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The Solution of Linear Probabilistic Recurrence Relations[1]

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Abstract. Linear probabilistic divide-and-conquer recurrence relations arise when analyzing the running time of divide-and-conquer randomized algorithms. We consider first the problem of finding the expected value of the random process $T(x)$, described as the output of a randomized recursive algorithm T. On input x, T generates a sample $(h_1, \ldots, h_k)$ from a given probability distribution on $[0, 1]^k$ and recurses by returning $g(x) + \sum_{i=1}^{k} c_i T(h_i x)$ until a constant is returned when x becomes less than a given number. Under some minor assumptions on the problem parameters, we present a closed-form asymptotic solution of the expected value of $T(x)$. We show that $E[T(x)] = \Theta(x^p + x^p \int_1^x (g(u)/u^{p+1})\,du)$, where p is the nonnegative unique solution of the equation $\sum_{i=1}^{k} c_i E[h_i^p] = 1$. This generalizes the result in [1] where we considered the deterministic version of the recurrence. Then, following [2], we argue that the solution holds under a broad class of perturbations including floors and ceilings that usually accompany the recurrences that arise when analyzing randomized divide-and-conquer algorithms.

Key Words. Randomized algorithms, Divide-and-conquer algorithms, Recurrence relations.

1. Introduction.

Linear divide-and-conquer recurrence relations arise when analyzing the running time of divide-and-conquer algorithms. In a deterministic setting, such recurrences are of the form

$$\begin{cases} T(x) = g(x) + \displaystyle\sum_{i=1}^{k} c_i T(h_i x) & \text{for} \quad x < x_0, \\ T(x) = c & \text{for} \quad 0 \le x \le x_0, \end{cases}$$

where $h_1, \ldots, h_k$ are constants in the interval (0, 1). We provided in [1] an asymptotic closed-form solution for this recurrence, and we argued that the result holds in the discrete setting. A simple inductive proof of the solution in [1] was presented in [2], and the result was extended to handle commonly occurring variations of the recurrence. In this work we first generalize the result in [1] to a probabilistic setting where $h_1, \ldots, h_k$ are random variables taking values from the interval (0, 1). More precisely, we consider first the problem of finding the expected value of the random process $T(x)$, described as

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the output of the following randomized recursive algorithm on input x:

T = "On input x:
1. if $x \leq x_0$, return c
2. generate a sample $(h_1, \ldots, h_k)$ from a given probability distribution on $[0, 1]^k$
3. return $g(x) + \sum_{i=1}^{k} c_i T(h_i x)$."

Under some minor assumptions on the problem parameters, we present a closed-form asymptotic solution of the expected value of $T(x)$. Then, following [2], we argue that the solution holds under a broad class of perturbations including floors and ceilings that usually accompany the recurrences that arise when analyzing randomized divide-and-conquer algorithms. Finally, we provide some illustrative applications and examples, and we conclude by discussing possible extensions of the work.

We are not aware of any general method for finding the expected value of such probabilistic recurrences. A related research report is [3], where a similar recursively defined random process is proved to fall within its expected value with high probability. Therefore, the result of [3]—when applicable—assures that $T(x)$ will fall with high probability within the closed-form solution we provide.

2. The Main Result

DEFINITION 2.1. Say that a function $g: \mathbb{R}^+ \to \mathbb{R}^+$ satisfies *the polynomial growth condition* if for each $u \in (0, 1)$, there exists $x_1 > 0$ and $k > 0$ such that $g(ux) \geq kg(x)$ for all $x > x_1$.

THEOREM 2.1. *Let $T(x)$ be the probabilistic recurrence described as the output of the following randomized recursive algorithm on input x*:

T = *"On input x (a nonnegative real number)*:
1. *if $x \leq x_0$, return c*
2. *generate a sample $(h_1, \ldots, h_k)$ from a given probability distribution on $[0, 1]^k$*
3. *return $g(x) + \sum_{i=1}^{k} c_i T(h_i x)$"*,

where

- *k is a strictly positive integer and $x_0, c, c_1, \ldots, c_k$ are strictly positive real numbers satisfying $\sum_{i=1}^{k} c_i \geq 1$*,
- *none of the h_i's takes the values 0 or 1 with a nonzero probability*,
- *$g: \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function that satisfies the polynomial growth condition*,

then

$$E[T(x)] = \Theta\left(x^p + x^p \int_1^x \frac{g(u)}{u^{p+1}}\, du\right),$$

where p is the nonnegative unique solution of the equation

$$\sum_{i=1}^{k} c_i E[h_i^p] = 1.$$

(If $p = 0$, the lower bound holds under the additional assumption that $E[-\log h_i] < \infty$ for $i = 1, \ldots, k$.)

Note. The random variables $h_1, \ldots, h_k$ need not be independent and are not assumed to have a probability density function.

EXAMPLE. Consider the case when on input x, T picks h from the uniform distribution on [0, 1], and recurses by returning $x + 0.5T(hx) + 0.5T((1-h)x) + T(x/2)$ until 1 is returned when x becomes less than 1. In this case $0.5E[h^p]+0.5E[(1-h)^p]+E[(\frac{1}{2})^p] = 1/(p+1) + 2^{-p}$. Solving for p in $1/(p+1) + 2^{-p} = 1$, we obtain $p = 1$, and thus $E[T(x)] = \Theta(x + x\int_1^x u^{-1}\,du) = \Theta(x \log x)$.

3. Proof of Theorem 2.1. We solve the probabilistic recurrence by deriving an integral equation satisfied by the expected value of the recurrence, and then solving the integral equation asymptotically. The resulting integral equation turns out to have a clean asymptotic solution, despite the fact that it belongs to a class of integral equations that are usually considered hard to solve exactly.

Let $\Phi(x) = E[T(x)]$. If $x \le x_0$, $\Phi(x) = c$. Else if $x > x_0$, $\Phi(x) = g(x) + \sum_{i=1}^{k} c_i E[T(h_i x)]$. Noting that $E[T(h_i x)] = E_{h_i}[\Phi(h_i x)] = \int_0^1 \Phi(tx)\,d\mu_{h_i}(t)$, where μ_{h_i} is the probability distribution of h_i, we obtain

$$\Phi(x) = g(x) + \int_0^1 \Phi(tx)\,d\sum_{i=1} c_i \mu_{h_i}(t).$$

In other words, Φ satisfies the integral equation of Theorem 3.2 below with

$$\alpha(t) = \sum_{i=1}^{k} c_i \mu_{h_i}(t) \qquad \text{for} \quad 0 \le t \le 1.$$

Observe that α satisfies the assumed conditions. Namely, $\alpha(1) - \alpha(0) = \alpha(1) = \sum_{i=1}^{k} c_i \ge 1$, and α is not constant on the open interval (0, 1) and is left continuous at 1 because none of the h_i's takes the values 0 or 1 with a nonzero probability.

THEOREM 3.2. *If Φ is a function satisfying the integral*[3] *equation*

$$\begin{cases} \Phi(x) = g(x) + \displaystyle\int_0^1 \Phi(xt)\,d\alpha(t) & \textit{for} \quad x > x_0, \\ \Phi(x) = c & \textit{for} \quad 0 \le x \le x_0, \end{cases}$$

[3] The integrals used are assumed to be Lebesgue integrals.

where

- α *is a real-valued nonnegative and nondecreasing function defined on the interval* $[0, 1]$ *such that* $\alpha(1) - \alpha(0) \geq 1$, α *is not constant on the open interval* $(0, 1)$, *and* α *is left continuous at* 1,
- g: $\mathbb{R}^+ \to \mathbb{R}^+$ *is a nondecreasing function that satisfies the polynomial growth condition*, *and* c *and* x_0 *are strictly positive real numbers*,

then

(i) Φ *is a nonnegative and nondecreasing function that satisfies the polynomial growth condition*,

(ii)
$$\Phi(x) = \Theta\left(x^p + x^p \int_{x_0}^{x} \frac{g(u)}{u^{p+1}}\, du\right),$$
where p *is the nonnegative unique solution of the equation*
$$\int_0^1 t^p \, d\alpha(t) = 1.$$
(*If* $p = 0$, *the lower bound holds under the additional assumption that* $\int_0^1 (-\log t)\, d\alpha(t) < \infty$.)

Part (i) is needed to establish (ii). The reader may want to skip the proof of (i) and go to that of (ii) in a first reading.

PROOF OF (i). We want to argue that Φ is a nonnegative and nondecreasing function that satisfies the polynomial growth condition. The proof uses Lemma 3.3 below. The nonnegativity of Φ follows directly from that lemma. To show that Φ is nondecreasing, we construct an integral equation for $\Psi_\delta(x) \equiv \Phi((1+\delta)x) - \Phi(x)$ and conclude from Lemma 3.3 that it is nonnegative for any $\delta \geq 0$. Consider any $\delta \geq 0$. By subtracting two instance of Φ's integral equation—the first with x as a variable and the second with $(1+\delta)x$ as a variable—we obtain
$$\begin{cases} \Psi_\delta(x) = g((1+\delta)x) - g(x) + \displaystyle\int_0^1 \Psi_\delta(xt)\, d\alpha(t) & \text{for} \quad x > x_0, \\ \Psi_\delta(x) = \Phi((1+\delta)x) - c & \text{for} \quad \dfrac{x_0}{1+\delta} < x \leq x_0, \\ \Psi_\delta(x) = 0 & \text{for} \quad 0 \leq x \leq \dfrac{x_0}{1+\delta}. \end{cases}$$
We know that $g((1+\delta)x) - g(x)$ is nonnegative because g is nondecreasing. If we can show that $\Phi((1+\delta)x) - c$ is nonnegative for $x_0/(1+\delta) < x < x_0$, the nonnegativity of Ψ_δ will follow from Lemma 3.3. We will show something stronger, we argue below that $\Phi(x) - c$ is nonnegative for any $x \geq 0$. Again we use Lemma 3.3 on $\Omega(x) \equiv \Phi(x) - c$ which satisfies
$$\begin{cases} \Omega(x) = g(x) + c\displaystyle\int_0^1 d\alpha(t) - c + \int_0^1 \Omega(xt)\, d\alpha(t) & \text{for} \quad x > x_0, \\ \Omega(x) = 0 & \text{for} \quad 0 \leq x \leq x_0, \end{cases}$$

and is thus nonnegative because $g(x) + c\int_0^1 d\alpha(t) - c = g(x) + c(\alpha(1) - \alpha(0) - 1) \geq g(x) \geq 0$.

To show that Φ satisfies the polynomial growth condition, consider any $u \in (0, 1)$. We know that g satisfies this condition, so let k_g and x_g be such that $g(ux) \geq k_g g(x)$ for all $x \geq x_g$. Now let $k = \min\{\Phi(0)/\Phi(x_g), k_g\}$. We argue below that $\Phi(ux) \geq k\Phi(x)$ for all $x \geq 0$. Let $\Xi(x) = \Phi(ux) - k\Phi(x)$. Ξ satisfies the following integral equation:

$$\begin{cases} \Xi(x) = g(ux) - kg(x) + \displaystyle\int_0^1 \Xi(xt)\, d\alpha(t) & \text{for} \quad x > x_g, \\ \Xi(x) \geq \Phi(0) - k\Phi(x_g) & \text{for} \quad 0 \leq x \leq x_g. \end{cases}$$

The nonnegativity of Ξ follows from Lemma 3.3 because we have selected k in such a way that $g(ux) - kg(x) \geq 0$ for all $x > x_g$ and $\Phi(0) - k\Phi(x_g) \geq 0$.

LEMMA 3.3. *If Δ is a function satisfying the integral inequality*

$$\begin{cases} \Delta(x) \geq \displaystyle\int_0^1 \Delta(xt)\, d\beta(t) & \textit{for} \quad x > x_0, \\ \Delta(x) \geq 0 & \textit{for} \quad 0 \leq x \leq x_0, \end{cases}$$

where x_0 is strictly positive, and β is a nonnegative and nondecreasing function defined on $[0, 1]$ *such that it is left continuous at* 1, *then Δ is nonnegative.*

PROOF. The proof is by contradiction. Assume that $\Delta(x) < 0$ for some $x \geq 0$. We consider two cases, the first when $\beta(1^-) \neq \beta(1)$ and the second when $\beta(1^-) = \beta(1)$.

Consider the case when $\beta(1^-) \neq \beta(1)$, and accordingly let t_0 in $(0, 1)$ be such that $0 < \beta(1) - \beta(t_0) < 1$. The existence of t_0 is based also on the fact that β is left continuous at 1. Let $\xi = \beta(1) - \beta(t_0)$, $x_1 = \inf\{x \geq 0\colon \Delta(x) < 0\}$, $M = \inf\{\Delta(x)\colon x \in [0, x_1/t_0)\}$, and $x_2 \in [0, x_1/t_0)$ such that $\Delta(x_2) < \xi M$. Observe that $x_1 > 0$ because $x_0 > 0$ and hence the interval $[0, x_1/t_0)$ is not empty. We have

$$\begin{aligned} \Delta(x_2) &\geq \int_0^1 \Delta(x_2 t)\, d\beta(t) \\ &= \int_0^{t_0} \Delta(x_2 t)\, d\beta(t) + \int_{t_0}^1 \Delta(x_2 t)\, d\beta(t) \\ &\geq \int_{t_0}^1 \Delta(x_2 t)\, d\beta(t) \qquad (1) \\ &\geq (\beta(1) - \beta(t_0)) \inf\{\Delta(x)\colon x \in [t_0 x_2, x_2]\} \\ &\geq \xi M \qquad (2) \\ &> \Delta(x_2), \end{aligned}$$

which is not possible. Note that (1) follows from the fact that $x_2 t < x_1$ when $0 < t < t_0$, and (2) follows from the fact that $[t_0 x_2, x_2] \subset [0, x_1/t_0)$.

Now consider the case when $\beta(1^-) = \beta(1)$, which is actually simpler. Let t_0 in $(0, 1)$ be such that $\beta(1) = \beta(t_0)$, $x_1 = \inf\{x \geq 0\colon\ \Delta(x) < 0\}$, and let x_2 in $[0, x_1/t_0)$ be such that $\Delta(x_2) < 0$. We have

$$\begin{aligned}
\Delta(x_2) &\geq \int_0^1 \Delta(x_2 t)\, d\beta(t) \\
&= \int_0^{t_0} \Delta(x_2 t)\, d\beta(t) + \int_{t_0}^1 \Delta(x_2 t)\, d\beta(t) \\
&\geq \int_{t_0}^1 \Delta(x_2 t)\, d\beta(t) \\
&= 0,
\end{aligned}$$

which is not possible. Here again the second inequality follows from the fact that $x_2 t < x_1$ when $0 < t < t_0$. □

This completes the proof of part (i) of Theorem 3.2. □

PROOF OF (ii). If $x > x_0$, then

$$\Phi(x) = g(x) + \int_0^1 \Phi(xt)\, d\alpha(t).$$

Multiplying both sides by $x^{-(p+1)}$ $(p \geq 0)$ and integrating with respect to x from x_0 to y we obtain

$$\int_{x_0}^y \Phi(x) x^{-(p+1)}\, dx = \int_{x_0}^y g(x) x^{-(p+1)}\, dx + \int_0^1 \int_{x_0}^y \Phi(xt) x^{-(p+1)}\, dx\, d\alpha(t).$$

We can expand the kernel of the last integral as follows:

$$\begin{aligned}
&\int_{x_0}^y \Phi(xt) x^{-(p+1)}\, dx \\
&\qquad = t^p \int_{tx_0}^{yt} \Phi(u) u^{-(p+1)}\, du \\
&\qquad = t^p \int_{x_0}^{y} \Phi(u) u^{-(p+1)}\, du - t^p \int_{yt}^{y} \Phi(u) u^{-(p+1)}\, du + t^p \int_{tx_0}^{x_0} \Phi(u) u^{-(p+1)}\, du \\
&\qquad = t^p \int_{x_0}^{y} \Phi(u) u^{-(p+1)}\, du - y^{-p} t^p \int_{t}^{1} \Phi(yu) u^{-(p+1)}\, du \\
&\qquad\quad + c x_0^{-p} t^p \int_t^1 u^{-(p+1)}\, du,
\end{aligned}$$

where the last term follows from the fact that $\Phi(u) = c$ if $u \leq x_0$. Replacing in the previous equation, we obtain after rearrangement

$$y^{-p} \int_0^1 t^p \int_t^1 \Phi(yu)u^{-(p+1)}\, du\, d\alpha(t)$$
$$= \left(\int_0^1 t^p\, d\alpha(t) - 1\right) \int_{x_0}^y \Phi(u)u^{-(p+1)}\, du$$
$$+ \int_{x_0}^y g(x)x^{-(p+1)}\, dx + cx_0^{-p} \int_0^1 t^p \int_t^1 u^{-(p+1)}\, du\, d\alpha(t).$$

We claim that there is one and only one nonnegative value of p for which $\int_0^1 t^p\, d\alpha(t) = 1$. To see why this is the case it is sufficient to observe that the function $s(p) = \int_0^1 t^p\, d\alpha(t)$ satisfies:

- $s(0) = \alpha(1) - \alpha(0) \geq 1$,
- $\lim_{p\to+\infty} s(p) = \int_0^1 (\lim_{p\to+\infty} t^p)\, d\alpha(t) = 0$,
- s is continuous and nonincreasing on $[0, \infty)$ because

$$\frac{d}{dp}s(p) = \int_0^1 \ln(t)t^p\, d\alpha(t) \leq 0.$$

If we set p to this unique solution, the previous integral equation reduces to

$$\int_0^1 t^p \int_t^1 \Phi(yu)u^{-(p+1)}\, du\, d\alpha(t)$$
$$= y^p \int_{x_0}^y g(x)x^{-(p+1)}\, dx + cx_0^{-p} y^p \int_0^1 t^p \int_t^1 u^{-(p+1)}\, du\, d\alpha(t).$$

Observe that Φ appears now only in the left-hand side of the equation. To solve the equation asymptotically, we show that

$$\int_0^1 t^p \int_t^1 \Phi(yu)u^{-(p+1)}\, du\, d\alpha(t) = \Theta(\Phi(y)).$$

This follows from Lemma 3.4 below which is applicable on Φ since, by part (i), Φ is a nonnegative and nondecreasing function that satisfies the polynomial growth condition.

It follows that

$$\Phi(y) = \Theta\left(cx_0ky^p + y^p \int_{x_0}^y \frac{g(u)}{u^{p+1}}\, du\right),$$

where $k = \int_0^1 t^p \int_t^1 u^{-(p+1)}\, du\, d\alpha(t)$, which is a positive constant by the argument in the the lemma below where it corresponds to $k(0)$.

LEMMA 3.4. *If Φ is real-valued nonnegative and nondecreasing function that satisfies the polynomial growth condition, α is a real-valued nonnegative and nondecreasing* *function defined on the interval* [0, 1] *such that it is not constant on the interval* (0, 1), *and* $p \geq 0$, *then*

$$\int_0^1 t^p \int_t^1 \Phi(yu)u^{-(p+1)}\, du\, d\alpha(t) = \Theta(\Phi(y)).$$

(*If* $p = 0$, *we need* $\int_0^1 (-\log t)\, d\alpha(t) < \infty$ *for the upper bound to hold*.)

PROOF. Let t_0 and t_1, $0 < t_0 < t_1 < 1$, be such that $\alpha(t_0) < \alpha(t_1)$, and let y_0 and k_0 be such that $\Phi(t_0 y) \geq k_0 \Phi(y)$, $\forall y \geq y_0$. For any $y > y_0$, and any $u \in [t_0, 1)$, we have

$$k_0\Phi(y) \leq k_0\Phi\left(y\frac{u}{t_0}\right) \leq \Phi\left(\left(y\frac{u}{t_0}\right)t_0\right) = \Phi(yu),$$

where the first inequality follows from the fact that Φ is nondecreasing. Integrating twice after scaling the kernels, we obtain

$$\begin{aligned} k_0\Phi(y)\int_{t_0}^1 t^p \int_t^1 u^{-(p+1)}\, du\, d\alpha(t) &\leq \int_{t_0}^1 t^p \int_t^1 \Phi(yu)u^{-(p+1)}\, du\, d\alpha(t) \\ &\leq \int_0^1 t^p \int_t^1 \Phi(yu)u^{-(p+1)}\, du\, d\alpha(t). \end{aligned}$$

Now, observe that because Φ is nondecreasing we also have

$$\int_0^1 t^p \int_t^1 \Phi(yu)u^{-(p+1)}\, du\, d\alpha(t) \leq \Phi(y)\int_0^1 t^p \int_t^1 u^{-(p+1)}\, du\, d\alpha(t).$$

Therefore

$$k_0 k(t_0)\Phi(y) \leq \int_0^1 t^p \int_t^1 \Phi(yu)u^{-(p+1)}\, du\, d\alpha(t) \leq k(0)\Phi(y), \tag{3}$$

where

$$k(a) = \int_a^1 t^p \int_t^1 u^{-(p+1)}\, du\, d\alpha(t).$$

We argue now that $k(0) < \infty$ and $k(t_0) > 0$.

We have two cases to consider; $p \neq 0$ and $p = 0$. If $p \neq 0$, then $t^p \int_t^1 u^{-(p+1)}\, du = (1 - t^p)/p$. Hence $k(0) \leq (\alpha(1) - \alpha(0))/p < \infty$, and

$$k(t_0) \geq \int_{t_0}^{t_1} \frac{1 - t^p}{p}\, d\alpha(t) \geq (\alpha(t_1) - \alpha(t_0))\frac{1 - t_1^p}{p} > 0.$$

Otherwise, $p = 0$, then $t^p \int_t^1 u^{-(p+1)}\, du = -\log t$. Hence $k(0) < \infty$ (by assumption), and as in the previous case $k(t_0) \geq (\alpha(t_1) - \alpha(t_0))(-\log(t_1)) > 0$. □

This completes the proof of part (ii) of Theorem 3.2. □

4. Handling Perturbations. In algorithms design, probabilistic divide-and-conquer recurrences that usually arise are variations of the recurrences that we have considered so far. For instance, a common situation is

$$\begin{cases} E[T(n)] = g(n) + \sum_{i=1}^{k} c_i E[T(v_i(n))] & \text{for} \quad n > n_0, \\ E[T(n)] = \Theta(1) & \text{for} \quad 0 \le n \le n_0, \end{cases}$$

where n is an integer, and the $v_i(n)$'s are integer valued random variables satisfying $0 \le v_i(n) \le n - 1$ and given by $v_i(n) = h_i n + \Theta(1)$, where the h_i's are real random variables from the interval [0, 1]. Such perturbations are due to floors, ceilings, and/or the structure of the algorithm. Following [2], where the deterministic solution in [1] was shown to hold under a broad class of perturbations that include floors and ceilings, we adapt the probabilistic situation to a similar class of perturbations.

Namely, we consider recurrences of the form

$$\begin{cases} E[T(n)] = g(n) + \sum_{i=1}^{k} c_i E[T(v_i(n))] & \text{for} \quad n > n_0, \\ E[T(n)] = \Theta(1) & \text{for} \quad 0 \le n \le n_0, \end{cases} \tag{4}$$

where each $v_i(n)$ is an integer-valued random variable satisfying

$$0 \le v_i(n) \le n - 1,$$

and is given by

$$v_i(n) = h_i n + e_i(n),$$

with h_i a random variable taking real values from the interval [0, 1], and $e_i(n)$ a real-valued random variable satisfying, for n large enough,

$$|e_i(n)| < n^{1-\xi}$$

for some ξ, $0 < \xi < 1$, common to all the v_i's. The other parameters $g(x), k, c_1, \ldots, c_k$ are as in the previous sections. Here also we assume that none of the h_i's takes the values 1 or 0 with a nonzero probability.

THEOREM 4.5. *If $T(n)$ satisfies* (4), *then*

$$E[T(n)] = \Theta\left(n^p + n^p \int_1^n \frac{g(u)}{u^{p+1}}\, du\right),$$

where p is the nonnegative unique solution of the equation

$$\sum_{i=1}^{k} c_i E[h_i^p] = 1.$$

(*If $p = 0$, the lower bound holds under the additional assumption* $E_{v_i}[-\log(v_i(n)/n)] < \infty$.)

The proof verifies by induction that the result of the previous section holds if such perturbations are allowed. We follow the technique introduced in [2] to make the induction work with the perturbations.

The argument is to some extent along the lines of the proof in the previous section. The differences are mainly that the argument in part (i) of Theorem 3.2 is not needed, but we need more technicalities to handle the discrete situation and the perturbations in general.

PROOF. First we establish the upper bound. We argue by induction on n that

$$E[T(n)] \leq a(1-n^{-\mu})\left(n^p + n^p \int_1^n \frac{g(u)}{u^{p+1}}\,du\right), \tag{5}$$

where a is a large enough constant selected so that the induction can start and will be tuned further to make the induction work, and

$$\mu = \begin{cases} \dfrac{\xi}{2} & \text{if} \quad p \geq 1, \\ \dfrac{p\xi}{2} & \text{if} \quad 0 \leq p < 1. \end{cases} \tag{6}$$

Assume that (5) holds for all m, $1 \leq m < n$, and assume that n is larger than the constants in Lemmas 4.6 and 4.7 below. We want to argue that (5) holds for n. Since $1 \leq v_i(n) < n$, we have

$$\begin{aligned} E[T(n)] \;&\leq\; g(n) + a\sum_i c_i E_{v_i}\left[(1-v_i(n)^{-\mu})\left(v_i(n)^p + v_i(n)^p \int_1^{v_i(n)} \frac{g(u)}{u^{p+1}}\,du\right)\right] \\ &=\; a\left(n^p + n^p\int_1^n \frac{g(n)}{u^{p+1}}\right)\sum_i c_i E_{v_i}\left[(1-v_i(n)^{-\mu})\left(\frac{v_i(n)}{n}\right)^p\right] \\ &\quad + g(n) - a\sum_i c_i E_{v_i}\left[(1-v_i(n)^{-\mu})v_i(n)^p\int_{v_i(n)}^n \frac{g(u)}{u^{p+1}}\,du\right]. \end{aligned}$$

Using Lemmas 4.6 and 4.7 below, we get

$$E[T(n)] \leq a\left(n^p + n^p\int_1^n \frac{g(n)}{u^{p+1}}\right)(1-n^{-\mu})\sum_i c_i E_{h_i}[h_i^p] + g(n) - akg(n),$$

and hence (5) by making a large enough and since p satisfies $\sum_i c_i E_{h_i}[h_i^p] = 1$. Note that the existence of such a unique p follows from the previous section and uses the fact that $\sum_i c_i \geq 1$.

LEMMA 4.6. *For n large enough*,

$$E_{v_i}\left[(1-n^{-\mu})h_i^p - (1-v_i(n)^{-\mu})\left(\frac{v_i(n)}{n}\right)^p\right] > 0.$$

PROOF. See the Appendix. □

LEMMA 4.7. *There is a constant $k > 0$ such that for n large enough,*

$$E_{v_i}\left[(1 - v_i(n)^{-\mu})v_i(n)^p \int_{v_i(n)}^n \frac{g(u)}{u^{p+1}}\, du\right] \geq kg(x).$$

PROOF. The proof is as in Lemma 3.4 with some additional technicalities to handle the perturbations. Let α be the probability distribution of h_i. Since h_i takes values other than 0 and 1 with a nonzero probability, let t_0 and t_1, $0 < t_0 < t_1 < 1$, be such that $\alpha(t_0) < \alpha(t_1)$. We can bound from below the expected value in the statement of the lemma by

$$\begin{aligned}\int_{t_0}^{t_1} &(1 - (tn - n^{1-\xi})^{-\mu})(tn - n^{1-\xi})^p \int_{tn+n^{1-\xi}}^n \frac{g(u)}{u^{p+1}}\, du\, d\alpha(t) \\ &\geq (1 - (t_0 n - n^{1-\xi})^{-\mu})(t_0 - n^{-\xi})^p \int_{t_0}^{t_1} n^p \int_{(t+\theta)n}^n \frac{g(u)}{u^{p+1}}\, du\, d\alpha(t),\end{aligned}$$

where $\theta > 0$ such that $t_1 + \theta < 1$ and n is assumed to be large enough so that $\theta n > n^{1-\xi}$. Since g satisfies the polynomial growth condition, let $k_0 > 0$ be such that $g((t_0+\theta)n) > k_0 g(n)$ for n large enough. For $t > t_0$, we have

$$n^p \int_{(t+\theta)n}^n \frac{g(u)}{u^{p+1}}\, du = \int_{t+\theta}^1 \frac{g(un)}{u^{p+1}}\, du \geq g((t_0+\theta)n) \int_{t+\theta}^1 \frac{du}{u^{p+1}} \geq g(n)k_0 \int_{t+\theta}^1 \frac{du}{u^{p+1}},$$

where the first inequality follows from the fact that g is nondecreasing. The lemma then follows with

$$k = k_0(1 - (t_0 n_0 - n_0^{1-\xi})^{-\mu})(t_0 - n_0^{-\xi})^p \int_{t_0}^{t_1} \int_{t+\theta}^1 \frac{du}{u^{p+1}}\, d\alpha(t),$$

where n_0 is large enough so that $k > 0$. Note that the fact that $\alpha(t_0) < \alpha(t_1)$ and $t_1+\theta < 1$ guarantees that the integral term of k is positive. Indeed, when $p > 0$, this integral term reduces to

$$\frac{1}{p}\int_{t_0}^{t_1} ((t+\theta)^{-p} - 1)\, d\alpha(t) \geq \frac{1}{p}(\alpha(t_1) - \alpha(t_0))((t_1+\theta)^{-p} - 1),$$

and when $p = 0$, it becomes $\int_{t_0}^{t_1} \log(t+\theta)^{-1}\, d\alpha(t) \geq (\alpha(t_1) - \alpha(t_0)) \log(t_1+\theta)^{-1}$. $\square$

Now the lower bound follows by suitably modifying the upper bound argument. We argue by induction on n that

$$E[T(n)] \geq a(1 + (n+1)^{-\mu})\left(n^p + n^p \int_1^n \frac{g(u)}{u^{p+1}}\, du\right), \tag{7}$$

where $a > 0$ is small enough so that the induction can start and will be tuned further as before to make the induction work, and μ is as in (6).

If we assume that (7) holds for all m, $1 \le m < n$, and assume that n is large enough so that Lemmas 4.8 and 4.9 below are applicable, we get

$$\begin{aligned} E[T(n)] \;&\ge\; g(n) \\ &\quad + a \sum_i c_i E_{v_i}\left[(1 + (v_i(n)+1)^{-\mu})\left(v_i(n)^p + v_i(n)^p \int_1^{v_i(n)} \frac{g(u)}{u^{p+1}}\,du\right)\right] \\ &= a\left(n^p + n^p \int_1^n \frac{g(n)}{u^{p+1}}\right)\sum_i c_i E_{v_i}\left[(1 + v_i(n)^{-\mu})\left(\frac{v_i(n)}{n}\right)^p\right] \\ &\quad + g(n) - a \sum_i c_i E_{v_i}\left[(1 + (v_i(n)+1)^{-\mu}) v_i(n)^p \int_{v_i(n)}^n \frac{g(u)}{u^{p+1}}\,du\right], \end{aligned}$$

and hence, from Lemmas 4.8 and 4.9 below,

$$E[T(n)] \ge a\left(n^p + n^p \int_1^n \frac{g(n)}{u^{p+1}}\right)(1 + (n+1)^{-\mu}) \sum_i c_i E_{h_i}[h_i^p] + g(n) - akg(n),$$

which lead us to (7) by making a small enough and using the fact that $\sum_i c_i E_{h_i}[h_i^p] = 1$.

LEMMA 4.8. *For n large enough*,

$$E_{v_i}\left[(1 + (n+1)^{-\mu}) h_i^p - (1 + (v_i(n)+1)^{-\mu})\left(\frac{v_i(n)}{n}\right)^p\right] < 0.$$

PROOF. See the Appendix. □

LEMMA 4.9. *There is a constant $k > 0$ such that for n large enough*,

$$E_{v_i}\left[(1 + (v_i(n)+1)^{-\mu}) v_i(n)^p \int_{v_i(n)}^n \frac{g(u)}{u^{p+1}}\,du\right] \le kg(x).$$

PROOF. Here again the proof follows Lemma 3.4. Since $g(n)$ is nondecreasing and $v_i(n) \ge 0$, the expected value under consideration is at most $g(n)k$, where

$$k = 2E_{v_i}\left[v_i(n)^p \int_{v_i(n)}^n \frac{1}{u^{p+1}}\,du\right].$$

To see why $k < \infty$, we consider the cases $p > 0$ and $p = 0$ separately. If $p > 0$, then

$$k = \frac{2}{p} E_{v_i}\left[1 - \left(\frac{v_i(n)}{n}\right)^p\right] \le \frac{2}{p},$$

else if $p = 0$, then $k = 2E_{v_i}[\log(n/v_i(n))] < \infty$ by assumption. □

This completes the proof of Theorem 4.5. □

5. Applications and Examples. Below are some illustrative applications following [3] and some examples. All the recurrences are assumed to be initially $\Theta(1)$, and we use the notation

$$\zeta_n(x) = \begin{cases} 1 & \text{if} \quad x < 0, \\ \lfloor x \rfloor & \text{if} \quad 0 \le x < n, \\ n-1 & \text{if} \quad x \ge n. \end{cases}$$

5.1. *QuickSort*. On input S, a set of n elements, QuickSort picks uniformly an element p of S and partitions S in linear time around p into $S_1 = \{x \in S\colon x \le p\} - \{p\}$ and $S_2 = \{x \in S\colon x > p\}$, then it recurses on S_1 and S_2 until S becomes sorted. If we let $T(n)$ be the running time of QuickSort, we obtain the following recursion for its expected value:

$$E[T(n)] = E[\Theta(n) + T(\zeta_n(hn + \Theta(1))) + T(\zeta_n((1-h)n + \Theta(1)))]$$

for n large enough, where h is a continuous random variable uniformly distributed on [0, 1]. Solving for p in the characteristic equation $E[h^p] + E[(1-h)^p] = 1/(p+1) + 1/(p+1) = 1$, we obtain $p = 1$, and thus $E[T(n)] = \Theta(n \log n)$.

5.2. *A Randomized Selection Algorithm*. Hoare's algorithm finds the kth-smallest element in a set S of n elements by first choosing uniformly an element p of S and partitioning S around p in linear time into S_1 and S_2 as described in the QuickSort case. Then depending on k, the index of p, and the size of S_1 the algorithm decides whether it should stop, recurse on S_1, or recurse on S_2. If we let $T(n)$ be the running time of the selection algorithm, we get $E[T(n)] \le E[an + T(\zeta_n(\max\{h, (1-h)\}n + \Theta(1)))]$ for n large enough, where $a > 0$ and h is a continuous random variable uniformly distributed on [0, 1]. Solving for p in $E[\max\{h, (1-h)\}^p] = 1$, we obtain $p = 0$, and thus $E[T(n)] = O(n)$.

5.3. *Random Permutations*. The number $T(n)$ of cycles in a randomly chosen n-permutation satisfies $E[T(n)] = E[1 + T(\lceil hn \rceil - 1)]$ if $n \ge 1$ and $E[T(0)] = 0$, where h is a continuous random variable uniformly distributed on [0, 1]. Solving for p in $E[h^p] = 1$, we obtain $p = 0$, and thus $E[T(n)] = O(\log n)$. Note that we cannot conclude that $E[T(n)] = \Omega(\log n)$ since we are now in the case $p = 0$ where $E_h[-\log((\lceil hn \rceil - 1)/n)] = \infty$ due to the ceiling.

5.4. *Examples*. **(1)** If $E[T(x)] = E[\sqrt{x} + T(hx) + T(x/3)]$ for x large enough, where h is uniform on [0, 1], then $E[h^p] + E[(\frac{1}{3})^p] = 1/(p+1) + (\frac{1}{3})^p = 1$ implies $p = 0.7626\ldots$ and thus $E[T(x)] = \Theta(x^{0.7626\ldots})$.

(2) If $E[T(x)] = E[x^5 \log x + 3T(hx) + 8T(h^3x)]$ for x large enough, where h is uniform on [0, 1] , then $3E[h^p] + 8E[(h^3)^p] = 3/(p+1) + 8/(3p+1) = 1$ implies $p = 5$ and thus $E[T(x)] = \Theta(x^5 \log^2 x)$.

(3) If $E[T(n)] = E[n^3 + \frac{3}{4}T(\zeta_n(h_1 n + \sqrt{n})) + 0.5T(\zeta_n(h_2 n - \sqrt{n})) + 0.5T(\lfloor n/2 \rfloor)]$ for n large enough, where $h_1 = \max\{r, s\}$, $h_2 = (s+t)/2$, and r, s are independent and uniform on [0, 1], then $\frac{3}{4}E[h_1^p] + 0.5E[h_2^p] + 0.5E[(\frac{1}{2})^p] = 1$ implies $p = 1$, and thus $E[T(n)] = \Theta(n^3)$.

6. Possible Extensions of the Work. The setting in which we analyzed the probabilistic recurrence is very general. However, one minor assumption that we did not attempt to investigate its necessity is the condition imposed on the h_i's or the v_i's to get a lower bound when $p = 0$, namely $\int_0^1 -\log t \, d\alpha(t) < \infty$ or $E_{v_i}[-\log(v_i(n)/n)] < \infty$ respectively in Theorems 3.2 and 4.5.

A more general question is whether the solution of the probabilistic recurrence holds under perturbations in the continuous situation, i.e., in the setting of Theorems 2.1 or 3.2. This sounds true, but it is not obvious how to establish it with reasonable restrictions on the h_i's.

A similar recursively defined random process was shown in [3] to fall within its expected value with high probability. Therefore, the result of [3]—when applicable—assures that $T(x)$ will fall with high probability within the closed-form solution we provided. Translated to our setting, the assumption made in [3] on $T(x)$ restricts the recurrence to the case where $E[T(x)] \geq E[\sum_{i=1}^{k} c_i T(h_{i0}x)]$ for each x and each possible value $(h_{10}, \ldots, h_{k0})$ of the random vector $(h_1, \ldots, h_k)$. As explained in [3], this assumption is violated by many probabilistic recurrences that arise in computational geometry and data structures analysis. A possible research direction might be in finding the necessary and sufficient conditions for the high-probability result to hold, or at least in weakening that assumption.

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Appendix

LEMMA 4.6. *For n large enough*,

$$E_{v_i}\left[(1 - n^{-\mu})h_i^p - (1 - v_i(n)^{-\mu})\left(\frac{v_i(n)}{n}\right)^p\right] > 0.$$

PROOF. Let

$$f(x, t) = (1 - x^{-\mu})t^p - (1 - \Upsilon_x(tx + x^{1-\xi})^{-\mu})(t + x^{-\xi})^p,$$

where

$$\Upsilon_x(y) = \begin{cases} 0 & \text{if} \quad x < 0, \\ y & \text{if} \quad 0 \leq y \leq x, \\ x & \text{if} \quad y > x, \end{cases}$$

and let α be the probability distribution of h_i. The lemma will follow if we can argue that $\int_0^1 f(x, t)\, d\alpha(t) > 0$ for x large enough since $(1 - n^{-\mu})h_i^p - (1 - v_i(n)^{-\mu})(v_i(n)/n)^p \geq f(n, h_i)$. We consider the cases $p \geq 1$ and $0 \leq p < 1$ separately.

Assume that $p \geq 1$. Then there is a constant $c > 0$ such that $(t + x^{-\xi})^p \leq t^p + cx^{-\xi}$ for x large enough and for all $0 \leq t \leq 1$. Since h_i is not 1 with probability 1, let t_1, $0 < t_1 < 1$, be such that $\int_0^{t_1} t^p \, d\alpha(t) > 0$.

If $0 \le t \le t_1$, we have

$$f(x,t) \ge ((t_1x + x^{1-\xi})^{-\mu} - x^{-\mu})t^p - (1 - \Upsilon_x(tx + x^{1-\xi})^{-\mu})cx^{-\xi},$$

else if $t_1 < t \le 1$, we use the bound

$$f(x,t) \ge -(1 - x^{-\mu})cx^{-\xi}.$$

Thus

$$\begin{aligned}\int_0^1 f(x,t)\,d\alpha(t) &\ge ((t_1x + x^{1-\xi})^{-\mu} - x^{-\mu}) \int_0^{t_1} t^p\,d\alpha(t) - 2cx^{-\xi}\\ &= x^{-\mu}((t_1 + x^{-\xi})^{-\mu} - 1) \int_0^{t_1} t^p\,d\alpha(t) - 2cx^{-\xi}\\ &> 0\end{aligned}$$

when x is large enough since $\mu < \xi$ by (6).

Otherwise $0 \le p < 1$. Then $(t + x^{-\xi})^p \le t^p + pt^{p-1}x^{-\xi}$. Since h_i takes values other than 0 and 1 with a nonzero probability let t_0, t_1, $0 < t_0, t_1 < 1$, be such that $\int_{t_0}^{t_1} t^p\,d\alpha(t) > 0$, and assume that x is large enough so that $x^{-\xi} \le t_0$. Note that if $x^{-\xi} \le t \le 1$, we have $(t + x^{-\xi})^p \le t^p + px^{-p\xi}$.

If $x^{-\xi} \le t \le t_1$,

$$f(x,t) \ge ((t_1x + x^{1-\xi})^{-\mu} - x^{-\mu})t^p - (1 - \Upsilon_x(tx + x^{1-\xi})^{-\mu})px^{-p\xi},$$

else if $t_1 < t \le 1$,

$$f(x,t) \ge -(1 - x^{-\mu})px^{-p\xi},$$

otherwise if $0 \le t < x^{-\xi}$,

$$f(x,t) \ge -(t + x^{-\xi})^p \ge -2^p x^{-p\xi}.$$

Thus

$$\begin{aligned}\int_0^1 f(x,t)\,d\alpha(t) &\ge ((t_1x + x^{1-\xi})^{-\mu} - x^{-\mu}) \int_{t_0}^{t_1} t^p\,d\alpha(t) - (2p + 2^p)x^{-p\xi}\\ &= x^{-\mu}((t_1 + x^{-\xi})^{-\mu} - 1) \int_{t_0}^{t_1} t^p\,d\alpha(t) - (2p + 2^p)x^{-p\xi}\\ &> 0\end{aligned}$$

when x is large enough since $\mu < p\xi$ by (6). □

LEMMA 4.8. *For n large enough*,

$$E_{v_i}\left[(1 + (n+1)^{-\mu})h_i^p - (1 + (v_i(n) + 1)^{-\mu})\left(\frac{v_i(n)}{n}\right)^p\right] < 0.$$

PROOF. Let

$$f(x,t) = (1+(x+1)^{-\mu})t^p - (1+(\Upsilon_x(tx+x^{1-\xi})+1)^{-\mu})\Upsilon_1(t-x^{-\xi})^p,$$

where Υ_x is as in Lemma 4.6, and let α be the probability distribution of h_i. Since $(1+(n+1)^{-\mu})h_i^p - (1+(v_i(n)+1)^{-\mu})(v_i(n)/n)^p \leq f(n,h_i)$, we can argue that $\int_0^1 f(x,t)\,d\alpha(t) < 0$ for x large enough. Here again, we consider the cases $p \geq 1$ and $0 \leq p < 1$ separately.

Assume that $p \geq 1$. Then there is a constant $c > 0$ such that $(t-x^{-\xi})^p \geq t^p - cx^{-\xi}$ when $x^{-\xi} \leq t \leq 1$. Let t_0, t_1, $0 < t_0, t_1 < 1$, be such that $\int_{t_0}^{t_1} t^p\,d\alpha(t) > 0$, and assume that x is large enough so that $x^{-\xi} < t_1$.

If $0 \leq t \leq x^{-\xi}$,

$$f(x,t) \leq (1+(x+1)^{-\mu})t^p \leq (1+(x+1)^{-\mu})x^{-p\xi},$$

else if $x^{-\xi} < t \leq t_1$,

$$f(x,t) \leq ((x+1)^{-\mu} - (t_1x+x^{1-\xi}+1)^{-\mu})t^p + (1+(\Upsilon_x(tx+x^{1-\xi})+1)^{-\mu})cx^{-\xi},$$

otherwise if $t_1 < t \leq 1$,

$$f(x,t) \leq (1+(x+1)^{-\mu})cx^{-\xi}.$$

Thus

$$\begin{aligned}
&\int_0^1 f(x,t)\,d\alpha(t)\\
&\qquad \leq ((x+1)^{-\mu} - (t_1x+x^{1-\xi}+1)^{-\mu})\int_{t_0}^{t_1} t^p\,d\alpha(t) + (2+4c)x^{-\xi}\\
&\qquad = -x^{-\mu}((t_1+x^{-\xi}+x^{-1})^{-\mu} - (1+x^{-1}))\int_{t_0}^{t_1} t^p\,d\alpha(t) + (2+4c)x^{-\xi}\\
&\qquad < 0
\end{aligned}$$

when x is large enough since $\mu < \xi$ by (6).

Otherwise $0 \leq p < 1$. Then $(t-x^{-\xi})^p \geq t^p - p(t-x^{-\xi})^{p-1}x^{-\xi}$ when $x^{-\xi} \leq t \leq 1$. As before, let t_0, t_1, $0 < t_0, t_1 < 1$, be such that $\int_{t_0}^{t_1} t^p\,d\alpha(t) > 0$, and assume that x is large enough so that $2x^{-\xi} \leq t_0$. Note that if $2x^{-\xi} \leq t \leq 1$, we have $(t-x^{-\xi})^p \geq t^p - px^{-p\xi}$.

If $2x^{-\xi} \leq t \leq t_1$,

$$f(x,t) \leq ((x+1)^{-\mu} - (t_1x+x^{1-\xi}+1)^{-\mu})t^p + px^{-p\xi}(1+(\Upsilon_x(tx+x^{1-\xi})+1)^{-\mu}),$$

else if $t_1 < t \leq 1$,

$$f(x,t) \leq (1+(x+1)^{-\mu})px^{-p\xi},$$

otherwise if $0 \leq t < 2x^{-\xi}$,

$$f(x,t) \leq (1+(x+1)^{-\mu})t^p \leq (1+(x+1)^{-\mu})2^px^{-p\xi}.$$

Thus

$$\begin{aligned}
&\int_0^1 f(x,t)\,d\alpha(t)\\
&\qquad \le ((x+1)^{-\mu} - (t_1 x + x^{1-\xi} + 1)^{-\mu}) \int_{t_0}^{t_1} t^p\,d\alpha(t) + (4p + 2^{p+1})x^{-p\xi}\\
&\qquad = x^{-\mu}((t_1 + x^{-\xi} + x^{-1})^{-\mu} - (1 + x^{-1})^{-\mu})\\
&\qquad\quad \times \int_{t_0}^{t_1} t^p\,d\alpha(t) + (4p + 2^{p+1})x^{-p\xi}\\
&\qquad < 0
\end{aligned}$$

when x is large enough since $\mu < p\xi$ by (6). □

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