \tag{3.21}$$

Because of the Brownian component v_t in (3.21) the variance of the innovation associated with y_t^3 grows without bound, thus in steady state there is no information in the y_t^3 measurement that is useful in estimating x_t. An identical problem occurs for all higher order tensors of the original observable in these problems. For this reason, little effort was expended in the study of a tensored observables approach.

A brief evaluation of the tensored observable scheme was made for a simple example. Covariance calculations were performed for a tensored observable filter using the original measurement y_t and the pseudo-observable y_t^3. The system considered was

$$dx_t = -2x_t dt + x_t dw_t \tag{3.22}$$

$$dy_t = x_t dt + dv_t \tag{3.23}$$

The process noise w_t was assumed independent of v_t, with w_t amd v_t independent of x_0, and

$$E[(dw_t)^2] = q\ dt \tag{3.24}$$

$$E[(dv_t)^2] = r\ dt \tag{3.25}$$

$$x_0 \text{ uniform on } [-10,10] \tag{3.26}$$

The initial condition y_0 was assumed to be equal to zero, and the method of [10] was used to deal with the singular observation at $t = 0$. For the case $q = 1$, $r = 10$, the error variance in the estimation of x_t during the transient period was less than one percent smaller than that obtained using the optimal linear filter. For the case $q = 1$, $r = 1$, the estimation error variance was identical to five decimal places. In addition, due to the problem explained above, the steady error variance was identical for the tensored observable filter and the linear filter in each of the cases studied.

As a result of the problems associated with the tensored observable filter, and its failure to provide a significant performance improvement over the linear filter (even during the transient period), further evaluation of this approach was considered unlikely to produce useful results.

3.3 Nonlinear Post-Processing

Again consider the bilinear system and observation

$$dx_t = F(t)x_t dt + \sum_{i=1}^{q} G_i(t)x_t dw_{i_t} \tag{3.6}$$

$$dy_t = H(t)x_t dt + \sum_{i=1}^{r} M_i(t)x_t dv_{i_t} \qquad (3.10)$$

One approach to improving on the best linear estimate of x_t given y_s, $0 \leqslant s \leqslant t$, might be to follow the linear filtering by a nonlinear memoryless post-processor. Denoting this nonlinear processor as $\psi(\cdot)$, the filter structure would be as shown in Figure 3.1.

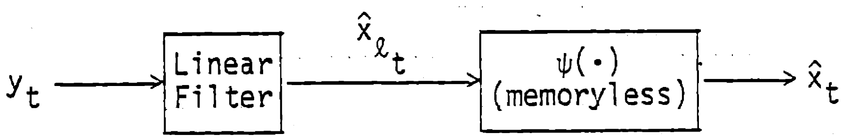

Figure 3.1. Post-Processing of Linear Estimate.

The criteria for our choice of $\psi(\hat{x}_{\ell_t})$ is that it should minimize the squared error loss function $E[(x_t - \psi(\hat{x}_{\ell_t}))^2]$. Clearly the proper choice of $\psi(\hat{x}_{\ell_t})$ is the conditional mean

$$\psi(\hat{x}_{\ell_t}) = E[x_t | \hat{x}_{\ell_t}] \qquad (3.27)$$

which is (at least in theory) computable using a Bayes rule formulation.

The approach taken in this research was to approximate the optimal $\psi(\hat{x}_{\ell_t})$ by the polynomial form

$$\psi(\hat{x}_{\ell_t}) \simeq \sum_{i=1}^{n} \alpha_i(t)\hat{x}_{\ell_t}^{\,i} \qquad (3.28)$$

For example, in the numerical studies to be discussed in Chapters 4 and 5, estimators of the form

$$\hat{x}_t = \alpha(t)\hat{x}_{\ell_t} + \beta(t)\hat{x}_{\ell_t}^{\,3} \qquad (3.29)$$

were used as approximations to $\psi(\hat{x}_{\ell_t})$.

The values of $\alpha(t)$ and $\beta(t)$ are calculated off-line in the following manner. Because the vector

$$z_t \triangleq \begin{bmatrix} x_t \\ \hat{x}_{\ell_t} \end{bmatrix} \tag{3.30}$$

satisfies a bilinear stochastic differential equation, its unconditional moments can be calculated by propagating a finite set of ordinary differential equations as discussed in Section 3.1. By solving for the second moments of

$$z_{[3]} = \begin{bmatrix} z_t \\ z_t^{[2]} \\ z_t^{[3]} \end{bmatrix} \tag{3.31}$$

we have all of the information necessary to choose the values of $\alpha(t)$ and $\beta(t)$ such that the mean square error

$$E[(x_t - \alpha(t)\hat{x}_{\ell_t} - \beta(t)\hat{x}_{\ell_t}{}^3)^2] \tag{3.32}$$

is minimized.

This calculation is accomplished as follows. First, the above expression for the mean square error in the estimation of x_t is differentiated with respect to $\alpha(t)$ and $\beta(t)$. These derivations are set equal to zero, and the resulting equations are used to solve for $\alpha(t)$ and $\beta(t)$ in terms of the moments of x_t and $\hat{x}_{\ell_t}$. Using this method, we obtain

$$\alpha(t) = \left(\overline{\hat{x}_{\ell_t}^{\,6}}\;\overline{x_t\hat{x}_{\ell_t}} - \overline{\hat{x}_{\ell_t}^{\,4}}\;\overline{x_t\hat{x}_{\ell_t}^{\,3}}\right)\Big/\Delta \tag{3.33}$$

$$\beta(t) = \left(-\overline{\hat{x}_{\ell_t}^{\,4}}\;\overline{x_t\hat{x}_{\ell_t}} + \overline{\hat{x}_{\ell_t}^{\,2}}\;\overline{x_t\hat{x}_{\ell_t}^{\,3}}\right)\Big/\Delta \tag{3.34}$$

where

$$\Delta = \overline{\hat{x}_{\ell_t}^{\,2}}\;\overline{\hat{x}_{\ell_t}^{\,6}} - \left(\overline{\hat{x}_{\ell_t}^{\,4}}\right)^2 \tag{3.35}$$

and the overbars denote unconditional expectation. Note that once we have solved for $\alpha(t)$ and $\beta(t)$, we can compute the mean square error in our estimate using (3.32).

Some insight into why estimators of the form (3.29) provide better performance than the optimal linear filter can be gained by briefly considering the standard linear filtering problem. Let x_t be a scalar Gauss-Markov process that satisfies

$$dx_t = -x_t dt + dw_t, \qquad E[x_0] = 0 \tag{3.36}$$

and take the observation to be

$$dy_t = x_t dt + dv_t \tag{3.37}$$

Assume that the observation y_t is Kalman filtered to obtain the optimal linear estimate $\hat{x}_{\ell_t}$. Now assume that we try to improve on the performance of this estimator with a nonlinear post-processor of the form

$$\hat{x}_t = \hat{x}_{\ell_t} + \frac{E[e_t \hat{x}_{\ell_t}^3]}{E[\hat{x}_{\ell_t}^6]} \hat{x}_{\ell_t}^3 \tag{3.38}$$

where

$$e_t = x_t - \hat{x}_{\ell_t} \tag{3.39}$$

The estimate $\hat{x}_t$ will be an improvement over $\hat{x}_{\ell_t}$ if the estimation error e_t is correlated with $\hat{x}_{\ell_t}^3$, that is

$$E[e_t \hat{x}_{\ell_t}^3] \neq 0 \tag{3.40}$$

or

$$E[x_t \hat{x}_{\ell_t}^3] \neq E[\hat{x}_{\ell_t}^4] \tag{3.41}$$

However, due to the fact that x_t and $\hat{x}_{\ell_t}$ are jointly Gaussian in the linear problem considered here, we can apply Gaussian moment factoring to show that

$$E[x_t \hat{x}_{\ell_t}^3] = 3(E[x_t \hat{x}_{\ell_t}])^2 \tag{3.42}$$

and that

$$E[\hat{x}_{\ell_t}^{\,4}] = 3(E[\hat{x}_{\ell_t}^{\,2}])^2 \qquad (3.43)$$

By the orthogonal projections theorem

$$E[x_t\hat{x}_{\ell_t}] = E[\hat{x}_{\ell_t}^{\,2}] \qquad (3.44)$$

since the estimation error is orthogonal to the estimate. Thus the inequality in (3.41) does not hold true, and the filter design in (3.38) provides no improvement over the optimal linear filter.

Now let us return to the bilinear filtering problem. In this case x_t and $\hat{x}_{\ell_t}$ are not jointly Gaussian (because x_t is not Gaussian). Therefore, Gaussian moment factoring cannot be applied. For the bilinear problem it seems unlikely that the condition

$$E[x_t\hat{x}_{\ell_t}^{\,3}] = E[\hat{x}_{\ell_t}^{\,4}] \qquad (3.45)$$

usually holds. In general it appears likely that all odd powers (tensors in the vector problem) of the linear estimate are correlated with the linear estimation error $x_t-\hat{x}_{\ell_t}$. Therefore in the bilinear problem one might expect that nonlinear post-processors of the form (3.28) will improve on the performance of the linear estimate.

Unfortunately, this author has been unable to prove the above conjecture for bilinear systems in general. However, nonlinear post-processing was used in the applications discussed in Chapters 4 and 5 and resulted in better performance than the linear filter with a minimum of added complexity.

3.4 Additional Applications of Tensor Methods

In Section 3.3 nonlinear post-processing of the state estimate from a linear filter was considered. A particular form was chosen for the approximation to the conditional expectation $E[x_t|\hat{x}_{\ell_t}]$. The purpose of this section is to briefly mention some other approaches to nonlinear filtering for bilinear systems that are made possible through tensor methods.

The first idea to be mentioned is quite closely related to the polynomial approximation to $E[x_t|\hat{x}_{\ell_t}]$ used in Section 3.3. The idea here is that we can post-process the best linear estimate $\hat{x}_{\ell_t}$ with a polynomial function of the form

$$\sum_{i=1}^{n} \alpha_i(t)(\hat{x}_{\ell_t})^i$$

to obtain an improved estimate for x_t as previously discussed, then the new estimate $\hat{x}_t$ can be updated again in a similar manner to produce an improved estimate $\hat{\hat{x}}_t$, etc., as suggested in the figure below.

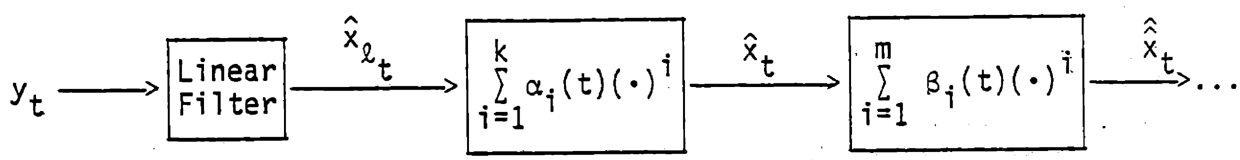

Figure 3.2. Multiple Stage Nonlinear Post-Processing.

The only drawback to this scheme is that in order to calculate the optimal values of the parameters α_i, β_i, etc. for each additional stage of post-processing, we must calculate higher moments of the vector

$$z_{[p]} = \begin{bmatrix} z_t \\ z_t^{[2]} \\ \vdots \\ z_t^{[p]} \end{bmatrix} \tag{3.46}$$

where

$$z_t = \begin{bmatrix} x_t \\ \hat{x}_{\ell_t} \end{bmatrix} \tag{3.47}$$

Obviously, dimensionality is the curse of this scheme. For example, in Section 3.3 we saw that a single stage cubic post-processor required the calculation of moments up to sixth order in x_t and $\hat{x}_{\ell_t}$. In order to implement a nonlinear post-processor of the form shown in Figure 3.2 with two stages of cubic post-processing, moments up to eighteenth order in x_t and $\hat{x}_{\ell_t}$ would be needed to calculate the parameters of the second post-processor. Of course, similar difficulties are encountered with high order polynomial single stage post-processors.

In fairness to the post-processing schemes considered here, it should be noted that the extensive moment calculations necessary to compute the parameters of the post-processors are off-line calculations. Therefore, the real time implementation of nonlinear post-processors is not burdensome from the standpoint of computing capability.

Another type of nonlinear filter that might prove useful is based on the idea of updating the innovations. Figure 3.3 illustrates the linear filter/nonlinear post-processor combination previously discussed. Observe that the linear filter makes no use of the improved state estimate $\hat{x}_t$ in its calculation of the innovations process $\dot{\nu}_t$. An idea that seems quite reasonable (but was not evaluated in this research) would be to update the innovations sequence using the nonlinear estimate $\hat{x}_t$ to produce a nonlinear filter of the form seen in Figure 3.4.

In conclusion, the nonlinear filters discussed in this chapter have only utilized a few of the many different applications of tensor methods in filtering for bilinear systems. Obviously, other variations are possible. The value of these methods is hard to predict without considering specific applications. Chapters 4 and 5 discuss the application of the nonlinear post-processor of Section 3.3 to two different example problems.

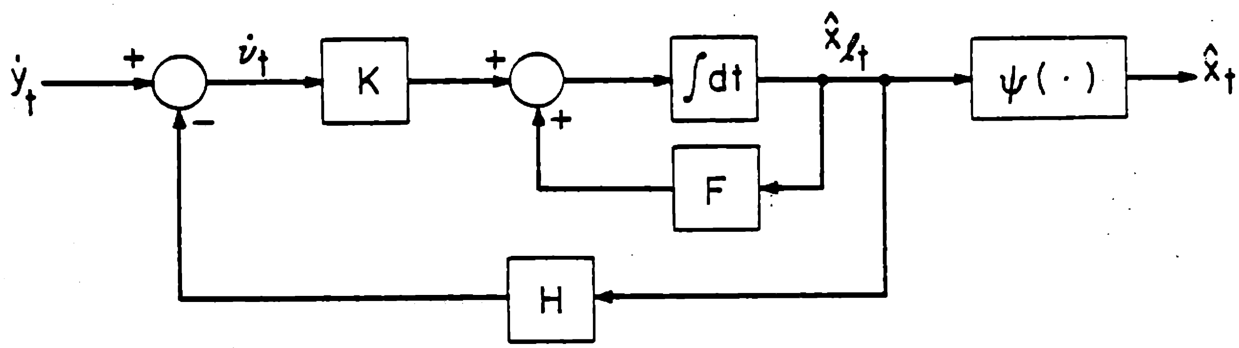

Figure 3.3. Optimal Linear Filter with Nonlinear Post-Processor.

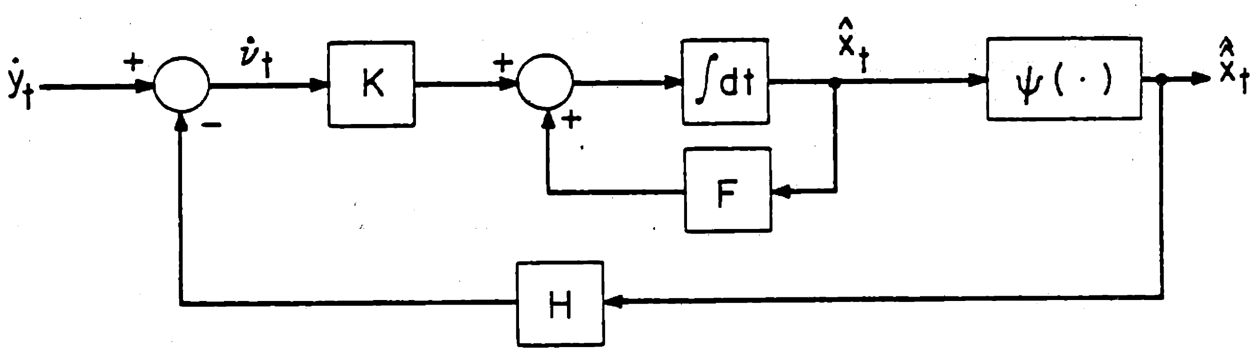

Figure 3.4. Nonlinear Filter Using Updated Innovations.

CHAPTER 4

APPLICATION OF TENSOR METHODS TO PHASE TRACKING

As an application of the filtering algorithms presented in Chapter 3, we consider an important class of phase tracking problem. Specifically we consider the estimation of a Brownian motion phase process θ_t given a nonlinear measurement corrupted by additive Gaussian white noise. This type of phase estimation problem is often encountered in the areas of navigation and communication. Section 4.1 formulates the problem precisely and describes the classical solution, the phase-lock loop. Section 4.2 demonstrates that through a change of coordinates, this phase tracking problem becomes a bilinear filtering problem. The application of tensor methods to the phase tracking problem is discussed. Finally, the results of numerical experiments applying the linear filter/nonlinear post-processor combination of Section 3.3 to the phase tracking problem are reported in Section 4.3.

4.1 Phase-Lock Loop

In this section we describe the phase tracking problem in detail and present the classical solution. We wish to estimate the phase process θ_t given the received signal

$$\dot{z}_t = \sin(\omega_c t + \theta_t) + \dot{n}_t \tag{4.1}$$

where $\dot{n}_t$ is zero mean Gaussian white noise of intensity r. It is assumed that θ_t is a Brownian motion process with variance parameter q, and that the θ and $\dot{n}$ processes are independent.

In [11] Van Trees demonstrates that $\dot{n}_t$ can be decomposed into

$$\dot{n}_t = \dot{n}_{1_t} \cos\omega_c t + \dot{n}_{2_t} \sin\omega_c t \tag{4.2}$$

where $\dot{n}_{1_t}$ and $\dot{n}_{2_t}$ are independent zero mean white Gaussian noise processes of intensity 2r. Then, by multiplying (heterodyning) the received signal $\dot{z}_t$ by $2\cos\omega_c t$ and $2\sin\omega_c t$ and then low pass filtering (to remove the resulting "$2\omega_c$" frequency terms) we obtain the vector measurement

$$\begin{bmatrix} \dot{z}_{1_t} \\ \dot{z}_{2_t} \end{bmatrix} = \begin{bmatrix} \sin\theta_t \\ \cos\theta_t \end{bmatrix} + \begin{bmatrix} \dot{n}_{1_t} \\ \dot{n}_{2_t} \end{bmatrix} \tag{4.3}$$

Because the measurements have been "stepped down" from the carrier frequency ω_c they are often called baseband measurements.

Similarly, the received signal $\dot{z}_t$ can be heterodyned with $2\cos(\omega_c t + \tilde{\theta}_t)$ and $2\sin(\omega_c t + \tilde{\theta}_t)$ for any function $\tilde{\theta}_t$ to produce the in-phase and quadrature measurements (at baseband)

$$\begin{bmatrix} \dot{z}_{I_t} \\ \dot{z}_{Q_t} \end{bmatrix} = \begin{bmatrix} \sin(\theta_t - \tilde{\theta}_t) \\ \cos(\theta_t - \tilde{\theta}_t) \end{bmatrix} + \begin{bmatrix} \dot{n}_{I_t} \\ \dot{n}_{Q_t} \end{bmatrix} \tag{4.4}$$

Van Trees also demonstrates that if $\tilde{\theta}_t$ is at least one integration removed from $\dot{z}_t$, then $\dot{n}_{I_t}$ and $\dot{n}_{Q_t}$ are zero mean Gaussian white noise processes of intensity 2r, independent of each other and $\tilde{\theta}_t$.

In an Ito calculus framework, the baseband measurements (4.3) and (4.4) are written

$$\begin{bmatrix} dz_{1_t} \\ dz_{2_t} \end{bmatrix} = \begin{bmatrix} \sin\theta_t \\ \cos\theta_t \end{bmatrix} dt + \begin{bmatrix} dn_{1_t} \\ dn_{2_t} \end{bmatrix} \tag{4.5}$$

and

$$\begin{bmatrix} dz_{I_t} \\ dz_{Q_t} \end{bmatrix} = \begin{bmatrix} \sin(\theta_t - \tilde{\theta}_t) \\ \cos(\theta_t - \tilde{\theta}_t) \end{bmatrix} dt + \begin{bmatrix} dn_{I_t} \\ dn_{Q_t} \end{bmatrix} \tag{4.6}$$

These measurements will be used as the inputs to the filters discussed in this chapter.

The classical solution to the phase tracking problem described above is known as the phase-lock loop and is discussed in Van Trees [11] and Viterbi [12]. The configuration of the PLL is illustrated in Figure 4.1. The gain K is chosen to minimize the mean square phase-estimation error. If the carrier frequency ω_c is sufficiently high so that low pass filtering does not affect the low frequency phase error signal, the equivalent baseband model shown in Figure 4.2 is accurate. Using the linearizing assumption

$$\sin(\theta_t - \tilde{\theta}_t) \sim \theta_t - \tilde{\theta}_t \tag{4.7}$$

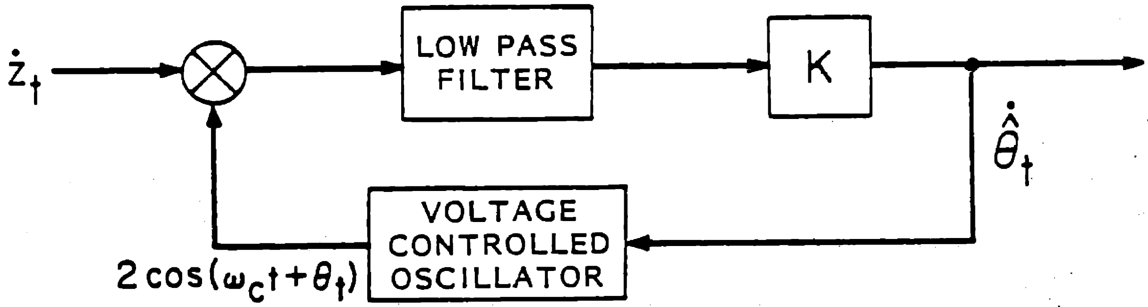

Figure 4.1. Classical Phase-Lock Loop Model.

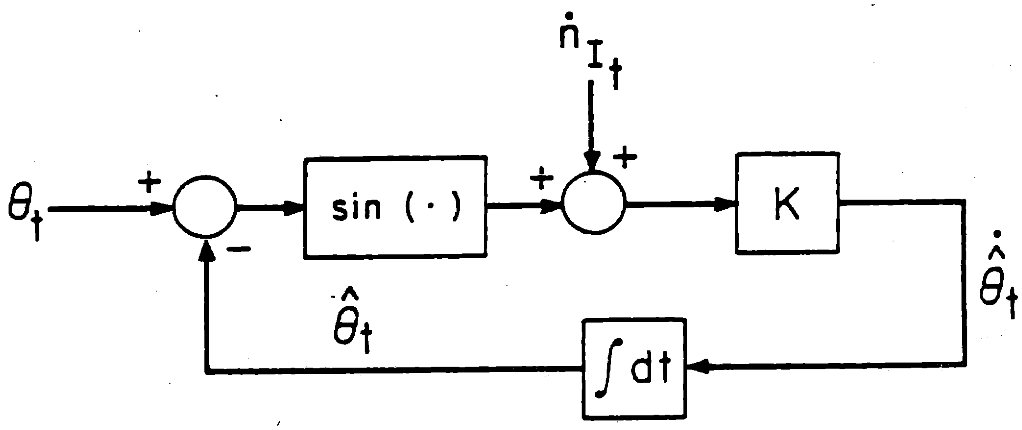

Figure 4.2. Baseband Phase-Lock Loop Model.

the value of K which minimizes the mean square error is

$$K = \sqrt{\frac{q}{2r}} \tag{4.8}$$

The resulting error variance is

$$P_{\theta_\ell} = \sqrt{2rq} \tag{4.9}$$

where the subscript ℓ refers to the linearizing assumption (4.7). Thus, from Figure 4.2, it can be seen that the classical phase-lock loop is described by the Ito equation

$$d\hat{\theta}_t = K\, dz_{I_t} \tag{4.10}$$

with K as given in (4.8).

An interesting feature of the classical PLL is that it is the extended Kalman filter for the phase tracking problem, as demonstrated in Mallinckrodt [13] and Eterno [14]. In Section 4.2 an alternative approach to phase tracking is discussed which leads to a bilinear filtering problem.

4.2 Phase Tracking in Rectangular Coordinates

The phase estimation problem posed in Section 4.1 has linear dynamics and a nonlinear measurement. By changing to a rectangular coordinate system, the problem can be transformed to one with bilinear dynamics and a linear measurement, as observed by Gustafson [2]. This is accomplished in the following manner. Define the two dimensional state vector

$$x_t = \begin{bmatrix} x_{1_t} \\ x_{2_t} \end{bmatrix} = \begin{bmatrix} \sin\theta_t \\ \cos\theta_t \end{bmatrix} \tag{4.11}$$

Since θ_t is Brownian motion, we can use the Ito differential rule to write the stochastic differential of x_t:

$$\begin{bmatrix} dx_{1_t} \\ dx_{2_t} \end{bmatrix} = \begin{bmatrix} -q/2 & 0 \\ 0 & -q/2 \end{bmatrix} \begin{bmatrix} x_{1_t} \\ x_{2_t} \end{bmatrix} dt + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_{1_t} \\ x_{2_t} \end{bmatrix} d\theta_t \tag{4.12}$$

$$E[(d\theta_t)^2] = q\ dt \tag{4.13}$$

The filtering problem specification is completed by the baseband measurement (4.5) (repeated here):

$$\begin{bmatrix} dz_{1_t} \\ dz_{2_t} \end{bmatrix} = \begin{bmatrix} x_{1_t} \\ x_{2_t} \end{bmatrix} dt + \begin{bmatrix} dn_{1_t} \\ dn_{2_t} \end{bmatrix} \tag{4.5}$$

with

$$E\left\{ \begin{bmatrix} dn_{1_t} \\ dn_{2_t} \end{bmatrix} [dn_{1_t} \quad dn_{2_t}] \right\} = \begin{bmatrix} 2r & 0 \\ 0 & 2r \end{bmatrix} dt \tag{4.14}$$

The goal is then to estimate $x_{1_t} = \sin\theta_t$ and $x_{2_t} = \cos\theta_t$ with the phase estimate given by

$$\hat{\theta}_t = \tan^{-1}\left(\frac{\hat{x}_{1_t}}{\hat{x}_{2_t}}\right) \tag{4.15}$$

These equations represent the phase tracking problem in a form suitable for application of the linear minimum variance filter. Furthermore, equations (4.12) and (4.13) are representative of a rich class of bilinear estimation problems in which the state evolves on an n-dimensional sphere of radius one. That is, with probability one,

$$\sum_{i=1}^{n} x_{i_t}^2 = 1 \text{ for all } t \tag{4.16}$$

Processes of this type are sometimes called rotational processes. Estimation for these systems is an interesting problem in its own right, independent of the connection with phase tracking. Further results on estimation for rotational processes can be found in [15].

Now that our estimation problem has been transformed into a bilinear filtering problem, the linear minimum variance filter and the tensor methods discussed in Chapter 3 are applicable. The optimal steady state linear filter is given by

$$\begin{bmatrix} d\hat{x}_{1_t} \\ d\hat{x}_{2_t} \end{bmatrix} = -\frac{q}{2}\begin{bmatrix} \hat{x}_{1_t} \\ \hat{x}_{2_t} \end{bmatrix} dt + K \begin{bmatrix} dz_{1_t} - \hat{x}_{1_t} dt \\ dz_{2_t} - \hat{x}_{2_t} dt \end{bmatrix} \tag{4.17}$$

where

$$K = \frac{\sqrt{rq(rq+1)}}{2r} - \frac{q}{2} \tag{4.18}$$

is obtained from the steady state solution of the Riccati equation (2.43). The phase estimate is given by

$$\hat{\theta}_t = \tan^{-1}\left(\frac{\hat{x}_{1_t}}{\hat{x}_{2_t}}\right) \tag{4.19}$$

as previously stated. Gustafson and Speyer [16] called this filter a linear quadrature filter (LQF). Numerical results on the performance of the LQF are given in the above noted reference. In general, this filter outperformed the classical PLL at high noise-to-signal ratios, and performed slightly worse at very low noise-to-signal ratios.

At this stage the tensor methods discussed in Chapter 3 are applicable. The approach chosen here was to operate on the linear estimates with a nonlinear post-processor of the form

$$\hat{\hat{x}}_t = \alpha(t)\hat{x}_t + \beta(t)\hat{x}_t^{\,3} \tag{4.20}$$

where $\hat{x}_t$ denotes the linear estimate. Steady state values of $\alpha(t)$ and $\beta(t)$ were calculated in the following manner. The linear estimates can be written

$$d\hat{x}_1 = (\tfrac{-q}{2} - K)\hat{x}_1 dt + Kx_1 dt + Kdn_1 \tag{4.21}$$

$$d\hat{x}_2 = (\tfrac{-q}{2} - K)\hat{x}_2 dt + Kx_2 dt + Kdn_2 \tag{4.22}$$

Combining the equation for the evolution of x_1 and x_2 (4.12) with (4.21) and (4.22), and defining

$$dw_1 = d\theta_t \tag{4.23}$$

$$dw_2 = dn_{1_t} \tag{4.24}$$

$$dw_3 = dn_{2_t} \tag{4.25}$$

and

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \hat{x}_1 \\ \hat{x}_2 \\ 1 \end{bmatrix} \tag{4.26}$$

we can write the bilinear stochastic differential equation

$$d\underline{x} = F\underline{x}\, dt + \sum_{i=1}^{3} G_i\underline{x}\, dw_i \tag{4.27}$$

If we now write an $x_{[p]}$ version of (4.27) with $p = 3$, we will have equations for all of the first, second, and third order tensors of $\underline{x}$. Symbolically, we have

$$d\underline{x}_{[3]} = A\underline{x}_{[3]}dt + \sum_{i=1}^{3} B_i\underline{x}_{[3]}dw_i \tag{4.28}$$

A listing of the elements of $\underline{x}_{[3]}$ and a definition of the elements of A, B_1, B_2, and B_3 for the phase tracking problem can be found in Appendix A. Next, we solve for the steady state value of

$$X(t) \triangleq E[\underline{x}_{[3]_t} \quad \underline{x}'_{[3]_t}] \tag{4.29}$$

by propagating the ordinary differential equation

$$\dot{X}(t) = AX(t) + X(t)A' + \sum_{i=1}^{3} B_i X(t) B_i' \tag{4.30}$$

At this point we have all of the moments necessary for the calculation of the steady state values of α and β, and thus to minimize

$$E[(x_t - \alpha\hat{x}_t - \beta\hat{x}_t^{\,3})^2]$$

This calculation is accomplished in a straightforward manner by differentiating the above expression with respect to α and β, equating to zero, and solving for α and β in terms of the moments of $\underline{x}_{[3]}$. Expressions for α and β in terms of these moments have been given in Section 3.3.

After estimates of the form $\hat{\hat{x}}_t = \alpha\hat{x}_t + \beta\hat{x}_t^{\,3}$ are obtained for x_1 and x_2, a new phase estimate is generated using

$$\hat{\hat{\theta}}_t = \tan^{-1}\left(\frac{\hat{\hat{x}}_{1_t}}{\hat{\hat{x}}_{2_t}}\right) \tag{4.31}$$

just as was done with the linear estimates in (4.15).

This completes the description of one approach to the design of a nonlinear post-processor for use in the phase tracking problem. Quantitative results comparing the performance of this linear filter/nonlinear post-processor algorithm with the LQF and the classical phase-lock loop can be found in Section 4.3.

4.3 Numerical Results

The first step in the numerical calculations necessary to simulate the linear filter/nonlinear post-processor combination was to determine the optimal values of α and β. The method used was described in the previous section. A byproduct of the calculation for α and β was the mean square error in the estimation of $x_1 = \sin\theta$ and $x_2 = \cos\theta$. The mean square error was obtained directly by using the calculated values of α and β in the expression

$$E[(x_i - \alpha\hat{x}_i - \beta\hat{x}_i^{\,3})^2]$$

(Note that all moments necessary for this calculation were already computed in order to select the optimal values of α and β.)

The results of these covariance calculations are illustrated in Figure 4.3. In this figure the reduction in mean square error obtained by nonlinear post-processing (with the performance of the linear filter as a reference) is plotted versus P_{θ_ℓ}. P_{θ_ℓ} is a useful (and commonly used) noise-to-signal ratio type parameter (see equation (4.9)). At this stage several comments should be made. First, note that the reduction in filtering error brought about by nonlinear post-processing varies with noise-to-signal ratio. At very low noise-to-signal ratios, the performance of the linear filter is nearly optimal, so the nonlinear filter has little room for improvement. On the other hand, in very high noise situations, there is just not much information about the state in the measurement, and thus the mean square error of both the linear and nonlinear filters approaches the a priori value.

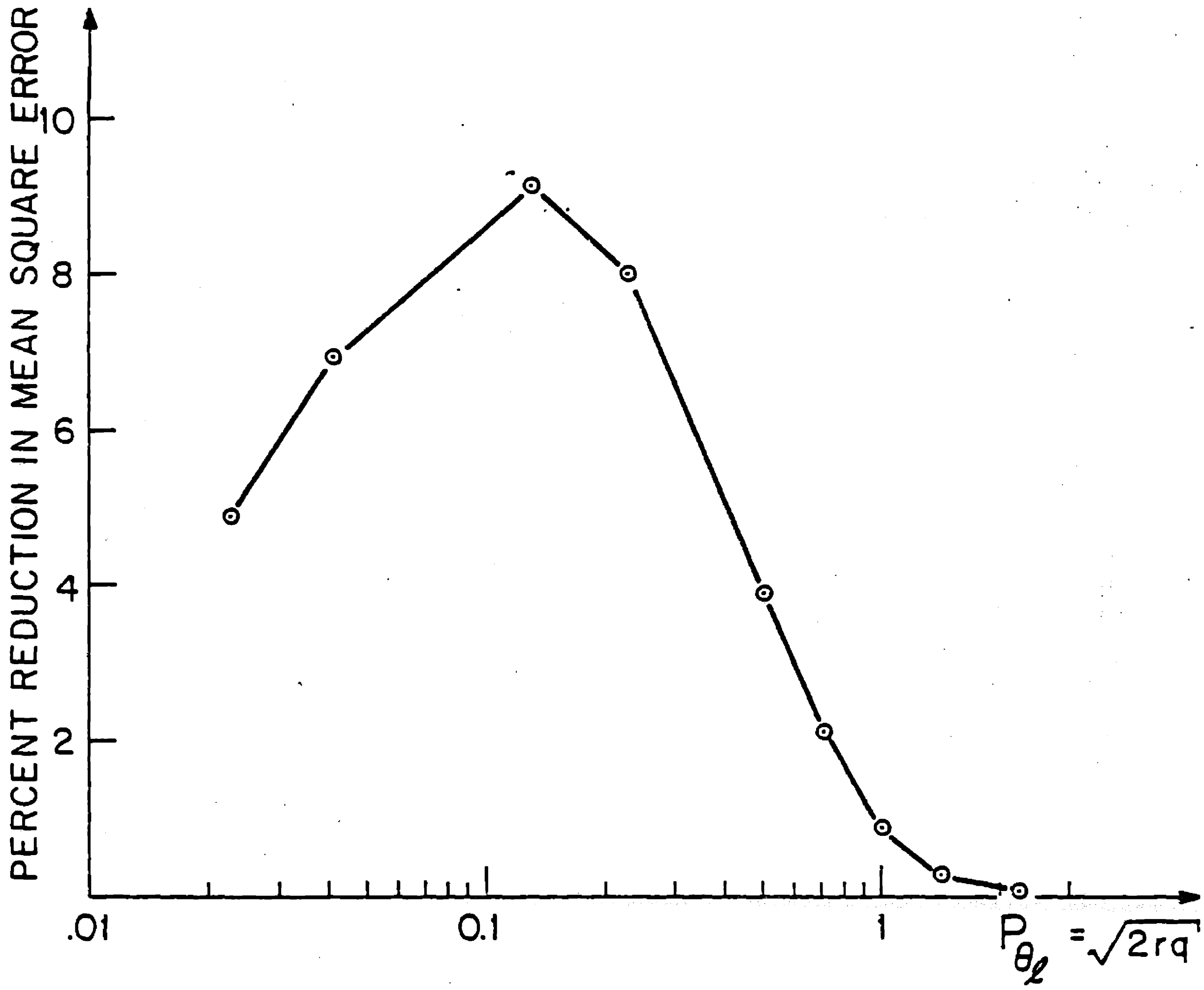

Figure 4.3. Reduction in Mean Square Error Using Nonlinear Post-Processor.

An encouraging aspect of the results given in Figure 4.3 is that at $P_{\theta_\ell} = .129$, the nonlinear post-processor reduced the mean square estimation error by over 9%. Improvements in filter performance of this order could be significant in some applications.

A plot of the steady state values of the parameters α and β for the nonlinear post-processor is given in Figure 4.4. It is not surprising that the filter makes very little use of the cube of the linear estimate ($|\beta|$ is small) at low values of P_{θ_ℓ}, because the linear filter is nearly optimal in low noise situations as previously noted.

Because the value $P_{\theta_\ell} = .129$ represented the point of maximum improvement by the nonlinear post-processor over the linear filter in the estimation of x_1 and x_2, this value was chosen for a filter comparison using a digital computer simulation. In the comparison, we shall denote the linear filter/nonlinear post-processor combination as a tensor filter. The phase-lock loop (PLL), linear quadrature filter (LQF), and the tensor filter (TF) were simulated using identical pseudo-noise sequences for each. Following Eterno [14], a fourth order Runge-Kutta integration routine with an integration step size of one percent of the PLL time constant (1/K) was used. 1900 effective degrees of freedom were obtained by running the filters for four runs of 500 time constants and discarding the data from the first 25 time constants in each run. This gives a three percent predicted standard deviation in the computed error variances, as demonstrated in [16].

The results of the simulation are given in Table 4.1. Several comments are appropriate. The PLL slightly outperformed the LQF as

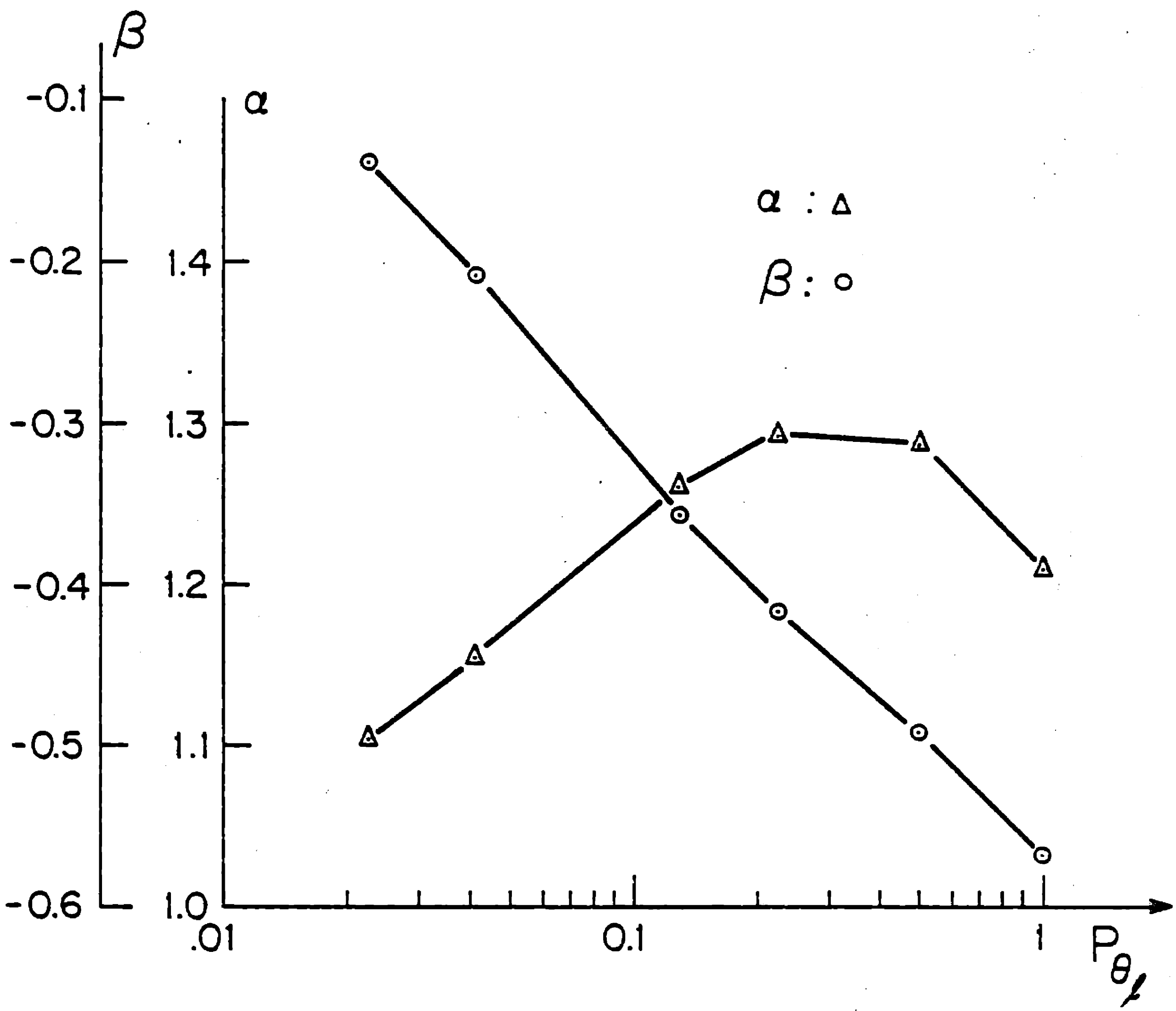

Figure 4.4. α and β Parameters for Nonlinear Post-Processor.

TABLE 4.1: Suboptimal Filter Comparison at $P_{\theta_\ell} = .129$

	Phase Estimation Error $\varepsilon = \theta - \hat{\theta}$		
	$E[\varepsilon^2]$ (rad^2)	RMS Error (deg)	$E[1 - \cos\varepsilon]$
PLL	.1401	21.4	.068
LQF	.1477	22.0	.071
TF	.1511	22.2	.073

expected. However, the fact that the tensor filter did not outperform the LQF in the estimation of the phase angle θ_t was disappointing. In trying to determine the cause of the difficulty, two things became apparent. First, the tensor filter was performing as expected in the estimation of x_1 and x_2. The computed reduction in mean square error (relative to the linear filter) in estimating x_1 was 7.3%, which agrees fairly well with the 9.2% figure determined during the off-line covariance calculations.

The second item discovered in the search for an explanation of the tensor filter's poor performance in estimating θ_t is that both the LQF and the TF produced biased phase estimates. In fact, in both cases the biases were large enough to account for the degradation in performance relative to the PLL. It should also be noted that the observed difference in performance (in estimating θ_t) between the LQF and the TF is well within the 3% sampling error on the computed error variances, and thus the significance of the observed performance difference is questionable.

In conclusion, it must be stated that the performance of the tensor filter was disappointing when applied to the phase tracking problem. However, as noted earlier, the tensor filter did provide up to 9% reduction in mean square error (compared to the optimal linear filter) in the estimation of x_1 and x_2. Thus, the tensor filter performed quite well for a class of estimation problem that has occupied many researchers--the problem of state estimation for processes evolving on the circle.

CHAPTER 5

THE CUBIC SENSOR PROBLEM

In Chapter 4 it was found that nonlinear post-processing of linear estimates was quite useful for the class of bilinear estimation problems in which the state evolves on a circle. However, the question of the applicability of this method to bilinear problems in general remains unanswered. As a small first step toward answering this question, we consider the usefulness of nonlinear post-processing in another estimation problem, commonly known as the cubic sensor.

Section 5.1 introduces the problem, and demonstrates that the cubic sensor can easily be formulated as a bilinear problem. The application of a tensor method, in particular the nonlinear post-processor, is discussed in Section 5.2. Numerical results are presented in Section 5.3.

5.1 Formulation as a Bilinear Filtering Problem

Consider the stationary Gauss-Markov process described as the steady state solution of

$$dx_t = -x_t dt + dw_t \tag{5.1}$$

We shall consider the nonlinear filtering problem with observation

$$dy_t = x_t^3 dt + dv_t \tag{5.2}$$

The Brownian motions w_t and v_t are independent, and independent of x_0, with

$$E[(dw_t)^2] = q\ dt \tag{5.3}$$

$$E[(dv_t)^2] = r\ dt \tag{5.4}$$

Note that if the Ito differential rule is used to write stochastic differential equations for x_t^2 and x_t^3, we obtain the bilinear estimation problem given below.

$$d\begin{bmatrix} x_t \\ x_t^2 \\ x_t^3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & q \\ 3q & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_t \\ x_t^2 \\ x_t^3 \\ 1 \end{bmatrix} dt + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_t \\ x_t^2 \\ x_t^3 \\ 1 \end{bmatrix} dw_t \tag{5.5}$$

$$dy_t = [0\quad 0\quad 1\quad 0]\begin{bmatrix} x_t \\ x_t^2 \\ x_t^3 \\ 1 \end{bmatrix} dt + dv_t \tag{5.6}$$

This problem is suitable for application of the optimal linear filter of Chapter 2. Furthermore, the nonlinear post-processor of Section 3.3 may be applied, as will be discussed in the next section.

5.2 Application of Nonlinear Post-Processor

Given the Ito equations for the dynamical model (5.5) and observation (5.6), equations can be written for the linear state estimate $\hat{x}_t$ and signal estimate $\widehat{x_t^3}$. We now define a new state vector

$$\underline{x}_t \triangleq \begin{bmatrix} x_t \\ x_t^2 \\ x_t^3 \\ \hat{x}_t \\ \widehat{x_t^3} \\ 1 \end{bmatrix} \tag{5.7}$$

Using the Ito differential rule, we can write a bilinear stochastic differential equation for $\underline{x}_{[3]}$ which consists of all first, second, and third order tensors of $\underline{x}_t$. We shall write this equation as

$$d\underline{x}_{[3]} = A\underline{x}_{[3]}dt + \sum_{i=1}^{2} B_i\underline{x}_{[3]}d\eta_i \tag{5.8}$$

where

$$d\eta_1 = dw_t \tag{5.9}$$

$$d\eta_2 = dv_t \tag{5.10}$$

The elements of $\underline{x}_{[3]}$, A, B_1, and B_2 are given in Appendix B.

At this point we can follow a procedure identical to that used in the phase tracking problem to solve for the values of $\alpha(t)$ and $\beta(t)$ to produce estimators of the form

$$\hat{\hat{x}}_t = \alpha_1(t)\hat{x}_t + \beta_1(t)\hat{x}_t^{\,3} \tag{5.11}$$

$$\widehat{\widehat{x_t^{\,3}}} = \alpha_2(t)\widehat{x_t^{\,3}} + \beta_2(t)(\widehat{x_t^{\,3}})^3 \tag{5.12}$$

where $\hat{x}_t$ and $\widehat{x_t^{\,3}}$ are the linear estimates. We note that the performance of the estimators (5.11) and (5.12) can be calculated off-line, in a procedure similar to the performance calculations associated with the Kalman filter. The results of these calculations are given in Section 5.3.

5.3 Numerical Results

By propagating the second moments of (5.8) using

$$\dot{X}(t) = AX(t) + X(t)A' + \sum_{i=1}^{2} B_i X(t) B_i' \tag{5.13}$$

with

$$X(t) = E[\underline{x}_{[3]}\ \underline{x}_{[3]}'] \tag{5.14}$$

the optimal values of the $\alpha(t)$ and $\beta(t)$ parameters in the estimates (5.11) and (5.12) and the mean square errors associated with these estimates can be calculated. The steady state mean square estimation errors for the state x_t and the signal $x_t^{\,3}$ are given in Tables 5.1 and 5.2. The linear filter/nonlinear post-processor combination, denoted the tensor filter, is compared with the optimal linear filter. Data for the case $q = 2$, $r = 1$ is provided in Table 5.1; data for $q = r = 1$ is found in Table 5.2. Numbers in parentheses in the tables denote the percent improvement of the tensor filter over the linear filter.

The steady state value of the Cramer-Rao based Bobrovsky-Zakai lower bound [18] for the estimation of x_t is included in the table. However, the bound is so loose (in comparison to the mean square error of the tensor and linear filters) that it is of little value in judging the performance of the tensor filter relative to the optimal nonlinear filter.

The improvements in performance by the tensor filter over the linear filter shown in Tables 5.1 and 5.2 are not startling. However, from the experience gained in studying the phase tracking problem (see Figure 4.3), it seems safe to conjecture that the tensor filter could be more valuable at noise-to-signal ratios lower than the values used here. Attempts to collect this performance data at lower noise-to-signal ratios were frustrated by exceedingly high computational costs. The causes of these high costs were the large linear filter gains associated with low noise-to-signal ratio cases. These large gains forced the use of very small time steps in order to accurately integrate the high dimensional differential equation (5.13) until steady state was reached. Nevertheless, because these large computational costs are encountered off-line (only in filter design, not implementation), it would be beneficial to examine the performance of nonlinear post-processing algorithms in a low noise-to-signal ratio environment.

TABLE 5.1: Suboptimal Filter Comparison; $q = 2$, $r = 1$

	Steady State Mean Square Error	
	$E[(x_t - \hat{x}_t)^2]$	$E[(x_t^3 - \widehat{x_t^3})^2]$
Bobrovsky-Zakai Lower Bound	.238	---
Linear Filter	.540 (ref.)	5.63 (ref.)
Tensor Filter	.522 (3.3%)	5.49 (2.5%)

TABLE 5.2: Suboptimal Filter Comparison; $q = r = 1$

	Steady State Mean Square Error	
	$E[(x_t - \hat{x}_t)^2]$	$E[(x_t^3 - \widehat{x_t^3})^2]$
Bobrovsky-Zakai Lower Bound	.264	---
Linear Filter	.378 (ref.)	1.33 (ref.)
Tensor Filter	.371 (1.9%)	1.31 (1.5%)

CHAPTER 6

CONCLUSIONS

The problem of filtering for bilinear systems has been considered in this report. Bilinear filtering problems are important for a variety of reasons. Many physical processes are naturally described by bilinear models. In addition, many problems that do not appear to be bilinear at first inspection can be easily transformed to this type (for example, the cubic sensor problem of Chapter 5). Another reason for studying bilinear models is that their close relationship with linear models provides many insights into their behavior. In a sense, filtering for bilinear systems can be considered the "simplest" of nonlinear filtering problems, and thus an important first step in this area of study.

Because the optimal filter for bilinear systems is infinite dimensional and therefore cannot be realized, suboptimal filters are the only alternative. One such filter is the optimal linear filter derived in Chapter 2. In addition to the derivation, asymptotic behavior of the filter was discussed. Conditions for filter stability and the existence of a finite steady state error covariance were given based on some results of Wonham [7] for the stochastic control problem.

One of the useful properties of bilinear systems is that the unconditional moments of the state vector can be propagated by a finite system of ordinary differential equations. This property distinguishes bilinear systems from the general nonlinear system, and makes possible the use of tensor methods in the design of suboptimal nonlinear filters. In Chapter 3 several applications of tensor methods were discussed, including tensoring of the original observable to create additional "pseudo-observables." Although this method originally seemed promising, it was rejected for several reasons. The most significant flaw with this approach was that in steady state there was no improvement in performance over the optimal linear filter.

Better results were obtained using another tensor method, in which the optimal linear filter was followed by a nonlinear post-processor. Basically the purpose of the nonlinear post-processor was to make use of correlations between the linear estimation error and higher order tensors of the linear estimate. Numerical results were obtained using this approach for the phase tracking problem and the cubic sensor problem. Results were somewhat disappointing for the phase tracking problem. However, in the related problem of estimation for bilinear processes evolving on the circle, the nonlinear post-processor provided up to 9 percent reduction in mean square error relative to the optimal linear filter. This improvement in performance is significant, and to the best of the author's knowledge is a new result. Performance improvements were not as substantial for the cubic sensor problem. Approximately 3 percent reduction in mean square error was obtained for one of the cases studied. From the experience gained in the phase tracking problem, it appears that nonlinear post-processing for the cubic sensor problem might prove more useful at lower noise-to-signal ratios.

Recommendations for Future Research

An important issue that was not resolved in this thesis is the question of when the tensor methods considered here will be useful. It cannot yet be concluded that these methods will be worthwhile for all bilinear systems. Furthermore, it appears that tensor methods (at least the nonlinear post-processor) provide significant performance gains over a limited range of noise-to-signal ratios. Additional work in this area is warranted.

Only the nonlinear post-processor approach to nonlinear filtering was numerically evaluated in this research. It would be quite interesting to evaluate some of the many other possible approaches to the use of tensor methods, in particular the updated innovations scheme illustrated in Figure 3.4.

In order to facilitate future research in the aforementioned areas, the first step should be the automation of the various moment calculations involved in the tensor methods. The calculations can be quite tedious even for low order tensor methods such as the cubic nonlinear post-processor evaluated in this research. For example, a computer program could be written to implement the Ito differential rule for high order tensors of bilinear processes. This type of automation would prove invaluable in any future research into the use of tensor methods.

Some possible areas for future research might be the following. First, it appears that it would be fairly straightforward to extend the use of tensor methods to the discrete time bilinear filtering problem. Second, the application of these methods to smoothing problems (both in continuous and discrete time) could be considered. Finally, as previously noted, the question of the general applicability and usefulness of tensor methods in bilinear estimation problems should be addressed.

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APPENDIX A

EQUATIONS FOR $\underline{x}_{[3]}$ VERSION OF PLL

For the nonlinear post-processor used in the phase tracking problem (Section 4.2), it is necessary to solve for the second moments of $\underline{x}_{[3]}$, where

$$\underline{x} = \begin{bmatrix} x_{1_t} \\ x_{2_t} \\ \hat{x}_{1_t} \\ \hat{x}_{2_t} \\ 1 \end{bmatrix} \tag{A.1}$$

The vector $\underline{x}_{[3]}$ is made up of all first, second, and third order tensors of $\underline{x}$. In Section 4.2 it was noted that $\underline{x}_{[3]}$ satisfied a bilinear stochastic differential equation of the form

$$d\underline{x}_{[3]} = A\underline{x}_{[3]}dt + \sum_{i=1}^{3} B_i\underline{x}_{[3]}dw_i \tag{A.2}$$

with

$$dw_1 = d\theta_t \tag{A.3}$$

$$dw_2 = dn_{1_t} \tag{A.4}$$

$$dw_3 = dn_{2_t} \tag{A.5}$$

The purpose of this appendix is to define the elements of $\underline{x}_{[3]}$, and the matrices A, B_1, B_2, and B_3. The remainder of the appendix supplies this information. The symbols K, q, and r used here are as defined in Section 4.2.

The elements of $\underline{x}_{[3]}$ in numerical order are:

1. x_1
2. x_2
3. $\hat{x}_1$
4. $\hat{x}_2$
5. x_1^2
6. x_2^2
7. x_1x_2
8. $\hat{x}_1^2$
9. $\hat{x}_2^2$
10. $\hat{x}_1\hat{x}_2$
11. $x_1\hat{x}_1$
12. $x_1\hat{x}_2$
13. $x_2\hat{x}_1$
14. $x_2\hat{x}_2$
15. x_1^3
16. $x_1^2x_2$
17. $x_1^2\hat{x}_1$
18. $x_1^2\hat{x}_2$
19. $x_1x_2^2$
20. $x_1\hat{x}_1^2$
21. $x_1\hat{x}_2^2$
22. $x_1x_2\hat{x}_1$
23. $x_1x_2\hat{x}_2$
24. $x_1\hat{x}_1\hat{x}_2$
25. x_2^3
26. $x_2^2\hat{x}_1$
27. $x_2^2\hat{x}_2$
28. $x_2\hat{x}_1^2$
29. $x_2\hat{x}_2^2$
30. $x_2\hat{x}_1\hat{x}_2$
31. $\hat{x}_1^3$
32. $\hat{x}_1^2\hat{x}_2$
33. $\hat{x}_1\hat{x}_2^2$
34. $\hat{x}_2^3$
35. 1

The nonzero elements of A are:

$A(1,1) = -q/2$

$A(2,2) = -q/2$

$A(3,1) = K$

$A(3,3) = -\frac{q}{2} - K$

$A(4,2) = K$

$A(4,4) = -\frac{q}{2} - K$

$A(5,5) = -q$

$A(5,6) = q$

$A(6,5) = q$

$A(6,6) = -q$

$A(7,7) = -2q$

$A(8,8) = -q - 2K$

$A(8,11) = 2K$

$A(8,35) = K^2r$

$A(9,9) = -q - 2K$

$A(9,14) = 2K$

$A(9,35) = K^2r$

$A(10,10) = -q - 2K$

$A(10,12) = K$

$A(10,13) = K$

$A(11,5) = K$

$A(11,11) = -q - 2K$

$A(12,7) = K$

$A(12,12) = -q - 2K$

$A(13,7) = K$

$A(13,13) = -q - 2K$

$A(14,6) = K$

$A(14,14) = -q - 2K$

$A(15,15) = -\frac{3}{2}q$

$A(15,19) = 3q$

$A(16,16) = -\frac{7}{2}q$

$A(16,25) = q$

$A(17,15) = K$

$A(17,17) = -\frac{3}{2}q - K$

$A(17,26) = q$

$A(18,16) = K$

$A(18,18) = -\frac{3}{2}q - K$

$A(18,27) = q$

$A(19,15) = q$

$A(19,19) = -\frac{7}{2}q$

$A(20,1) = K^2r$

$A(20,17) = 2K$

$A(20,20) = -\frac{3}{2}q - 2K$

$A(21,1) = K^2r$

$A(21,21) = -\frac{3}{2}q - 2K$

$A(21,23) = 2K$

$A(22,16) = K$

$A(22,22) = -\frac{5}{2}q - K$

$A(23,19) = K$

$A(23,23) = -\frac{5}{2}q - K$

$A(24,18) = K$

$A(24,22) = K$

$A(24,24) = -\frac{3}{2}q - 2K$

$A(25,16) = 3q$

$A(25,25) = -\frac{3}{2}q$

$A(26,17) = q$

$A(26,19) = K$

$A(26,26) = -\frac{3}{2}q - K$

$A(27,18) = q$

$A(27,25) = K$

$A(27,27) = -\frac{3}{2}q - K$

$A(28,2) = K^2r$

$A(28,22) = 2K$

$A(28,28) = -\frac{3}{2}q - 2K$

$A(29,2) = K^2r$

$A(29,27) = 2K$

$A(29,29) = -\frac{3}{2}q - 2K$

$A(30,23) = K$

$A(30,26) = K$

$A(30,30) = -\frac{3}{2}q - 2K$

$A(31,3) = 3K^2r$

$A(31,20) = 3K$

$A(31,31) = -\frac{3}{2}q - 3K$

$A(32,4) = K^2r$

$A(32,24) = 2K$

$A(32,28) = K$

$A(32,32) = -\frac{3}{2}q - 3K$

$A(33,3) = K^2r$

$A(33,21) = K$

$A(33,30) = 2K$

$A(33,33) = -\frac{3}{2}q - 3K$

$A(34,4) = 3K^2r$

$A(34,29) = 3K$

$A(34,34) = -\frac{3}{2}q - 3K$

The nonzero elements of B_1 are:

$B_1(1,2) = 1$

$B_1(2,1) = 1$

$B_1(5,7) = 2$

$B_1(6,7) = -2$

$B_1(7,5) = -1$

$B_1(7,6) = 1$

$B_1(11,13) = 1$

$B_1(12,14) = 1$

$B_1(13,11) = -1$

$B_1(14,12) = -1$

$B_1(15,16) = 3$

$B_1(16,15) = -1$

$B_1(16,19) = 2$

$B_1(17,22) = 2$

$B_1(18,23) = 2$

$B_1(19,16) = -2$

$B_1(19,25) = 1$

$B_1(20,28) = 1$

$B_1(21,29) = 1$

$B_1(22,17) = -1$

$B_1(22,26) = 1$

$B_1(23,18) = -1$

$B_1(23,27) = 1$

$B_1(24,30) = 1$

$B_1(25,19) = -3$

$B_1(26,22) = -2$

$B_1(27,23) = -2$

$B_1(28,20) = -1$

$B_1(29,21) = -1$

$B_1(30,24) = -1$

The nonzero elements of B_2 are:

$B_2(3,35) = K$

$B_2(8,3) = 2K$

$B_2(10,4) = K$

$B_2(11,1) = K$

$B_2(13,2) = K$

$B_2(17,5) = K$

$B_2(20,11) = 2K$

$B_2(22,7) = K$

$B_2(24,12) = K$

$B_2(26,6) = K$

$B_2(28,13) = 2K$

$B_2(30,14) = K$

$B_2(31,8) = 3K$

$B_2(32,10) = 2K$

$B_2(33,9) = K$

The nonzero elements of B_3 are:

$B_3(4,35) = K$

$B_3(9,4) = 2K$

$B_3(10,3) = K$

$B_3(12,1) = K$

$B_3(14,2) = K$

$B_3(18,5) = K$

$B_3(21,12) = 2K$

$B_3(23,7) = K$

$B_3(24,11) = K$

$B_3(27,6) = K$

$B_3(29,14) = 2K$

$B_3(30,13) = K$

$B_3(32,8) = K$

$B_3(33,10) = 2K$

$B_3(34,9) = 3K$

APPENDIX B

EQUATIONS FOR $\underline{x}_{[3]}$ VERSION OF CUBIC SENSOR

For the design of a nonlinear post-processor for the cubic sensor problem (Section 5.2), it is necessary to solve for the second moments of $\underline{x}_{[3]}$, where

$$\underline{x} = \begin{bmatrix} x_t \\ x_t^2 \\ x_t^3 \\ \hat{x}_t \\ \widehat{x_t^3} \\ 1 \end{bmatrix} \quad \text{(B.1)}$$

The vector $\underline{x}_{[3]}$ is made up of all first, second, and third order tensors of $\underline{x}$. In Section 5.2 it was noted that $\underline{x}_{[3]}$ satisfied a bilinear stochastic differential equation of the form

$$d\underline{x}_{[3]} = A\underline{x}_{[3]}dt + \sum_{i=1}^{2} B_i\underline{x}_{[3]}d\eta_i \quad \text{(B.2)}$$

with

$$d\eta_1 = dw_t \quad \text{(B.3)}$$

$$d\eta_2 = dv_t \quad \text{(B.4)}$$

The remainder of the appendix is used to define $\underline{x}_{[3]}$, A, B_1, and B_2. The symbols q and r used here are defined in Section 5.1. The symbols K_1 and K_2 are the linear filter gains for the cubic sensor problem, that is

$$d\hat{x}_t = \hat{x}_t dt + K_1(dy_t - \widehat{x_t^3}dt) \tag{B.5}$$

$$d\widehat{x_t^3} = (3q\hat{x}_t - 3\widehat{x_t^3})dt + K_2(dy_t - \widehat{x_t^3}dt) \tag{B.6}$$

The elements of the state vector $\underline{x}_{[3]}$ in numerical order are:

1. x
2. x^2
3. x^3
4. x^4
5. x^5
6. x^6
7. x^7
8. x^8
9. x^9
10. $x\hat{x}$
11. $x^2\hat{x}$
12. $x^3\hat{x}$
13. $x^4\hat{x}$
14. $x^5\hat{x}$
15. $x^6\hat{x}$
16. $x\widehat{x^3}$
17. $x^2\widehat{x^3}$
18. $x^3\widehat{x^3}$
19. $x^4\widehat{x^3}$
20. $x^5\widehat{x^3}$
21. $x^6\widehat{x^3}$
22. $x\hat{x}^2$
23. $x^2\hat{x}^2$
24. $x^3\hat{x}^2$
25. $x(\widehat{x^3})^2$
26. $x^2(\widehat{x^3})^2$
27. $x^3(\widehat{x^3})^2$
28. $x\hat{x}\widehat{x^3}$
29. $x^2\hat{x}\widehat{x^3}$
30. $x^3\hat{x}\widehat{x^3}$
31. $\hat{x}$
32. $\widehat{x^3}$
33. $\hat{x}^2$
34. $\hat{x}\widehat{x^3}$
35. $(\widehat{x^3})^2$
36. $\hat{x}^3$
37. $\hat{x}^2\widehat{x^3}$
38. $\hat{x}(\widehat{x^3})^2$
39. $(\widehat{x^3})^3$
40. 1

The nonzero elements of A are given by

$A(1,1) = -1$

$A(2,2) = -2$

$A(2,40) = q$

$A(3,1) = 3q$

$A(3,3) = -3$

$A(4,2) = 6q$

$A(4,4) = -4$

$A(5,3) = 10q$

$A(5,5) = -5$

$A(6,4) = 15q$

$A(6,6) = -6$

$A(7,5) = 21q$

$A(7,7) = -7$

$A(8,6) = 28q$

$A(8,8) = -8$

$A(9,7) = 36q$

$A(9,9) = -9$

$A(10,4) = K_1$

$A(10,10) = -2$

$A(10,16) = -K_1$

$A(11,5) = K_1$

$A(11,11) = -3$

$A(11,17) = -K_1$

$A(11,31) = q$

$A(12,6) = K_1$

$A(12,10) = 3q$

$A(12,12) = -4$

$A(12,18) = -K_1$

$A(13,7) = K_1$

$A(13,11) = 6q$

$A(13,13) = -5$

$A(13,19) = -K_1$

$A(14,8) = K_1$

$A(14,12) = 10q$

$A(14,14) = -6$

$A(14,20) = -K_1$

$A(15,9) = K_1$

$A(15,13) = 15q$

$A(15,15) = -7$

$A(15,21) = -K_1$

$A(16,4) = K_2$

$A(16,10) = 3q$

$A(16,16) = -4 - K_2$

$A(17,5) = K_2$

$A(17,11) = 3q$

$A(17,17) = -5 - K_2$

$A(17,32) = q$

$A(18,6) = K_2$

$A(18,12) = 3q$

$A(18,16) = 3q$

$A(18,18) = -6 - K_2$

$A(19,7) = K_2$

$A(19,13) = 3q$

$A(19,17) = 6q$

$A(19,19) = -7 - K_2$

$A(20,8) = K_2$

$A(20,14) = 3q$

$A(20,18) = 10q$

$A(20,20) = -8 - K_2$

$A(21,9) = K_2$

$A(21,15) = 3q$

$A(21,19) = 15q$

$A(21,21) = -9 - K_2$

$A(22,1) = K_1^2 r$

$A(22,13) = 2K_1$

$A(22,22) = -3$

$A(22,28) = -2K_1$

$A(23,2) = K_1^2 r$

$A(23,14) = 2K_1$

$A(23,23) = -4$

$A(23,29) = -2K_1$

$A(23,33) = q$

$A(24,3) = K_1^2 r$

$A(24,15) = 2K_1$

$A(24,22) = 3q$

$A(24,24) = -5$

$A(24,30) = -2K_1$

$A(25,1) = K_1^2 r$

$A(25,19) = 2K_2$

$A(25,25) = -7 - 2K_2$

$A(25,28) = 6q$

$A(26,2) = K_2^2 r$

$A(26,20) = 2K_2$

$A(26,26) = -8 - 2K_2$

$A(26,29) = 6q$

$A(26,35) = q$

$A(27,3) = K_2^2 r$

$A(27,21) = 2K_2$

$A(27,25) = 3q$

$A(27,27) = -9 - 2K_2$

$A(27,30) = 6q$

$A(28,1) = K_1 K_2 r$

$A(28,13) = K_2$

$A(28,19) = K_1$

$A(28,22) = 3q$

$A(28,25) = -K_1$

$A(28,28) = -5 - K_2$

$A(29,2) = K_1 K_2 r$

$A(29,14) = K_2$

$A(29,20) = K_1$

$A(29,23) = 3q$

$A(29,26) = -K_1$

$A(29,29) = -6 - K_2$

$A(29,34) = q$

$A(30,3) = K_1K_2r$

$A(30,15) = K_2$

$A(30,21) = K_1$

$A(30,24) = 3q$

$A(30,27) = -K_1$

$A(30,28) = 3q$

$A(30,30) = -7 - K_2$

$A(31,3) = K_1$

$A(31,31) = -1$

$A(31,32) = -K_1$

$A(32,3) = K_2$

$A(32,31) = 3q$

$A(32,32) = -3 - K_2$

$A(33,12) = 2K_1$

$A(33,33) = -2$

$A(33,34) = -2K_1$

$A(33,40) = K_1^2r$

$A(34,12) = K_2$

$A(34,18) = K_1$

$A(34,33) = 3q$

$A(34,34) = -4 - K_2$

$A(34,35) = -K_1$

$A(34,40) = K_1K_2r$

$A(35,18) = 2K_2$

$A(35,34) = 6q$

$A(35,35) = -6 - 2K_2$

$A(35,40) = K_2^2r$

$A(36,24) = 3K_1$

$A(36,31) = 3K_1^2r$

$A(36,36) = -3$

$A(36,37) = -3K_1$

$A(37,24) = K_2$

$A(37,30) = 2K_1$

$A(37,31) = 2K_1K_2r$

$A(37,32) = K_1{}^2r$

$A(37,36) = 3q$

$A(37,37) = -5 - K_2$

$A(37,38) = -2K_1$

$A(38,27) = K_1$

$A(38,30) = 2K_2$

$A(38,31) = K_1{}^2r$

$A(38,32) = 2K_1K_2r$

$A(38,37) = 6q$

$A(38,38) = -7 - 2K_2$

$A(38,39) = -K_1$

$A(39,27) = 3K_2$

$A(39,32) = 3K_2{}^2r$

$A(39,38) = 9q$

$A(39,39) = -9 - 3K_2$

The nonzero elements of B_1 are:

$B_1(1,40) = 1$

$B_1(2,1) = 2$

$B_1(3,2) = 3$

$B_1(4,3) = 4$

$B_1(5,4) = 5$

$B_1(6,5) = 6$

$B_1(7,6) = 7$

$B_1(8,7) = 8$

$B_1(9,8) = 9$

$B_1(10,31) = 1$

$B_1(11,10) = 2$

$B_1(12,11) = 3$

$B_1(13,12) = 4$

$B_1(14,13) = 5$

$B_1(15,14) = 6$

$B_1(16,32) = 1$

$B_1(17,16) = 2$

$B_1(18,17) = 3$

$B_1(19,18) = 4$

$B_1(20,19) = 5$

$B_1(21,20) = 6$

$B_1(22,33) = 1$

$B_1(23,22) = 2$

$B_1(24,23) = 3$

$B_1(25,35) = 1$

$B_1(26,25) = 2$

$B_1(27,26) = 3$

$B_1(28,34) = 1$

$B_1(29,28) = 2$

$B_1(30,29) = 3$

The nonzero elements of B_2 are:

$B_2(21,6) = K_2$

$B_2(22,10) = 2K_1$

$B_2(23,11) = 2K_1$

$B_2(24,12) = 2K_1$

$B_2(25,16) = 2K_2$

$B_2(26,17) = 2K_2$

$B_2(27,18) = 2K_2$

$B_2(28,10) = K_2$

$B_2(28,16) = K_1$

$B_2(29,11) = K_2$

$B_2(29,17) = K_1$

$B_2(30,12) = K_2$

$B_2(30,18) = K_1$

$B_2(31,40) = K_1$

$B_2(32,40) = K_2$

$B_2(33,31) = 2K_1$

$B_2(34,31) = K_2$

$B_2(34,32) = K_1$

$B_2(35,32) = 2K_2$

$B_2(36,33) = 3K_1$

$B_2(37,33) = K_2$

$B_2(37,34) = 2K_1$

$B_2(38,34) = 2K_2$

$B_2(38,35) = K_1$

$B_2(39,35) = 3K_2$