SIAM J. CONTROL
Vol. 13, No. 1, January 1975

STABILITY AND THE INFINITE-TIME QUADRATIC COST PROBLEM FOR LINEAR HEREDITARY DIFFERENTIAL SYSTEMS*

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Abstract. This paper studies the infinite-time quadratic cost control problem for a general class of linear autonomous hereditary differential systems. It uses an approach which clarifies the system-theoretic relationship between stabilizability, stability and existence of a solution of an associated operator equation of Riccati type. For this purpose the stability problem is studied and an operator equation of the Lyapunov type is derived. In both cases we obtain equations which characterize the kernels of the Lyapunov and the Riccati equations.

1. Introduction. In a previous paper (cf. Delfour–Mitter [8]) we have studied the quadratic cost optimal control problem over a finite time interval for a general class of linear hereditary differential systems. In particular we have characterized the optimal controller as a linear feedback controller acting on the "state" of the system. The feedback operator is determined by the solution of an operational differential equation of Riccati type. The main objective of the present paper is to study the infinite-time quadratic cost problem for a general class of linear autonomous hereditary differential systems. In undertaking this study we insist on an approach which clarifies the system-theoretic relationship between controllability, stabilizability, stability and existence of a solution of an associated operator equation of Riccati type.

For systems described by ordinary differential equations the infinite-time quadratic cost problem is well-studied (cf. R. W. Brockett [1], R. E. Kalman [13], J. C. Willems [22], W. M. Wonham [23]). This problem has been studied for certain classes of infinite-dimensional systems. J. L. Lions [15] has studied this problem for abstract evolution equations of parabolic type and given a complete solution to the problem. Lukes and Russell [16] have studied this problem for abstract evolution equations of the type

$$(1.1)\qquad \begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\ x(0) &= x_0 \in \mathscr{D}(A), \end{aligned}$$

where A is an unbounded spectral operator (cf. Dunford and Schwartz [11]) and B is also an unbounded operator satisfying certain conditions. Lukes and Russell also allow unbounded operators in the cost function. Using an approach originally

* Received by the editors January 17, 1972, and in final revised form October 18, 1973.

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due to R. E. Kalman [13] they obtain an operational differential equation of Riccati type to characterize the time-varying feedback gain in th finite time case. They also show that under an appropriate stabilizability hypothesis the solution to the infinite-time quadratic cost problem can be obtained in feedback form, where the "feedback gain" is characterized by the solution of an operator equation of quadratic type. The same problem has also been studied by R. Datko [4]. Unfortunately, R. Datko [4] does not characterize the solution as a feedback controller acting on the "state" of the system.

It is felt that the contributions of the present paper are the following:

(i) We present a complete detailed solution to the infinite-time quadratic cost problem for a general class of linear hereditary differential systems. Other than the parabolic case solved by J. L. Lions [15], this appears to be the only other case (so far) where the problem can be solved in a way which is satisfactory from the system-theoretic point of view (that is, no ad hoc mathematical assumptions need to be made).

(ii) The approach we use here is different from that of Lukes and Russell [16] as well as R. Datko [3], [4] and constitutes a synthesis of the work of J. L. Lions [15] and Delfour and Mitter [6], [7], [8].

(iii) The detailed results we obtain exploit the structure of hereditary differential systems in an essential way.

(iv) It gives rigorous derivations of earlier incomplete results of Ross and Flügge-Lotz [19] for a more specialized problem.

The results on the equations for the kernel of the solution of the Lyapunov equation have been announced in 1972 (cf. Delfour [5]).

2. Notation, terminology and preliminary definitions. Let $\mathbb{R}$ be the field of all real numbers and let $a > 0$ be given.

Let X and Y be real Hilbert spaces with norms $|\cdot|_X$, $|\cdot|_Y$ and inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$ respectively.

Let $\mathscr{L}^2(-a,0;X)$ be the vector space of all m-measurable (m denoting the complete Lebesgue measure on $\mathbb{R}$) maps $[-a,0] \to X$ which are square integrable and let $L^2(-a,0;X)$ denote the natural Hilbert space associated with $\mathscr{L}^2(-a,0;X)$ with norm $\|\cdot\|_2$. Consider the space $\mathscr{L}^2(-a,0;X)$ endowed with the seminorm

$$\|f\|_{M^2} = [|f(0)|_X^2 + \|f\|_2^2]^{1/2}. \tag{2.1}$$

The quotient space of $\mathscr{L}^2(-a,0;X)$ by the linear subspace of all f such that $\|f\|_{M^2} = 0$ is denoted by $M^2(-a,0;X)$. $M^2(-a,0;X)$ endowed with the norm (2.1) and inner product

$$(f,g)_{M^2} = (f(0),g(0))_X + \int_{-a}^{0} (f(\theta),g(\theta))_X \, d\theta \tag{2.2}$$

is a Hilbert space isometrically isomorphic to $X \times L^2(-a,0;X)$ endowed with the norm

$$\|h\| = \left[|h^0|_X^2 + \int_{-a}^{0} |h^1(\theta)|_X^2 \, d\theta \right]^{1/2} \tag{2.3}$$

and inner product

$$(2.4)\qquad (h,k) = (h^0,k^0)_X + \int_{-a}^{0} (h^1(\theta),k^1(\theta))_X\,d\theta.$$

The isomorphism is denoted by κ, where $\kappa(h) = (h(0), h)$. For simplicity we shall often identify h and the pair (h^0, h^1). For the motivation in introducing M^2, see Delfour and Mitter [6].

For all $t \in [0, \infty)$, we denote by $W^{1,2}(0, t; X)$ the vector space of all absolutely continuous maps $[0, t] \to X$ with a distributional derivative Dx in $L^2(0, t; X)$. $W^{1,2}(0, t; X)$ endowed with the norm

$$(2.5)\qquad \|x\|_{W^{1,2}} = \left[\int_0^t (|x(s)|_X^2 + |Dx(s)|_X^2)\,ds\right]^{1/2}$$

is a Hilbert space.

We denote by $L^2_{\mathrm{loc}}(0, \infty; X)$ the Fréchet space of measurable maps $[0, \infty) \to X$ which are square integrable on every compact subset of $[0, \infty)$. $W^{1,2}_{\mathrm{loc}}(0, \infty; X)$ denotes the Fréchet space of all absolutely continuous maps $[0, \infty) \to X$ with derivatives in $L^2_{\mathrm{loc}}(0, \infty; X)$, and $C_{\mathrm{loc}}(0, \infty; X)$ denotes the Fréchet space of all continuous maps $[0, \infty) \to X$.

Let $\mathscr{L}(X, Y)$ denote the real Banach space of all continuous linear maps $\Lambda : X \to Y$ endowed with the natural norm $\|\Lambda\|$. The adjoint of Λ in $\mathscr{L}(X, Y)$ will be denoted by $\Lambda^* \in \mathscr{L}(Y, X)$. When $X = Y$, we write $\mathscr{L}(X)$ instead of $\mathscr{L}(X, X)$. $\Lambda \in \mathscr{L}(X)$ will be said to be self-adjoint if $\Lambda = \Lambda^*$. A self-adjoint Λ will be said to be positive and written $\Lambda \geqq 0$ if $(\Lambda x, x) \geqq 0$ for all $x \in X$ and positive definite and written $\Lambda > 0$ if $(\Lambda x, x) > 0$, $x \neq 0$. The identity in $\mathscr{L}(X)$ is denoted by I_X.

For an operator $\Lambda \in \mathscr{L}(M^2)$ we can exploit the isomorphism between M^2 and $X \times L^2$ to decompose Λ into a matrix of operators

$$(2.6)\qquad \begin{pmatrix} \Lambda^{00} & \Lambda^{01} \\ \Lambda^{10} & \Lambda^{11} \end{pmatrix},$$

where $\Lambda^{00} \in \mathscr{L}(X)$, $\Lambda^{01} \in \mathscr{L}(L^2(-a, 0; X), X)$, $\Lambda^{10} \in \mathscr{L}(X, L^2(-a, 0; X))$ and $\Lambda^{11} \in \mathscr{L}(L^2(-a, 0; X))$ are defined in the obvious way.

Let $A : \mathscr{D}(A) \to X$ be a closed linear operator with dense domain $\mathscr{D}(A)$ in X. The operator A is said to be bounded from below (resp. above) by $\alpha \in \mathbb{R}$ if for all $x \in \mathscr{D}(A)$, $(Ax, x) \geqq \alpha\|x\|^2$ (resp. $(Ax, x) \leqq \alpha\|x\|^2$).

3. Summary of previous results. Let $N \geqq 1$ be an integer, let $a > 0$ and $-a = \theta_N < \cdots < \theta_1 < \theta_0 = 0$ be real numbers, let $X = \mathbb{R}^n$ be the Euclidean real Hilbert space of finite dimension n and let U be an arbitrary real Hilbert space.

Consider the autonomous hereditary differential system

$$(\mathrm{L})\qquad \begin{aligned} \frac{dx}{dt}(t) &= A_{00}x(t) + \sum_{i=1}^{N} A_i \begin{cases} x(t+\theta_i), & t+\theta_i \geqq 0 \\ h^1(t+\theta_i), & t+\theta_i < 0 \end{cases} \\ &\quad + \int_{-a}^{0} A_{01}(\theta) \begin{cases} x(t+\theta), & t+\theta \geqq 0 \\ h^1(t+\theta), & t+\theta < 0 \end{cases} d\theta \\ &\quad + Bv(t), \quad \text{a.e. in } [0,\infty), \\ x(0) &= h^0, \quad h = (h^0, h^1) \quad \text{in } M^2(-a, 0; X), \end{aligned}$$

where A_{00}, A_i $(i = 1, 2, \cdots, N)$ are elements of $\mathscr{L}(X)$, $A_{01} \in L^\infty(-a, 0; \mathscr{L}(X))$, $v \in L^2_{\mathrm{loc}}(0, \infty; U)$ and $B \in \mathscr{L}(U, X)$.

It was shown in Delfour and Mitter [8], [10] that the system (L) can be equivalently described by an evolution equation in $M^2(-a, 0; X)$. For this purpose we define the *state at time t* as an element

$$\tilde{x}(t; h, v) = (\tilde{x}(t; h, v)^0, \tilde{x}(t; h, v)^1) \in M^2(-a, 0; X) \tag{3.1}$$

in terms of $h = (h^0, h^1)$ and the solution $x(\cdot\,; h, v)$ of system (L):

$$\tilde{x}(t; h, v)^0 = x(t; h, v), \quad \tilde{x}(t; h, v)^1(\theta) = \begin{cases} x(t + \theta; h, v), & t + \theta \geqq 0 \\ h^1(t + \theta), & \text{otherwise} \end{cases}. \tag{3.2}$$

We define

$$V = \{(h(0), h) | h \in W^{1,2}(-a, 0; X)\} \tag{3.3}$$

and $\tilde{A}_0 : V \to X$, $\tilde{A}_1 : V \to L^2(-a, 0; X)$ and $\tilde{A} : V \to M^2(-a, 0; X)$ as follows:

$$\tilde{A}_0 h = A_{00} h(0) + \sum_{i=1}^{N} A_i h(\theta_i) + \int_{-a}^{0} A_{01}(\theta) h(\theta)\, d\theta, \tag{3.4}$$

$$(\tilde{A}_1 h)(\theta) = \frac{dh}{d\theta}(\theta), \tag{3.5}$$

and

$$[\tilde{A}h]^0 = \tilde{A}_0 h, \quad [\tilde{A}h]^1 = \tilde{A}_1 h. \tag{3.6}$$

Let $v(t) = 0$ in $[0, \infty)$ in (L). We then have (cf. Delfour and Mitter [8], [10]) the following.

THEOREM 3.1. *The map* $t \mapsto \tilde{x}(t; h, 0)$ *given by* (3.1) *generates a one-parameter semigroup* $\{\tilde{\Phi}(t)\}$ *in* $\mathscr{L}(M^2)$ *satisfying the following properties*:

(i) *for all h in* M^2, $t \mapsto \tilde{\Phi}(t)h : [0, \infty) \to M^2$ *is continuous*;

(ii) $\tilde{\Phi}(0) = I_{M^2}$;

(iii) *for* $t \geqq a$, $\tilde{\Phi}(t)$ *is compact* (*i.e., maps bounded sets into relatively compact sets*);

(iv) *for all h in V the map* $t \mapsto \tilde{\Phi}(t)h : [0, \infty) \to V$ *is continuous*;

(v) *the operator* $\tilde{A}$ *defined by* (3.4)–(3.6) *is the infinitesimal generator of the semigroup* $\tilde{\Phi}(t)$.

Now define the operator $\tilde{B} \in \mathscr{L}(U, M^2(-a, 0; X))$ as

$$\tilde{B}u = (Bu, 0). \tag{3.7}$$

Consider the controlled evolution equation

$$\frac{d\tilde{x}}{dt}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \qquad \tilde{x}(0) = h. \tag{$\tilde{\mathrm{L}}$}$$

We then have the following theorem.

THEOREM 3.2.

(i) *For all h in V and v in* $L^2_{loc}(0, \infty; U)$, *system* ($\tilde{L}$) *has a unique solution in*

$$W_{loc}(0, \infty; V, M^2) = \{z \in L^2_{loc}(0, \infty; V) | Dz \in L^2_{loc}(0, \infty; M^2)\} \tag{3.8}$$

which coincides with the state $\tilde{x}(\cdot\,; h, v)$ *constructed from h and* $x(\cdot\,; h, v)$.

(ii) *The map* $(h, v) \mapsto \Lambda(h, u) = \tilde{x}(\cdot\,; h, v): V \times L^2_{loc}(0, \infty; U) \to W_{loc}(0, \infty; V, M^2)$ *is linear and continuous when V is endowed with the* $W^{1,2}$*-topology; it can be lifted to a unique continuous linear map* $\tilde{\Lambda}: M^2 \times L^2_{loc}(0, \infty; U) \to C_{loc}(0, \infty; M^2)$.

Consider the control system (L) and fix the final time $T \in (0, \infty)$ and the initial time t in $[0, T)$. With a pair (h, v) we associate the cost function

$$J_T^t(v, h) = \int_t^T [(x(s; h, v), Qx(s; h, v)) + (v(s), Nv(s))]\, ds, \tag{3.9}$$

where $Q \in \mathscr{L}(X)$ self-adjoint, $Q \geqq 0$, $N \in \mathscr{L}(U)$ self-adjoint, $(u, Nu) \geqq c|u|^2, c > 0$.

Consider the optimal control problem of minimizing (3.9) in the interval $[t, T]$. For each h, it can be shown that there exists a unique u in $L^2(t, T; U)$ which minimizes (3.9) over all v in $L^2(t, T; U)$. We can then show that there exists a unique operator $\Pi_T(t) \in \mathscr{L}(M^2)$ which is self-adjoint and positive such that

$$(h, \Pi_T(t)h)_{M^2} = \min\{J_T^t(v, h) | v \in L^2(t, T; U)\}. \tag{3.10}$$

Moreover the optimal control is given by

$$u(s) = -N^{-1}\tilde{B}^*\Pi_T(s)\tilde{x}(s; h), \tag{3.11}$$

where $\tilde{x}(\cdot\,; h)$ is the solution of

$$\begin{aligned} \frac{dy(s)}{ds} &= [\tilde{A} - \tilde{B}N^{-1}\tilde{B}^*\Pi_T(s)]y(s) \quad \text{a.e. in } [t, T], \\ y(t) &= h. \end{aligned} \tag{3.12}$$

The operator $\Pi_T(s)$ can be shown to satisfy an operator differential equation of Riccati type which (when interpreted appropriately) has a unique solution in $[0, T]$ (cf. Delfour and Mitter [8]).

In the sequel we shall abbreviate $M^2(-a, 0; X)$ by M^2.

4. Formulation of the infinite-time problem. We now associate with the control system (L) (or equivalently $\tilde{L}$) the quadratic cost J_∞ which is equal to the quadratic cost (3.9) where $T = \infty$ and $t = 0$. Our objective is to study the problem:

$$\text{Minimize } J_\infty(v, h) \quad \text{over all } v \in L^2_{loc}(0, \infty; U). \tag{4.1}$$

Our main result may be summarized as follows: Under certain stabilizability hypotheses for each $h \in M^2(-a, 0; X)$, there exists a unique $u \in L^2_{loc}(0, \infty, U)$ which minimizes $J_\infty(v, h)$ over all $v \in L^2_{loc}(0, \infty; U)$. Moreover, the minimizing control u can be expressed in "feedback form" in terms of an operator Π for which an operator Riccati equation can be obtained. Under further hypotheses on Q, the resulting closed-loop control is also stable.

The theory is thus as complete as the theory for the corresponding ordinary differential equation case.

5. Solution of the infinite-time problem. The solution to the infinite-time problem proceeds in three parts:

(i) We first have to make sure that the problem is well-posed in the sense that there exists a constant $c > 0$ and for each h a control v_h such that the corresponding cost $J_\infty(v_h, h)$ is bounded by $c\|h\|_{M^2}^2$. This naturally leads to a study of the stability and stabilizability of linear hereditary systems.

(ii) We then study the behavior of $J_T^t(v, h)$ and the feedback operator $\Pi_T(t)$ as $T \to \infty$. We show in particular that $\Pi_T(t)$ converges to an operator Π.

(iii) Finally we characterize Π and study the stability of the resulting closed-loop system.

5.1. Stability. In this section we shall denote by $x(s; h)$ the solution $x(s; h, 0)$ of (L).

DEFINITION 5.1. The uncontrolled system (L) is said to be *L^2-stable* if

$$\lim_{t\to\infty} \int_0^t (x(s;h), x(s;h))_X \, ds < \infty \quad \forall h \in M^2. \tag{5.1}$$

By virtue of the choice of M^2 as the space of initial conditions it is easy to show that (5.1) is equivalent to

$$\lim_{t\to\infty} \int_0^t (\tilde{x}(s;h), \tilde{x}(s;h))_{M^2} \, ds < \infty \quad \forall h \in M^2. \tag{5.2}$$

DEFINITION 5.2. An operator $R \in \mathscr{L}(M^2)$ is said to be *positive definite on X* if

$$(h^0, R^{00}h^0)_X > 0 \quad \forall\, h^0 \neq 0, \tag{5.3}$$

where $R^{00} \in \mathscr{L}(X)$ is defined by

$$R^{00}h^0 = [R(h^0, 0)]^0 \quad \forall h^0 \in X.$$

Using the techniques of R. Datko [2] we can state the following equivalent conditions for L^2-stability.

THEOREM 5.3. *Let $R \geqq 0$ in $\mathscr{L}(M^2)$ and $Q > 0$ in $\mathscr{L}(X)$ be given. The following statements are equivalent:*

(i) *(L) is L^2-stable.*

(ii) *For all h in M^2,*

$$\lim_{t\to\infty} \int_0^t [(R\tilde{x}(s;h), \tilde{x}(s;h))_{M^2} + (Qx(s;h), x(s;h))_X] \, ds < \infty. \tag{5.4}$$

(iii) *There exists a self-adjoint operator $B \geqq 0$ in $\mathscr{L}(M^2)$ such that*

$$(\tilde{A}h, Bk) + (h, B\tilde{A}k) + (h, \tilde{I}k) = 0 \quad \forall\, h, \forall\, k \in V, \tag{5.5}$$

where

$$(\tilde{I}h) = (h^0, 0). \tag{5.6}$$

(iv) *There exists a self-adjoint operator $B \geqq 0$ such that*

$$(\tilde{A}h, Bk) + (h, B\tilde{A}k) + (h, Rk) + (h, \tilde{Q}k) = 0 \quad \forall\, h, \forall\, k \in V, \tag{5.7}$$

where

$$\tilde{Q}h = (Qh^0, 0).$$

(v) *There exist* $\tilde{\omega} > 0$ *and* $\tilde{M} \geqq 1$ *such that*

$$\|\tilde{x}(t; h)\|_{M^2} \leqq \tilde{M} \exp(-\tilde{\omega}t)\|h\|_{M^2} \quad \forall\, t \geqq 0. \tag{5.8}$$

(vi) *There exist* $\omega > 0$ *and* $M > 1$ *such that*

$$|x(t; h)|_X \leqq M \exp(-\omega t)\|h\|_{M^2} \quad \forall\, t \geqq 0. \tag{5.9}$$

(vii) *There exists* $\alpha < 0$ *such that the spectrum* $\sigma(\tilde{A})$ *of* $\tilde{A}$ *lies entirely in* $\{\lambda \in \mathbb{C} | \operatorname{Re} \lambda \leqq \alpha\}$, *where* $\mathbb{C}$ *is the field of all complex numbers*, $\sigma(\tilde{A}) = \{\lambda \in \mathbb{C} | \det \Delta(\lambda) = 0\}$ *and* $\det \Delta(\lambda)$ *is the determinant of the matrix*

$$\Delta(\lambda) = \lambda I - \sum_{i=1}^{N} A_i \exp(\lambda\theta_i) - \int_{-a}^{0} A_{01}(\theta) \exp(\lambda\theta)\, d\theta. \tag{5.10}$$

Proof. The equivalence of conditions (i) through (vi) can be easily proved by using the results and techniques of R. Datko [2] and the remark following Definition 5.1. As for condition (vii) it is a straightforward application of the results of J. K. Hale [12] with the space $M^2(-a, 0; X)$ in place of the space $C(-a, 0; X)$. □

Remark. (i) Equation (5.5) can be rewritten as an equation in $\mathscr{L}(V, V^*)$ (V^*, the topological dual of V):

$$\tilde{A}^*B + B\tilde{A} + \tilde{I} = 0. \tag{5.11}$$

This is the generalization of Lyapunov's equation in the finite-dimensional case. This condition is much sharper than R. Datko's condition (see [2])

$$2(B\tilde{A}x, x) = -|x|^2 \quad \forall\, x \in V, \tag{5.12}$$

but obviously equivalent.

(ii) Notice also that a straightforward application of R. Datko's results (see [3]) would have yielded the Lyapunov equation

$$\tilde{A}^*B + B\tilde{A} + I = 0, \tag{5.13}$$

where I is the identity in $\mathscr{L}(M^2)$, or equivalently

$$\tilde{A}^*B + B\tilde{A} + Q = 0 \tag{5.14}$$

for some positive self-adjoint Q in $\mathscr{L}(M^2)$ which is *bounded below by some positive nonzero constant.* Conditions (iii) and (iv) are different and make use of the special structure of hereditary systems (cf. remark following Definition 5.1). It is this *subtle difference* that will enable us to solve the infinite-time quadratic cost problem.

In Proposition 5.4 and Theorem 5.5 we further characterize the solutions of equations (5.5) and (5.7).

PROPOSITION 5.4. *Let the hypotheses of Theorem* 5.3 *be true. If equation* (5.5) *resp.* (5.7)) *has a positive self-adjoint solution B in* $\mathscr{L}(M^2)$, *it is unique and for all* h *and* k *in* M^2,

$$(Bh, k)_{M^2} = \int_0^{\infty} (x(s; h), x(s; k))_X\, ds \tag{5.15}$$

(5.16) $$(resp. (Bh, k)_{M^2} = \int_0^\infty ([R + \tilde{Q}]\tilde{x}(s; h), \tilde{x}(s; k))_{M^2}\, ds)$$

and B is positive definite on X.

Proof. We prove the proposition only for equation (5.5).

(i) Let B_1 and B_2 be two solutions of (5.5) and let $D = B_2 - B_1$. Then for all h and k in V,

$$(\tilde{A}h, Dk)_{M^2} + (Dh, \tilde{A}k)_{M^2} = 0.$$

Thus for all $t \geqq 0$ and h and k in V,

(5.17) $$(\tilde{x}(t; h), D\tilde{x}(t; k))_{M^2} = (h, Dk)_{M^2}.$$

Since the system is L^2-stable, the left-hand side of (5.17) is 0.

(ii) Similarly from equation (5.5) we obtain for all $t \geqq 0$, h and k in V,

$$(\tilde{A}\tilde{x}(t; h), B\tilde{x}(t; k)) + (B\tilde{x}(t; h), \tilde{A}\tilde{x}(t; k)) + (x(t; h), x(t; k)) = 0.$$

This yields

$$(h, Bk) = \int_0^t (x(s; h), x(s; k))\, ds + (\tilde{x}(t; h), B\tilde{x}(t; k))$$

and since the system is L^2-stable, $\tilde{x}(t; h) \to 0$ and we obtain (5.15) as t goes to infinity.

(iii) Finally for all $h^0 \neq 0$ in X,

$$(B^{00}h^0, h^0)_X = \int_0^\infty |x(s; (h^0, 0))|_X^2\, ds > 0$$

since the map $s \mapsto x(s; (h^0, 0))$ is continuous and $x(0; (h^0, 0)) = h^0$. □

For linear hereditary differential systems we can exploit the particular structure of the system to further characterize the solution of Lyapunov's equation (5.5).

THEOREM 5.5. *Let $B \geqq 0$ in $\mathscr{L}(M^2)$ be the solution of* (5.5) *in condition* (iii) *of Theorem* 5.3. *It is completely characterized by its matrix of operators*

(5.18) $$\begin{bmatrix} B^{00} & B^{01} \\ B^{10} & B^{11} \end{bmatrix}, \quad \begin{matrix} B^{00} \in \mathscr{L}(X), & B^{01} \in \mathscr{L}(L^2(-a, 0; X), X), \\ B^{10} \in \mathscr{L}(X, L^2(-a, 0; X)), & B^{11} \in \mathscr{L}(L^2(-a, 0; X)). \end{matrix}$$

B^{00} is characterized by the equation

(5.19) $$\begin{aligned} &B^{00}A_{00} + A_{00}^*B^{00} + B^{10}(0) + B^{10}(0)^* + I = 0, \\ &B^{00} = (B^{00})^* \geqq 0. \end{aligned}$$

B^{10} is characterized in the following way:

(5.20) $$(B^{10}h^0)(\alpha) = B^{10}(\alpha)h^0,$$

where the map

(5.21) $$\alpha \mapsto B^{10}(\alpha): [-a, 0] \to \mathscr{L}(X)$$

is piecewise absolutely continuous with jumps at $\alpha = \theta_i$ of height $A_i^*B^{00}$, $i = 1, \cdots,$

$N-1$. *Moreover the map* (5.21) *is itself characterized by the differential equation*

$$\frac{dB^{10}}{d\alpha}(\alpha) = B^{10}(\alpha)A_{00} + A_{01}(\alpha)^*B^{00} + \sum_{i=1}^{N-1} A_i^*B^{00}\delta(\alpha - \theta_i)$$
$$+ B^{11}(\alpha,0), \quad \text{a.e. in } [-a,0], \tag{5.22}$$
$$B^{10}(-a) = A_N^*B^{00},$$

where $\delta(\alpha - \theta_i)$ *is the* δ*-function at* $\alpha = \theta_i$.

B^{01} *is obtained from* B^{10}:

$$B^{01}h^1 = \int_{-a}^{0} B^{10}(\alpha)^*h^1(\alpha)\,d\alpha. \tag{5.23}$$

B^{11} *is characterized in the following way*:

$$(B^{11}h^1)(\alpha) = \int_{-a}^{0} B^{11}(\alpha,\beta)h^1(\beta)\,d\beta, \tag{5.24}$$

where the map

$$(\alpha,\beta) \mapsto B^{11}(\alpha,\beta):[-a,0] \times [-a,0] \to \mathscr{L}(X) \tag{5.25}$$

is piecewise absolutely continuous in each variable with jumps of height $A_i^*B^{10}(\beta)^*$ *at* $\alpha = \theta_i$, $i = 1, \cdots, N-1$ (*resp.* $B^{10}(\alpha)A_j$ *at* $\beta = \theta_j$, $j = 1, \cdots, N-1$). *Moreover* $B^{11}(\alpha,\beta)$ *is the solution of*

$$\left[\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right]B^{11}(\alpha,\beta) = A_{01}(\alpha)^*B^{10}(\beta)^* + B^{10}(\alpha)A_{01}(\beta)$$
$$+ \sum_{i=1}^{N-1} A_i^*B^{10}(\beta)^*\delta(\alpha - \theta_i) \tag{5.26}$$
$$+ \sum_{j=1}^{N-1} B^{10}(\alpha)A_j\delta(\beta - \theta_j)$$

with boundary conditions

$$B^{11}(-a,\beta) = A_N^*B^{10}(\beta)^*, \quad B^{11}(\alpha,-a) = B^{10}(\alpha)A_N, \tag{5.27}$$

and symmetry property $B^{11}(\alpha,\beta) = B^{11}(\beta,\alpha)^*$.

The solution of the above differential system is

$$B^{11}(\alpha,\beta) = \begin{Bmatrix} B^{10}(\alpha - \beta - a)A_N, & \alpha \geqq \beta \\ A_N^*B^{10}(\beta - \alpha - a)^*, & \alpha < \beta \end{Bmatrix}$$
$$+ \sum_{i=1}^{N-1} \begin{Bmatrix} A_i^*B^{10}(\beta - \alpha + \theta_i)^*, & -a \leqq \beta - \alpha + \theta_i, \theta_i < \alpha \\ 0 & , \text{otherwise} \end{Bmatrix}$$
$$+ \sum_{j=1}^{N-1} \begin{Bmatrix} B^{10}(\alpha - \beta + \theta_j)A_j, & -a \leqq \alpha - \beta + \theta_j, \theta_j < \beta \\ 0 & , \text{otherwise} \end{Bmatrix} \tag{5.28}$$

(*cont.*)

$$+\int_{-a}^{\beta}\begin{cases} B^{10}(\alpha-\beta+\xi)A_{01}(\xi), & -a \leqq \alpha-\beta+\xi \\ 0 &, \ otherwise \end{cases} d\xi$$

$$+\int_{-a}^{\alpha}\begin{cases} A_{01}(\theta)^*B^{10}(\beta-\alpha+\theta)^*, & -a \leqq \beta-\alpha+\theta \\ 0 &, \ otherwise \end{cases} d\theta.$$

Proof. See Appendix A.

5.2. Stabilizability. In the control theory of linear ordinary differential equations there is an important result which says that if the system is completely controllable then it is stabilizable, i.e., there exists a constant feedback matrix K such that the resulting closed-loop system matrix can be made to have its eigenvalues strictly in the left half-plane. For hereditary systems we first need a definition of stabilizability.

DEFINITION 5.6. The controlled system (L) (or ($\tilde{\text{L}}$)) is said to be *stabilizable* if there exists some operator G in $\mathscr{L}(V, U)$ of the form

$$Gh = G_{00}h(0) + \sum_{i=1}^{M} G_i h(\tau_i) + \int_{-a}^{0} G_{01}(\theta)h(\theta)\,d\theta \tag{5.29}$$

(for some integer $M \geqq 1$, some real numbers $-a = \tau_M < \cdots < \tau_1 < \tau_0 = 0$, some G_{00}, G_i $(i = 1, \cdots, M)$ in $\mathscr{L}(X, U)$ and $G_{01}:[-a, 0] \to \mathscr{L}(X, U)$ strongly measurable and bounded) such that the resulting *closed-loop system*

$$\begin{aligned} \dot{\tilde{x}}(t) &= [\tilde{A} + \tilde{B}G]\tilde{x}(t), \quad \text{a.e. in } [0, \infty), \\ \tilde{x}(0) &= h, \quad h \text{ in } V, \end{aligned} \tag{5.30}$$

is L^2-stable.

It is extremely important to notice that for operators of the form (5.29) the map

$$h \mapsto \tilde{x}_G(\cdot\,; h): M^2 \to C_{\text{loc}}(0, \infty; M^2)$$

is continuous, where for each h in V, $\tilde{x}_G(\cdot\,; h)$ denotes the solution of (5.30). This is not true of all operators in $\mathscr{L}(V, U)$. This definition opens the way to the investigation of stabilizability by feedback of a delayed signal (cf. V. M. Popov [18]).

Using the spectral properties of $\tilde{A}$ (cf. J. K. Hale [12]) an analogue of the ordinary differential equation result cited above could be obtained for a linear hereditary differential system. For a study of this question see Y. S. Osipov [17] and H. F. Vandevenne [20], [21].

The importance of the concept of stabilizability and a theorem relating controllability and stabilizability is that it provides us with a verifiable condition for asserting that there exists a constant $c > 0$ and for each h at least one control v such that $J_\infty(v, h) \leqq c\|h\|^2$. Thus the infinite-time problem is well-posed.

5.3. Asymptotic behavior of $\Pi_T(t)$ as $T \to \infty$. We know that for the quadratic cost problem over $[0, T]$ the optimal control $u^*(s)$ is given by

$$u^*(s) = -N^{-1}\tilde{B}^*\Pi_T(s)\tilde{x}(s; h), \qquad s \in [0, T],$$

and the optimal cost by

$$J_T^0(u^*, h) = (h, \Pi_T(0)h)_{M^2}.$$

We now show the following.

THEOREM 5.7. *Assume that* ($\tilde{\text{L}}$) *is stabilizable. Then*:

(i) *For all h in* M^2, $\lim_{t<T\to\infty}\Pi_T(t)h = \Pi h, t \geqq 0$.

(ii) *For all h in* M^2,

$$(5.31)\qquad (\Pi h, h)_{M^2} = \int_0^\infty ([\tilde{Q} + \Pi\tilde{R}\Pi]\tilde{x}(s), \tilde{x}(s))\, ds,$$

where $R = BN^{-1}B^*$,

$$(5.32)\qquad \tilde{R}h = (Rh^0, 0),$$

and $\tilde{x}$ *is the solution of*

$$(5.33)\qquad \begin{aligned} &\frac{dy}{dt}(t) = (\tilde{A} - \tilde{R}\Pi)y(t), \quad \text{a.e. in } [0,\infty), \\ &y(0) = h, \end{aligned}$$

with initial datum h.

(iii) *For all h in* M^2,

$$(5.34)\qquad (\Pi h, h)_{M^2} = J_\infty(-N^{-1}\tilde{B}^*\Pi\tilde{x}, h).$$

Proof. (i) Consider the optimal control problem on the interval $[s, T]$. By virtue of the stabilizability hypothesis there exists a feedback operator G of the type described in Definition 5.6 such that the operator $\tilde{A} + \tilde{B}G$ is L^2-stable. Let $\tilde{\Phi}_G$ be the semigroup generated by this operator. For all $T > s \geqq 0$,

$$\begin{aligned} (\Pi_T(s)h, h)_{M^2} &= \inf\{J_T^s(v, h) | v \in L^2(s, T, U)\} \\ &\leqq \int_s^T [(\tilde{Q}\tilde{\Phi}_G(t-s)h, \tilde{\Phi}_G(t-s)h) + (NG\tilde{\Phi}_G(t-s)h, G\tilde{\Phi}_G(t-s)h)]\, dt \\ &\leqq \int_0^T [(\tilde{Q}\tilde{\Phi}_G(t)h, \tilde{\Phi}_G(t)h) + (NG\tilde{\Phi}_G(t)h, G\tilde{\Phi}_G(t)h)]\, dt \\ &\leqq \|Q\| \int_0^T |z(t)|_X^2\, dt + \|{\cdot}N\| \int_0^T |G\tilde{z}(t)|_U\, dt, \end{aligned}$$

where z is the solution of

$$\begin{aligned} &\dot{z}(t) = (\tilde{A}_0 + BG)\tilde{z}(t), \quad \text{a.e. in } [0,\infty), \\ &\tilde{z}(0) = h, \end{aligned}$$

and $\tilde{z}$ is the state constructed from h and z. But

$$\begin{aligned} \left[\int_0^T |G\tilde{z}(t)|^2\, dt\right]^{1/2} &\leqq \|G_{00}\| \left[\int_0^T |z(t)|^2\, dt\right]^{1/2} \\ &\quad + \sum_{i=1}^M \|G_i\| \left[\int_{\tau_i}^0 |h(\theta)|^2\, d\theta + \int_0^T |z(t)|^2\, dt\right]^{1/2} \\ &\quad + \|G_{01}\|_\infty a^{1/2} \left[\int_{-a}^0 |h(\theta)|^2\, d\theta + \int_0^T |z(t)|^2\, dt\right]^{1/2}, \end{aligned}$$

where z is the solution of (5.30). Finally since (5.30) is L^2-stable there exists a constant $c > 0$ (independent of h, T and s) such that.

$$(\Pi_T(s)h, h)_{M^2} \leqq c\|h\|_{M^2}^2 \quad \forall\, h, \forall\, T \geqq s \geqq 0.$$

It is now easy to show the following:

(a) $\Pi_{T_2}(s) \geqq \Pi_{T_1}(s)$, $T_2 \geqq T_1 \geqq s$, where $\geqq$ denotes the natural partial ordering of positive operators, and

(b) there exists $c > 0$ such that $\|\Pi_T(s)\|_{\mathscr{L}(M^2)} \leqq c$ for all $T \geqq s$.

Then by a well-known theorem on positive operators (cf. Kantorovich and Akilov [14, p. 189]), for all h in M^2, $\Pi_T(s)h$ converges to $\Pi(s)h$, for some positive self-adjoint operator $\Pi(s)$ in $\mathscr{L}(M^2)$.

Now for $0 < T_1 - s_1 = T_2 - s_2$, $s_1 \geqq s_2 \geqq 0$,

$$(h, \Pi_{T_1}(s_1)h) = (h, \Pi_{T_2}(s_2)h)$$

and hence $\Pi_{T_1}(s_1) = \Pi_{T_2}(s_2)$. In particular, for all $s_1 < s_2$ and h in M^2,

$$\Pi(s_1)h = \lim_{T_1 \to \infty} \Pi_{T_1}(s_1)h = \lim_{T_1 \to \infty} \Pi_{T_1 + s_2 - s_1}(s_2)h = \Pi(s_2)h$$

and

$$\lim_{T \to \infty} \Pi_T(s)h = \Pi h \quad \forall\, s \geqq 0.$$

(ii) We now consider the control problem in the interval $[0, \infty)$. Let $\tilde{z}$ denote the solution of (5.30) corresponding to the stabilizing feedback control law G, let $\tilde{x}$ be the solution of (5.33) in $[0, \infty)$ and let $\tilde{x}_T$ be the solution of

$$\text{(5.35)} \qquad \begin{aligned} &\frac{d\tilde{x}_T}{ds}(s) = (\tilde{A} - \tilde{R}\Pi_T(s))\tilde{x}_T(s), \quad \text{a.e. in } [0, T], \\ &\tilde{x}_T(0) = h. \end{aligned}$$

We first show that for all $t_1 > 0$,

$$\text{(5.36)} \qquad \lim_{t_1 < T \to \infty} \tilde{x}_T(t) \to \tilde{x}(t) \quad \text{uniformly in } [0, t_1].$$

Fix $t_1 > 0$ and consider T, $T > t_1$. Let

$$y_T(t) = x_T(t) - x(t) \quad \text{in } [0, t_1].$$

Then

$$\begin{aligned} &\frac{d\tilde{y}_T}{dt}(t) = \tilde{A}\tilde{y}_T(t) + \tilde{R}[\Pi\tilde{x}(t) - \Pi_T(t)\tilde{x}_T(t)], \quad \text{a.e. in } [0, t_1], \\ &\tilde{y}_T(0) = 0, \end{aligned}$$

where

$$\tilde{y}_T(s)(\theta) = \begin{cases} y_T(s + \theta), & s + \theta \geqq 0, \\ 0 \qquad , & \text{otherwise}. \end{cases}$$

As a result there exists $c(t_1) > 0$ such that for all $0 \leqq t \leqq t_1$,

$$|\tilde{y}_T(t)| \leqq c(t_1) \int_0^t |\Pi\tilde{x}(s) - \Pi_T(s)\tilde{x}_T(s)|\,ds$$

$$\leqq c(t_1) \int_0^t [|(\Pi - \Pi_T(s))\tilde{x}(s)| + |\Pi_T(s)\tilde{y}_T(s)|]\,ds$$

and we can find $c'(t_1) > 0$ such that

$$\|\tilde{y}_T\|_{C(0,t_1;M^2)} \leqq c'(t_1) \int_0^{t_1} |(\Pi - \Pi_T(s))\tilde{x}(s)|\,ds.$$

But $\tilde{x} \in L^1(0, t_1; M^2)$. Then $f_T(s) = \Pi_T(s)\tilde{x}(s)$ and $f(s) = \Pi\tilde{x}(s)$ belong to $L^1(0, t_1; M^2)$. Both f_T and f are bounded by the L^1-function $c|\tilde{x}(s)|$ and for almost all t,

$$f_T(t) = \Pi_T(t)\tilde{x}(t) \to f(t) = \Pi\tilde{x}(t) \quad \text{as } T \to \infty.$$

By the Lebesgue dominated convergence theorem, $f_T \to f$ in $L^1(0, t_1; M^2)$. This shows that $\tilde{y}_T \to 0$ and proves (5.36). This also shows that $\tilde{x}_T(t)$ is uniformly bounded in $[0, t_1]$ by a constant independent of T.

We know that for all $T > 0$ (cf. Delfour and Mitter [8])

$$(\Pi_T(0)h, h) = \int_0^T ([\tilde{Q} + \Pi_T(s)\tilde{R}\Pi_T(s)]\tilde{x}_T(s), \tilde{x}_T(s))\,ds. \tag{5.37}$$

The left-hand side of (5.37) converges to $(\Pi h, h)$ as T goes to infinity. We now show that the right-hand side of (5.37) converges to

$$\int_0^\infty ([\tilde{Q} + \Pi\tilde{R}\Pi]\tilde{x}(s), \tilde{x}(s))\,ds.$$

For this purpose we define

$$g_T(t) = \begin{cases} ([\tilde{Q} + \Pi_T(t)\tilde{R}\Pi_T(t)]\tilde{x}_T(t), \tilde{x}_T(t)), & 0 \leqq t \leqq T, \\ 0 & , \text{ otherwise}, \end{cases}$$

$$g(t) = ([\tilde{Q} + \Pi\tilde{R}\Pi]\tilde{x}(t), \tilde{x}(t)).$$

From previous considerations it is now clear that

$$g_T(t) \to g(t) \quad \text{pointwise in } [0, \infty) \text{ as } T \to \infty.$$

By Fatou's lemma,

$$\int_0^\infty g(t)\,dt \leqq \varliminf_{T\to\infty} \int_0^\infty g_T(t)\,dt = \lim_{T\to\infty} (\Pi_T(0)h, h) = (\Pi h, h),$$

and for all $T > 0$,

$$\int_0^T g(t)\,dt = J_T^0(-N^{-1}\tilde{B}^*\tilde{x}, h) \geqq (\Pi_T(0)h, h).$$

(iii) Finally (5.34) has been established at the end of (ii). □

5.4. Solution to the infinite-time problem.

THEOREM 5.8. *Assume that* ($\tilde{\text{L}}$) *is stabilizable. Then for each* h *in* M^2, *there exists a control function* u^* *in* $L^2_{\text{loc}}(0, \infty; U)$ *such that*

$$(5.38)\qquad J_\infty(u^*, h) = \inf\{J_\infty(v, h) | v \in L^2_{\text{loc}}(0, \infty; U)\} = (h, \Pi h).$$

Moreover,

$$(5.39)\qquad u^*(t) = -N^{-1}\tilde{B}^*\Pi\tilde{x}(s),$$

where $\tilde{x}$ *is the solution of*

$$\frac{d\tilde{x}}{ds}(s) = (\tilde{A} - \tilde{R}\Pi)\tilde{x}(s), \quad \text{a.e. in } [0, \infty),$$

$$\tilde{x}(0) = h.$$

Proof. The control function u^* defined by (5.38) is clearly an element of $L^2_{\text{loc}}(0, \infty; U)$. Consider any $v \in L^2_{\text{loc}}(0, \infty; U)$. Then for all $T > 0$,

$$(h, \Pi_T(0)h) = \min_{v \in L^2(0,T;U)} J^0_T(v, h) \leqq \int_0^T [(Qx(s; v), x(s; v)) + (Nv(s), v(s))]\, ds,$$

where $x(\cdot\,; v)$ is the solution of (L) corresponding to h and v. Therefore,

$$(h, \Pi h) \leqq \int_0^\infty [(Qx(s; v), x(s; v)) + (Nv(s), v(s))]\, ds,$$

and the result follows from Theorem 5.7 (iii). □

5.5. Characterization of Π and stability of the closed-loop system.

THEOREM 5.9. *Let* $Q > 0$. *Then*:

(i) ($\tilde{\text{L}}$) *is stabilizable if and only if there exists a positive self-adjoint operator* Π *in* $\mathscr{L}(M^2)$ *which is a solution to the operator equation of Ricatti type*

$$(5.40)\qquad (\tilde{A}h, \Pi k) + (h, \Pi\tilde{A}k) - (h, \Pi\tilde{R}\Pi k) + (h, \tilde{Q}k) = 0 \quad \forall\, h, k \text{ in } V.$$

(ii) *If a positive self-adjoint solution of* (5.40) *exists, it is unique and equal to the* Π *of Theorem* 5.7. *The operator* $\tilde{A} - \tilde{R}\Pi$ *is* L^2-*stable, the operator* $G^* = -N^{-1}\tilde{B}^*\Pi$ *defines a stable feedback law and* Π *is positive definite on* X.

Proof. (i) Assume that system ($\tilde{\text{L}}$) is stabilizable. Then equation (5.31) of Theorem 5.7(ii) is true for all h in M^2. Since $Q > 0$ and $\Pi\tilde{R}\Pi \geqq 0$ we can use Theorem 5.3(i) and (ii) to conclude that the operator $\tilde{A} - \tilde{R}\Pi$ is L^2-stable. Since Q, Π and $\tilde{Q} + \Pi\tilde{R}\Pi$ are positive and self-adjoint, equation (5.31) implies that for all h and k in M^2,

$$(\Pi h, k) = \int_0^\infty ([\tilde{Q} + \Pi\tilde{R}\Pi]\tilde{x}_h(s), \tilde{x}_k(s))\, ds,$$

where $\tilde{x}_h$ (resp. $\tilde{x}_k$) is the solution of equation (5.33) with initial datum h (resp. k).

Let $\tilde{\Phi}(s)$ be the strongly continuous semigroup generated by $\tilde{A} - \tilde{R}\Pi$, that is, $\tilde{x}(s) = \tilde{\Phi}(s)h$. For all h and k in V,

$$(\Pi(\tilde{A} - \tilde{R}\Pi)h, k) = \int_0^\infty ((\tilde{Q} + \Pi\tilde{R}\Pi)\tilde{\Phi}(s)(\tilde{A} - \tilde{R}\Pi)h, \tilde{\Phi}(s)k)\, ds,$$

$$(\Pi h, (\tilde{A} - \tilde{R}\Pi)k) = \int_0^\infty ((\tilde{Q} + \Pi\tilde{R}\Pi)\tilde{\Phi}(s)h, \tilde{\Phi}(s)(\tilde{A} - \tilde{R}\Pi)k)\, ds,$$

$$\begin{aligned}(\Pi(\tilde{A} - \tilde{R}\Pi)h, k) + (\Pi h, (\tilde{A} - \tilde{R}\Pi)k) &= \int_0^\infty \frac{d}{ds}((\tilde{Q} + \Pi\tilde{R}\Pi)\tilde{x}_h(s), \tilde{x}_k(s))\, ds \\ &= -((\tilde{Q} + \Pi\tilde{R}\Pi)h, k),\end{aligned}$$

since $\tilde{A} - \tilde{R}\Pi$ is L^2-stable and $\tilde{x}(s) \to 0$ as $s \to \infty$. Finally,

$$\begin{aligned}0 &= \Pi(\tilde{A} - \tilde{R}\Pi) + (\tilde{A} - \tilde{R}\Pi)^*\Pi + \Pi\tilde{R}\Pi + \tilde{Q} \\ &= \Pi\tilde{A} + \tilde{A}^*\Pi - \Pi\tilde{R}\Pi + \tilde{Q}.\end{aligned}$$

Conversely assume that there exists a solution Π to the operator Riccati equation (5.40) which is self-adjoint and positive. Equation (5.40) can be rewritten as

$$(\tilde{A} - \tilde{R}\Pi)^*\Pi + \Pi(\tilde{A} - \tilde{R}\Pi) + \Pi\tilde{R}\Pi + \tilde{Q} = 0.$$

By Theorem 5.3(iv), this means that the system defined by the operator $\tilde{A} - \tilde{R}\Pi$ is L^2-stable. It is now a simple matter to check that the stabilizing feedback law is $G^* = -N^{-1}B^*\Pi^0$.

(ii) If a positive self-adjoint solution of (5.40) exists, we have shown that system $(\tilde{\mathrm{L}})$ is stabilizable, that Π is a solution of (5.40), that the operator $\tilde{A} - \tilde{R}\Pi$ is L^2-stable and that G^* is a stable feedback law. By Proposition 5.4 we can also say that Π is positive definite on X. It remains to prove uniqueness. Assume that there exist two solutions $\Pi_1 \geqq 0$ and $\Pi_2 \geqq 0$ to the Riccati equation (5.40). Let $P = \Pi_1 - \Pi_2$. Then necessarily

$$(\tilde{A}h, Pk) + (Ph, \tilde{A}k) + (h, \Pi_2\tilde{R}\Pi_2 k) - (h, \Pi_1\tilde{R}\Pi_1 k) = 0$$

or

$$((\tilde{A} - \tilde{R}\Pi_2)h, Pk) + (h, P(\tilde{A} - \tilde{R}\Pi_1)k) = 0.$$

Hence

$$\begin{aligned}\frac{d}{ds}(\tilde{\Phi}_2(s)h, P\tilde{\Phi}_1(s)k) &= ((\tilde{A} - \tilde{R}\Pi_2)\tilde{\Phi}_2(s)h, P\tilde{\Phi}_1(s)k) \\ &\quad + (\tilde{\Phi}_2(s)h, P(\tilde{A} - \tilde{R}\Pi_1)\tilde{\Phi}_1(s)k) = 0,\end{aligned}$$

where $\tilde{\Phi}_2$ (resp. $\tilde{\Phi}_1$) is the semigroup generated by $\tilde{A} - \tilde{R}\Pi_2$ (resp. $\tilde{A} - \tilde{R}\Pi_1$). Then

$$(h, Pk) = (\tilde{\Phi}_2(s)h, P\tilde{\Phi}_1(s)k) \to 0 \quad \text{as } s \to \infty,$$

since $\tilde{\Phi}_2$ and $\tilde{\Phi}_1$ are L^2-stable. Finally $P = 0$ and equation (5.40) has a unique solution which is necessarily equal to the Π of Theorem 5.7.

Remark. Note that the hypothesis $Q > 0$ implies that the pair $(\tilde{A}, Q^{1/2})$ is observable since the map $h^0 \mapsto Q^{1/2}\Phi^0(\cdot)h^0$ is injective (cf. Delfour and Mitter [8, Def. 3.11 and Prop. 3.13]).

6. Detailed characterization of Π. One can exploit the structure of the space M^2 and the fact that Π is a matrix of operators to give a detailed characterization of Π. This is done in the following theorem.

THEOREM 6.1. *Let $\Pi \geqq 0$ in $\mathscr{L}(M^2)$ be the solution of* (5.40). *Then*

$$(h, \Pi k) = \int_0^\infty (\tilde{\Phi}(t)h, [\tilde{Q} + \Pi\tilde{R}\Pi]\tilde{\Phi}(t)k)\,dt. \tag{6.1}$$

It is completely characterized by its matrix of operators

$$\begin{bmatrix} \Pi_{00} & \Pi_{01} \\ \Pi_{10} & \Pi_{11} \end{bmatrix}, \quad \begin{array}{ll} \Pi_{00} \in \mathscr{L}(X), & \Pi_{01} \in \mathscr{L}(L^2(-a,0;X),X), \\ \Pi_{10} \in \mathscr{L}(X, L^2(-a,0;X)), & \Pi_{11} \in \mathscr{L}(L^2(-a,0;X)). \end{array} \tag{6.2}$$

Π_{00} *is characterized by the equation*

$$\begin{aligned} &\Pi_{00}A_{00} + A_{00}^*\Pi_{00} + \Pi_{10}(0) + \Pi_{10}(0)^* + Q - \Pi_{00}R\Pi_{00} = 0, \\ &\Pi_{00}^* = \Pi_{00} \geqq 0. \end{aligned} \tag{6.3}$$

Π_{10} *is characterized in the following way*:

$$(\Pi_{10}h^0)(\alpha) = \Pi_{10}(\alpha)h^0, \tag{6.4}$$

where the map

$$\alpha \mapsto \Pi_{10}(\alpha): [-a, 0] \to \mathscr{L}(X) \tag{6.5}$$

is piecewise absolutely continuous with jumps at $\alpha = \theta_i$ of height $A_i^*\Pi_{00}$, $i = 1, \cdots, N-1$. Moreover the map (6.5) *is characterized by the differential equation*

$$\begin{aligned} \frac{d\Pi_{10}}{d\alpha}(\alpha) &= \Pi_{10}(\alpha)[A_{00} - R\Pi_{00}] + \sum_{l=1}^{N-1} A_i^*\Pi_{00}\delta(\alpha - \theta_i) + A_{01}(\alpha)^*\Pi_{00} \\ &\quad + \Pi_{11}(\alpha, 0), \quad \text{a.e. in } [-a, 0], \\ \Pi_{10}(-a) &= A_N^*\Pi_{00}, \end{aligned} \tag{6.6}$$

where $\delta(\alpha - \theta_i)$ is the delta function at $\alpha = \theta_i$.

Π_{01} *is obtained from* Π_{10}:

$$\Pi_{01}h^1 = \int_{-a}^0 \Pi_{10}(\alpha)^*h^1(\alpha)\,d\alpha. \tag{6.7}$$

Π_{11} *is characterized in the following way*:

$$(\Pi_{11}h^1)(\alpha) = \int_{-a}^0 \Pi_{11}(\alpha, \beta)h^1(\beta)\,d\beta, \tag{6.8}$$

where the map

$$(\alpha, \beta) \mapsto \Pi_{11}(\alpha, \beta): [-a, 0] \times [-a, 0] \to \mathscr{L}(X) \tag{6.9}$$

is piecewise absolutely continuous in each variable with jumps of height $A_i^*\Pi_{10}(\beta)^*$ *at* $\alpha = \theta_i, i = 1, \cdots, N-1$ *(resp.* $\Pi_{10}(\alpha)A_j$ *at* $\beta = \theta_j, j = 1, \cdots, N-1$*). Moreover* $\Pi_{11}(\alpha, \beta)$ *is the solution of*

$$\begin{aligned}\left[\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right]\Pi_{11}(\alpha,\beta) &= A_{01}(\alpha)^*\Pi_{10}(\beta)^* + \Pi_{10}(\alpha)A_{01}(\beta)\\ &\quad + \sum_{i=1}^{N-1} A_i^*\Pi_{10}(\beta)^*\delta(a-\theta_i) + \sum_{j=1}^{N-1}\Pi_{10}(\alpha)A_j\delta(\beta-\theta_j)\\ &\quad - \Pi_{10}(\alpha)R\Pi_{10}(\beta)^*\end{aligned} \tag{6.10}$$

with boundary conditions

$$\Pi_{11}(-a,\beta) = A_N^*\Pi_{10}(\beta)^*, \qquad \Pi_{11}(\alpha,-a) = \Pi_{10}(\alpha)A_N, \tag{6.11}$$

and symmetry property

$$\Pi_{11}(\alpha,\beta) = \Pi_{11}(\beta,\alpha)^*.$$

The solution of the above differential system is

$$\begin{aligned}\Pi_{11}(\alpha,\beta) &= \begin{Bmatrix}\Pi_{10}(\alpha-\beta-a)A_N, & \alpha \geqq \beta\\ A_N^*\Pi_{10}(\beta-\alpha-a)^*, & \alpha < \beta\end{Bmatrix}\\ &\quad + \sum_{i=1}^{N-1}\begin{Bmatrix}A_i^*\Pi_{10}(\beta-\alpha+\theta_i)^*, & -a \leqq \beta-\alpha+\theta_i, \theta_i < \alpha\\ 0 & , \text{otherwise}\end{Bmatrix}\\ &\quad + \sum_{j=1}^{N-1}\begin{Bmatrix}\Pi_{10}(\alpha-\beta+\theta_j)A_j, & -a \leqq \alpha-\beta+\theta_j, \theta_j < \beta\\ 0 & , \text{otherwise}\end{Bmatrix}\\ &\quad + \int_{-a}^{\alpha}\begin{Bmatrix}A_{01}(\xi)^*\Pi_{10}(\xi-\alpha+\beta)^*, & \xi \geqq \alpha-\beta-a\\ 0 & , \text{otherwise}\end{Bmatrix}d\xi\\ &\quad + \int_{-a}^{\beta}\begin{Bmatrix}\Pi_{10}(\theta-\beta+\alpha)A_{01}(\theta), & \theta \geqq \beta-\alpha-a\\ 0 & , \text{otherwise}\end{Bmatrix}d\theta\\ &\quad - \begin{Bmatrix}\displaystyle\int_{-a}^{\beta}\Pi_{10}(\alpha-\beta+\theta)R\Pi_{10}(\theta)^*\,d\theta, & \alpha \geqq \beta\\ \displaystyle\int_{-a}^{\alpha}\Pi_{10}(\xi)R\Pi_{10}(\beta-\alpha+\xi)^*\,d\xi, & \alpha < \beta\end{Bmatrix}.\end{aligned} \tag{6.12}$$

Proof. See Appendix B.

Appendix A.

Proof of Theorem 5.5. The reader can find the definitions of Φ^0, Φ^1 and $\tilde{\Phi}$ in Delfour and Mitter [7], [8] and [10]. We first rewrite equation (5.15) in terms of $\tilde{\Phi}$:

$$(h, \Pi k) = \int_0^\infty (\tilde{\Phi}(t)h, [\tilde{Q} + \Pi\tilde{R}\Pi]\tilde{\Phi}(t)k)\,dt. \tag{A.1}$$

We shall also use the identity

$$[\tilde{\Phi}(t)h]^0 = \Phi^0(t)h^0 + \Phi^1(t)h^1. \tag{A.2}$$

We first study B^{00} and the kernels $B^{10}(\alpha)$ and $B^{11}(\alpha, \beta)$ of the operators B^{10} and B^{11}. Since we know where the discontinuities can occur we derive differential equations for $B^{10}(\alpha)$ and $B^{11}(\alpha, \beta)$. Finally we solve the equation for $B^{11}(\alpha, \beta)$ and give an explicit expression of $B^{11}(\alpha, \beta)$ in terms of $B^{10}(\cdot)$.

(i) Let $h = (h^0, 0)$ and $k = (k^0, 0)$ in (A.1). Then

$$B^{00} = \int_0^\infty \Phi^0(t)^*\Phi^0(t)\,dt. \tag{A.3}$$

Let $h = (0, h^1)$, $k = (k^0, 0)$ in (A.1). Then

$$(h^1, B^{10}k^0) = \int_0^\infty (\Phi^1(t)h^1, \Phi^0(t)k^0)\,dt. \tag{A.4}$$

But (cf. Delfour and Mitter [7] and [8])

$$\Phi^1(t)h^1 = \int_{-a}^0 \Phi^1(t, \alpha)h^1(\alpha)\,d\alpha, \tag{A.5}$$

$$(h^1, B^{10}k^0) = \int_{-a}^0 (h^1(\alpha), \int_0^\infty \Phi^1(t, \alpha)^*\Phi^0(t)k^0)\,dt\,d\alpha, \tag{A.6}$$

and

$$B^{10}(\alpha) = \int_0^\infty \Phi^1(t, \alpha)^*\Phi^0(t)\,dt. \tag{A.7}$$

We now substitute for $\Phi^1(t, \alpha)$ the expression (cf. Delfour and Mitter [7] and [8])

$$\sum_{i=1}^N \left\{\begin{matrix} \Phi^0(t - \alpha + \theta_i)A_i, & t \geqq \alpha - \theta_i \geqq 0 \\ 0 & , \text{ otherwise} \end{matrix}\right\} + \int_{\max\{-a, \alpha - t\}}^\alpha \Phi^0(t - \alpha + \theta)A_{01}(\theta)\,d\theta. \tag{A.8}$$

Identity (A.7) can now be rewritten in the form

$$B^{10}(\alpha) = \sum_{i=1}^N \left\{\begin{matrix} A_i^* \int_{\alpha - \theta_i}^\infty \Phi^0(t - \alpha + \theta_i)^*\Phi^0(t)\,dt, & \theta_i \leqq \alpha \\ 0 & , \theta_i > \alpha \end{matrix}\right\} + \int_0^\infty dt \int_{\max\{-a, \alpha - t\}}^\alpha d\theta\, A_{01}(\theta)^*\Phi^0(t - \alpha + \theta)^*\Phi^0(t). \tag{A.9}$$

Finally we change the order of integration of the last term in (A.9) to obtain

$$B^{10}(\alpha) = \sum_{i=1}^{N} A_i^* \begin{Bmatrix} \int_{\alpha-\theta_i}^{\infty} \Phi^0(t-\alpha+\theta_i)^*\Phi^0(t)\,dt, & \theta_i \leqq \alpha \\ 0 & , \theta_i > \alpha \end{Bmatrix}$$

$$+ \int_{-a}^{\alpha} d\theta\, A_{01}(\theta)^* \int_{\alpha-\theta}^{\infty} dt\, \Phi^0(t-\alpha+\theta)^*\Phi^0(t). \tag{A.10}$$

By inspection it is readily seen that $B^{10}(\alpha)$ has jumps at $\alpha = \theta_i$, $i = 1, \cdots, N-1$, of respective heights $A_i^*B^{00}$. Moreover

$$B^{10}(-a) = A_N^*B^{00}. \tag{A.11}$$

Let $h = (0, h^1)$ and $k = (0, k^1)$ in (A.1). Then

$$(h^1, B^{11}k^1) = \int_0^{\infty} (\Phi^1(t)h^1, \Phi^1(t)k^1)\,dt. \tag{A.12}$$

In view of (A.5),

$$(h^1, B^{11}k^1) = \int_0^{\infty} \left(\int_{-a}^{0} \Phi^1(t,\alpha)h^1(\alpha)\,d\alpha, \int_{-a}^{0} \Phi^1(t,\beta)k^1(\beta)\,d\beta \right) dt \tag{A.13}$$

and

$$B^{11}(\alpha,\beta) = \int_0^{\infty} \Phi^1(t,\alpha)^*\Phi^1(t,\beta)\,dt. \tag{A.14}$$

We again use (A.8) to express $B^{11}(\alpha, \beta)$ in terms of Φ^0:

$$B^{11}(\alpha,\beta) = \int_0^{\infty} \left[\sum_{i=1}^{N} \begin{Bmatrix} A_i^*\Phi^0(t-\alpha+\theta_i)^*, & t \geqq \alpha-\theta_i \geqq 0 \\ 0 & , \text{otherwise} \end{Bmatrix} + \int_{\max\{-a,\alpha-t\}}^{\alpha} A_{01}(\theta)^*\Phi^0(t-\alpha+\theta)^*\,d\theta \right].$$

$$\left[\sum_{j=1}^{N} \begin{Bmatrix} \Phi^0(t-\beta+\theta_j)A_j, & t \geqq \beta-\theta_j \geqq 0 \\ 0 & , \text{otherwise} \end{Bmatrix} + \int_{\max\{-a,\beta-t\}}^{\beta} \Phi^0(t-\beta+\xi)A_{01}(\xi)\,d\xi \right] dt$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^{\infty} \begin{Bmatrix} A_i^*\Phi^0(t-\alpha+\theta_i)^*\Phi^0(t-\beta+\theta_j)A_j, & t \geqq \alpha-\theta_i \geqq 0, t \geqq \beta-\theta_j \geqq 0 \\ 0 & , \text{otherwise} \end{Bmatrix} dt$$

$$+ \sum_{i=1}^{N} \int_0^{\infty} dt \begin{Bmatrix} A_i^*\Phi^0(t-\alpha+\theta_i)^*, & t \geqq \alpha-\theta_i \geqq 0 \\ 0 & , \text{otherwise} \end{Bmatrix} \int_{\max\{-a,\beta-t\}}^{\beta} \Phi^0(t-\beta+\xi)A_{01}(\xi)\,d\xi$$

$$+ \sum_{j=1}^{N} \int_0^{\infty} dt \begin{Bmatrix} \int_{\max\{-a,\alpha-t\}}^{\alpha} d\theta\, A_{01}(\theta)^*\Phi^0(t-\alpha+\theta)^*\Phi^0(t-\beta+\theta_j)A_j, & t \geqq \beta-\theta_j \geqq 0 \\ 0 & , \text{otherwise} \end{Bmatrix}$$

$$+ \int_0^{\infty} dt \int_{\max\{-a,\alpha-t\}}^{\alpha} d\theta \int_{\max\{-a,\beta-t\}}^{\beta} d\xi\, A_{01}(\theta)^*\Phi^0(t-\alpha+\theta)^*\Phi^0(t-\beta+\xi)A_{01}(\xi) \tag{A.15}$$

(cont.)

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \begin{cases} \displaystyle\int_{\alpha-\theta_i}^{\infty} dt\, A_i^* \Phi^0(t-\alpha+\theta_i)^* \Phi^0(t-\beta+\theta_j) A_j, & \alpha-\theta_i \geqq \beta-\theta_j \geqq 0 \\ \displaystyle\int_{\beta-\theta_j}^{\infty} dt\, A_i^* \Phi^0(t-\alpha+\theta_i)^* \Phi^0(t-\beta+\theta_j) A_j, & \beta-\theta_j \geqq \alpha-\theta_i \geqq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$+ \sum_{i=1}^{N} \begin{cases} \displaystyle\int_{\alpha-\theta_i}^{\infty} dt \int_{\max\{-a,\beta-t\}}^{\beta} d\xi\, A_i^* \Phi^0(t-\alpha+\theta_i)^* \Phi^0(t-\beta+\xi) A_{01}(\xi), & \alpha-\theta_i \geqq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$+ \sum_{j=1}^{N} \begin{cases} \displaystyle\int_{\beta-\theta_j}^{\infty} dt \int_{\max\{-a,\alpha-t\}}^{\alpha} d\theta\, A_{01}(\theta)^* \Phi^0(t-\alpha+\theta)^* \Phi^0(t-\beta+\theta_j) A_j, & \beta-\theta_j \geqq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$+ \int_0^{\infty} dt \int_{\max\{-a,\alpha-t\}}^{\alpha} d\theta \int_{\max\{-a,\beta-t\}}^{\beta} d\xi\, A_{01}(\theta)^* \Phi^0(t-\alpha+\theta)^* \Phi^0(t-\beta+\xi) A_{01}(\xi)$$

$$= \sum_{i=1}^{N} \begin{cases} \displaystyle\sum_{j=1}^{N} \begin{cases} \begin{cases} \displaystyle\int_{\alpha-\theta_i}^{\infty} dt\, A_i^* \Phi^0(t-\alpha+\theta_i)^* \Phi^0(t-\beta+\theta_j) A_j, & \beta-\theta_j-\alpha+\theta_i \leqq 0 \\ \displaystyle\int_{\beta-\theta_j}^{\infty} dt & , \ \beta-\theta_j-\alpha+\theta_i > 0 \end{cases}, & \beta \geqq \theta_j \\ 0 & , \ \beta < \theta_j \end{cases}, & \alpha \geqq \theta_i \\ 0 & , \ \alpha < \theta_i \end{cases}$$

$$+ \sum_{i=1}^{N} \begin{cases} \displaystyle\int_{0}^{\infty} dt \int_{\max\{-a,\beta-t-\alpha+\theta_i\}}^{\beta} d\xi\, A_i^* \Phi^0(t)^* \Phi^0(t+\alpha-\theta_i-\beta+\xi) A_{01}(\xi), & \alpha-\theta_i \geqq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$+ \sum_{j=1}^{N} \begin{cases} \displaystyle\int_{0}^{\infty} dt \int_{\max\{-a,\alpha-t-\beta+\theta_j\}}^{\alpha} d\theta\, A_{01}(\theta)^* \Phi^0(t+\beta-\theta_j-\alpha+\theta)^* \Phi^0(t) A_j, & \beta-\theta_j \geqq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$+ \int_{-a}^{\alpha} d\theta \int_{\alpha-\theta}^{\infty} dt \int_{\max\{-a,\beta-t\}}^{\beta} d\xi\, A_{01}(\theta)^* \Phi^0(t-\alpha+\theta)^* \Phi^0(t-\beta+\xi) A_{01}(\xi)$$

$$= \text{①} + \text{②} + \text{③} + \text{④}.$$

Given α, term ① has jumps at $\beta = \theta_j, j = 1, \cdots, N-1$, of height

(A.16)
$$\sum_{i=1}^{N} \begin{cases} \displaystyle\int_{\alpha-\theta_i}^{\infty} dt\, A_i^* \Phi^0(t-\alpha+\theta_i)^* \Phi^0(t) A_j, & \alpha \geqq \theta_i \\ 0 & , \ \alpha < \theta_i \end{cases}.$$

Given β, term ① has jumps at $\alpha = \theta_i, i = 1, \cdots, N-1$, of height

(A.17)
$$\sum_{j=1}^{N} \begin{cases} \displaystyle\int_{\beta-\theta_j}^{\infty} dt\, A_i^* \Phi^0(t)^* \Phi^0(t-\beta+\theta_j) A_j, & \beta \geqq \theta_j \\ 0 & , \ \beta < \theta_j \end{cases}.$$

Given β, term ② has jumps at $\alpha = \theta_i, i = 1, \cdots, N-1$, of height

$$\int_{-a}^{\beta} d\xi \int_{\beta-\xi}^{\infty} dt\, A_i^*\Phi^0(t)^*\Phi^0(t-\beta+\xi)A_{01}(\xi). \tag{A.18}$$

Given α, term ③ has jumps at $\beta = \theta_j, j = 1, \cdots, N-1$, of height

$$\int_{-a}^{\alpha} d\theta \int_{\alpha-\theta}^{\infty} dt\, A_{01}(\theta)^*\Phi^0(t-\alpha+\theta)^*\Phi^0(t)A_j. \tag{A.19}$$

Given α, term ② has no jumps. Given β, term ③ has no jumps. Term ④ has no jumps. Finally, given α the map $\beta \mapsto B^{11}(\alpha, \beta)$ has jumps at $\beta = \theta_j, j = 1, \cdots, N-1$, of height $B^{10}(\alpha)A_j$ and given β the map $\alpha \mapsto B^{11}(\alpha,\beta)$ has jumps at $\alpha = \theta_i$, $i = 1, \cdots, N-1$, of height $A_i^*B^{10}(\beta)^*$. Moreover,

$$B^{11}(-a,\beta) = \sum_{i=1}^{N} \begin{Bmatrix} \displaystyle\int_{\beta-\theta_j}^{\infty} A_N^*\Phi^0(t)^*\Phi^0(t-\beta+\theta_j)A_j, & \beta \geqq \theta_j \\ 0 & , \quad \beta < \theta_j \end{Bmatrix}$$
$$+ \int_0^{\infty} dt \int_{\max\{-a,\beta-t\}}^{\beta} d\xi\, A_N^*\Phi^0(t)^*\Phi^0(t-\beta+\xi)A_{01}(\xi)$$

and

$$B^{11}(-a,\beta) = A_N^*B^{10}(\beta)^*. \tag{A.20}$$

We now express $B^{11}(\alpha,\beta)$ in terms of $B^{10}(\cdot)$. To do this we consider separately each of the four terms in (A.15).

$$① = \sum_{j=1}^{N} \left\{ \begin{array}{ll} \displaystyle\sum_{i=1}^{N} \left\{ \begin{array}{ll} \left\{ \begin{array}{ll} \displaystyle\int_{\alpha-\theta_i}^{\infty} dt\, A_i^*\Phi^0(t-\alpha+\theta_i)^*\Phi^0(t-\beta+\theta_j)A_j, & \beta-\theta_j-\alpha+\theta_i \leqq 0 \\ 0 & \text{otherwise} \end{array} \right\} & \alpha \geqq \theta_i \\ 0 & \alpha < \theta_i \end{array} \right\}, & \beta \geqq \theta_j \\ 0 & , \quad \beta < \theta_j \end{array} \right\}$$

$$+ \sum_{i=1}^{N} \left\{ \begin{array}{ll} \displaystyle\sum_{j=1}^{N} \left\{ \begin{array}{ll} \left\{ \begin{array}{ll} 0 & , \quad \beta-\theta_j-\alpha+\theta_i \leqq 0 \\ \displaystyle\int_{0}^{\infty} dt\, A_i^*\Phi^0(t-\alpha+\theta_i)^*\Phi^0(t-\beta+\theta_j)A_j, & \beta-\theta_j-\alpha+\theta_i > 0 \end{array} \right\}, & \beta \geqq \theta_j \\ 0 & , \quad \beta < \theta_j \end{array} \right\}, & \alpha \geqq \theta_i \\ 0 & , \quad \alpha < \theta_i \end{array} \right\}$$

$$= \sum_{j=1}^{N} \left\{ \begin{array}{ll} \displaystyle\sum_{i=1}^{N} \left\{ \begin{array}{ll} \displaystyle\int_{\alpha-\beta+\theta_j-\theta_i}^{\infty} dt\, A_i^*\Phi^0(t-\alpha+\beta-\theta_j+\theta_i)^*\Phi^0(t)A_j, & \begin{array}{c} -\alpha+\beta-\theta_j+\theta_i \leqq 0 \\ \alpha \geqq \theta_i \end{array} \\ 0 & , \quad \text{otherwise} \end{array} \right\}, & \beta \geqq \theta_j \\ 0 & , \quad \beta < \theta_j \end{array} \right\}$$

$$+ \sum_{i=1}^{N} \left\{ \begin{array}{ll} \displaystyle\sum_{j=1}^{N} \left\{ \begin{array}{ll} \displaystyle\int_{\beta-\alpha+\theta_i-\theta_j}^{\infty} dt\, A_i^*\Phi^0(t)^*\Phi^0(t-\beta+\alpha-\theta_i+\theta_j), & \begin{array}{c} -\beta+\alpha-\theta_i+\theta_j \leqq 0 \\ \beta \geqq \theta_j \end{array} \\ 0 & , \quad \text{otherwise} \end{array} \right\}, & \alpha \geqq \theta_i \\ 0 & , \quad \alpha < \theta_i \end{array} \right\},$$

$$② = \sum_{i=1}^{N} \begin{cases} \int_{-a}^{\beta} d\xi \begin{cases} \int_{\beta-\alpha+\theta_i-\xi}^{\infty} dt\, A_i^* \Phi^0(t)^* \Phi^0(t+\alpha-\theta_i-\beta+\xi) A_{01}(\xi), & \beta-\alpha+\theta_i \geqq \xi \\ \int_0^{\infty} dt & , \beta-\alpha+\theta_i < \xi \end{cases}, & \alpha \geqq \theta_i \\ 0 & , \alpha < \theta_i \end{cases}$$

$$= \sum_{i=1}^{N} A_i^* \begin{cases} \int_{-a}^{\beta-\alpha+\theta_i} d\xi \int_{\beta-\alpha+\theta_i-\xi}^{\infty} dt\, \Phi^0(t)^* \Phi^0(t+\alpha-\theta_i-\beta+\xi) A_{01}(\xi), & -a \leqq \beta-\alpha+\theta_i, \alpha \geqq \theta_i \\ 0 & , \text{otherwise} \end{cases}$$

$$+ \sum_{i=1}^{N} A_i^* \begin{cases} \int_{\beta-\alpha+\theta_i}^{\beta} d\xi \int_0^{\infty} dt\, \Phi^0(t)^* \Phi^0(t+\alpha-\theta_i-\beta+\xi) A_{01}(\xi), & -a \leqq \beta-\alpha+\theta_i, \alpha \geqq \theta_i \\ \int_{-a}^{\beta} d\xi & , -a > \beta-\alpha+\theta_i, \alpha \geqq \theta_i \\ 0 & , \text{otherwise} \end{cases}$$

$$= \sum_{i=1}^{N} A_i^* \begin{cases} \int_{-a}^{\beta-\alpha+\theta_i} d\xi \int_{\beta-\alpha+\theta_i-\xi}^{\infty} dt\, \Phi^0(t)^* \Phi^0(t-\beta+\alpha-\theta_i+\xi) A_{01}(\xi), & -a \leqq \beta-\alpha+\theta_i, \alpha \geqq \theta_i \\ 0 & , \text{otherwise} \end{cases}$$

$$+ \int_{-a}^{\beta} d\xi \sum_{i=1}^{N} \begin{cases} A_i^* \int_{\alpha-\beta+\xi-\theta_i}^{\infty} dt\, \Phi^0(t-\alpha+\beta-\xi+\theta_i)^* \Phi^0(t), & \xi \geqq \beta-\alpha+\theta_i, \alpha \geqq \theta_i \\ 0 & , \text{otherwise} \end{cases} A_{01}(\xi).$$

Notice that we can drop $\alpha \geqq \theta_i$ in the last term since

$$\beta \geqq \xi \quad \text{and} \quad \xi \geqq \beta - \alpha + \theta_i \Rightarrow \alpha \geqq \theta_i.$$

By symmetry

$$③ = \sum_{j=1}^{N} \begin{cases} \int_{-a}^{\alpha-\beta+\theta_j} d\theta\, A_{01}(\theta)^* \int_{\alpha-\beta+\theta_j-\theta}^{\infty} dt\, \Phi^0(t-\alpha+\beta-\theta_j+\theta)^* \Phi^0(t) A_j, & -a \leqq \alpha-\beta+\theta_j, \beta \geqq \theta_j \\ 0 & , \text{otherwise} \end{cases}$$

$$+ \int_{-a}^{\alpha} d\theta\, A_{01}(\theta)^* \sum_{j=1}^{N} \begin{cases} \int_{\beta-\alpha+\theta-\theta_j}^{\infty} dt\, \Phi^0(t)^* \Phi^0(t-\beta+\alpha-\theta) A_j, & \theta \geqq \alpha-\beta+\theta_j \\ 0 & , \text{otherwise} \end{cases}.$$

Finally,

$$④ = \int_{-a}^{\alpha} d\theta \int_{-a}^{\beta} d\xi \begin{cases} \int_{\alpha-\theta}^{\infty} dt\, A_{01}(\theta)^* \Phi^0(t-\alpha+\theta)^* \Phi^0(t-\beta+\xi) A_{01}(\xi), & \alpha-\theta \geqq \beta-\xi \\ \int_{\beta-\xi}^{\infty} dt\, A_{01}(\theta)^* \Phi^0(t-\alpha+\theta)^* \Phi^0(t-\beta+\xi) A_{01}(\xi), & \alpha-\theta < \beta-\xi \end{cases}$$

$$= \int_{-a}^{\beta} d\xi \begin{cases} \int_{-a}^{\alpha-\beta+\xi} d\theta \int_{\alpha-\beta+\xi-\theta}^{\infty} dt\, A_{01}(\theta)^* \Phi^0(t-\alpha+\beta-\xi+\theta)^* \Phi^0(t) A_{01}(\xi), & \alpha-\beta+\xi \geqq -a \\ 0 & , \text{otherwise} \end{cases}$$

(cont.)

$$+\int_{-a}^{\alpha} d\theta \begin{cases} \displaystyle\int_{-a}^{\beta-\alpha+\theta} d\theta \int_{\beta-\alpha+\theta-\xi}^{\infty} dt\, A_{01}(\theta)^*\Phi^0(t)^*\Phi^0(t-\beta+\alpha-\theta+\xi)A_{01}(\xi), & \beta-\alpha+\theta \geqq -a \\ 0 & \text{, otherwise} \end{cases}$$

(ii) We now derive equations (5.19), (5.22) and (5.26). Our starting point is the Lyapunov equation

$$\text{(A.21)} \qquad 0 = (\tilde{A}h, Bk) + (Bh, \tilde{A}k) + (\tilde{I}h, k) \quad \forall\, h, k \text{ in } V$$

or

$$\text{(A.22)} \qquad \begin{aligned} &\left(A_{00}h(0) + \sum_{i=1}^{N} A_i h(\theta_i) + \int_{-a}^{0} A_{01}(\alpha)h(\alpha)\,d\alpha,\, B^{00}k(0) + \int_{-a}^{0} B^{01}(\theta)k(\theta)\,d\theta\right) \\ &+ \int_{-a}^{0} \left(\frac{dh}{d\alpha}(\alpha),\, B^{10}(\alpha)k(0) + \int_{-a}^{0} B^{11}(\alpha,\theta)k(\theta)\,d\theta\right) d\alpha \\ &+ \left(B^{00}h(0) + \int_{-a}^{0} B^{01}(\alpha)h(\alpha)\,d\alpha,\, A_{00}k(0) + \sum_{i=1}^{N} A_i k(\theta_i) \right. \\ &\qquad \left. + \int_{-a}^{0} A_{01}(\theta)k(\theta)\,d\theta\right) \\ &+ \int_{-a}^{0} \left(B^{10}(\theta)h(0) + \int_{-a}^{0} B^{11}(\theta,\alpha)h(\alpha)\,d\alpha,\, \frac{dk}{d\theta}(\theta)\right) d\theta + (h(0), k(0)) = 0. \end{aligned}$$

Let

$$h_n(\theta) = \begin{cases} h^0\left(1 + n\dfrac{\theta}{a}\right), & -\dfrac{a}{n} \leqq \theta \leqq 0, \\ 0, & \text{, otherwise,} \end{cases}$$

where n is chosen in such a way that $n > a\theta_1^{-1}$. Then

$$h_n(0) \to h^0 \quad \text{and} \quad h_n \to 0 \text{ in } L^2(-a, 0; X).$$

Let k_i be chosen in $W^{1,2}(-a, 0; X)$ in such a way that

$$\operatorname{supp} k_i \subset (\theta_i, \theta_{i-1}) \cup (\theta_1, 0].$$

Let $h = h_n$ and $k = k_i$ in (A.22):

$$\text{(A.23)} \qquad \begin{aligned} 0 = &\left(A_{00}h^0 + \int_{\theta_1}^{0} A_{01}(\theta)h_n(\theta)\,d\theta,\, B^{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\} B^{01}(\theta)k_i(\theta)\,d\theta\right) \\ &+ \int_{\theta_1}^{0} \left(\frac{dh_n}{d\alpha}(\alpha),\, B^{01}(\alpha)k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\} B^{11}(\alpha,\theta)k_i(\theta)\,d\theta\right) d\alpha \\ &+ \left(B^{00}h^0 + \int_{\theta_1}^{0} B^{01}(\alpha)h_n(\alpha)\,d\alpha,\, A_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\} A_{01}(\theta)k_i(\theta)\,d\theta\right) \\ &+ \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\} \left(B^{10}(\theta)h^0 + \int_{\theta_1}^{0} B^{11}(\theta,\alpha)h_n(\alpha)\,d\alpha,\, \frac{dk_i}{d\theta}(\theta)\right) d\theta + (h^0, k_i(0)). \end{aligned}$$

Since $\alpha \mapsto \Pi_{01}(\alpha)$, $\alpha \mapsto \Pi_{11}(\alpha, \theta)$ and $\theta \mapsto \Pi_{11}(\alpha, \theta)$ are absolutely continuous in (θ_i, θ_{i-1}) and $(\theta_1, 0)$ we can now integrate by parts.

Equation (A.23) now reduces to

$$
\begin{aligned}
0 = &\left(A_{00}h^0 + \int_{\theta_1}^0 A_{01}(\theta)h_n(\theta)\,d\theta,\, B^{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\} B^{01}(\theta)k_i(\theta)\,d\theta\right) \\
&+ \left(h_n(0),\, B^{10}(0)k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\} B^{11}(0,\theta)k_i(\theta)\,d\theta\right) \\
&- \int_{\theta_1}^0 \left(h_n(\alpha),\, B^{10}(\alpha)k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\} B^{11}(\alpha,\theta)k_i(\theta)\,d\theta\right) \\
&+ \left(B^{00}h^0 + \int_{\theta_1}^0 B^{01}(\alpha)h_n(\alpha)\,d\alpha,\, A_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\} A_{01}(\theta)k_i(\theta)\,d\theta\right) \\
&+ \left(B^{10}(0)h^0 + \int_{\theta_1}^0 B^{11}(0,\alpha)h_n(\alpha)\,d\alpha,\, k_i(0)\right) - \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}\left(\frac{dB^{01}}{d\theta}(\theta)h^0\right. \\
&\left. + \int_{\theta_1}^0 \frac{d\Pi_{11}}{d\theta}(\theta,\alpha)h_n(\alpha)\,d\alpha,\, k_i(\theta)\right) d\theta + (h^0, k_i(0)).
\end{aligned} \tag{A.24}
$$

Notice that

$$\int_{-a}^0 |A_{01}(\theta)h_n(\theta)|\,d\theta \leqq \left[\int_{-a}^0 |A_{01}(\theta)|^2\,d\theta\right]^{1/2}\left[\int_{-a}^0 |h_n(\theta)|^2\,d\theta\right]^{1/2}$$

and

$$\lim_{n\to\infty} |h_n|_{L^2(-a,0;X)} = 0$$

imply that

$$\int_{-a}^0 |A_{01}(\theta)h_n(\theta)|\,d\theta \to 0 \quad \text{as } n \to \infty.$$

Similarly given any f in $L^2(-a, 0; X)$,

$$\left|\int_{-a}^0 (h_n(\theta), f(\theta))\,d\theta\right| \leqq |h_n|_{L^2}|f|_{L^2}$$

and

$$\lim_{n\to\infty} \int_{-a}^0 (h_n(\theta), f(\theta))\,d\theta \to 0 \quad \text{as } n \to \infty.$$

As a result equation (A.24) yields

$$
\begin{aligned}
0 = &\left(A_{00}h^0, B^{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\} B^{01}(\theta)k_i(\theta)\, d\theta\right) \\
&+ \left(h^0, B^{10}(0)k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\} B^{11}(0,\theta)k_i(\theta)\, d\theta\right) \\
&+ \left(B^{00}h^0, A_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\} A_{01}(\theta)k_i(\theta)\, d\theta\right) \\
&+ (B^{10}(0)h^0, k_i(0)) - \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\}\left(\frac{dB^{01}}{d\theta}(\theta)h^0, k_i(\theta)\right) d\theta + (h^0, k_i(0)).
\end{aligned} \tag{A.25}
$$

To obtain equation (5.19) we use

$$
k_i(\theta) = k_m(\theta) = \begin{cases} k^0\left(1 + m\dfrac{\theta}{a}\right), & -\dfrac{a}{m} \leqq \theta \leqq 0 \\ 0 & , \text{ otherwise} \end{cases},
$$

where m is chosen in such a way that $m > a\theta_1^{-1}$. When we take the limit of equation (A.25) as m goes to infinity we obtain

$$
([B^{00}A_{00} + B^{10}(0)^* + A_{00}^*B^{00} + B^{10}(0) + I]h^0, k^0) = 0
$$

for all h^0 and k^0 in X.

To obtain equation (5.22) in the open interval (θ_i, θ_{i-1}) we choose k_i such that

$$
\operatorname{supp} k_i \subset (\theta_i, \theta_{i-1}).
$$

Then equation (A.25) yields

$$
0 = \int_{\theta_i}^{\theta_{i-1}} \left(\left[B^{01}(\theta)^*A_{00} + B^{11}(0,\theta)^* + A_{01}(\theta)^*B^{00} - \left(\frac{dB^{01}}{d\theta}(\theta)\right)^*\right]h^0, k_i(\theta)\right) d\theta.
$$

By density of the set of absolutely continuous maps with support in (θ_i, θ_{i-1}) in $L^2(\theta_i, \theta_{i-1}; X)$ and the properties

$$
B^{10}(\theta) = B^{01}(\theta)^*, \qquad B^{11}(\alpha, \theta)^* = B^{11}(\theta, \alpha), \tag{A.26}
$$

the above equation yields for all h^0 in X

$$
\left[-\frac{dB^{10}}{d\theta}(\theta) + B^{10}(\theta)A_{00} + A_{01}(\theta)^*B^{00} + B^{11}(\theta, 0)\right]h^0 = 0,
$$

almost everywhere in (θ_i, θ_{i-1}).

To obtain (5.26) in the region

$$
\{(\alpha, \theta) \in [-a, 0] \times [-a, 0] | \alpha \in (\theta_i, \theta_{i-1}), \theta \in (\theta_j, \theta_{j-1})\}
$$

we choose

$$
h = h_i, \quad \operatorname{supp} h_i \subset (\theta_i, \theta_{i-1}),
$$
$$
k = k_j, \quad \operatorname{supp} k_j \subset (\theta_j, \theta_{j-1})
$$

and substitute in (A.22) which reduces to the following expression:

$$
\begin{aligned}
0 = &\left(\int_{\theta_i}^{\theta_{i-1}} A_{01}(\alpha)h_i(\alpha)\,d\alpha, \int_{\theta_j}^{\theta_{j-1}} B^{01}(\theta)k_j(\theta)\,d\theta\right) \\
&+ \int_{\theta_i}^{\theta_{i-1}} \left(\frac{dh_i}{d\alpha}(\alpha), \int_{\theta_j}^{\theta_{j-1}} B^{11}(\alpha,\theta)k_j(\theta)\,d\theta\right) d\alpha \\
&+ \left(\int_{\theta_i}^{\theta_{i-1}} B^{01}(\alpha)h_i(\alpha)\,d\alpha, \int_{\theta_j}^{\theta_{j-1}} A_{01}(\theta)k_j(\theta)\,d\theta\right) \\
&+ \int_{\theta_j}^{\theta_{j-1}} \left(\int_{\theta_i}^{\theta_{i-1}} B^{11}(\theta,\alpha)h_i(\alpha)\,d\alpha, \frac{dk_j}{d\theta}(\theta)\right) d\theta.
\end{aligned}
\tag{A.27}
$$

The two terms with a derivative can be integrated by parts:

$$
\begin{aligned}
&\int_{\theta_i}^{\theta_{i-1}} \left(\frac{dh_i}{d\alpha}(\alpha), \int_{\theta_j}^{\theta_{j-1}} B^{11}(\alpha,\theta)k_j(\theta)\,d\theta\right) d\alpha \\
&\qquad = -\int_{\theta_i}^{\theta_{i-1}} \left(h_i(\alpha), \int_{\theta_j}^{\theta_{j-1}} \frac{\partial}{\partial\alpha} B^{11}(\alpha,\theta)k_j(\theta)\,d\theta\right) d\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\theta_j}^{\theta_{j-1}} \left(\int_{\theta_i}^{\theta_{i-1}} B^{11}(\theta,\alpha)h_i(\alpha)\,d\alpha, \frac{dk_j}{d\theta}(\theta)\right) d\theta \\
&\qquad = -\int_{\theta_j}^{\theta_{j-1}} \int_{\theta_i}^{\theta_{i-1}} \left(\frac{\partial}{\partial\theta} B^{11}(\theta,\alpha)h_i(\alpha)\,d\alpha, k_j(\theta)\right) d\theta.
\end{aligned}
$$

Finally equation (A.27) takes the form

$$
\begin{aligned}
\int_{\theta_i}^{\theta_{i-1}} d\alpha \int_{\theta_j}^{\theta_{j-1}} d\theta \Bigg(\Bigg[& B^{01}(\theta)^* A_{01}(\alpha) - \left(\frac{\partial B^{11}}{\partial\alpha}(\alpha,\theta)\right)^* + A_{01}(\theta)^* B^{01}(\alpha) \\
& - \frac{\partial B^{11}}{\partial\theta}(\theta,\alpha)\Bigg] h_i(\alpha), k_j(\theta)\Bigg).
\end{aligned}
$$

By using relations (A.26) and the density argument we obtain

$$
\frac{\partial B^{11}}{\partial\theta}(\alpha,\theta) + \frac{\partial B^{11}}{\partial\alpha}(\alpha,\theta) = A_{01}(\alpha)^* B^{10}(\theta)^* + B^{10}(\alpha)A_{01}(\theta)
$$

for almost all (α,θ) in $(\theta_i,\theta_{i-1}) \times (\theta_j,\theta_{j-1})$.

(iii) We now solve equation (5.26) with boundary conditions (5.27). We let $\eta = \alpha - \beta$ and consider two cases. First let $a \geqq \eta \geqq 0$; then

$$
-a \leqq \beta \leqq 0 \Rightarrow \eta - a \leqq \alpha \leqq 0.
$$

If we change the variable β to $\eta = \alpha - \beta$, equation (5.26) becomes

$$
\begin{aligned}
\frac{d}{d\alpha} B^{11}(\alpha,\alpha-\eta) = {} & A_{01}(\alpha)^* B^{10}(\alpha-\eta)^* + B^{10}(\alpha)A_{01}(\alpha-\eta) \\
& + \sum_{i=1}^{N-1} A_i^* B^{10}(\alpha-\eta)^* \delta(\alpha-\theta_i) + \sum_{j=1}^{N-1} B^{10}(\alpha)A_j\delta(\alpha-\eta-\theta_j).
\end{aligned}
$$

This last equation can be integrated from $\eta - a$ to α:

$$\begin{aligned}B^{11}(\alpha,\alpha-\eta) &= B^{11}(\eta-a,-a)\\&\quad+\int_{\eta-a}^{\alpha} A_{01}(\xi)^*B^{10}(\xi-\eta)^*\,d\xi+\int_{\eta-a}^{\alpha} B^{10}(\xi)A_{01}(\xi-\eta)\,d\xi\\&\quad+\sum_{i=1}^{N-1}\begin{Bmatrix}A_i^*B^{10}(\theta_i-\eta)^*, & \eta-a\leqq\theta_i<\alpha\\0 &, \text{ otherwise}\end{Bmatrix}\\&\quad+\sum_{j=1}^{N-1}\begin{Bmatrix}B^{10}(\eta+\theta_j)A_j, & \eta-a\leqq\eta+\theta_j<\alpha\\0 &, \text{ otherwise}\end{Bmatrix}.\end{aligned}$$

Finally for $\alpha \geqq \beta$,

$$\begin{aligned}B^{11}(\alpha,\beta) &= B^{10}(\alpha-\beta-a)A_N+\sum_{j=1}^{N-1}\begin{Bmatrix}B^{10}(\alpha-\beta+\theta_j)A_j, & \theta_j<\beta\\0 &, \text{ otherwise}\end{Bmatrix}\\&\quad+\sum_{i=1}^{N-1}\begin{Bmatrix}A_i^*B^{10}(\beta-\alpha+\theta_i)^*, & -a\leqq\beta-\alpha+\theta_i,\theta_i<\alpha\\0 &, \text{ otherwise}\end{Bmatrix}\\&\quad+\int_{-a}^{\alpha}\begin{Bmatrix}A_{01}(\xi)^*B^{10}(\beta-\alpha+\xi)^*, & \xi\geqq\alpha-\beta-a\\0 &, \text{ otherwise}\end{Bmatrix}d\xi\\&\quad+\int_{-a}^{\beta}B^{10}(\alpha-\beta+\theta)A_{01}(\theta)\,d\theta,\end{aligned}$$

$$\begin{aligned}B^{11}(\alpha,\beta) &= B^{10}(\alpha-\beta-a)A_N\\&\quad+\sum_{j=1}^{N-1}\begin{Bmatrix}B^{10}(\alpha-\beta+\theta_j)A_j, & \theta_j<\beta,\ -a\leqq\alpha-\beta+\theta_j\\0 &, \text{ otherwise}\end{Bmatrix}\\&\quad+\sum_{i=1}^{N-1}\begin{Bmatrix}A_i^*B^{10}(\beta-\alpha+\theta_i)^*, & -a\leqq\beta-\alpha+\theta_i,\theta_i<\alpha\\0 &, \text{ otherwise}\end{Bmatrix}\\&\quad+\int_{-a}^{\alpha}\begin{Bmatrix}A_{01}(\xi)^*B^{10}(\beta-\alpha+\xi)^*, & \xi\geqq\alpha-\beta-a\\0 &, \text{ otherwise}\end{Bmatrix}d\xi\\&\quad+\int_{-a}^{\beta}\begin{Bmatrix}B^{10}(\alpha-\beta+\theta)A_{01}(\theta), & \theta\geqq\beta-\alpha-a\\0 &, \text{ otherwise}\end{Bmatrix}d\theta.\end{aligned}$$

Notice that in the above expression for $B^{11}(\alpha,\beta)$ all terms but the first are symmetrical. Hence for $\alpha \leqq \beta$ we shall obtain the same expression with the exception of the first term which will be equal to

$$A_N^*B^{10}(\beta-\alpha-a)^*.$$

But

$$\lim_{\alpha\leqq\beta,\beta\to\alpha} B^{10}(\alpha-\beta-a)A_N = B^{10}(-a)A_N = A_N^*B^{00}A_N$$

and

$$\lim_{\beta\leqq\alpha,\beta\to\alpha} A_N^*B^{10}(\beta-\alpha-a)^* = A_N^*B^{10}(-a)^* = A_N^*B^{00}A_N$$

imply that this first term is continuous at (α, α), $-a \leqq \alpha < \theta_{N-1}$. This makes it possible to write the first term as follows:

$$\begin{aligned} &B^{10}(\alpha - \beta - a)A_N, && \alpha \geqq \beta, \\ &A_N^* B^{10}(\beta - \alpha - a)^*, && \alpha < \beta. \end{aligned}$$

This yields identity (5.28).

Appendix B.

Proof of Theorem 6.1. The reader can find the definitions of Φ^0, Φ^1 and $\tilde{\Phi}$ in Delfour and Mitter [7], [8], [10].

We first study Π_{00} and the kernels $\Pi_{10}(\alpha)$ and $\Pi_{11}(\alpha, \beta)$ of the operators Π_{10} and Π_{11}. Since we know where the discontinuities can occur we derive differential equations for $\Pi_{10}(\alpha)$ and $\Pi_{11}(\alpha, \beta)$. Finally we solve the equation for $\Pi_{11}(\alpha, \beta)$ and give an explicit expression for $\Pi_{11}(\alpha, \beta)$ in terms of $\Pi_{10}(\cdot)$. We shall use the following results (cf. Delfour and Mitter [7], [8], [10]):

$$[\tilde{\Phi}(t)h]^0 = \Phi^0(t)h^0 + \Phi^1(t)h^1, \tag{B.1}$$

$$\begin{aligned} [\Pi\tilde{\Phi}(t)h]^0 = \Pi_{00}\Phi^0(t)h^0 &+ \int_{-a}^0 \Pi_{01}(\alpha)\begin{Bmatrix} \Phi^0(t+\alpha)h^0, & t+\alpha \geqq 0 \\ 0 \quad, & \text{otherwise} \end{Bmatrix} d\alpha \\ + \Pi_{00}\Phi^1(t)h^1 &+ \int_{-a}^0 \Pi_{01}(\alpha)\begin{Bmatrix} \Phi^1(t+\alpha)h^1, & t+\alpha \geqq 0 \\ h^1(t+\alpha) \quad, & \text{otherwise} \end{Bmatrix} d\alpha. \end{aligned} \tag{B.2}$$

(i) Let $h = (h^0, 0)$ and $k = (k^0, 0)$ in (6.1). Then

$$\begin{aligned} (\Pi_{00}h^0, k^0) = \int_0^\infty \Big\{ &(Q\Phi^0(t)h^0, \Phi^0(t)k^0) \\ &+ \Big(R\Big[\Pi_{00}\Phi^0(t)h^0 + \int_{-\min(t,a)}^0 \Pi_{01}(\alpha)\Phi^0(t+\alpha)h^0\, d\alpha\Big], \\ &\qquad \Pi_{00}\Phi^0(t)k^0 + \int_{-\min(t,a)}^0 \Pi_{01}(\beta)\Phi^0(t+\beta)k^0\, d\beta\Big)\Big\}\, dt. \end{aligned} \tag{B.3}$$

Notice that

$$\int_0^\infty dt \int_{-\min(a,t)}^0 d\alpha = \int_{-a}^0 d\alpha \int_{-\alpha}^\infty dt$$

and

$$\int_0^\infty dt \int_{-\min(a,t)}^0 d\beta \int_{-\min(a,t)}^0 d\alpha = \int_{-a}^0 d\beta \int_{-a}^0 d\alpha \begin{Bmatrix} \displaystyle\int_{-\alpha}^\infty dt, & \alpha \leqq \beta \\ \displaystyle\int_{-\beta}^\infty dt, & \alpha > \beta \end{Bmatrix}.$$

Hence

$$\Pi_{00} = \int_0^\infty \Phi^0(t)^*[Q + \Pi_{00}R\Pi_{00}]\Phi^0(t)\,dt$$

$$+\int_{-a}^0 d\beta \int_{-\beta}^\infty dt\, \Phi^0(t+\beta)^*\Pi_{01}(\beta)^*R\Pi_{00}\Phi^0(t)$$

$$\text{(B.4)} \qquad +\int_{-a}^0 d\alpha \int_{-\alpha}^\infty dt\, \Phi^0(t)^*\Pi_{00}R\Pi_{01}(\alpha)\Phi^0(t+\alpha)$$

$$+\int_{-a}^0 d\alpha \int_{-a}^0 d\beta \begin{Bmatrix} \displaystyle\int_{-\alpha}^\infty dt\, \Phi^0(t+\beta)^*\Pi_{01}(\beta)^*R\Pi_{01}(\alpha)\Phi^0(t+\alpha), & \alpha \leqq \beta \\ \displaystyle\int_{-\beta}^\infty dt\, \Phi^0(t+\beta)^*\Pi_{01}(\beta)^*R\Pi_{01}(\alpha)\Phi^0(t+\alpha), & \alpha > \beta \end{Bmatrix}.$$

Let $h = (0, h^1)$, $k = (k^0, 0)$ in (6.1). Then

$$\Pi_{01}h^1 = \int_0^\infty \Phi^0(t)^*Q\Phi^1(t)h^1\,dt$$

$$+\int_0^\infty \left[\Phi^0(t)^*\Pi_{00}^* + \int_{-\min(t,a)}^0 \Phi^0(t+\theta)^*\Pi_{01}(\theta)^*\,d\theta\right]$$

$$\cdot R\left[\Pi_{00}\Phi^1(t)h^1 + \int_{-a}^0 \Pi_{01}(\alpha)\begin{Bmatrix}\Phi^1(t+\alpha)h^1, & t+\alpha \geqq 0 \\ h^1(t+\alpha)\quad, & \text{otherwise}\end{Bmatrix} d\alpha\right] dt$$

$$\text{(B.5)} \qquad = \int_0^\infty \Phi^0(t)^*Q\Phi^1(t)h^1\,dt$$

$$+\int_0^\infty \left[\Phi^0(t)^*\Pi_{00}^* + \int_{-\min(t,a)}^0 \Phi^0(t+\theta)^*\Pi_{01}(\theta)^*\,d\theta\right]$$

$$\cdot R\left[\Pi_{00}\Phi^1(t)h^1 + \int_{-\min(t,a)}^0 \Pi_{01}(\alpha)\Phi^1(t+\alpha)h^1\,d\alpha\right] dt$$

$$+\int_0^\infty \left[\Phi^0(t)^*\Pi_{00}^* + \int_{-\min(t,a)}^0 \Phi^0(t+\theta)^*\Pi_{01}(\theta)^*\,d\theta\right]$$

$$\cdot R\int_{-a}^{-\min(t,a)} \Pi_{01}(\alpha)h^1(t+\alpha)\,d\alpha\,dt.$$

In view of (B.3) and (B.4) and the fact that

$$\Phi^1(t+\alpha)h^1 = \int_{-a}^{0} \Phi^1(t+\alpha,\xi)h^1(\xi)\,d\xi,$$

$$\begin{aligned}
\Pi_{01}(\xi) = {} & \int_0^\infty dt\, \Phi^0(t)^*[Q + \Pi_{00}R\Pi_{00}]\Phi^1(t,\xi) \\
& + \int_{-a}^{0} d\theta \int_{-\theta}^{\infty} dt\, \Phi^0(t+\theta)^*\Pi_{01}(\theta)^*R\Pi_{00}\Phi^1(t,\xi) \\
& + \int_{-a}^{0} d\alpha \int_{-\alpha}^{\infty} dt\, \Phi^0(t)^*\Pi_{00}^*R\Pi_{01}(\alpha)\Phi^1(t+\alpha,\xi) \\
& + \int_{-a}^{0} d\alpha \int_{-a}^{0} d\theta \begin{Bmatrix} \displaystyle\int_{-\alpha}^{\infty} dt\, \Phi^0(t+\theta)^*\Pi_{01}(\theta)^*R\Pi_{01}(\alpha)\Phi^1(t+\alpha,\xi), & \alpha \leqq \theta \\ \displaystyle\int_{-\theta}^{\infty} dt\, \Phi^0(t+\theta)^*\Pi_{01}(\theta)^*R\Pi_{01}(\alpha)\Phi^1(t+\alpha,\xi), & \alpha > \theta \end{Bmatrix} \\
& + \int_{-a}^{\xi} d\alpha \left[\Phi^0(\xi-\alpha)^*\Pi_{00}^* + \int_{\alpha-\xi}^{0} \Phi^0(\xi-\alpha+\theta)^*\Pi_{01}(\theta)^*\,d\theta\right] R\Pi_{01}(\alpha),
\end{aligned} \tag{B.6}$$

where the last term is obtained from the last term in (B.5) after changes in the order of integration and changes of variable:

$$\int_0^\infty dt \int_{-a}^{-\min(a,t)} d\alpha = \int_{-a}^{0} d\alpha \int_0^{-\alpha} dt = \int_{-a}^{0} d\alpha \int_\alpha^0 d\xi \quad \text{with } \xi = t+\alpha,$$

$$\int_{-a}^{0} d\alpha \int_\alpha^0 d\xi \int_{-\min(a,\xi-\alpha)}^{0} = \int_{-a}^{0} d\xi \int_{-a}^{\xi} d\alpha \int_{-\min(a,\xi-\alpha)}^{0} d\theta = \int_{-a}^{0} d\xi \int_{-a}^{\xi} d\alpha \int_{\alpha-\xi}^{0} d\theta,$$

since $0 \leqq \xi - \alpha \leqq a + \xi \leqq a$.

We can now use the identity (cf. Delfour and Mitter [7], [8])

$$\begin{aligned}
\Phi^1(t,\xi) = {} & \sum_{i=1}^{N} \begin{Bmatrix} \Phi^0(t-\xi+\theta_i)A_i, & t \geqq \xi - \theta_i \geqq 0 \\ 0 \quad , & \text{otherwise} \end{Bmatrix} \\
& + \int_{\max\{-a,\xi-t\}}^{\xi} \Phi^0(t-\xi+\theta)[A_{01}(\theta) - R\Pi_{01}(\theta)]\,d\theta
\end{aligned} \tag{B.7}$$

to eliminate Φ^1 of the expression (B.6) for $\Pi_{01}(\xi)$. The term $\Phi^1(t,\xi)$ will always be integrated with respect to t and the only discontinuities that can occur are at $\xi = \theta_i$ where $\Phi^1(t,\xi)$ has a jump of height $\Phi^0(t)A_i$, $i = 1, \cdots, N-1$. This will produce a jump in $\Pi_{01}(\xi)$ of height

$$\begin{aligned}
&\int_0^\infty dt\, \Phi^0(t)^*[Q + \Pi_{00}R\Pi_{00}]\Phi^0(t)A_i \\
&\quad + \int_{-a}^0 d\theta \int_{-\theta}^\infty dt\, \Phi^0(t+\theta)^*\Pi_{01}(\theta)^*R\Pi_{00}\Phi^0(t)A_i \\
&\quad + \int_{-a}^0 d\alpha \int_{-a}^\infty dt\, \Phi^0(t)^*\Pi_{00}^*R\Pi_{01}(\alpha)\Phi^0(t+\alpha)A_i \\
&\quad + \int_{-a}^0 d\alpha \int_{-a}^0 d\theta \begin{cases} \displaystyle\int_{-\alpha}^\infty dt\, \Phi^0(t+\theta)^*\Pi_{01}(\theta)^*R\Pi_{01}(\alpha)\Phi^0(t+\alpha)A_i, & \alpha \leqq \theta \\ \displaystyle\int_{-\theta}^\infty dt\, \Phi^0(t+\theta)^*\Pi_{01}(\theta)^*R\Pi_{01}(\alpha)\Phi^0(t+\alpha)A_i, & \alpha > \theta \end{cases} \\
&= \Pi_{00}A_i
\end{aligned} \tag{B.8}$$

at the points $\xi = \theta_i, i = 1, \cdots, N-1$. Moreover as $\xi \to -a$,

$$\lim_{\xi \to -a} \Phi^1(t,\xi) = \begin{cases} \Phi^0(t)A_N, & t \geqq 0, \\ 0 & , \text{ otherwise}, \end{cases}$$

and if we let

$$\Pi_{01}(-a) = \lim_{\xi \to -a} \Pi_{01}(\xi) \tag{B.9}$$

we obtain in a similar way

$$\Pi_{01}(-a) = \Pi_{00}A_N. \tag{B.10}$$

Let $h = (0, h^1)$ and $k = (0, k^1)$ in (6.1). Then

$$\begin{aligned}
(\Pi_{11}h^1, k^1) &= \int_0^\infty (Q\Phi^1(t)h^1, \Phi^1(t)k^1)\, dt \\
&\quad + \int_0^\infty dt \left(R\left[\Pi_{00}\Phi^1(t)h^1 + \int_{-a}^0 \Pi_{01}(\alpha)\begin{Bmatrix} \Phi^1(t+\alpha)h^1, & t+\alpha \geqq 0 \\ h^1(t+\alpha) & , \text{ otherwise} \end{Bmatrix} d\alpha\right],\right. \\
&\qquad \left. \Pi_{00}\Phi^1(t)k^1 + \int_{-a}^0 \Pi_{01}(\beta)\begin{Bmatrix} \Phi^1(t+\beta)k^1, & t+\beta \geqq 0 \\ k^1(t+\beta) & , \text{ otherwise} \end{Bmatrix} d\beta\right).
\end{aligned} \tag{B.11}$$

The right-hand side of (B.11) can be rewritten in the following form:

$$\begin{aligned}
&\int_0^\infty ([Q + \Pi_{00}R\Pi_{00}]\Phi^1(t)h^1, \Phi^1(t)k^1)\, dt \\
&\quad + \int_0^\infty dt \int_{-a}^0 d\alpha \left(R\Pi_{01}(\alpha)\begin{Bmatrix} \Phi^1(t+\alpha)h^1, & t+\alpha \geqq 0 \\ h^1(t+\alpha) & , \text{ otherwise} \end{Bmatrix}, \Pi_{00}\Phi^1(t)k^1\right) \\
&\quad + \int_0^\infty dt \int_{-a}^0 d\beta \left(R\Pi_{00}\Phi^1(t)h^1, \Pi_{01}(\beta)\begin{Bmatrix} \Phi^1(t+\beta)k^1, & t+\beta \geqq 0 \\ k^1(t+\beta) & , \text{ otherwise} \end{Bmatrix}\right)
\end{aligned}$$

(*cont.*)

$$+\int_0^\infty dt \int_{-a}^0 d\alpha \int_{-a}^0 d\beta \left(R\Pi_{01}(\alpha) \begin{Bmatrix} \Phi^1(t+\alpha)h^1, & t+\alpha \geqq 0 \\ h^1(t+\alpha) \quad , & \text{otherwise} \end{Bmatrix}, \right.$$
$$\left. \Pi_{01}(\beta) \begin{Bmatrix} \Phi^1(t+\beta)k^1, & t+\beta \geqq 0 \\ h^1(t+\beta) \quad , & \text{otherwise} \end{Bmatrix} \right)$$

and

$$(\Pi_{11}h^1, k^1)$$
$$= \int_0^\infty dt([Q + \Pi_{00}R\Pi_{00}]\Phi^1(t)h^1, \Phi^1(t)k^1)$$
$$+ \int_0^\infty dt \int_{-\min(a,t)}^0 d\alpha (R\Pi_{01}(\alpha)\Phi^1(t+\alpha)h^1, \Pi_{00}\Phi^1(t)k^1)$$
$$+ \int_0^\infty dt \int_{-\min(a,t)}^0 d\beta (R\Pi_{00}\Phi^1(t)h^1, \Pi_{01}(\beta)\Phi^1(t+\beta)k^1)$$
$$+ \int_0^\infty dt \int_{-\min(a,t)}^0 d\alpha \int_{-\min(a,t)}^0 d\beta (R\Pi_{01}(\alpha)\Phi^1(t+\alpha)h^1, \Pi_{01}(\beta)\Phi^1(t+\beta)k^1)$$

$$\text{(B.12)} \quad + \int_0^\infty dt \int_{-a}^{-\min(a,t)} d\alpha (R\Pi_{01}(\alpha)h^1(t+\alpha), \Pi_{00}\Phi^1(t)k^1)$$
$$+ \int_0^\infty dt \int_{-a}^{-\min(a,t)} d\beta (\Pi_{00}\Phi^1(t)h^1, \Pi_{01}(\beta)k^1(t+\beta))$$
$$+ \int_0^\infty dt \int_{-a}^{-\min(a,t)} d\alpha \int_{-\min(a,t)}^0 d\beta (R\Pi_{01}(\alpha)h^1(t+\alpha), \Pi_{01}(\beta)\Phi^1(t+\beta)k^1)$$
$$+ \int_0^\infty dt \int_{-\min(a,t)}^0 d\alpha \int_{-a}^{-\min(a,t)} d\beta (R\Pi_{01}(\alpha)\Phi^1(t+\alpha)h^1, \Pi_{01}(\beta)k^1(t+\beta))$$
$$+ \int_0^\infty dt \int_{-a}^{-\min(a,t)} d\alpha \int_{-a}^{-\min(a,t)} d\beta (R\Pi_{01}(\alpha)h^1(t+\alpha), \Pi_{01}(\beta)k^1(t+\beta)).$$

We number ①, ②, $\cdots$, ⑤ the last five terms in the right-hand side of (B.12).

Since

$$\int_0^\infty dt \int_{-a}^{-\min(a,t)} d\alpha = \int_{-a}^0 d\alpha \int_0^{-\alpha} dt = \int_{-a}^0 d\alpha \int_\alpha^0 d\xi = \int_{-a}^0 d\xi \int_{-a}^\xi d\alpha$$

with the change of variable t to $\xi = t + \alpha$,

$$\text{(B.13)} \qquad ① = \int_{-a}^0 d\xi \int_{-a}^\xi d\alpha (R\Pi_{01}(\alpha)h^1(\xi), \Pi_{00}\Phi^1(\xi - \alpha)k^1).$$

Similarly,

$$\text{(B.14)} \qquad ② = \int_{-a}^0 d\theta \int_{-a}^\theta d\beta (R\Pi_{00}\Phi^1(\theta - \beta)h^1, \Pi_{01}(\beta)k^1(\theta)).$$

Also,

$$\int_0^\infty dt \int_{-a}^{-\min(a,t)} d\alpha \int_{-\min(a,t)}^0 d\beta = \int_0^a dt \int_{-a}^{-t} d\alpha \int_{-t}^0 d\beta = \int_{-a}^0 d\alpha \int_0^{-\alpha} dt \int_{-t}^0 d\beta$$

and the change of variable t to $\xi = t + \alpha$ yields

(B.15)
$$③ = \int_{-a}^{0} d\alpha \int_{\alpha}^{0} d\xi \int_{\alpha-\xi}^{0} d\beta (R\Pi_{01}(\alpha)h^1(\xi), \Pi_{01}(\beta)\Phi^1(\xi - \alpha + \beta)k^1)$$
$$= \int_{-a}^{0} d\xi \int_{-a}^{\xi} d\alpha \int_{\alpha-\xi}^{0} d\beta (R\Pi_{01}(\alpha)h^1(\xi), \Pi_{01}(\beta)\Phi^1(\xi - \alpha + \beta)k^1).$$

Similarly,

$$\text{(B.16)} \quad ④ = \int_{-a}^{0} d\theta \int_{-a}^{\theta} d\beta \int_{\beta-\theta}^{0} d\alpha (R\Pi_{01}(\alpha)\Phi^1(\theta - \beta + \alpha)h^1, \Pi_{01}(\beta)k^1(\theta)).$$

Finally,

$$\int_{0}^{\infty} dt \int_{-a}^{-\min(a,t)} d\alpha \int_{-a}^{-\min(a,t)} d\beta = \int_{-a}^{0} d\alpha \int_{\alpha}^{0} d\xi \int_{-a}^{\alpha-\xi} d\beta$$

with the change of the variable t to $\xi = t + \alpha$ and

$$⑤ = \int_{-a}^{0} d\alpha \int_{\alpha}^{0} d\xi \int_{-a}^{\alpha-\xi} d\beta (R\Pi_{01}(\alpha)h^1(\xi), \Pi_{01}(\beta)k^1(\xi - \alpha + \beta)).$$

We change the variable β to $\theta = \xi - \alpha + \beta$,

$$⑤ = \int_{-a}^{0} d\alpha \int_{\alpha}^{0} d\xi \int_{\xi-\alpha-a}^{0} d\theta (R\Pi_{01}(\alpha)h^1(\xi), \Pi_{01}(\alpha - \xi + \theta)k^1(\theta)),$$

and change the order of integration. But

$$\int_{-a}^{0} d\alpha \int_{\alpha}^{0} d\xi \int_{\zeta-\alpha-a}^{0} d\theta = \int_{-a}^{0} d\xi \int_{-a}^{\zeta} d\alpha \int_{\zeta-\alpha-a}^{0} d\theta$$
$$= \int_{-a}^{0} d\xi \int_{-a}^{0} d\theta \begin{Bmatrix} \displaystyle\int_{-a}^{\xi} d\alpha, & \xi \leqq \theta \\ \displaystyle\int_{\xi-\theta-a}^{\xi} d\alpha, & \xi > \theta \end{Bmatrix},$$

and by changing once more the variable α to $\alpha - \xi$ we finally obtain

(B.17)
$$⑤ = \int_{-a}^{0} d\xi \int_{-a}^{0} d\theta \begin{Bmatrix} \displaystyle\int_{-a-\xi}^{0} d\alpha (R\Pi_{01}(\alpha + \xi)h^1(\xi), \Pi_{01}(\alpha + \theta)k^1(\theta)), & \xi \leqq \theta \\ \displaystyle\int_{-a-\theta}^{0} d\alpha (R\Pi_{01}(\alpha + \xi)h^1(\xi), \Pi_{01}(\alpha + \theta)k^1(\theta)), & \xi > 0 \end{Bmatrix}.$$

By analogy with equations (B.3) and (B.4) and with the help of equations (B.13) to (B.17), identity (B.12) yields

$$
\begin{aligned}
\Pi_{11}(\xi,\theta) \\
&= \int_0^\infty \Phi^1(t,\xi)^*[Q+\Pi_{00}R\Pi_{00}]\Phi^1(t,\theta)\,dt \\
&+ \int_{-a}^0 d\alpha \int_{-\alpha}^\infty dt\,\Phi^1(t+\alpha,\xi)^*\Pi_{01}(\alpha)^*R\Pi_{00}\Phi^1(t,\theta) \\
&+ \int_{-a}^0 d\beta \int_{-\beta}^\infty dt\,\Phi^1(t,\xi)^*\Pi_{00}R\Pi_{01}(\beta)\Phi^1(t+\beta,\theta) \\
&+ \int_{-a}^0 d\alpha \int_{-a}^0 d\beta \begin{Bmatrix} \int_{-\alpha}^\infty dt\,\Phi^1(t+\alpha,\xi)^*\Pi_{01}(\alpha)^*R\Pi_{01}(\beta)\Phi^1(t+\beta,\theta), & \alpha \leqq \beta \\ \int_{-\beta}^\infty dt\,\Phi^1(t+\alpha,\xi)^*\Pi_{01}(\alpha)^*R\Pi_{01}(\beta)\Phi^1(t+\beta,\theta), & \alpha > \beta \end{Bmatrix} \\
&+ \int_{-a}^\xi d\alpha\,\Pi_{01}(\alpha)^*R\Pi_{00}\Phi^1(\xi-\alpha,\theta) + \int_{-a}^\theta d\beta\,\Phi^1(\theta-\beta,\xi)^*\Pi_{00}R\Pi_{01}(\beta) \\
&+ \int_{-a}^\xi d\alpha \int_{\alpha-\xi}^0 d\beta\,\Pi_{01}(\alpha)^*R\Pi_{01}(\beta)\Phi^1(\xi-\alpha+\beta,\theta) \\
&+ \int_{-a}^\theta d\beta \int_{\beta-\theta}^0 d\alpha\,\Phi^1(\theta-\beta+\alpha,\xi)^*\Pi_{01}(\alpha)^*R\Pi_{01}(\beta) \\
&+ \begin{Bmatrix} \int_{-a-\xi}^0 d\alpha\,\Pi_{01}(\alpha+\xi)^*R\Pi_{01}(\alpha+\theta), & \xi \leqq \theta \\ \int_{-a-\theta}^0 d\alpha\,\Pi_{01}(\alpha+\xi)^*R\Pi_{01}(\alpha+\theta), & \xi < \theta \end{Bmatrix}.
\end{aligned} \tag{B.18}
$$

In the light of identity (B.7), $\Phi^1(t,\xi)$ has discontinuities of height $\Phi^0(t)A_i$ at $\xi = \theta_i$, $i = 1, \cdots, N-1$, and

$$\lim_{\xi \to -a} \Phi^1(t,\xi) = \Phi^0(t)A_N. \tag{B.19}$$

Fix θ and consider the map $\xi \mapsto \Pi_{11}(\xi,\theta)$. Since everywhere $\Phi^1(t,\xi)$ is integrated with respect to t, discontinuities can only occur at $\xi = \theta_i$, $i = 1, \cdots, N-1$. At $\xi = \theta_i$, $\Pi_{11}(\xi,\theta)$ has a jump of height

$$
\begin{aligned}
\Pi_{11}(\xi,\theta) \\
&= \int_0^\infty A_i^*\Phi^0(t)^*[Q+\Pi_{00}R\Pi_{00}]\Phi^1(t,\theta)\,dt \\
&+ \int_{-a}^0 d\alpha \int_{-\alpha}^\infty dt\,A_i^*\Phi^0(t+\alpha)^*\Pi_{01}(\alpha)^*R\Pi_{00}\Phi^1(t,\theta) \\
&+ \int_{-a}^0 d\beta \int_{-\beta}^\infty dt\,A_i^*\Phi^0(t)^*\Pi_{00}R\Pi_{01}(\beta)\Phi^1(t+\beta,\theta)
\end{aligned} \tag{B.20}
$$

(*cont.*)

$$+\int_{-a}^{0} d\alpha \int_{-a}^{0} d\beta \begin{Bmatrix} \int_{-\alpha}^{\infty} dt\, A_i^*\Phi^0(t+\alpha)^*\Pi_{01}(\alpha)^*R\Pi_{01}(\beta)\Phi^1(t+\beta,\theta), & \alpha \leqq \beta \\ \int_{-\beta}^{\infty} dt\, A_i^*\Phi^0(t+\alpha)^*\Pi_{01}(\alpha)^*R\Pi_{01}(\beta)\Phi^1(t+\beta,\theta), & \alpha > \beta \end{Bmatrix}$$

$$+\int_{-a}^{\theta} d\beta \int_{\beta-\theta}^{0} d\alpha\, A_i^*\Phi^0(\theta-\beta+\alpha)^*\Pi_{01}(\alpha)^*R\Pi_{01}(\beta)$$

$$+\int_{-a}^{\theta} d\beta\, A_i^*\Phi^0(\theta-\beta)^*\Pi_{00}R\Pi_{01}(\beta) = A_i^*\Pi_{01}(\theta) = A_i^*\Pi_{10}(\theta)^*.$$

By symmetry for each ξ the map $\theta \mapsto \Pi_{11}(\xi, \theta)$ has jump discontinuities of height $\Pi_{10}(\xi)A_j$ at $\theta = \theta_j$, $j = 1, \cdots, N-1$. As for the boundary conditions we fix θ and evaluate

$$\lim_{\xi \to -a} \Pi_{11}(\xi, \theta) = \Pi_{11}(-a, \theta)$$

using (B.19). This yields

$$\text{(B.21)} \qquad \Pi_{11}(-a,\theta) = A_N^*\Pi_{01}(\theta) = A_N^*\Pi_{10}(\theta)^*,$$

and by symmetry

$$\text{(B.22)} \qquad \Pi_{11}(\xi, -a) = \Pi_{11}(-a, \xi)^* = \Pi_{01}(\xi)A_N.$$

(ii) Now that we know where the jumps are we can derive equations (6.3), (6.6) and (6.10). Our starting point is the Riccati equation

$$\text{(B.23)} \qquad 0 = (\tilde{A}h, \Pi k) + (\Pi k, \tilde{A}k) - (\tilde{R}\Pi h, \Pi k) + (\tilde{Q}h, k),$$

or in expanded form

$$\begin{aligned}
&\left(A_{00}h(0) + \sum_{i=1}^{N} A_i h(\theta_i) + \int_{-a}^{0} A_{01}(\alpha)h(\alpha)\,d\alpha, \Pi_{00}k(0) + \int_{-a}^{0} \Pi_{01}(\theta)k(\theta)\,d\theta\right) \\
&\quad + \int_{-a}^{0} \left(\frac{dh}{d\alpha}(\alpha), \Pi_{10}(\alpha)k(0) + \int_{-a}^{0} \Pi_{11}(\alpha,\theta)k(\theta)\,d\theta\right) d\alpha \\
&\quad + \left(\Pi_{00}h(0) + \int_{-a}^{0} \Pi_{01}(\alpha)h(\alpha)\,d\alpha, A_{00}k(0) + \sum_{i=1}^{N} A_i k(\theta_i) + \int_{-a}^{0} A_{01}(\theta)k(\theta)\,d\theta\right) \\
\text{(B.24)} &\quad + \int_{-a}^{0} \left(\Pi_{10}(\theta)h(0) + \int_{-a}^{0} \Pi_{11}(\theta,\alpha)h(\alpha)\,d\alpha, \frac{dk}{d\theta}(\theta)\right) d\theta \\
&\quad - \left(\Pi_{00}h(0) + \int_{-a}^{0} \Pi_{01}(\alpha)h(\alpha)\,d\alpha, R\left[\Pi_{00}k(0) + \int_{-a}^{0} \Pi_{01}(\theta)k(\theta)\,d\theta\right]\right) \\
&\quad + (Qh(0), k(0)).
\end{aligned}$$

Let

$$\text{(B.25)} \qquad h_n(\theta) = \begin{Bmatrix} h^0\left(1 + n\dfrac{\theta}{a}\right), & -\dfrac{a}{n} \leqq \theta < 0 \\ 0 \qquad\qquad , & \text{otherwise} \end{Bmatrix},$$

where n is chosen in such a way that $n > a\theta_1^{-1}$. Then

$$h_n(0) \to h^0 \quad \text{and} \quad h_n \to 0 \text{ in } L^2(-a, 0; X).$$

Let k_i be chosen in $W^{1,2}(-a, 0; X)$ in such a way that

$$\operatorname{supp} k_i \subset (\theta_i, \theta_{i-1}) \cup (\theta_1, 0].$$

Let $h = h_n$ and $k = k_i$ in (B.24):

$$\begin{aligned}
&\left(A_{00}h^0 + \int_{\theta_1}^0 A_{01}(\theta)h_n(\theta)\,d\theta, \Pi_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}\Pi_{01}(\theta)k_i(\theta)\,d\theta\right) \\
&\quad + \int_{\theta_1}^0 \left(\frac{dh_n}{d\alpha}(\alpha), \Pi_{01}(\alpha)k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}\Pi_{11}(\alpha,\theta)k_i(\theta)\,d\theta\right) d\alpha \\
&\quad + \left(\Pi_{00}h^0 + \int_{\theta_1}^0 \Pi_{01}(\alpha)h_n(\alpha)\,d\alpha, A_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}A_{01}(\theta)k_i(\theta)\,d\theta\right) \\
\text{(B.26)}\quad &\quad + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}\left(\Pi_{10}(\theta)h^0 + \int_{\theta_1}^0 \Pi_{11}(\theta,\alpha)h_n(\alpha)\,d\alpha, \frac{dk_i}{d\theta}(\theta)\right) d\theta - \left(\Pi_{00}h^0\right. \\
&\quad \left. + \int_{\theta_1}^0 \Pi_{01}(\alpha)h_n(\alpha)\,d\alpha, R\left[\Pi_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}\Pi_{01}(\theta)k_i(\theta)\,d\theta\right]\right) \\
&\quad + (Qh^0, k_i(0)) = 0.
\end{aligned}$$

Since $\alpha \mapsto \Pi_{01}(\alpha)$, $\alpha \mapsto \Pi_{11}(\alpha, \theta)$ and $\theta \mapsto \Pi_{11}(\alpha, \theta)$ are absolutely continuous in (θ_i, θ_{i-1}) and $(\theta_1, 0)$ we can now integrate by parts.

Equation (B.26) now reduces to

$$\begin{aligned}
&\left(A_{00}h^0 + \int_{\theta_1}^0 A_{01}(\theta)h_n(\theta)\,d\theta, \Pi_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}\Pi_{01}(\theta)k_i(\theta)\,d\theta\right) \\
&\quad + \left(h^0, \Pi_{10}(0)k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}\Pi_{11}(0,\theta)k_i(\theta)\,d\theta\right) \\
&\quad - \int_{\theta_1}^0 \left(h_n(\alpha), \Pi_{10}(\alpha)k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}\Pi_{11}(\alpha,\theta)k_i(\theta)\,d\theta\right) \\
&\quad + \left(\Pi_{00}h^0 + \int_{\theta_1}^0 \Pi_{01}(\alpha)h_n(\alpha)\,d\alpha, A_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}A_{01}(\theta)k_i(\theta)\,d\theta\right) \\
\text{(B.27)}\quad &\quad + \left(\Pi_{10}(0)h^0 + \int_{\theta_1}^0 \Pi_{11}(\theta,\alpha)h_n(\alpha)\,d\alpha, k_i(\theta)\right) \\
&\quad - \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^0\right\}\left(\frac{d\Pi_{01}}{d\theta}(\theta)h^0 + \int_{\theta_1}^0 \frac{\partial \Pi_{11}}{\partial\theta}(\theta,\alpha)h_n(\alpha), k_i(\theta)\right) d\theta - \left(\Pi_{00}h^0\right.
\end{aligned}$$

(*cont.*)

$$+\int_{\theta_1}^{0} \Pi_{01}(\alpha)h_n(\alpha)\,d\alpha,\quad R\left[\Pi_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\}\Pi_{01}(\theta)k_i(\theta)\,d\theta\right]\Bigg)$$

$$+(Qh^0, k_i(0)) = 0.$$

Notice that

$$\int_{-a}^{0} |A_{01}(\theta)h_n(\theta)|\,d\theta \leqq \left[\int_{-a}^{0} |A_{01}(\theta)|^2\,d\theta\right]^{1/2}\left[\int_{-a}^{0} |h_n(\theta)|^2\,d\theta\right]^{1/2}$$

and

$$\lim_{n\to\infty} |h_n|_{L^2(-a,0;X)} = 0$$

imply that

$$\int_{-a}^{0} |A_{01}(\theta)h_n(\theta)|\,d\theta \to 0 \quad \text{as } n \to \infty.$$

Similarly given an f in $L^2(-a, 0; X)$,

$$\left|\int_{-a}^{0} (h_n(\theta), f(\theta))\,d\theta\right| \leqq |h_n|_{L^2}|f|_{L^2}$$

and

$$\lim_{n\to\infty}\int_{-a}^{0} (h_n(\theta), f(\theta))\,d\theta \to 0 \quad \text{as } n \to \infty.$$

As a result equation (B.27) yields

$$\begin{aligned}
&\left(A_{00}h^0, \Pi_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\}\Pi_{01}(\theta)k_i(\theta)\,d\theta\right)\\
&\quad + \left(h^0, \Pi_{10}(0)k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\}\Pi_{11}(0,\theta)k_i(\theta)\,d\theta\right)\\
&\quad + \left(\Pi_{00}h^0, A_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\}A_{01}(\theta)k_i(\theta)\,d\theta\right)\\
\text{(B.28)}\quad&\quad + \left(\Pi_{10}(0)h^0, k_i(0)\right) - \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\}\left(\frac{d\Pi_{01}}{d\theta}(\theta)h^0, k_i(\theta)\right)d\theta\\
&\quad - \left(\Pi_{00}h^0, R\left[\Pi_{00}k_i(0) + \left\{\int_{\theta_i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right\}\Pi_{01}(\theta)k_i(\theta)\,d\theta\right]\right)\\
&\quad + (Qh^0, k_i(0)) = 0.
\end{aligned}$$

To obtain equation (6.3) we use

$$k_i(\theta) = k_m(\theta) = \begin{cases} k^0\left(1 + m\dfrac{\theta}{a}\right), & -\dfrac{a}{m} \leqq \theta \leqq 0 \\ 0 \quad , & \text{otherwise} \end{cases}$$

where m is chosen in such a way that $m > a\theta_1^{-1}$. When we take the limit of equation (B.28) as m goes to infinity we obtain

(B.29) $$([\Pi_{00}A_{00} + \Pi_{10}(0)^* + A_{00}^*\Pi_{00} + \Pi_{10}(0) - \Pi_{00}R\Pi_{00} + Q]h^0, k^0) = 0$$

for all h^0 and k^0 in X.

To obtain equation (6.6) in the open interval (θ_i, θ_{i-1}) we choose k_i such that

$$\operatorname{supp} k_i \subset (\theta_i, \theta_{i-1}).$$

The equation (B.28) yields

$$0 = \int_{\theta_i}^{\theta_{i-1}} \left(\left[\Pi_{01}(\theta)^*A_{00} + \Pi_{11}(0,\theta)^* + A_{01}(\theta)^*\Pi_{00} - \left(\frac{d\Pi_{01}}{d\theta}(\theta)\right)^* - \Pi_{01}(\theta)^*R\Pi_{00}\right]h^0, k_j(\theta)\right) d\theta.$$

By density of the set of absolutely continuous maps with support in (θ_i, θ_{i-1}) in $L^2(\theta_i, \theta_{i-1}; X)$ and the properties

(B.30) $$\Pi_{10}(\theta) = \Pi_{01}(\theta)^*, \quad \Pi_{11}(\alpha, \theta)^* = \Pi_{11}(\theta, \alpha),$$

the above equation yields for h^0 in X,

$$\left[-\frac{d\Pi_{10}}{d\theta}(\theta) + \Pi_{10}(\theta)[A_{00} - R\Pi_{00}] + A_{01}(\theta)^*\Pi_{00} + \Pi_{11}(\theta, 0)\right]h^0 = 0,$$

a.e. in (θ_i, θ_{i-1}).

To obtain (6.10) in the region

$$\{(\alpha, \theta) \in [-a, 0] \times [-a, 0] \,|\, \alpha \in (\theta_i, \theta_{i-1}), \theta \in (\theta_j, \theta_{j-1})\},$$

we choose

$$h = h_i, \quad \operatorname{supp} h_i \subset (\theta_i, \theta_{i-1}),$$
$$k = k_j, \quad \operatorname{supp} k_j \subset (\theta_j, \theta_{j-1})$$

and substitute in (B.24) which reduces to the following expression:

(B.31)
$$\begin{aligned}
&\left(\int_{\theta_i}^{\theta_{i-1}} A_{01}(\alpha)h_i(\alpha)\,d\alpha, \int_{\theta_j}^{\theta_{j-1}} \Pi_{01}(\theta)k_j(\theta)\,d\theta\right)\\
&+ \int_{\theta_i}^{\theta_{i-1}} \left(\frac{dh_i}{d\alpha}(\alpha), \int_{\theta_j}^{\theta_{j-1}} \Pi_{11}(\alpha, \theta)k_j(\theta)\,d\theta\right) d\alpha\\
&+ \left(\int_{\theta_i}^{\theta_{i-1}} \Pi_{01}(\alpha)h_i(\alpha)\,d\alpha, \int_{\theta_j}^{\theta_{j-1}} A_{01}(\theta)k_j(\theta)\,d\theta\right)\\
&+ \int_{\theta_j}^{\theta_{j-1}} \left(\int_{\theta_i}^{\theta_{i-1}} \Pi_{11}(\theta, \alpha)h_i(\alpha)\,d\alpha, \frac{dk_j}{d\theta}(\theta)\right) d\theta\\
&- \left(\int_{\theta_i}^{\theta_{i-1}} \Pi_{01}(\alpha)h_i(\alpha)\,d\alpha, R\int_{\theta_j}^{\theta_{j-1}} \Pi_{01}(\theta)k_j(\theta)\,d\theta\right).
\end{aligned}$$

The two terms with a derivative can be integrated by parts:

$$\int_{\theta_i}^{\theta_{i-1}} \left(\frac{dh_i}{d\alpha}(\alpha), \int_{\theta_j}^{\theta_{j-1}} \Pi_{11}(\alpha,\theta)k_j(\theta)\,d\theta\right) d\alpha$$

$$= -\int_{\theta_i}^{\theta_{i-1}} \left(h_i(\alpha), \int_{\theta_j}^{\theta_{j-1}} \frac{\partial \Pi_{11}}{\partial \alpha}(\alpha,\theta)k_j(\theta)\,d\theta\right) d\alpha$$

and

$$\int_{\theta_j}^{\theta_{j-1}} \left(\int_{\theta_i}^{\theta_{i-1}} \Pi_{11}(\theta,\alpha)h_i(\alpha)\,d\alpha, \frac{dk_j}{d\theta}(\theta)\right) d\theta$$

$$= -\int_{\theta_j}^{\theta_{j-1}} \int_{\theta_i}^{\theta_{i-1}} \left(\frac{\partial \Pi_{11}}{\partial \theta}(\theta,\alpha)h_i(\alpha)\,d\alpha, k_j(\theta)\right) d\theta.$$

Finally equation (B.31) takes the form

$$\int_{\theta_i}^{\theta_{i-1}} d\alpha \int_{\theta_j}^{\theta_{j-1}} d\theta \Bigg(\Bigg[\Pi_{01}(\theta)^* A_{01}(\alpha) - \left(\frac{\partial \Pi_{11}}{\partial \alpha}(\alpha,\theta)\right)^* + A_{01}(\theta)^*\Pi_{01}(\alpha)$$

$$- \frac{\partial \Pi_{11}}{\partial \theta}(\theta,\alpha) - \Pi_{01}(\theta)^* R \Pi_{01}(\alpha)\Bigg] h_i(\alpha), k_i(\theta)\Bigg).$$

By using relations (B.30) and the density argument we obtain

$$\frac{\partial \Pi_{11}}{\partial \theta}(\theta,\alpha) + \frac{\partial \Pi_{11}}{\partial \alpha}(\theta,\alpha) = \Pi_{10}(\theta)A_{01}(\alpha) + A_{01}(\theta)^*\Pi_{10}(\alpha)^* - \Pi_{10}(\theta)R\Pi_{10}(\alpha)^*$$

for almost all (α, θ) in $(\theta_i, \theta_{i-1}) \times (\theta_j, \theta_{j-1})$.

(iii) We now solve equation (6.10) with boundary conditions (6.11). We let $\eta = \alpha - \beta$ and consider two cases. First let $a \geqq \eta \geqq 0$; then

$$-a \leqq \beta \leqq 0 \Rightarrow \eta - a \leqq \alpha \leqq 0.$$

If we change the variable β to $\eta = \alpha - \beta$, equation (6.10) becomes

$$\frac{d}{d\alpha}\Pi_{11}(\alpha,\alpha-\eta) = A_{01}(\alpha)^*\Pi_{10}(\alpha-\eta)^* + \Pi_{10}(\alpha)A_{01}(\alpha-\eta) - \Pi_{10}(\alpha)R\Pi_{10}(\alpha-\eta)$$

$$+ \sum_{i=1}^{N-1} A_i^*\Pi_{10}(\alpha-\eta)^*\delta(\alpha-\theta_i) + \sum_{j=1}^{N-1} \Pi_{10}(\alpha)A_j\delta(\alpha-\eta-\theta_j).$$

This last equation can be integrated from $\eta - a$ to α:

$$\Pi_{11}(\alpha,\alpha-\eta) = \Pi_{11}(\eta-a,-a) + \int_{\eta-\alpha}^{\alpha} A_{01}(\xi)^*\Pi_{10}(\xi-\eta)^*\,d\xi$$

$$+ \int_{\eta-a}^{\alpha} \Pi_{10}(\xi)[A_{01}(\xi-\eta) - R\Pi_{10}(\xi-\eta)^*]\,d\xi$$

$$+ \sum_{i=1}^{N-1} \begin{Bmatrix} A_i^*\Pi_{10}(\theta_i-\eta)^*, & \eta - a \leqq \theta_i < \alpha \\ 0 & , \text{ otherwise} \end{Bmatrix}$$

$$+ \sum_{j=1}^{N-1} \begin{Bmatrix} \Pi_{10}(\eta+\theta_j)A_j, & \eta - a \leqq \eta + \theta_j < \alpha \\ 0 & , \text{ otherwise} \end{Bmatrix}.$$

Finally for $\alpha \geqq \beta$,

$$\Pi_{11}(\alpha,\beta) = \Pi_{10}(\alpha-\beta-a)A_N + \sum_{j=1}^{N-1}\begin{cases}\Pi_{10}(\alpha-\beta+\theta_j)A_j, & \theta_j < \beta \\ 0 & \text{, otherwise}\end{cases}$$
$$+\sum_{i=1}^{N-1}\begin{cases}A_i^*\Pi_{10}(\beta-\alpha+\theta_i)^*, & -a \leqq \beta-\alpha+\theta_i, \theta_i < \alpha \\ 0 & \text{, otherwise}\end{cases}$$
$$+\int_{\alpha-\beta-a}^{\alpha} A_{01}(\xi)^*\Pi_{10}(\xi-\alpha+\beta)^*\,d\xi$$
$$+\int_{-a}^{\beta} \Pi_{10}(\alpha-\beta+\theta)[A_{01}(\theta)-R\Pi_{10}(\theta)^*]\,d\theta$$

and

$$\Pi_{11}(\alpha,\beta) = \Pi_{10}(\alpha-\beta-a)A_N$$
$$+\sum_{j=1}^{N-1}\begin{cases}\Pi_{10}(\alpha-\beta+\theta_j)A_j, & -a \leqq \alpha-\beta+\theta_j, \theta_j < \beta \\ 0 & \text{, otherwise}\end{cases}$$
$$+\sum_{i=1}^{N-1}\begin{cases}A_i^*\Pi_{10}(\beta-\alpha+\theta_i)^*, & -a \leqq \beta-\alpha+\theta_i, \theta_i < \alpha \\ 0 & \text{, otherwise}\end{cases}$$
$$+\int_{-a}^{\alpha}\begin{cases}A_{01}(\xi)^*\Pi_{10}(\xi-\alpha+\beta)^*, & \xi \geqq \alpha-\beta-a \\ 0 & \text{, otherwise}\end{cases}d\xi$$
$$+\int_{-a}^{\beta}\begin{cases}\Pi_{10}(\theta-\beta+\alpha)A_{01}(\theta), & \theta \geqq \beta-\alpha-a \\ 0 & \text{, otherwise}\end{cases}d\theta$$
$$-\begin{cases}\displaystyle\int_{-a}^{\beta}\Pi_{10}(\alpha-\beta+\theta)R\Pi_{10}(\theta)^*\,d\theta, & \alpha \geqq \beta \\ \displaystyle\int_{-a}^{\alpha}\Pi_{10}(\xi)R\Pi_{10}(\beta-\alpha+\xi)^*\,d\xi, & \alpha < \beta\end{cases}.$$

Notice that in the above expression for $B^{11}(\alpha,\beta)$ all terms but the first are symmetrical. Hence for $\alpha \leqq \beta$ we shall obtain the same expression with the exception of the first term which will be equal to

$$A_N^*\Pi_{10}(\beta-\alpha-a)^*.$$

But

$$\lim_{\alpha\leqq\beta,\beta\to\alpha}\Pi_{10}(\alpha-\beta-a)A_N = \Pi_{10}(-a)A_N = A_N^*\Pi_{00}A_N$$

and

$$\lim_{\beta\leqq\alpha,\beta\to\alpha}A_N^*\Pi_{10}(\beta-\alpha-a)^* = A_N^*\Pi_{10}(-a)^* = A_N^*\Pi_{00}A_N$$

imply that this first term is continuous at (α,α), $-a \leqq \alpha < \theta_{N-1}$. This makes it possible to write the first term as follows:

$$\begin{aligned} &\Pi_{10}(\alpha - \beta - a)A_N, && \alpha \geqq \beta, \\ &A_N^*\Pi_{10}(\beta - \alpha - a)^*, && \alpha < \beta. \end{aligned}$$

This yields identity (6.12).

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