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# Duality of Linear Input-Output Maps

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Dedicated to James L. Massey on the occasion of his 60th birthday.

**Abstract**

This paper is concerned with the duality of linear input-output maps and makes precise in what sense the dual of a linear input-output map gives rise to a dual linear state-space system and how reachability and observability of the original system corresponds to observability and reachability of the dual system.

## I Introduction

The objective of this technical note is to close a gap in the module theory of stationary finite-dimensional linear systems as developed by R.E. Kalman. This is concerned with the duality of linear input-output maps and makes precise in what sense the dual of a linear input-output map gives rise to a dual linear state-space system and how reachability and observability of the original system corresponds to observability and reachability of the dual system.

It is appropriate that this note be dedicated to Jim Massey. He more than anybody else has investigated the deep connections that exist between systems theory and coding (decoding) theory. We see ample demonstration of this in his early work on the Berlekamp-Massey algorithm and its connections to Partial Realization Theory, his joint work with Sain on inverses of linear sequential systems and in his recent joint work with his students Loeliger and Mittelholzer on the relationship between the behavioural theory of linear systems and coding theory.

On a more personal note, I have admired and continue to admire Jim's devotion to the science of engineering, his constant search for clarity of thought and exposition and his intellectual integrity. In a scientific and technological world over-crowded with opportunists and charlatans the values that Jim stands for have provided strength and sustenance to his friends and undoubtedly to his students.

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## II Problem Formulation and Preliminaries

We shall follow the notation and terminology of R.E. Kalman (see KALMAN-FALB-ARBIB [2]).

Consider the linear stationary finite-dimensional system:

$$(\Sigma)\left\{\begin{array}{l} x(t+1)=Fx(t)+Gu(t); \quad u(-\infty)=0 \\ y(t)=Hx(t)\end{array}\right.$$

where $U$, $X$ and $Y$ are finite-dimensional topological $K$-vector spaces and $F: X \to X$, $G: U \to X$ and $H: X \to Y$ are $K$-linear continuous maps.

Let

$$U[z]=\{u=u_{-k}z^k+u_{-k+1}z^{k-1}+\cdots+u_0 \mid u_{-i}\in U, i=-k,-k+1,\cdots 0\}$$

and

$$Y[[z^{-1}]]=\{y=y_1z^{-1}+y_2z^{-2}+\cdots \mid y_i\in Y, i=1,2,\cdots\}$$

The input-ouput map corresponding to $\Sigma$ is defined by solving the equation defining $(\Sigma)$ recursively:

$$\begin{aligned} f_\Sigma &: U[z]\to Y[[z^{-1}]] \\ &: u\mapsto f_\Sigma(u) \end{aligned} \tag{2.1}$$

The map $f_\Sigma$ is a Hankel map which is a module homomorphism from the $K[z]$-module $U[z]$ to the $K[z]$-module $Y[[z^{-1}]]$. Now suppose that $U[z]$ and $Y[[z^{-1}]]$ can be made into topological $K$-vector spaces. Let $(U[z])^*$ and $(Y[[z^{-1}]])^*$ be their topological duals.

Define the dual map

$$f_\Sigma^*:(Y[[z^{-1}]])^*\longrightarrow (U[z])^*$$

as follows:

Let $Q: Y[[z^{-1}]]\to K$ be $K$-linear and continuous. Then

$$f_\Sigma^*\circ Q=Q\circ f_\Sigma$$

Now it is not a-priori clear that $f_\Sigma^*$ is an input-output map, that is, maps polynomials into formal power series. We wish to show in this note that appropriate topologies can be put on $Y[[z^{-1}]]$ and $U[z]$ such that there exists $K[z^{-1}]$-module isomorphisms $\varphi$ and $\psi$ such that the following diagram commutes:

$$\begin{tikzcd} (Y[[z^{-1}]])^* \arrow[r, "f_\Sigma^*"] \arrow[d, leftrightarrow, "\varphi"'] & (U[z])^* \arrow[d, leftrightarrow, "\psi"] \\ Y^*[z^{-1}] \arrow[r, "f_{\bar\Sigma}"] & U^*[[z]]. \end{tikzcd}$$

with $f_\Sigma^*$ and $f_{\bar\Sigma}$ being $K[z^{-1}]$-module homomorphisms. In the above $Y^*$ and $U^*$ are dual spaces of $Y$ and $U$ and $f_{\bar\Sigma}$ is the input-output map of the dual system:

$$(\bar\Sigma)\left\{\begin{array}{l} \xi(t-1)=F^*\xi(t)+H^*\eta(t); \xi(+\infty)=0 \\ \chi(t)=G^*\xi(t), \text{ where}\end{array}\right.$$

$F^*$, $H^*$, $G^*$ are the dual linear maps corresponding to $F$, $G$, $H$ defined in the natural way. We shall treat the following two cases:



(i) $K$ is either $R$ or $C$ and $U$ and $Y$ are finite-dimensional $K$-vector spaces with the euclidean topology.

(ii) $K$ is a finite field with the discrete topology and $U$ and $Y$ are finite-dimensional $K$-vector spaces with discrete topologies.

## III Main Results

We first look at Case (i). The ideas that we use are well-known in Functional Analysis (cf. TREVES [3], pp. 227–231). For the sake of completeness we describe this here.

Let $\mathcal{P}$ denote the vector space of all polynomials in one indeterminate with complex coefficients (the real case is similar). Let $\mathcal{P}_k$ denote the vector subspace of polynomials with degree $\leq k, k=0,1,2,\ldots$. Each $\mathcal{P}_k$ is finite-dimensional. We provide $\mathcal{P}$ with the locally convex topology which is the inductive limit of the topologies of the Hausdorff finite-dimensional space (we can choose this to be the euclidian topology) $\mathcal{P}_k, k=0,1,\ldots$ (cf. TREVES [3], p. 130). $\mathcal{P}$ with this topology is a so-called $LF$-space.

On the other hand, let $\mathcal{F}$ denote the vector space of formal power series in one-indeterminate. We put on $\mathcal{F}$ the topology of convergence of each coefficient.

This topology can be defined by the sequence of seminorms:

$$u=\sum_{p\in N}u_pz^p \rightsquigarrow \sup_{p\leq k}|u_p|, k=0,1,2,\ldots$$

This topology converts $\mathcal{F}$ into a Fréchet space (cf. TREVES [3], p. 91).

There is a natural duality between polynomials and formal power series which can be expressed by the bracket

$$< P,u >=\sum_{p\in\mathrm{N}}P_pu_p, \text{ where}$$

$$P=\sum_p P_pz^p \text{ and } u=\sum_p u_pz^p.$$

This is well-defined since all coefficients $P_p$, except possibly a finite number of them are equal to zero. We then have:

**Theorem 3.1** *(TREVES [3], p. 228, Th. 22.1).*

*(a) The map. $u\rightsquigarrow (P\rightsquigarrow< P,u >)$ is an isomorphism for the structures of topological vector spaces of the Fréchet space of formal power series $\mathcal{F}$ onto the strong dual of the $LF$-space $\mathcal{P}$ of polynomials.*

*(b) The map. $P\rightsquigarrow (u\rightsquigarrow< P,u >)$ is an isomorphism of $\mathcal{P}$ onto the strong dual of $\mathcal{F}$.*

**Remark 3.1** *If we disregard the multiplicative structure of $\mathcal{P}$ and $\mathcal{F}$ and we write $u=(u_p)_{p\in\mathrm{N}}$, then $\mathcal{F}$ is the space of complex functions on $\mathcal{N}$ and $\mathcal{P}$ is the space of functions on $\mathcal{N}$ which vanish outside a finite set.*

*Then*

$$\mathcal{F}=\prod_{p\in\mathrm{N}}C_p \qquad C_p\simeq C$$

*and $\mathcal{P}$ can be regarded as a direct sum of the $C_p$'s.*

In fact, the topology of simple convergence of the coefficients of $u \in \mathcal{F}$ is precisely the product topology of the $C_p$'s. Furthermore $\mathcal{P}$ is continuously embedded in $\mathcal{F}$ and is dense in $\mathcal{F}$.

We can apply the above ideas to our problem. Firstly, we can make $Y[[z^{-1}]]$ into a topological vector space by assigning to each $\underline{y} \in Y[[z^{-1}]]$ a basis of a filter.

Let $\underline{y} = \sum_{i=1}^{\infty} y_i z^i$, and define

$$Q_{m,n} = \{y \in Y[[z^{-1}]] | \ \forall\, i \leq n, |y_i| \leq \frac{1}{m}\}, n = 1, 2, \cdots, m = 1, 2, \cdots$$

It is easily checked that $(Q_{m,n})$ is a basis for a filter on $\underline{0}$. Define now $\{\underline{y}+Q_{m,n}\}$ to be a basis of a filter for any $\underline{y} \in Y[[z^{-1}]]$. We can now easily show that this filter assignment defines a topology on $Y[[z^{-1}]]$. We now claim that with this topology $Y[[z^{-1}]]$ is a topological vector space. For this purpose we need to check the following:

(i) $f_{y_0} : Y[[z^{-1}] \to Y[[z^{-1}]] : y \mapsto y_0 + y$ is a homeomorphism, i.e., the topology is translation invariant.

(ii) If $V \in (Q_{m,n})$, $\exists\, U \in (Q_{m,n})$ such that $U + U \subset V$.

(iii) $V \in (Q_{m,n})$ is absorbing.

(iv) $\exists\, \lambda \in \mathbf{C}, 0 < \lambda < 1$, such that $\lambda Q_{m,n} \in (Q_{m,n})$.

The above steps are easily carried out.

We now topologize $U[z]$. Let $U_m[z] = \{u \in U[z] | \ \deg(u) \leq m\}$. Clearly $U_m[z]$ is a finite-dimensional vector space. Endow $U^m[z]$ with the topology given by the norm

$$||u||_m = (|u_0|^2 + |u_{-1}|^2 + \cdots + |u_{-m}|^2)^{1/2}.$$

Now $U[z] = \bigcup_m U_m[z]$. Endow $U[z]$ with the inductive limit of the Hausdorff topology on the finite-dimensional spaces $U_m[z]$. This makes $U[z]$ into a topological vector space.

There is a natural $K[z^{-1}]$-module structure on $Y^*[z^{-1}]$ and $U^*[[z]]$. Define multiplication of elements of $Y^*[z^{-1}]$ by a polynomial as follows:

For

$$a(z^{-1}) = \sum_{i=0}^{n} a_i z^{-i}, a_i \in \mathbf{C}, a_i = 0, i > n, f = \sum_{j=0}^{m} f_j z^{-j}, f_j \in Y^*$$

$$a(z^{-1}) \cdot f = \sum_{\ell=0}^{n+m} g_\ell z^{-\ell} = g$$

where $g_\ell = \sum_{k=0}^{\ell} a_{\ell-k} f_k$.

This multiplication is well-defined and $g \in Y^*[z^{-1}]$. The module axioms are easily checked. As far as $U^*[[z]]$ is concerned, define multiplication of elements of $U^*[[z]]$ by a polynomial as follows:



For $a(z^{-1}) = \sum_{i=0}^{n} a_i z^{-i}$, $a_i \in \mathbf{C}$, and $\underline{f} = \sum_{j=1}^{\infty} f_{-j} z^j$ with $a_i = 0$ for all other $i \in Z$ and $f_j = 0$, for all other $j \in Z$,

$$a(z^{-1}) \cdot \underline{f} = \sum_{\ell=1}^{\infty} g_{-\ell} z^\ell = g$$

where $g_{-\ell} = \sum_{k=0}^{\ell+n} a_{-\ell+k} f_{-k}, \ell \geq 1$ and $g_{-\ell} = 0$, for $\ell < 1$. Again the module axioms are easily verified. Let $(Y[[z^{-1}]])^*$ and $(U[z])^*$ denote the strong duals of $Y[[z^{-1}]]$ and $U[z]$ respectively. For the definition of strong dual, see TREVES [3], p. 198.

Define the pairings

(i)

$$\begin{aligned} <\cdot,\cdot>_1 \ &: \ Y^*[z^{-1}] \times Y[[z^{-1}]] \longrightarrow K \\ &: \ (f, \underline{y}) \longmapsto \sum_{i=o}^{n} f_i y_{i+1}, \end{aligned}$$

where

$$f = \sum_{i=0}^{n} f_i z^{-i} \text{ and } \underline{y} = \sum_{i=1}^{\infty} y_i z^{-i}$$

(ii)

$$\begin{aligned} <\cdot,\cdot>_2 \ &: \ U^*[[z]] \times U[z] \longrightarrow K \\ &: \ (\underline{f}, u) \longmapsto \sum_{i=1}^{\infty} f_{-i} u_{-i+1} \end{aligned}$$

where

$$\underline{f} = \sum_{i=1}^{\infty} f_{-i} z^i, u = \sum_{i=0}^{n} u_{-i} z^i.$$

**Theorem 3.2** *(a) The map*

$$\begin{aligned} \phi \ &: \ Y^*[z^{-1}] \longrightarrow (Y[[z^{-1}])^* \\ &\ \ \ \underline{f} \rightsquigarrow (\underline{y} \rightsquigarrow < f, \underline{y} >_1) \end{aligned}$$

*is a $K[z^{-1}]$-module isomorphism from $Y^*[z^{-1}]$ to the strong dual of $Y[[z^{-1}]]$.*

*The map $\phi$ is an isomorphism for the structures of topological vector spaces on $Y^*[z^{-1}]$ and $Y[[z^{-1}]]^*$*

*(b) The map*

$$\begin{aligned} \psi \ &: \ U^*[[z]] \longrightarrow (U[z])^* \\ &: \ f \longmapsto (u \rightsquigarrow < \underline{f}, u >_2) \end{aligned}$$

*is a $K[z^{-1}]$-module isomorphism. It is also an isomorphism for the structures of topological vector spaces on $U^*[[z]]$ and $(U[z])^*$.*

**Proof.** We make $(Y[[z^{-1}]])^*$ into a $K[z^{-1}]$ module in the following way:

For $f \in Y^*[z^{-1}]$, let $\varphi(f)$ be denoted as $Q_f$ which belongs to $Y[[z^{-1}]]^*$. Define multiplication of $Q_f$ by $a(z^{-1}) \in K[z^{-1}]$ as

$$a(z^{-1}) \circ Q_f = Q_g \text{ where } g = a(z^{-1}) \circ f$$

Now we can easily check that $\varphi$ is $K$-linear and the module axioms are satisfied. Using the $K$-linearity and the above definition of multiplication it is easily checked that $\varphi$ is $K[z^{-1}]$-linear.

In a similar way we make $(U[z])^*$ into a $K[z^{-1}]$-module. The proof is now carried out by proving $\varphi$ and $\psi$ are continuous, injective, and surjective. This follows the proof of Theorem 3.1 as given by Treves.

As far as $\varphi$ and $\psi$ being isomorphisms of the topological vector space structures, the proof is carried by proving that $\varphi^{-1}$ and $\psi^{-1}$ are continuous as in Treves' proof of Theorem 3.1.

∎

**Theorem 3.3** *There exists module isomorphism $\varphi$ and $\psi$ as in the previous theorem such that the following diagram commutes:*

$$\begin{array}{ccc} (Y[[z^{-1}]])^* & \xrightarrow{f_\Sigma^*} & (U[z])^* \\ \uparrow \varphi & & \uparrow \psi \\ Y^*[z^{-1}] & \xrightarrow{f_\Sigma} & U^*[[z]] \end{array}$$

*where $f_\Sigma^*$ is defined as, $f_\Sigma^* \circ Q = Q \circ f_\Sigma$ and $Q : Y[[z^{-1}]] \to K$ is $K$-linear, $f_\Sigma^*$ is $K[z^{-1}]$-linear, and*

$$f_{\bar{\Sigma}} : f_n z^{-n} + f_{n-1} z^{-n+1} + \cdots + f_0 \longmapsto g_{-1} z^1 + g_{-2} z^2 + \cdots.$$

*with $f_i \in Y^*$ and $g_{-i} = G^* F^{*i-1} H^* f_0 + \cdots + G^* F^{*n+i-1} H^* f_n$.*

**Proof.** The proof is constructed by showing that

$$f_\Sigma^*[\varphi(f)](u) = \psi[f_{\bar{\Sigma}}(f)](u), u \in U[z] \tag{3.1}$$

where $f = f_n z^n + f_{n-1} z^{-n+1} + \cdots + f_0, f_i \in Y^*, i = 0, \cdots, n$.

Now $f_\Sigma^*[\varphi(f)](u) = \phi(f)[f_\Sigma(u)]$.

By solving the recurrence relation corresponding to $(\Sigma)$, we get

$$\begin{aligned} \varphi(f)[f_\Sigma(u)] &= f_0(HGu_0) + f_1(HFGu_0) + \cdots + f_n(HF^nGu_0) \\ &\quad + f_0(HFGu_{-1}) + f_1(HF^2Gu_{-1}) + \cdots + f_n(HF^{n+1}Gu_{-1}) \\ &\quad + \cdots \\ &\quad + f_0(HF^KGu_{-k}) + f_1(HF^{k+1}Gu_{-k}) + \cdots + f_n(HF^{n+k}Gu_{-k}) \\ &= (G^*H^*f_0)(u_0) + (G^*F^*H^*f_1)(u_0) + \cdots + (G^*F^{*n}Hf_n)(u_0) \\ &\quad + (G^*F^*H^*f_0)(u_{-1}) + (G^*F^{*2}H^*f_1)(u_{-1}) \\ &\quad + \cdots + (G^*F^{*n+1}H^*f_n)(u_1) + \ldots \\ &\quad + (G^*F^{k*}H^*f_0)(u_{-k}) + (G^*F^{k+1^*}H^*f_1)(u_{-1}) + \cdots + \\ &\quad \cdots (G^*F^{*n+k}H^*f_n)(u_{-k}) \end{aligned} \tag{3.2}$$



On the other hand if,

$$\begin{aligned} \underline{g} &= g_{-1}z^1 + g_{-2}z^2 + \cdots, \quad g_{-i} \in U^* \\ \psi(\underline{g})(u) &= \sum_{i=1}^{\infty} g_{-i}(u_{-i+1}) \end{aligned} \tag{3.3}$$

By solving the recurrence relation corresponding to $(\bar{\Sigma})$, we get for $\underline{g} = f_{\bar{\Sigma}}(f)$, that

$$\begin{aligned} &g_{-1} = G^*H^*f_0 + \cdots + (G^*F^{*n}Hf_n) \\ &\vdots \\ &g_{-i} = G^*F^{*i-1}H^*f_0 + \cdots + (G^*F^{*n+i-1}H^*f_n) \\ &\vdots \end{aligned}$$

Therefore $\phi[f_{\bar{\Sigma}}(f)](u)$ is precisely the right side of (3.1). ∎

<u>The Case where $K$ is a Finite Field.</u> Let $K$ be a finite field with the discrete topology and $U$ and $Y$ finite-dimensional $K$-vector spaces with the discrete topology. Let the topologies on $K$ and $U, Y$ be generated by a norm $|\cdot|_K, |\cdot|_U, |\cdot|_Y$ where $|v| = 0 \Leftrightarrow v = 0$ and $|v| = 1$ if and only if $v \neq 0$. Here $v \in K$, $U$ or $Y$.

Let us define

$$M^n = \{\underline{y} \in Y[[z^{-1}]] | y_1 = y_2 = \cdots y_{n-1} = 0\}, n = 1, 2, \ldots 3$$

The family of sets $(M^n)$ is the same as the family of set $Q_{m,n}$ defined earlier.

In the same way as before $(\underline{y} + M^n)$ is the assignment of a filter based on $\underline{y}$ and generates a topology. We can now check that addition and multiplication are continuous and hence $Y[[z^{-1}]]$ becomes a topological $K$-vector space.

On $U[z]$ we put the discrete topology. With these choices, we can proceed in the same way as before and prove Theorem 3.3 in the case where $K$ is a finite field.

## IV Final Remarks

We may proceed using realization theory instead of starting with an explicit state-space realization. Thus given an input-output map $f_\Sigma : U[z] \to Y[[z^{-1}]]$ obtain a minimal (reachable and observable) realization via the canonical factorization

$$\begin{array}{ccc} U[z] & \xrightarrow{f_\Sigma} & Y[[z^{-1}]] \\ \mathcal{R} \searrow & & \nearrow 0 \\ & U[z]/\ker f_\Sigma! = X & \end{array}$$

where the reachability operator $\mathcal{R}$ and the observability operator 0 are defined by

$$\begin{aligned} \mathcal{R} &: U[z] \to X : u \mapsto [u] \quad ([\cdot] \text{ denotes equivalence}) \\ O &: X \to Y[[z^{-1}]] : [u] \mapsto f_\Sigma(u). \end{aligned}$$

Let $F : X \to X, G : U \to X$ and $H : X \to Y$ be $K$-linear maps defining the corresponding minimal state space realization. Now define

$$f_{\bar{\Sigma}} : Y^*[z^{-1}] \longrightarrow U[[z]] \text{ by}$$
$$f_{\Sigma}^* \circ \varphi = \phi \circ f_{\bar{\Sigma}}.$$

Then as in Theorem 3.3, we can check that the state-space system defined by

$$F^* : X^* \to X^*$$
$$H^* : Y^* \to X^*$$
$$G^* : X^* \to U^*$$

realizes the map $f_{\bar{\Sigma}}$ (note the time-reversal). We can explicitly compute the reachability operator $\bar{\mathcal{R}}$ and the obvervability operator $\bar{\mathcal{O}}$ corresponding to $f_{\bar{\Sigma}}$. We have that $(\Sigma)$ is reachable if and only if $(\bar{\Sigma})$ is observable and $(\Sigma)$ is observable if and only if $(\bar{\Sigma})$ is reachable.

These ideas have relevance towards the development of a theory of duality for linear systems defined in a behavioural framework (cf. WILLEMS [4]). To make connection with the input-output setting we need to work with controllable systems in the sense of Willems. These ideas also have relevance on the duality theory of Abelian Group Codes and Systems. See FORNEY-TROTT[1].

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# Cut-off Rate Channel Design

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### Abstract

The eloquence with which Massey advocated the use of the cut-off rate parameter for the coordinated design of modulation and coding in communication systems caused many to redirect their thinking about how communication systems should be designed. Underlying his recommendation is the view that modulation and demodulation should be designed to realize a good discrete channel for encoding and decoding, rather than the prevailing view at the time, and still the view of many, that bit error-probability should be optimized. In this short paper, some of the research influenced by Massey's insightful suggestions on this subject is reviewed.

## I Introduction

The use of the cut-off rate parameter $R_0$ in the study of single-user coded communication systems was first advocated by Wozencraft and Kennedy [11] in 1966. Unfortunately, their proposal to use $R_0$ as a criterion for the design of the modulation system remained largely unheeded. Then, in a remarkable paper in 1974, Massey [4] gave an eloquent argument in favor of the cut-off rate parameter as a criterion for the coordinated design of modulation and coding in a communication system. Calling to discard the heretofore popular "error probability" criterion on the grounds that it was apposite only for uncoded systems, he resurrected the earlier proposal of Wozencraft and Kennedy and showed that the $R_0$ criterion led to a rich "communication theory" of its own for coded communications. In particular, by interpreting $R_0$ as a function of the modulator and demodulator, Massey [4] demonstrated how it could be used to design the best discrete channel as seen by the encoder and decoder. He crowned his arguments by establishing that a simplex signal set maximized the cut-off rate of an infinite-bandwidth, additive, white Gaussian-noise channel for infinitely soft decisions. The "optimality" of such a signal set with respect to the error probability criterion has remained a conjecture for many years.

Massey's paper [4] opened the floodgates for a plethora of publications employing the cut-off rate criterion to assess the performance of coding and modulation schemes for a variety of applications ranging from optical communications to spread-spectrum systems to an extension to multiaccess channels. A fair citation of this field is beyond the scope of this paper; good sources of relevant publications are the *IEEE Transactions on Information Theory and Communications*.