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COMPUTING METHODS IN OPTIMIZATION PROBLEMS — 2

NUMERICAL SOLUTION OF A DISTRIBUTED IDENTIFICATION PROBLEM VIA A DIRECT METHOD

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A. Introduction. We consider the state identification problem for systems whose state evolution process is described by a linear parabolic Partial Differential Equation, wherein all parameters are known.* The results derived are also applicable to systems whose state evolution process is described by linear, second order, hyperbolic partial differential equations (1).

The method of approach is a variational one and, under suitable hypothesis, the results obtained have a certain "stochastic respectability." In abstract, the problem is formulated as follows:

Given:

(1) A state evolution process S
(2) Input measurements I which are inexact--that is, there are measurement errors, and
(3) Output measurements O which are incomplete and/or inexact, the former qualification being a consequence of a certain physical realizability condition.

Problem

Recover an estimate of the "true state of nature of the process S, on the basis of I,O and the state evolution process associated with S, which is optimal in some sense. This problem is equivalent to determining optimal estimates of

(a) The initial condition with which the state evolution process began, and
(b) The environmental interaction or inputs.

Denoting this vector of optimal estimates by $\underline{u}$ and an arbitrary estimate by $\underline{v}$, the criterion of optimality chosen is

* An extension is proposed for distributed parameter estimation.

that of "least squares", that is,

$$J(\underline{u}) = \inf_{\underline{v}\in V} J(\underline{v}) \qquad (1)$$

where $J(\underline{v})$ is an appropriate quadratic error functional incorporating the measurements I and O, and V is the space of admissible estimates.

The identification problem as phrased is therefore a variational one, that of minimizing a quadratic functional. Characterization of extremals to the quadratic functional is achieved via the method of variational inequalities, due to Lions and Stampaccia (2). We employ a direct numerical method which uses the aforementioned characterization to generate conjugate directions of search on the quadratic functional, which lead, in the limit, to the sought for extremals. We remark that the numerical technique (considered in the sequel) is applicable to a braoder class of problems than those arising in the context of identification. In particular, we recognize an immediate application to problems in distributed optimal control.

Remark

The foregoing implies that we solve a smoothing problem, and thus the solution is made in an "off-line" sense. The filtering problem for the class of systems introduced here has been presented elsewhere (1),(3).

B. Mathematical Statement of the Problem

Notation:

Let Ω be a simply connected open set in R^r. Points of Ω are denoted by $x = (x_1,\ldots,x_r)$. The boundary of Ω is denoted by Γ which is assumed to be regular. Let t denote time. Define the sets

$$\Sigma = \Gamma x(0,T]$$

$$Q = \Gamma x(0,T] \ .$$

Let $L^2(\Omega)$, $L^2(\Sigma)$ denote square integrable functions (equivalence classes) and define $V = L^2(\Sigma)\ xL^2(\Omega)$. Evidently $L^2(\Sigma)$, $L^2(\Omega)$ and V are Hilbert spaces under the usual inner product and norm.

State Evolution Process:

$$\frac{\partial y}{\partial t}(x,t;\underline{u})+A[y(x,t;\underline{u})] = f(x,t) \qquad x,t \in Q \qquad (1)$$

$$y(s,t) = u_1(s,t) \qquad s,t \in \Sigma \qquad (2)$$

$$y(x,o) = u_2(x) \qquad x \in \Omega \qquad (3)$$

where the hypothesis on $f(x,t)$, $u_1(s,t)$ and $u_2(x)$ is: $f(\cdot,\cdot) \in L^2(Q)$, $u_1(\cdot,\cdot) \in L^2(\Sigma)$ and $u_2(\cdot) \in L^2(\Omega)$. In addition, $A[\cdot]$ is an elliptic partial differential operator:

$$A[\psi] = -\sum_{i,j=1}^{r} \frac{\partial}{\partial x_i}[a_{ij}(x)\frac{\partial}{\partial x_j}\psi(x,t)] + a_o(x)\,\psi(x,t)$$

with $a_{ij}(x)$, $a_o(x)$ bounded, measurable and

$$\sum_{i,j=1}^{r} a_{ij}(x)\,\xi_i\,\xi_j \geq \alpha(\xi_1^2 + \ldots + \xi_r^2) \ , \ \alpha > o$$

$$a_o(x) > \alpha$$

for almost every $x \in \Omega$.

Quadratic (cost) Functional:

Define

$$\underline{v} = [v_1(s,t) \vdots v_2(x)]^T \ ; \ \underline{v} \in V$$

$$\underline{u}^* = [u_1^*(s,t) \vdots u_2^*(x)]^T \ ; \ \underline{u}^* \in V$$

$$\underline{z} = \underline{u}^* + \underline{K}^T\,\underline{N}(t) \ ; \ \underline{z} \in V$$

$$z(x,t) = y(x,t;\underline{u}^*)+K_o N_o(t) \ ; \ z(\cdot,\cdot) \in L^2(Q)$$

In the above the K's are constants and $\underline{N}(t)$ and $N_o(t)$ are error processes in appropriate Hilbert spaces.

Remark

$\underline{u}^*$ is a vector of the true initial and boundary conditions for the system S. $\underline{z}$ and $z(x,t)$ are the input (I) and output (O) measurements, respectively. The measurement $z(x,t)$ given is physically unrealistic but is taken for simplicity. Parallel results have been obtained for the discrete measurement case, that is,

$$z(x,t) = \sum_{i=1}^{\nu} y(x^i,t;\underline{u}^*) + K_o\,N_o(t)$$

and will be reported.

The quadratic functional $J(\underline{v})$ is given by:

$$J(\underline{v}) = ||y(x,t;\underline{v})-z(x,t)||^2_{L^2(Q)}+||\underline{v}-\underline{z}||^2_V \qquad (4)$$

If we define

$$a(\underline{v},\underline{v}) = ||y(x,t;\underline{v})-y(x,t;\underline{o})||^2_{L^2(Q)}+||\underline{v}||^2_V \qquad (5)$$

$$\ell(\underline{v}) = -\{(y(x,t;\underline{v})-y(x,t;\underline{o}),y(x,t;\underline{o})-z(x,t))_{L^2(Q)} - (\underline{v},\underline{z})_V\} \qquad (6)$$

$$c = ||y(x,t;\underline{o})-z(x,t)||^2_{L^2(Q)}+||\underline{z}||^2_V \qquad (7)$$

Then (4) is equivalent to:

$$J(\underline{v}) = a(\underline{v},\underline{v}) - 2\ell(\underline{v}) + c \qquad (8)$$

We have the following:

Proposition 1

$a(\underline{v},\underline{w})$ is a coercive, continuous, bilinear form on V. $\ell(v)$ is a continuous linear form on V.

Using the results of (2), it is possible to give the following

characterization of the optimal estimate $\underline{u}$, stated as a theorem.

Theorem 1

There exists one and only one $\underline{u} \in V$ such that $J(\underline{u}) = \inf_{v \in V} J(\underline{v})$ and it is characterized by

$$a(\underline{u},\underline{v}) = \ell(\underline{v}) \quad \text{for all } v \in V \tag{9}$$

Remark

(9) is the (formal) differential of the functional J evaluated at $\underline{u}$, denoted here by $\delta J(\underline{u})$. In terms of the gradient $\underline{G}(\underline{u})$,

$$\delta J(\underline{u}) = (\underline{G}(\underline{u}),\underline{v})_V$$

An equation for $\underline{G}(\underline{u})$ is given in the following Proposition:

Proposition 2

$\underline{G}(\underline{u})$ is given by

$$\underline{G}(\underline{u}) = \begin{matrix} -\frac{\partial p}{\partial \nu_{A^*}}(s,t)+u_1(s,t)-z_1(s,t) \\ p(x,o)+u_2(x) - z_2(x) \end{matrix} \tag{10}$$

where $p(\cdot,t) \in H_0^1(\Omega)$ (the first Sobolev space) and is the unique solution of

$$\begin{aligned} &-\frac{\partial p}{\partial t}(x,t;\underline{u})+A^*[p(x,t;\underline{u})]=y(x,t;\underline{u})-z(x,t) \\ &p(s,t) = 0 \\ &p(T) = 0 \end{aligned} \tag{11}$$

Remark

It is convenient to define the gradient $G(\underline{u})$ in terms of Eqs. (10) and (11). For, if we put equations (5) and (6) into Eq. (9), we obtain

$$\begin{aligned} &\int_Q [y(x,t;\underline{u})-z(x,t)][y(x,t;\underline{v})-y(x,t;\underline{o})]dxdt \\ &+\int_\Sigma [u_1(s,t)-z_1(s,t)]v_1(s,t)dsdt \\ &+\int_\Omega [u_2(x)-z_2(x)]v_2(x)dx = 0 \end{aligned} \tag{12}$$

While (12) characterizes the optimum choice of $\underline{u}$, the numerical selection of $\underline{u}$ is obscure. Noting that the variable adjoint to $y(x,t;\underline{u})$ evolves according to (11), and that the R.H.S. of the evolution equation for $p(x,t;u)$ appears in (12), we substitute where appropriate and obtain

$$\begin{aligned} &\int_\Sigma [-\frac{\partial p}{\partial \nu_{A^*}}(s,t)+u_1(s,t)-z_1(s,t)]v_1(s,t)dsdt \\ &+ \int [p(x,o)+u_2(x)-z_2(x)]v_2(x)dx=0 \end{aligned} \tag{13}$$

Comparing (13) with the definition of $\delta J(\underline{u})$, we immediately extract $\underline{G}(\underline{u})$ given by (10). The choice of $\underline{u}$ is now conceptually clear, since

$$\underline{G}(\underline{u}) = 0 \tag{14}$$

is equivalent to Eq. (9). We attend now to an algorithm which accomplishes (14) by successive approximation.

C. The Programming Algorithm

The determination of $\underline{u}$ which satisfies (9) (and which solves the identification problem) is made by generating a minimizing sequence $\{\underline{u}^i\}$ which converges (weakly) to $\underline{u} \in V$. This sequence is generated by employing conjugate directions of search $\{\underline{s}^i\}$ on the functional $J(\underline{v})$, (4), (5). The algorithm for generating $\{\underline{s}^i\}$ and $\{\underline{u}^i\}$ together with the convergence property are now considered.

Algorithm:

(1) Select $\underline{u}^o \in V$ (Initial guess)

(2) Evaluate $G(\underline{u}^o)$. If $G(\underline{u}^o) \neq 0$, then for the (i+1)st iteration, (i = 0,1,2...) proceed as follows:

(3)
$$\begin{aligned} &\underline{u}^{i+1} = \underline{u}^i + \alpha^i \underline{s}^i \\ &\underline{s}^o = -\underline{G}(\underline{u}^o) \\ &\underline{s}^{i+1} = -\underline{G}(\underline{u}^{i+1}) + \beta^i \underline{s}^i \\ &\beta^i = \frac{(\underline{G}(\underline{u}^{i+1}),\underline{G}(\underline{u}^{i+1}))_V}{((\underline{G}(\underline{u}^i),\underline{G}(\underline{u}^i))_V} \end{aligned}$$

In addition, α^i is chosen so that

$$J(\underline{u}^i + \alpha^i \underline{s}^i) = \inf_{\gamma^i \in R^1} J(\underline{u}^i + \gamma^i \underline{s}^i)$$

It is possible to obtain an explicit expression for α^i:

$$\alpha^i = \frac{(\underline{G}(\underline{u}^i),\underline{G}(\underline{u}^i))_V}{a(\underline{s}^i,\underline{s}^i)}$$

Theorem 2

The sequence $\{\underline{u}^i\}$ converges weakly to a unique $\underline{u} \in V$ and that limit $\underline{u}$ has the property that

$$J(\underline{u}) = \inf_{\underline{v} \in V} J(\underline{v})$$

We remark that the computation of $\underline{G}(\underline{u}^i)$ involves the sequential solution of (1) forwards in time (starting from $u_2^i(x)$ and evolving under the influence of $u_1^i(s,t)$) and then the solution of (11) backwards in time to recover $p(\cdot,\cdot;u^i)$ and hence $\underline{G}(\underline{u}^i)$. These solution "directions" are numerically stable. Thus to obtain $\underline{G}(\underline{u}^i)$, it is necessary to solve the PDE's given by (1), (2),(3), and (11). We approximate these solutions by the method of Galerkin, in the manner suggested by Lions (6). The Galerkin approximations $y_m(x,t;\underline{u})$ and $p_m(x,t;\underline{u})$ are given by

$$y_m(x,t;u)= \sum_{i=1}^{m} y_i(t)w_i(x); \quad \{w_i(\cdot)\}_{i=1,2..} \text{ a basis in } L^2(\Omega)$$

$$p_m(x,t;\underline{u}) = \sum_{i=1}^{m} p_i(t)w_i(x)$$

where $y_i(t)$ and $p_i(t)$ satisfy the following ordinary differential equations.

$$\frac{dy_i}{dt}(t;\underline{u})+\lambda_i y_i(t;\underline{u})=f_i(t)-u_{1i}(t);\quad y_i(0)=u_{2i}\quad t\in(0,T)$$

$$-\frac{dp_i}{dt}(t;\underline{u})+\lambda_i p_i(t;\underline{u})=y_i(t;\underline{u})-z_i(t);\quad p_i(T)=0\quad t\in[0,T)$$

the λ_i are the eigenvalues of the following Sturm Lionville equation:

$$A[w(x)] - \lambda w(x) = 0 \qquad x \in \Omega$$
$$w(0) = 0$$
$$w(1) = 0$$

and

$$f_i(t) = (f,w_i)_{L^2(\Omega)}$$
$$z_i(t) = (z,w_i)_{L^2(\Omega)}$$
$$u_{1i}(t) = (z_1,\frac{\partial w_i}{\partial \nu_{A^*}})_{L^2(\Gamma)}$$
$$u_{2i} = (z_2,w_i)_{L^2(\Omega)}$$

Remark

The choice of the basis functions $\{w_i(x)\}_{i=1,2..}$ is a question of significant importance. Experience with those reported here indicate poor approximation of $y(\cdot,\cdot;\underline{u})$ and $p(\cdot,\cdot;\underline{u})$ unless m is large. However, because of the decoupling afforded they are convenient. For, we found that for small m.

$$G_m(\underline{u}) \to 0$$

where $G_m(\underline{u})$ is computed according to Eq. (15). (Note that $\underline{u}$ is not approximated but the definition of the gradient is). There are other selections for $\{w_i\}$ possible, perhaps the most appealing of which are the splines of interpolation (1).

It should be emphasized that $G_m(\underline{u}^i)$ determined from the Galerkin approximation is an approximation to $\underline{G}(\underline{u}^i)$, given by

$$G_m(\underline{u}^1) = \begin{matrix} -\sum_{j=1}^{m} p_j(t)\frac{\partial w_j(s)}{\partial \nu_{A^*}} + u_1^i(s,t)-z_1(s,t) \\ \sum_{j=1}^{m} p_j(o)\, w_j(x) + u_2^1(x) - z_2(x) \end{matrix} \qquad (15)$$

The question of convergence of $G_m \to G$ will be reported in a later paper (7). Experience with non trivial examples indicate a striking convergence of the overall algorithm. In

particular, numerical results have been obtained for the following system:

$$\frac{\partial y(x,t)}{\partial t} - \frac{\partial^2 y(x,t)}{\partial x^2} = 212.0 \qquad 0 < x < 1 \;;\; 0 \le t \le T$$
$$y(s,t) = u_1(s,t) \qquad s = 0,\ 1 \;;\; 0 < t \le T$$
$$y(x,o) = u_2(x) \qquad 0 < x < 1 \;;\; t = 0$$

Input Measurements

$$z_1(0,t) = u_1^*(0,t) + k_1 N_1(t) \qquad t \in (0,1]$$
$$z_1(1,t) = u_1^*(1,t) + k_2 N_2(t) \qquad t \in (0,1]$$
$$z_2(x) = u_2^*(x) + k_3 \qquad x \in (0,1).$$

where u^* is the true state of nature and is defined by:

$$u_1^*(0,t) = 70 + 10 \sin 2\pi t$$
$$u_1^*(1,t) = 54.5$$
$$u_2^*(x) = 70\, e^{-0.25x}$$

and $N_1(t)$, $N_2(t)$ are random telegraph signals with amplitude ±1.0. k_1 and k_2 are variables chosen to affect the signal to noise ratio and k_3 is an arbitrary bias on the initial condition, also chosen to affect the signal to noise.

Output Measurements

(1) $z(x,t) = y(x,t;\underline{u}^*) + k_o N_o(t) \qquad x,t \in (0,1) \times (0,1)$

where $N_o(t)$ is a random telegraph signal with amplitude ±1.0 and k_o is a constant which alters the signal to noise ratio. It is assumed that $N_o(t)$, $N_1(t)$, $N_2(t)$ are mutually independent random variables.

(2) Measurements were taken at 4 points of the spatial domain.

$$z(x^i,t) = y(x^i,t;\underline{u}^*) + k_o^i N_o^i(t),\ (i = 1,2,3,4)$$
$$x^1 = 0.2,\ x^2 = 0.4,\ x^3 = 0.6,\ x^4 = 0.8.$$

The minimization of $J(\underline{v})$ is displayed in Figure 1 for both measurement processes A and B. The behaviour of one of the elements of $G_m(\underline{u})$, namely $G(u_1)$ is displayed in Figure 2. It was found that three iterations were sufficient to determine $\underline{u}$ for which $G_m(\underline{u}) = 0$ under measurement A, and five iterations solved case B.

D. Extensions to Parameter Identification

The conventional expedient of converting a linear parameter identification problem into a nonlinear state identification problem can be considered in the context of our results, providing the following approximations are employed:

(1) Linearize the resulting nonlinear state equations

(2) Reduce the result to cannonical parabolic form by introducing, where appropriate, "diffusion terms" which are removed in the limit, as suggested in (8). That is, we make the linear equations "quasi reversible".

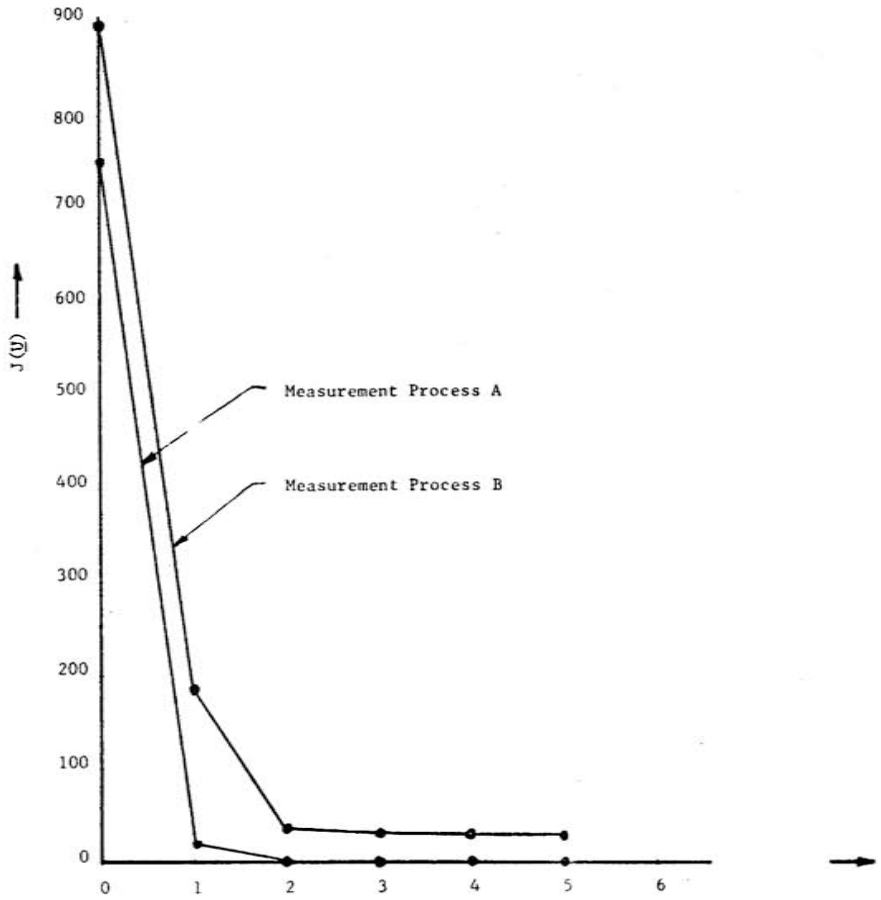

Figure 1: Minimization of the error functional.

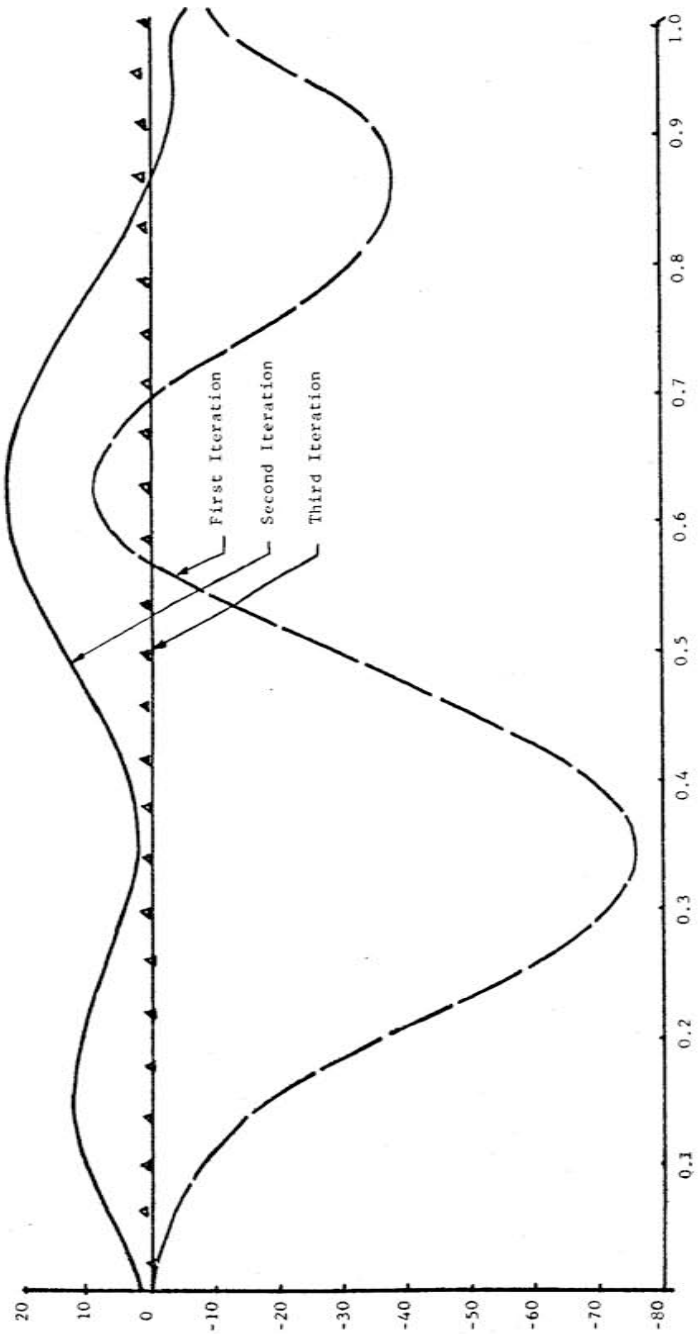

Figure 2: $G_M(U_1)$ under measurement process B with M = 8.

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