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OPTIMAL CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

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1. Introduction

The optimal control of systems governed by ordinary differential equations has reached a satisfactory state of completion. A comprehensive account of this theory may be found in the recent book of Lee and Markus [1]. Very general necessary conditions of optimality for abstract variational problems have recently been obtained [2], [3], [4]. These necessary conditions contain as special cases all known necessary conditions of optimality (including the Pontryagin Maximum Principle) for systems governed by ordinary differential equations as well as for certain classes of systems governed by functional differential equations [5], [6]. As yet however, it is not clear whether these necessary conditions for abstract variational problems can be specialized to obtain results for systems governed by partial differential equations. Even for systems governed by linear partial differential equations (for example of the parabolic type) with a general cost function there is no result analogous to the maximal principle. In this paper we shall attempt to summarize some of the results of optimal control that are known for systems governed by linear parabolic and hyperbolic equations. We shall present results on the existence of optimal controls, necessary conditions of optimality as well as some computational techniques for obtaining the optimal control that are currently available. The most important work here is undoubtedly due to Lions and his coworkers. An account of this work may be found in the recent book of Lions [7] and in the forthcoming thesis of Bensoussan [8]. Contributions have also been made by Balakrishnan [9], [10], [11]; Fattorini [12, [13], [14], [15], [16]; Friedman [17], [18], [19]; Phillipson-Mitter [20], [21]; Phillipson [22]; Russell [23], [24], [25] in the U.S.A. For an account of work in the U.S.S.R. see Egorov [26] and the recent survey article by Butkovsky, Egorov and Lurie [27]. For formal derivations of some of the results presented in this paper, see for example, Wang [28], [29]; Erzberger and Kim [30], Kim and Erzberger [31].

To obtain results that are not purely formal, it is first necessary to have a precise theory for the solution of boundary value problems for partial differential equations. The situation here is infinitely more complicated than in the ordinary differential case, where the Caratheodory existence theorem (see for example, [32]) furnishes us with an absolutely continuous solution even if the forcing term is measurable. Moreover in the ordinary differential equation case, we are dealing with initial value problems which are intrinsically simpler than boundary value problems. In the partial differential equation case, almost each case corresponding to the differential equation and the boundary condition has to be treated separately, care has to be taken that the formulated problem is well posed and finally that the solution is in the appropriate space (needless to say that the state of the system is infinite dimensional). Further problems that are important from a practical point of view, problems where control is exercised through the boundary and observations are made on the boundary require techniques which are sophisticated and delicate. Nevertheless once certain difficult theorems are accepted, the main results can be understood; and it is the author's belief that some of the results can be used for the solution of practical problems. This paper may be divided into none sections. In section 2 we consider minimization problems for convex functionals defined on a real Hilbert space, obtain an existence theorem and derive a maximum principle for such problems. In section 3, we consider applications of the theory developed in section 2 to optimal control problems for systems governed by ordinary differential equations. We also indicate the steps that need to be followed to solve optimal control problems for partial differential equations.

The study of control of partial differential equations begins in section 4. The notation as well as the function spaces to be used in the sequel are defined in section 4.1; examples of some representative physical systems whose control problems may be treated using the results of this paper are presented.

Section 5 is concerned with the optimal control of a class of linear parabolic equations (The differential operator satisfying a strong ellipticity condition) with a quadratic cost functional. The section begins with a study of optimal control problems for abstract evolution equations. The results are then illustrated by considering distributed and boundary control problems as well as estimation problems. Subsection 5.4 is devoted to obtaining a feedback controller for the linear quadratic problem. Subsection 5.9 presents various results on the time-optimal control problem.

Section 6 deals with the control of linear second order hyperbolic partial differential equations.

In section 7 we make some remarks on the existence question, while in section 8 we present some results on the controllability and observability of partial differential equations.

The final two sections are devoted to some comments on extensions of the present theory and to the numerical solution of optimal control problems for partial differential equations.

2. Minimization of Convex Functionals

Let U be a real Hilbert space and let U_{ad} be a closed convex subset of U. Let $v \to J(v)$ be a convex function from U_{ad} into R such that the following hold:

(2.1) $\quad J(v) \to +\infty \quad$ as $\quad \|v\| \to +\infty$, $v \in U_{ad}$,

(2.2) $\quad v \to J(v)$ is strongly lower semi-continuous.

We then have,

Theorem 2.1

There exists a $u \in U_{ad}$ such that

(2.3) $$J(u) = \operatorname*{Inf.}_{v \in U_{ad}} J(v). \blacksquare$$

If further, the function $v \to J(v)$ is strictly convex, then the minimizing element is unique. In many problems the set U_{ad} will be bounded. Assumption 2.1 is then unnecessary. The proof of the theorem essentially follows from the fact that under assumption (2.2), $v \to J(v)$ is weakly lower semi-continuous and from the well known fact that a closed convex set in a locally convex space is also weakly closed. Having obtained an existence result, it is necessary to characterize the minimizing element. In this direction we have,

Theorem 2.2

Assume that the function $v \to J(v)$ is strictly convex, differentiable (see Dieudonné [33], Chapter 8), and satisfies (2.1). Then the unique element $u \in U_{ad}$ satisfying $J(u) = \operatorname*{Inf}_{v \in U_{ad}} J(v)$ is characterized by,

(2.4) $\quad J'(u) \cdot (v-u) \geq 0 \quad \forall\, v \in U_{ad}. \blacksquare$

Remark 2.3
The above results are true when V is a reflexive Banach space. $\blacksquare$

In this paper we shall be mainly interested in a special class of convex functionals, positive definite quadratic functionals. Let us suppose that we are given the following data:

a) a continuous bi-linear form on U, which is symmetric,
$u,v \to \pi(u,v)$, $\pi(u,v) = \pi(v,u) \quad \forall\, u,v \in U$,

b) a continuous linear form $v \to L(v)$ on U,

c) and let U_{ad}, as before be a closed, convex subset of U.

We are now interested in minimizing J(v), where

(2.5) $\quad J(v) = \pi\,(v,v) - 2L(v).$

We need a further assumption:

π is coercive on U, that is,

(2.6) $\pi(v,v) \geq c \|v\|^2 \quad \forall v \in U$, where $c > o$ is a constant and $\|\cdot\|$ denotes the norm on U (in case of confusion the norm on the space X will be denoted by $\|\cdot\|_X$).

It is now easily verified that the function $v \to J(v)$ defined by (2.5) (together with assumption 2.6) is continuous, strictly convex and satisfies (2.1). From Theorem 2.1 it follows that there exists a unique $u \in U_{ad}$ minimizing $J(v)$. In this case the derivative $J'(u)$ can be explicitly calculated and we obtain,

Theorem 2.4.

The unique $u \in U_{ad}$ which minimizes $J(v)$ is characterized by,

(2.7) $\pi(u,v-u) \geq L(v-u) \quad \forall \; v \in U_{ad}$.

An inequality of the type (2.7) is termed a variational inequality. For a study of such inequalities, see Lions-Stampacchia [34]▌

Remark 2.5

Inequalities (2.4) and (2.7) are in effect maximum principles for the respective abstract variational problems.▌

The special case where $U_{ad} = U$, the whole space, is of interest. In this case in (2.7) we may take $v = u \pm \phi$, where ϕ is arbitrary giving us,

(2.8) $\pi(u,\phi) = L(\phi) \quad \forall \phi \in U.$▌

In case $\pi(u,v)$ is not necessarily symmetric and only satisfies,

(2.8) $\pi(v,v) \geq 0 \quad \forall v \in U$,

then we no longer have uniqueness (the set of minimizing solutions X may be an empty set; if U_{ad} is bounded then $X \neq \phi$). Nevertheless we have,

Theorem 2.6.

The set X of solutions $u \in U_{ad}$ satisfying

(2.9) $\pi(u,v-u) \geq L(v-u) \; \forall v \in U_{ad}$,

is a closed, convex subset of U.▌

3. Applications to Optimal Control Problems for Systems Governed by Ordinary Differential Equations

Before considering problems of partial differential equations, let us illustrate the theory of the previous section by considering some optimal control problems for ordinary differential equations.

3.1 Notation and Problem Formulation

Let us consider the differential equation,

$$(3.1) \qquad \frac{dy}{dt} + A(t)y = B(t)u$$

with the data,

(3.2)

- $y(t)$: state of the system, $y(t) \in R^N$,
- $A(t)$: linear operator from $R^N \to R^N$ ($N \times N$ matrix) the elements $a_{ij}(t)$ are bounded, measurable functions of $t \in [0,T]$, $T > 0$ given.
- Space of Controls : H a Hilbert space on R,
- Constraint Set : $U_{ad} \subset H$, closed, bounded, convex.
- $B(.)$: linear operator from $H \to R^N$
  $\forall h \in H$, $t \to B(t)h$ bounded measurable function,
- $y(o) = y_o = 0$ given vector in R^N.

For a given control $v \in U_{ad}$, equation (3.1) admits a unique solution $y(t;v)$.

Let,

$$(3.3) \qquad \begin{cases} y_d \text{ be a given function in } L^2(0,T,R^N)^{(1)}, \\ \chi \text{ be a given vector in } R^N, \\ \alpha,\beta \text{ be constants } \geq 0. \end{cases}$$

Let the cost function be,

$$(3.4) \qquad J(v) = \alpha \int_0^T |y(t;v) - y_d(t)|^2_{R^N}\, dt + \beta\, |y(T;v) - \chi|^2_{R^N}.$$

Problem : Find $u \in U_{ad}$ such that $J(u) = \inf_{v \in U_{ad}} J(v)$.

In order to solve the problem, we shall transform it to a problem of minimizatic of a coercive form on H.

3.2 Transformation of the Problem

Let $v \to y(t;v)$ be a continuous linear map of $H \to L^2(0,T,R^N)$ and let $y(.\,;u)$ be the function $t \to y(t;v)$ such that

$$(3.5) \qquad y(.\,;v) = Cv, \text{ where}$$

C is a continuous linear operator of $H \to L^2(0,T,R^N)$.

Similarly, let

$$(3.6) \qquad y(T;v) = Dv, \text{ where}$$

D is a continuous linear operator of $H \to R^N$.

Then (3.4) may be equivalently written as,

$$(3.4 \text{ bis}) \quad J(v) = \alpha \| Cv - yd \|^2_{L^2(0,T,R^N)} + \beta\, |Dv - \chi|^2_{R^N}.$$

Expanding the right hand side of (3.4), we may write (3.4 bis) as

$$(3.7) \qquad J(v) = \pi(v,v) - 2(f,v)_H, \quad \text{where}$$

$$\pi(v,v) = \alpha\, \|Cv\|^2_{L^2(0,T,R^N)} + \beta\, |Dv|^2_{R^N} \geq 0,$$

(1) $L^2(o,T,R^N)$ = space (equivalence class) of n-vector-valued functions $f(t)$ such that $\int_o^T \|f(t)\|^2_{R^N} dt < \infty$.

and $f = \alpha C^* y_d + \beta D^* \chi \in H$ (given).

Here $C^*: L^2(0,T,R^N) \to H$ is the adjoint of C (identifying $H = H'$, $L^2(0,T,R^N) \equiv (L^2(0,T,R^N))'$, prime denoting the dual space).

3.3 Necessary and Sufficient Conditions

Using Theorem 2.6 and Theorem (2.4), the optimal control u is characterized by

$$(3.8) \quad \alpha \int_0^T (y(t;u), y(t;v)-y(t;u))_{R^N}\, dt + \beta\, (y(T;u), y(T;v)-y(T;u))_{R^N}$$
$$\geq \alpha \int_0^T (y_d(t), y(t;v)-y(t;u))_{R^N}\, dt + \beta\, (\chi, y(T;v)-y(T;u))_{R^N} \quad \forall\, v \in U_{ad}.$$

Let us now introduce the adjoint state $p(t;u)$ (p depends on u via y) by the equation,

$$(3.9) \quad \begin{cases} \dfrac{dp}{dt} + A^*(t)p = \alpha y(t;u) - \alpha y_d \\ p(T;u) = \beta y(T;u) - \beta\chi. \end{cases}$$

$p(t,u)$ is thus uniquely defined.

Let us multiply both sides of the differential equation (3.9) by $y(t;v)-y(t;u)$, and integrate from 0 to T. Then using integration by parts, we obtain,

$$(3.10) \quad \alpha \int_0^T (y(t;u)-y_d(t), y(t;v)-y(t;u))_{R^N}\, dt + \beta\, (y(T;u)-\chi, y(T;v)-y(T;u))_{R^N}$$
$$= \int_0^T (p(t;u), B(t)v(t)-B(t)u(t))_{R^N}\, dt.$$

Hence (3.8) may be equivalently written as,

$$(3.11) \quad \int_0^T (p(t;u), B(t)v - B(t)u)_{R^N}\, dt \geq 0 \quad \forall\, v \in U_{ad}.$$

(3.11) is the Integral Maximal Principle for the problem.

Defining,

$$B(.) : H \to L^2(0,T;R^N)$$

$$B(.)^* : L^2(0,T,R^N) \to H \quad \text{(identifying dual spaces)},$$

(3.11) may finally be written as,

$$(3.12) \quad (B^*(.)\, p(.;u), v-u)_H \geq 0 \quad \forall\, v \in U_{ad}.$$

Summarizing,

Theorem 3.1

A necessary and sufficient condition for u to be an optimal control is:

$$(3.13) \quad (B^*(.)p(.;u),u)_H = \operatorname*{Inf.}_{v \in U_{ad}} (B^*(.)p(.;v),v)_H,$$

where p is defined by,

(3.14) $$\frac{dy}{dt} + A(t)y = B(t)u$$

$$y(o) = 0$$

(3.15) $$-\frac{dp}{dt} + A^*(t)p = \alpha\ (y(t;u)-y_d)$$

$$p(T;u) = \beta\ (y(T;u)-\chi). \ \blacksquare$$

Example 3.2

Let the hypotheses and data of the previous theorem hold and let

(3.16) $$U_{ad} = \left\{v \;\middle|\; \|v\|_H \leq M\right\}.$$

Then from Theorem (3.1),

(3.17) $$u = -\frac{M}{\|B^*(.)p(.;u)\|_H} B^*p(.;u) .$$

We may then eliminate u from equation (3.14), giving us

$$\frac{dy}{dt} + A(t)y + \frac{M}{\|B^*\ p\|_H} B\ B^*p = 0 \quad ; \quad y(o) = 0$$

$$\frac{dp}{dt} + A^*(t)p = \alpha\ (y-y_d) \ ; \ p(T) = \beta(y(T)-\chi).$$

If $\{y,p\}$ is a solution (not necessarily unique) of the above system,

$$u = -\frac{MB^*p}{\|B^*p\|_H} . \ \blacksquare$$

3.3 Local Constraints

In many problems the control constraint set is specified locally instead of globally as in (3.2). For such problems (3.13) may be transformed into a local maximal principle. We first need a lemma which can be easily proved.

Lemma 3.3

Let U be a Hilbert space, $H = L_2(0,T;U)$ and K_U be a closed, bounded convex subset of U. Let,

(3.18) $$U_{ad} = \left\{v \;\middle|\; v\epsilon H,\ v(t)\epsilon K_U \text{ almost everywhere}\right\}.$$

Theorem 3.4

Let the assumptions of Theorem 3.1 be true and let U_{ad} be given by (3.18). Then a necessary and sufficient condition for u to be an optimal control is

(3.19) $$(p(t;u),\ B(t)u(t))_{R^N} = \inf_{k\epsilon K_U} (p(t;u),\ B(t)k)_{R^N} \text{ almost everywhere,}$$

where p(t;u) is given by (3.14), (3.15).

Proof:

We shall show that (3.13) (or 3.11) and (3.19) are equivalent conditions.

(3.19) clearly implies (3.13) (or 3.11).

To show the equivalence on the other side, let us recapitulate a few facts.

Let Y be a separable Hilbert space and let $g : [0,T] \to Y$ be such that $g \in L(0,T;Y)$. Let O_j be a neighborhood of t for $t \in]o,T[$. Then,

$$\frac{1}{\text{mes } O_j} \int_{O_j} g(\sigma)d\sigma \to g(t) \quad \text{almost everywhere as mes}O_j \to 0.$$

The points at which the above is satisfied are termed the Lebesgue points of g.

Let $s \varepsilon]o,T[$ and let O_j be a neighborhood of s. Define

$$u_j = \begin{cases} k \text{ in } O_j, \ k \text{ arbitrary in } K_U \\ u \text{ in }]o,T[\sim O_j \end{cases}$$

Hence $u_j \in U_{ad}$ and $u_j(t) \in K_U$ almost everywhere. (3.11) reduces to

$$\frac{1}{\text{mes}O_j} \int_{O_j} (p(t;u), B(t)(k-u(t)))_{R^N} \, dt \geq 0.$$

Let us now choose s to be a Lebesgue point of $(p(t;u), B(t)u(t))_{R^N}$ and of $B^*(t)p(t;u)$. Let us suppose that as $j\to\infty$, $\text{mes}O_j \to o$.
Hence,

$$(p(s;u), B(s)k)_{R^N} - (p(s;u), B(s)u(s))_{R^N} \geq 0 \text{ , } \forall s \text{ so chosen.}$$

But the complement of the set of points that are Lebesgue points has measure zero and hence (3.19) is satisfied almost everywhere.▌

Example 3.5

If $K_U = \{k \mid \|k\|_{R^N} \leq M\}$, then (3.19) gives us

$$(3.20) \qquad u(t) = -\frac{M\, B^*(t)}{\|B^*(t)p(t;u)\|_{R^N}} \quad \text{provided } B^*(t)p(t;u) \neq 0.▌$$

Remark 3.4 (Ordinary Differential Equations in a Hilbert Space)

Let the state space Y be a Hilbert space instead of R^N, and let A be a mapping of $[o,T]$ into the Banach space $L(Y;Y)$ of continuous linear mappings of Y into itself and let B(.) be a linear operator of $H \to Y$. Then if A and B(t)u be regulated in $[0,T]$ all the previous considerations hold(1).▌

Remark 3.5 (Partial Differential Equations)

When we are considering control problems for partial differential equations, A will in general be an unbounded operator and the situation is somewhat more complicated. Nevertheless, by following the steps indicated

(1) See for example Dieudonne [33], Chapter X, section 6. for appropriate existence theorem for differential equation.

below we shall show that the techniques used in this section for ordinary differential equations can be effectively adapted to deal with interesting problems in the control of partial differential equations. The steps to be followed in sequence are:

a) Obtain existence and uniqueness theorem for partial differential equation describing the evolution of the state of the system.
b) Transform the original optimal control problem into an abstract variational problem of minimizing a convex functional on a closed, convex set.
c) Use the theory of Variational Inequalities to characterize the optimal control.
d) Introduce the adjoint equation and study the existence, uniqueness and regularity properties of solution.
e) For boundary control and boundary observation problems, use appropriate restriction theorems due to Lions-Magenes [35] to ensure that the state and adjoint functions as well as their normal derivatives are in the appropriate function space.
f) Transform the result of step c) to a concrete maximum principle using Green's Formula (which needs to be justified, see Lions-Magenes [35]).
g) Study the system of necessary conditions.

We shall carry out this program for a large class of control problems for linear parabolic and hyperbolic equations in the following sections. We shall also present results on the linear quadratic problem and show how a feedback controller may be synthesized for this problem.∎

Remark 3.6

It should be remarked that the results presented in this paper may also be applied to study optimal control problems for functional differential equations with a quadratic cost function.∎

4. Control of Systems Governed by Partial Differential Equations

4.1 Notation

Let $x = \{x_1, ---, x_n\}$ denote the space variable; x ranges over an open set $\Omega \subset R^n$ with boundary Γ: Γ is assumed to be sufficiently regular.

t denotes time; in general $t \in]o,T[$, $T < \infty$.

We set,

$$Q = \Omega \times]o,T[\ , \ \Sigma = \Gamma \times]o,T[.$$

The controls will be denoted by u,v etc. and will in general be an element of a real Hilbert space U; U_{ad} is a closed, convex subset of U and represents the set of admissible controls.

The state of the system corresponding to a control v is denoted by y(v); in the cases we consider in the paper, y(v) is a function of $x \in \Omega$ and $t \in]o,T[$, i.e. $y(v) = y(x,t;v)$.

p(v) denotes the adjoint state (co-state) and $p(v) = p(x,t;v)$.

We shall have occasion to use the following function spaces:

$C^k(\bar{\Omega})$ = space of functions which are k-times continuously differentiable on $\bar{\Omega}$ (the closure of Ω), $k \geq o$ integer: We clearly have analogous notation for Q, Γ, Σ.

$L^2(\Omega)$ = space (equivalence class) of functions which are square integrable on Ω.

$H^m(\Omega)$ = Sobolev space of order m
= space of functions ϕ such that

$$\phi \in L^2(\Omega), \frac{\partial \phi}{\partial x_i} \in L^2(\Omega), ---, D^\alpha \phi \in L^2(\Omega) \ \forall \alpha, |\alpha| \leq m, \quad \alpha = \{\alpha_1, ---, \alpha_n\}, |\alpha| = \alpha_1 + \alpha_2 + --- + \alpha_n.$$

$H^m_o(\Omega) = \{\phi \mid \phi \in H^m(\Omega), D\phi = 0 \text{ on } \Gamma, \ \alpha \leq m-1\}$.

$L^2(o,T,E)$ = space (equivalence class) of functions f defined on [o,T] with values in a Hilbert space E such that

$$\int_o^T \| f(t) \|_E^2 dt < + \infty .$$

$L^\infty(o,T,E)$ = space (equivalence class) of functions defined on [0,T] with values in E which are essentially bounded.

$D(\Omega)$ = space of infinitely differentiable functions in Ω with compact support in Ω, endowed with the inductive limit topology of L. Schwartz (1).

$D'(\Omega)$ = Dual space of $D(\Omega)$ = space of distributions on Ω.

$D'(]0,T[;X)$ = space of distributions on $]0,T[$ with values in a Banach space X.

(1) See for example Horvath [36].

Remark 4.1

For many control problems related to partial differential equations, the use of Sobolev spaces and distributions are unavoidable. Let us recall that $D'(\Omega)$ = space of distributions on Ω = space of continuous linear forms on $D(\Omega)$; if $f \in D(\Omega)$, its derivative $\frac{\partial f}{\partial x_i}$ is the unique element of $D'(\Omega)$ defined by,

$$\left\langle \frac{\partial f}{\partial x_i}, \phi \right\rangle = - \left\langle f, \frac{\partial \phi}{\partial x_i} \right\rangle \quad \forall \phi \in D(\Omega)$$

where $\langle , \rangle$ denotes the scalar product between $D'(\Omega)$ and $D(\Omega)$.
In general, we shall write distributions as functions; if $f \in D'(\Omega)$, $\phi \in D(\Omega)$, we shall write

$$\langle f, \phi \rangle = \int_\Omega f(x)\phi(x)dx .$$

In particular, if $a \in \Omega$ the Dirac mass + 1 at the point a is denoted by $\delta(x-a)$ and defined by

$$\langle \delta(x-a), \phi \rangle = \int_\Omega (x-a)\phi(x)dx = \phi(a).$$

In the sequel derivatives will often have to be interpreted in the distribution sense.

4.2 Representative Physical Systems

We now consider examples of some physical systems whose control problems can be treated with the theory to be developed in later sections.

(i) Parabolic Equation:

Temperature $y(x,t)$ at a point $x \in \Omega =]0,1[$ and $t \in]0,1[$ of a medium which is exchanging heat in a predominently diffusive manner with its environment, which at that point in space and at that time, is at a temperature $f(x,t)$:

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = f(x,t)^{(1)}. \tag{4.1}$$

Let the initial data be $y(x,o) = y_o(x)$, $0<x<1$.

The boundary data may, for example, be:

$$\left.\begin{array}{ll} y(o,t) = u_1(o,t) & o<t<1 \\ y(1,t) = u_1(1,t) & o<t<1 \end{array}\right\} \quad \text{Dirichlet}$$

(1) The "transport" term $\frac{\partial y}{\partial x}$ occurring in the description of many heat exchange problems can be avoided by using a linear transformation leading to the form given in (4.1).

or $-\frac{\partial y}{\partial x}(o,t) = v(t)$; $\frac{\partial y}{\partial x}(1,t) = 0$, $0<t<1$ – Neumann

The boundary condition may also be mixed.
The Dirichlet boundary condition corresponds to the case where there is no insulation, the Neumann to the case where there is full insulation and the mixed to the case where there is partial insulation.

(ii) Hyperbolic Equation: Vibrations in an elastic medium.
Consider for example the case of longitudinal displacement $y(x,t)$ at a point x in an elastic rod, at a certain time t, experiencing a dynamic anial load at the ends:

$$\frac{\partial^2 y}{\partial t^2} - \frac{P}{EA}\frac{\partial^2 y}{\partial x^2} = 0. \tag{4.2}$$

The load is applied in one of three ways:
(a) Direct Axial Loading
(b) Transverse Bending
(c) Loading through an elastic support

These three situations correspond to Dirichlet, Neumann and Mixed boundary conditions.

5. Optimal Control of Linear Parabolic Equations

In this section we shall study some representative optimal control problems for certain classes of linear parabolic equations where the control may be distributed control or boundary control. To do this, it is convenient to first study the optimal control of abstract evolution equations. We shall follow in sequence the steps we have indicated in Remark 3.5.

5.1 Equations of Evolution

5.1.1 Problem Formulation and Existence Theorem

We are given Hilbert spaces V and H respectively. V' is the dual space of V. We assume

(5.1) $V \subset H$, the injection V into H is continuous and V is dense in H.

H is identified with its dual space, and H may be identified with a subspace of V' and we have

$$V \subset H \subset V'. \tag{5.2}$$

A family of bi-linear forms on V is given:

$$\phi, \Psi \to a(t;\phi,\psi) \quad \text{for each } t \in]0,T[.$$

For this family we assume,

(5.3) $\forall\, \phi,\psi \in V$, the function $t \to a(t;\phi,\psi)$ is measurable on $]0,T[$ and $a(t;\phi,\Psi) \le c\,\|\phi\| \cdot \|\psi\|$

and there exists a λ such that

$$(5.4) \qquad a(t;\phi,\Psi) + \lambda |\phi|^2 \geq \alpha \|\phi\|^2, \ \alpha > 0, \ \forall\phi\epsilon V, \ t \ \epsilon \ ]0,T[,$$

where $|.|$ denotes the norm on H and $\|.\|$ denotes the norm of V. For each t, we may write,

$$(5.5) \qquad a(t;\phi,\Psi) = (A(t)\phi,\psi) \quad , \quad A(t)\phi \ \epsilon \ V',$$

the bracket denoting the scalar product between V and V'.

It may be seen that

$$(5.6) \qquad A(t)' \epsilon \ L \ (L^2(0,T;V); \ L^2(0,T;V')), \text{ that is}$$

if $f \ \epsilon \ L^2(0,T;V)$, $A(t)f$ is the function $t \to A(t)f(t) \ \epsilon \ V'$.

If $f \ \epsilon \ L^2(0,T;V)$ its derivative $\frac{df}{dt}$ may be considered to be continuous linear map from $D(]0,T[) \to V$ and defined by

$$\phi \to \frac{df}{dt}(\phi) = - f\left(\frac{d\phi}{dt}\right), \text{ where}$$

$$(5.7) \qquad f(\phi) = \int_0^T f(t)\phi(t)dt$$

(writing distributions as functions; the integral being a generalized integral with values in V). We may then show that $\frac{df}{dt}$ may be considered to be an element of $D'(]0,T[;V)$.

We now introduce the space,

$$(5.8) \qquad W(0,T) = \{f \mid f \ \epsilon \ L^2(0,T;V), \ \frac{df}{dt} \ \epsilon \ L^2(0,T;V')\}.$$

Endowed with the norm,

$$(5.9) \qquad \|f\|_{W(0,T)} = \left(\int_0^T \|f(t)\|^2 dt + \int_0^T \left\|\frac{df}{dt}\right\|^2_{V'} dt\right)^{1/2},$$

W(0,T) is a Hilbert space. It may be shown that all functions $f \ \epsilon \ W(0,T)$ are with eventual modification on a set of measure zero continuous from $[0,T] \to H$.

Consider now the <u>problem of evolution</u> : find $y \ \epsilon \ W(0,T)$ such that

$$(5.10) \qquad A(t)y + \frac{dy}{dt} = f, \ f \text{ given in } L^2(0,T;V)$$

and

$$(5.11) \qquad y(0) = y_o, \ y_o \text{ given in H.}$$

For this problem, we have the following existence and uniqueness theorem:

<u>Theorem 5.1</u>

Under the hypotheses (5.3) and (5.4) the above evolution problem admits a <u>unique solution</u> in W(0,T).

Furthermore the solution depends <u>continuously on the data</u> : the bi-linear map $f, y_o \to y$ is continuous map from

$$L^2(0,T;V') \times H \to W(0,T).$$

Sketch of Proof

The existence proof is a constructive proof. It consists in choosing a base in V (assuming it is separable), obtaining a finite dimensional approximation to (5.10) and then showing $y_m(t) \rightarrow y(t)$ as $t \rightarrow \alpha$ in an appropriate sense. The key idea in the limiting argument is to show that the sequence $\{y_m(t)\}$ is bounded and hence a weakly convergent subsequence may be extracted.

The uniqueness part uses standard arguments. ▌

5.1.2 The Control Problem

Armed with an existence theorem, we can now formulate an optimal control problem:

Let U be the Hilbert space of controls and let

(5.12) $\quad B \in L\,(U;\ L^2(0,T;V')).$

We assume that (5.3), (5.4) holds and let us denote by $y(v)$ the solution of

(5.13) $\quad \frac{dy}{dt} + A(t)y = f + Bv$

(5.14) $\quad y(v)\big|_{t=o} = y_o\ ,$

(5.15) $\quad y(v) \in L^2(0,T;V).$

Let the observation equation be,

(5.16) $\quad z\,(v) = Cy(v)$, where
$C \in L\ (W(0,T);\ H).$

Let the cost function be ,

(5.17) $\quad J(v) = \|Cy(v)-z_d\|^2_H + (Nv,v)_U$, where

(5.18) $\quad N \in L(U;U)\ ,\ (Nu,u)_U \geq u \quad u^2_U\ ,\ u > 0.$

Remark 5.2

Notice the analogy of this problem with the finite dimensional control problem considered in section 3: See formula (3.4 bis) with a cost of control term added. Our subsequent development will be along the lines of section 3, the techniques being more sophisticated (that is, various steps have to be mathematically justified). ▌

We are also given a set of admissible controls,

(5.19) $\quad U_{ad}$ = closed, convex subset of U.

We seek,

(5.20) $\quad \underset{v \in U_{ad}}{\text{Inf.}}\ J(v)$. ▌

5.1.3 Necessary and Sufficient Conditions

Let us first write J(v) in the form,

$$J(v) = \| C(y(v)-y_o) + Cy_o - z_d \|_H^2 + (Nv,v)_U .$$

If we set,

(5.21) $\pi(u,v) = (C[y(u)-y_o], C[y(v)-y_o]_H) + (Nu,v)_U$

(5.22) $L(v) = (z_d - Cy_o, C[y(v)-y_o])_H,$

the form $\pi(u,v)$ is a continuous bi-linear form on U and the form L(v) is a continuous linear form on U and we have,

(5.23) $J(v) = \pi(v,v) - 2L(v) + \| z_d - Cy(o) \|_H^2 .$

Since $\| z_d - Cy(o) \|_H^2$ is clearly ≥ 0, we have from (5.18)

Hence from Theorem 2.1 (and remarks following) we obtain,

Theorem 5.3 (Existence)

There exists a unique $u \in U_{ad}$ such that $J(u) = \inf_{v \in U_{ad}} J(v)$.]

From Theorem 2.4 we obtain,

Theorem 5.4 (Necessary and Sufficient Condition)

The unique $u \in U_{ad}$ is characterized by

(5.25) $(Cy(u)-z_d, C[y(v)-y(u)])_H + (Nu,v-u)_U \geq 0 \quad \forall\, v \in U_{ad}$.]

If we define,

(5.26) $\begin{cases} \Lambda = \text{canonical isomorphism of } H \text{ onto } H', \\ \Lambda_U = \text{canonical isomorphism of } U \text{ onto } U', \end{cases}$

the above formula reduces to,

(5.27) $(C^*\Lambda[Cy(u)-z_d], y(v)-y(u)) + (Nu,v-u)_U \geq 0 \quad \forall\, v \in U_{ad},$

where the first bracket denotes the duality between W(0,T) and W(0,T)'.

Introduce the adjoint state by,

(5.28) $-\frac{d}{dt} p(v) + A^*(t)p(v) = C^*\Lambda[Cy(v)-z_d] \quad \text{in }]o,T[$

(5.29) $p(T;v) = 0$

(5.30) $p(v) \in L^2(0,T;V).$

Then Theorem 5.1 (with time reversed) asserts the existence and uniqueness of a solution p(v) in W(0,T).

Now multiply both sides of (5.28) by $y(v)-y(u)$ and integrate from $[0,T]$. Using integration by parts (which is valid in $W(0,T)$ see Lions-Magenes [35]), we transform (5.27) (exactly as in the finite dimensional case) into

$$(5.31) \qquad (\Lambda_U^{-1} B^*p(u) + Nu, v-u)_U \geq 0 \quad \forall\, v \in U_{ad}.$$

Remark 5.5

Let us remark that we have in effect calculated the Frêchet derivative $\frac{1}{2} J'(u)$ to be equal to $B^*p(u) + \Lambda_U Nu$. ∎

Summarizing,

Theorem 5.6 (Necessary and Sufficient Conditions)

Under the assumptions of (5.3), (5.4) and (5.18), $u \in U_{ad}$ is an optimal control(1) if and only if

$$(5.32) \qquad (\Lambda_U^{-1} B^*p(u) + Nu, v-u)_U \geq 0 \quad \forall\, v \in U_{ad},$$

$$u \in U_{ad},$$

where $p(u)$ is defined by,

$$(5.33) \qquad \frac{dy}{dt}(u) + A(t)y(u) = f + Bu,$$

$$y(o;u) = y_o,$$

$$(5.34) \qquad -\frac{dp}{dt}(u) + A^*(t)p(u) = C^*\Lambda\,[Cy(u)-z_d],$$

$$p(T;u) = 0,$$

and

$$(5.35) \qquad y(u) \in L^2(0,T;V)\ ;\ p(u) \in L^2(0,T;V).$$ ∎

5.1.4 The Standard Linear Quadratic Problem

In this case $U_{ad} = U$ and (5.32) reduces to,

$$(5.36) \qquad \Lambda_U^{-1} B^*p(u) + Nu = 0,$$

and since N is invertible,

$$(5.37) \qquad u = -N^{-1}\Lambda_U^{-1} B^*p.$$

Also using (5.37) we may now eliminate u from equation (5.33), to give us a two-point boundary value problem analogous to the well known reduction of optimal control problems to two-point boundary value problems for finite-dimensional control problems. ∎

(1) A control u which minimizes $J(v)$.

5.1.5 The Case Where U_{ad} is a Ball

Let $U_{ad} = \{v \mid \|v\|_U \leq M\}$

Then if we set,

$\xi = \Lambda_U^{-1} B^* p(u) + Nu$ and if we assume $\xi \neq 0$, then

$$u = -M \frac{\xi}{\|\xi\|} \quad (5.38)$$

5.2 Example of a Mixed Dirichlet Problem with Distributed Control

Let $V = H_o^1(\Omega)$, $U = L^2(Q)$, B = identity map, and let $A(t)$ be defined as follows:

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y}{\partial x_j}\right), \text{ where} \quad (5.39)$$

$$\begin{cases} a_{ij} \in L^\infty(Q), \\ \sum_{i,j=1}^{n} a_{ij}(x,t)\, \xi_i \xi_j \geq \alpha\, (\xi_1^2 + \cdots + \xi_n^2),\ \alpha > o,\ \xi_i \in R \text{ almost everywhere in } Q. \end{cases} \quad (5.40)$$

$A(t)$ is thus a second order elliptic differential operator satisfying (5.40).

The state of the system is given by,

$$\begin{cases} \frac{\partial y}{\partial t} + A(t)y = f + u \quad \text{in } Q \\ y\big|_\Sigma = 0 \\ y(x,o) = y_o(x) \quad x \in \Omega. \end{cases} \quad (5.41)$$

The control is therefore distributed control.
If we take C = injection map of $L^2(0,T;V) \to L^2(Q)$, then

$H = L^2(Q) = H'$, Λ = identity map,
and equation (5.34) reduces to

$$\begin{cases} -\frac{\partial p}{\partial t} + A^*(t)p = y - z_d \quad \text{in } Q \\ p\big|_\Sigma = 0 \\ p\,(x,T) = 0. \end{cases} \quad (5.42)$$

(5.32) reduces to,

$$\int_o^T\int_\Omega (p(u) + Nu)(v-u)\,dx\,dt \geq 0\ ,\ \forall\ v \in U_{ad}\ ,\ u \in U_{ad}.$$

5.3 Example of a Mixed Neumann Problem with Boundary Control

Let $V = H^1(\Omega)$, $U = L^2(\Sigma)$ and let $A(t)$ be as in the previous example.
Let the state of the system be given by,

(5.43) $\frac{\partial}{\partial t} y(u) + A(t)y(u) = f \text{ in } Q$,

(5.44) $\frac{\partial}{\partial \nu_A} y(u) = u \text{ on } \Sigma$ (1)

(5.45) $y(x,o;u) = y_o(x)$, $x \in \Omega$.

Let C = injection map of $L^2(0,T;V) \to L^2(Q)$.

Then the cost function is

(5.46) $J(v) = \int_0^T\int_\Omega (y(v)-z_d)^2 dxdt + (Nv,v)_{L^2(\Sigma)}$, $z_d \in L^2(Q)$.

The adjoint state is now given by ,

$$(5.47) \quad \begin{cases} -\frac{\partial}{\partial t} p + A^*(t)p = y - z_d \text{ in } Q , \\ \frac{\partial p}{\partial \nu_{A^*}} = 0 \text{ on } \Sigma \\ p(x,T) = 0 \quad x \in \Omega . \end{cases}$$

The optimality condition is,

(5.48) $\int_0^T\int_\Omega (p + Nu)(v-u)d\Sigma \geq 0 \quad \forall\, v \in U_{ad}$. ∎

5.4 Decoupling of the Two-point Boundary Value Problem for the Linear Quadratic Problem

We now wish to study the two-point boundary value problem for the linear-quadratic problem (section 5.1.4) :

(5.49) $\frac{dy}{dt} + A(t)y = f - B\, N^{-1}\Lambda_U^{-1}B^*p$

(5.50) $-\frac{dp}{dt} + A^*(t)p = C^*\Lambda(y-z_d)$

(5.51) $y(o) = y_o$, $p(T) = 0$.

Let us make the assumptions:

$$(5.52) \quad \begin{aligned} &U = L^2(0,T;E) \text{ , } E = \text{separable Hilbert space ,} \\ &H = L^2(0,T;F) \text{ , } F = \text{separable Hilbert space ;} \\ &B(t) \in \mathcal{L}(E;V') \text{ ; } B \equiv B(t) \in \mathcal{L}(L^2(0,T;E);\ L^2(0,T;V')) \text{ ;} \\ &f(t) \in L^2(0,T;V') ; \\ &C(t) \in \mathcal{L}(V;F) \text{ ; } C \equiv C(t) \in \mathcal{L}(L^2(0,T;V) \text{ ; } L^2(0,T;F)). \end{aligned}$$

(1) $\frac{\partial}{\partial \nu_A}$ denotes the "normal derivative in accordance with A".
$\frac{\partial u}{\partial \nu_A} = \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \cos(n,x_j)$ on Γ , $\cos(n,x_i)$ = i^{th} direction cosine of n, normal exterior to Ω at Γ. For precise meaning of derivative see Lions-Magenes[35].

Let us set,

$$(5.53) \quad \begin{cases} D_1(t) = B(t)N^{-1}(t)\Lambda_E^{-1} B^*(t) \\ D_2(t) = C^*(t)\Lambda_F C(t) \end{cases}$$

Then $D_1(t) \in L(V;V')$, $D_2(t) \in L(V;V')$

and $D_1(t) = D_1^*(t)$, $D_2(t) = D_2^*(t)$.

Hence we have to study the system of equations,

$$(5.54) \quad \begin{aligned} &\frac{dy}{dt} + A(t)y + D_1(t)p = f \quad , \quad t \in]s,T[\ , \\ &-\frac{dp}{dt} + A^*(t)p - D_2(t)y = g \quad , \quad t \in]s,T[\ , \end{aligned}$$

where,

$$(5.55) \quad g(t) = - C^*(t)\Lambda_F z_d(t) \ ,$$

with the boundary conditions,

$$(5.56) \quad y(o) = y_o \ , \ p(T) = 0 \ .$$

Since the system of equations (5.54) and the boundary condition (5.56) arose from an optimal control problem which admits a unique solution, we immediately have,

Lemma 5.7

The Linear two-point boundary value problem (5.54) - (5.56) admits a unique solution for all given y_o .▌

Further we may prove,

Lemma 5.8

The mapping $y_o \to \{y,p\}$ = solution of (5.54) - (5.56)

is a continuous affine mapping of $H \to W(0,T) \times W(0,T)$.▌

Corollary 5.9

For $y_o \in H$, let $\{y,p\}$ be a solution of (5.54). The mapping $y_o \to p(s)$ is a continuous affine map of H into H.▌

Combining the above, we have,

Corollary 5.10

The mapping

$$(5.57) \quad y_o \to p(s)$$

may be uniquely written as,

$$(5.58) \quad p(s) = P(s)y_o + r(s) \text{ , where}$$

$P(s) \in L(H;H)$ and $r(s) \in H.$ ▌

The next lemma relates p(s) and y(s) by means of a continuous affine map and shows how the map P(s) and the element r(s) are defined.

Lemma 5.11

Let $\{y,p\}$ be a solution of (5.54) – (5.56). We have

$$(5.59) \quad p(t) = P(t)y(t) + r(t) \quad \forall\, t \in \,]0,T[$$

where P(t) and r(t) are defined in the following way :

(i) We solve,

$$(5.60) \quad \begin{cases} \dfrac{d\beta}{dt} + A(t)\beta + D_1(t)\gamma = 0 & \text{in }]s,T[\text{ ,} \\ -\dfrac{d\gamma}{dt} + A^*(t)\gamma - D_2(t)\beta = 0 & \text{in }]s,T[\text{ ,} \\ \beta(s) = h, \quad \gamma(T) = 0 \text{ ;} \end{cases}$$

then

$$(5.61) \quad P(s)h = \gamma(s) \text{ ;}$$

(ii) we solve,

$$(5.62) \quad \begin{cases} \dfrac{d\eta}{dt} + A(t)\eta + D_1(t)\xi = f & \text{in }]s,T[\text{ ,} \\ -\dfrac{d\xi}{dt} + A^*(t)\xi - D_2(t)\eta = g & \text{in }]s,T[\text{ ,} \\ \eta(s) = 0 \text{ , } \xi(T) = 0 \text{ ;} \end{cases}$$

then

$$(5.63) \quad r(s) = \xi(s) \text{ .} \ ▌$$

Some properties of P(s) can now be established :

$$(5.64) \begin{cases} \text{(i)} & P^*(t) = P(t) \text{ ;} \\ \text{(ii)} & \text{the mapping } t \to (P(t)h,\bar{h}) \text{ is continuous on } [0,T] \text{ , } \forall\, h,\bar{h} \in H. \\ \text{(iii)} & \text{there exists a constant } C_1 \text{ such that} \\ & \qquad P(s)h \le C_1 |h| \quad \forall\, h \in H, \forall\, s \in [0,T]. \\ \text{(iv)} & (P(s)h,h) \ge 0 \quad \forall\, h \in H \text{ ,} \end{cases}$$

that is P(s) is a continuous self-adjoint positive operator.

Proceeding formally, we shall now show that $P(t)$ satisfies a non-linear partial differential equation of the Riccati type and $r(t)$ satisfies a linear parabolic differential equation. This is done exactly as in the finite-dimensional case to obtain,

$$(5.65) \qquad -\frac{dP}{dt} + PA + A^*P + PD_1P = D_2 \quad \text{in }]0,T[$$

$$(5.66) \qquad -\frac{dr}{dt} + A^*r + PD_1r = Pf + g \quad \text{in }]0,T[$$

$$(5.67) \qquad P(T) = 0 \text{ , } r(T) = 0.$$

To justify the above formal calculations, we may choose a basis in V and use the Galerkin technique (that is, obtain a finite-dimensional approximation) and then pass to the limit to show convergence in an appropriate sense (one always uses weak convergence arguments). In this way we arrive rigorously at the following result.

Theorem 5.10

Under the appropriate indicated hypotheses, if $\{y,p\}$ is a solution (unique) of the two-point boundary value problem (5.54)-(5.56), then

$$(5.68) \qquad p = Py + r \text{ , where } P \text{ and } r \text{ have the following properties,}$$

$$(5.68) \qquad \begin{cases} P(t) \in \mathcal{L}(H;H) \text{ , } P^*(t) = P(t); \\ \text{if } \eta \in W(0,T) \text{ with } \frac{d\eta}{dt} + A(t)\eta \in L^2(0,T;H) \text{ then} \\ P(t)\eta \in W(0,T); \end{cases}$$

P satisfies (5.65) in the sense that

$$(5.69) \qquad \begin{cases} -\left(\frac{d}{dt}P(t)\right)\eta + PA\eta + A^*P\eta + PD_1P\eta = D_2\eta \\ \text{for all } \eta \text{ given by (5.68) with } A\eta \in L^2(0,T;H), \end{cases}$$

and

$$(5.70) \qquad \begin{cases} P(T) = 0 \text{ ,} \\ r(t) \text{ is the solution in } W(0,T) \text{ of (5.66),} \\ P \text{ and } r \text{ are unique.} \end{cases}$$

In this way, we solve the feedback synthesis problem to obtain,

$$(5.71) \qquad u(t) = -N^{-1}(t)\Lambda_E^{-1}B^*(t)[P(t)y(t) + r(t)] \text{ .}$$

What remains to be done is to consider particular examples so that we may give the operator $P(t)$ a concrete representation.

5.5 Mixed Dirichlet Problem with Distributed Control

This is the problem we considered in section 5.2. In terms of the notation of section 5.4, $E = H = L^2(\Omega)$, $B(t)$ = identity, $N(t) = N$, $C(t)$ = injection of $V \to H$, $D_2(t)$ = identity, $D_1(t) = N^{-1}$.

The key idea here is to use the Schwartz Kernel Theorem [37], to obtain a representation for $P(t)$,

$$(5.72) \qquad P(t)\phi = \int_\Omega P(x,\xi,t)\phi(\xi)d\xi \quad \forall\, \phi \in \mathcal{D}(\Omega),$$

where $P(x,\xi,t)$ = kernel of $P(t)$ = distribution on $\Omega_x \times \Omega_\xi$ defined uniquely by $P(t)$. Then $P(t)$ characterized in Theorem (5.10) satisfies

$$(5.73) \qquad \frac{\partial P}{\partial t}(x,\xi,t) + (A_x^* + A_\xi^*)P(x,\xi,t) + \int_\Omega P(x,\xi_1,t)\, N^{-1}P(\xi_1,\xi,t)d\xi_1 = \delta(x-\xi),$$

$$(5.74) \qquad P(x,\xi,t) = P(\xi,x,t),$$

$$(5.75) \qquad P(x,\xi,t) = 0 \quad \text{if} \quad x \in \Gamma,\ \xi \in \Omega,$$

$$(5.76) \qquad P(x,\xi,T) = 0. \blacksquare$$

Remark 5.11

The Galerkin technique used to justify the formal calculations provides a constructive method of solving the Riccati equation. ▌

5.6 An Estimation Problem

Consider the following state estimation problem.
Let the state of the system y be defined by,

$$(5.77) \qquad \begin{aligned} &\frac{\partial y}{\partial t} + A(t)\, y = 0 \\ &\left.\frac{\partial y}{\partial \nu_A}\right|_\Sigma = \xi(t) \\ &y(o) = \alpha . \end{aligned}$$

Here $f = o$ and $E = L^2(\Gamma)$. The observation is defined by,

$$(5.78) \qquad z(t) = y|_\Sigma + \eta(t).$$

The boundary and initial conditions $\xi(t)$ and α as well as the error $\eta(t)$ are assumed to be Gaussian with mean values $\bar{\xi}(t)$, $\bar{a}$ and covariance operators $\sigma_1^2 I$, $\sigma_2^2 I$ and $\sigma_3^2 I$ respectively (this statement has to be mathematically defined properly; see for example Bensoussan [8]). For our purposes we consider them to be functions in appropriate L^2 spaces.

We desire to obtain an optimal estimate $\hat{y}_T$ of $y(T)$, the state of the system at time T, on the basis of the observation $z(t)$ on the time interval $[0,T]$. The estimate is optimal in the least squares sense.

For this purpose, we define the obvious least squares criterion $J(\alpha,\xi)$.

We minimize $J(\alpha,\xi)$ to obtain the optimal α,ξ denoted by $\hat{\alpha},\hat{\xi}$. Then we may calculate $\hat{y}_T = y(T;\hat{\alpha},\hat{\xi})$ and this is the best estimate.

The theory developed in the previous section can now be applied to this problem giving us,

$$(5.79)\quad \begin{cases} \frac{\partial y_T}{\partial t} + A(t)y_T + \sigma_2^2\, \gamma^*\gamma p_T = \gamma^*\bar{\xi}(t) \\ \frac{\partial p_T}{\partial t} + A^*(t)p_T - \frac{\gamma\,\gamma}{\sigma_1^2}\, y_T = -\frac{\gamma^* z(t)}{\sigma_1^2} \\ y_T(o) = \bar{\alpha} - \sigma_3^2 p_T(o) \\ p_T(T) = 0 \end{cases}$$

The above equations can now be decoupled giving us the following equation for the kernel of the covariance operator $P(t)$:

$$(5.80)\quad \begin{cases} \frac{\partial P}{\partial t}(x,\xi,t) + (A_x + A_\xi)P(x,\xi,t) + \frac{1}{\sigma_3^2} \quad P(x,\xi_1,t)P(\xi_1,\xi,t)d\xi_1 \\ \qquad\qquad = \sigma_1^2\delta(x-\xi) \\ P(x,\xi,t) = P(\xi,x,t)\ , \\ P(x,\xi,t) = 0 \quad f\ x \in \Gamma\ ,\ \xi\in\Omega\ , \\ P(x,\xi,o) = \sigma_3^2\delta(x-\xi). \end{cases}$$

For analogous considerations for other state estimation problems (for example, with point-wise measurements), see Phillipson-Mitter [20], Mitter-Phillipson [37].

5.7 Remarks on Decoupling and the Riccati Equation

a) It should be remarked that the techniques used in deriving the Riccati equation and the equation for the kernel can be adapted to obtain results on the feedback synthesis of systems governed by differential-delay equations. To do this one considers the differential-delay equation as an abstract evolution equation and then results of Baiocchi [38] can be used. A study of this problem may be found in the forthcoming paper [39].

b) An important part of the linear quadratic theory is the study of the behavior of the optimal control $u(t)$ as well as the operator $P(t)$ as the upper limit T of the interval $[0,T]$ tends to ∞. Under the assumption that the operator $A(t)$ satisfies

$$(5.81)\quad (A(t)\phi,\phi) \geq \alpha\|\phi\|^2\ ,\ \forall t \geq o,\ \forall\phi\in V\ ,\ \alpha>0.$$

If we now say that all operators concerned are time-independent, the asymptotic behavior of the two-point boundary value problem, and of the equations for $\gamma(t)$ and $P(t)$ can be studied and it may be shown that as $T\to\infty$,

$$u_T \to u_\infty\ ,\ y_T \to y_\infty\ ,\ p_T \to p_\infty \quad \text{and} \quad P_T \to P_\infty$$

in an appropriate topology [7].

c) In the finite dimensional linear-quadratic problem one uses the property of complete controllability to establish the existence of a stabilizing control giving finite cost for the infinite time problem. For an interesting study in this direction for infinite dimensional problems, see Russell [41], [42].

5.8 A Boundary Control Problem; Non-Homogeneous Mixed Dirichlet Problem

In section 5.3 we considered a boundary control problem. Let us now consider a different boundary control problem which leads to the study of non-homogeneous Dirichlet Problems [35].

Let the state $y(v)$ be given by ,

$$(5.82) \qquad \frac{\partial}{\partial t} y(v) + A(t)y(v) = f \quad \text{in} \quad Q ,$$

$$(5.83) \qquad y(v)\Big|_{\Sigma} = v \quad \text{on} \quad \Sigma ,$$

$$(5.84) \qquad y(x,o;v) = y_o(x) \ , \ x \in \Omega .$$

The operator $A(t)$ is given by (5.39), satisfies the strong ellipticity condition (5.40) and the coefficients a_{ij} are assumed to be C^{∞} in $\bar{Q}$.

Problems of this type with non-homogeneous boundary conditions (control is exercised through the lateral boundary) require special treatment. But one can show that taking $U = L^2(\Sigma)$, for example, that the above equations admits a unique solution $y(v) \in L^2(Q)$ (1).

Let the cost function be,

$$(5.85) \qquad J(v) = \int_Q \left| y(v) - z_d \right|^2 dxdt + (Nv,v)_U \ , \quad z_d \text{ given in } L^2(Q) \ ,$$

where $N \in L(U;U)$ and N is positive definite.

Let U_{ad} = closed, convex subset of U.

In this case the <u>adjoint problem</u> is

$$(5.86) \qquad -\frac{\partial p}{\partial t} + A^*(t)p = y(u) - z_d \quad \text{in } Q$$

$$(5.87) \qquad p\Big|_{\Sigma} = 0$$

$$(5.88) \qquad p(x,T;u) = 0$$

The solution $p(u)$ may be shown to belong to the Sobolev space $H^{2,1}(Q)$, where $H^{2,1}(Q) = \{p \mid p, \frac{\partial p}{\partial x_i}, \frac{\partial^2 p}{\partial x_i \partial x_j}, \frac{\partial p}{\partial t} \in L^2(Q)\}$.

In the manner we have indicated in the previous section, the optimal control u is characterized by,

(1) Actually one can show that $y(v) \in H^{1/2,1/4}(Q)$ – fractional Sobolev space [35].

(5.89) $$\int_Q (y(u) - z_d)(y(v) - y(u))dxdt + (Nu, v-u)_U \geq 0 \quad \forall\, v \in U_{ad}.$$

We now have to use the adjoint system and Green's Formula to transform (5.89) into a more convenient form.

Using (5.86) , we see that the integral in (5.89) is equal to,

(5.90) $$\int_Q \left(- \frac{\partial p(u)}{\partial t} + A^*(t)p(u)\right)(y(v)-y(u))dxdt \ .$$

Using Green's Formula which has to be justified in this case, since $y(u)$ is a generalized solution, we can show that (5.90) is equal to

(5.91) $$- \int_\Sigma \frac{\partial p}{\partial \nu_{A^*}}(u)\ (y(v)-y(u))d\Sigma \quad ,$$

where $\frac{\partial p}{\partial \nu_{A^*}}$ is the exterior normal derivative to Σ associated with A^* .

The above integral makes sense since $\frac{\partial p}{\partial \nu_{A^*}}$ can be shown to be in an appropriate Sobolev space.

Since $y(v) = v$ on Σ , we finally obtain,

(5.92) $$\left(Nu + \frac{\partial p}{\partial \nu_{A^*}}(u)\ ,\ v-u\right)_U \geq 0 \quad \forall\, v \in U_{ad} \ .$$

If now $U_{ad} = U$, that is there are no constraints,

(5.93) $$Nu + \frac{\partial p}{\partial \nu_{A^*}}(u) = 0 \ ,$$

and hence,

(5.94) $$u = - N^{-1} \frac{\partial p}{\partial \nu_{A^*}}(u) \quad .$$

For problems with both control and observation on the boundary, see Lions [7]. ∎

5.9 Time Optimal Control

For time optimal control problems, some results analogous to those for finite dimensional problems have been obtained.

Consider the system,

$$\frac{d}{dt} y(t;v) + A(t)y(t;v) = f + Bv \tag{5.95}$$

$$y(o;v) = yo \tag{5.96}$$

Let U_{ad} be a closed, convex subset of U and let y_1 be a given element in H. We assume that the following controllability assumption holds:

$$\begin{cases} \text{there exists a } v \in U_{ad} \text{ such that } y(\tau;v) = y_1 \text{ for some finite} \\ \tau. \end{cases} \tag{5.97}$$

Let $\tau_o = \inf [\, \tau : \text{ such that (5.98) holds}]$.

We first have the following existence theorem.

Theorem 5.11

Under the assumptions (5.3), (5.4) on the operator $A(t)$, the assumption (5.12) on the operator B and the controllability assumption there exists a $u \in U_{ad}$ and a τ_o such that $y(\tau_o;u) = y_1$. ∎

The above theorem may be applied to boundary control problems--for example, of the type shown below.

Example

Let $U = L^2(\Sigma)$ and let the state $y(v)$ be given by ,

$$\begin{cases} \dfrac{\partial}{\partial t} y(v) + A(t)y(v) = f \\ \dfrac{\partial y}{\partial \nu_A}(v) = v \\ y(x,o;v) = y_o(x) . \end{cases} \tag{5.98}$$

We now wish to proceed further and characterize the optimal control for a special constraint set.

Let,

$$U_{ad} = \{v \mid |v(t)| \leq 1 \text{ almost everywhere}\} . \tag{5.99}$$

The operator A be time independent and be the infinitesimal generator of a strongly continuous semi-group G(t) in H.

We then have,

Theorem 5.12 (Bang-Bang Principle)

Let all the above assumptions on the system hold. Let u be an optimal control satisfying $y(\ _o; u) = y_1$. Then,

(5.100) $u(t) = 1$ almost everywhere on $[0,T_o]$.

This result is due to Fattorini [14].
For additional results when G(t) is a group see Lions [7].
See also the recent result of Conti [ ].

6. Optimal Control of Systems Governed by Hyperbolic Partial Differential Equations

A theory of optimal control for systems governed by second order hyperbolic equations in a manner analogous to that developed for parabolic equations may be developed. The main difference here is that we have to begin with a study of a different abstract evolution problem.

We use the same notation as in the previous section.

6.1 Evolution Equation

6.1.1 Problem Formulation and Existence Theorem

The basic evolution problem to be studied is :

Find y satisfying

(6.1) $y \in L^2(0,T;V)$, $\frac{dy}{dt} \in L^2(0,T;H)$,

(6.2) $\frac{d^2y}{dt^2} + A(t)y = f$ in $]o,T[$, f given in $L^2(0,T;H)$

together with the initial data ,

(6.3) $y(o) = y_o$, y_o given in V

(6.4) $\frac{dy}{dt}(o) = y_1$, y_1 given in H.

The hypothesis on the family of operators $A(t) \in L(V;V')$ are : setting

(6.5) $(A(t)\phi,\Psi) = a(t;\phi,\Psi) \quad \forall\phi,\Psi \in V$,

we assume

(6.6) $\forall\phi,\Psi \in V$, the function $t \to a(t;\phi,\psi)$ is continuously differentiable in $[0,T]$,

(6.7) $a(t;\phi,\Psi) = a(t;\Psi,\phi) \quad \forall\phi,\psi \in V$;

(6.8) there exists a $\lambda \in R$ such that

$$a(t;\phi,\psi) + \lambda |\phi|^2 \geq \alpha \|\phi\|^2 \quad \forall\phi\in V, \alpha>0 \text{ , } t \in [0,T] \text{ .}$$

We then have,

Theorem 6.1

Under the above assumptions, the evolution problem (6.1)-(6.4) admits a unique solution. The mapping

(6.9) $\{f,y_o,y_1\} \to \{y,\frac{dy}{dt}\}$

is a continuous linear mapping of $L^2(0,T;H)xVxH \to L^2(0,T;V)x\, L^2(0,T;H)$.

6.1.2 The Control Problem

Let $B \in L(U; L^2(0,T;H))$(1). A control problem exactly similar to that formulated in section 5.1.2 (clearly there are other possibilities for defining the observation equation) can be formulated for the hyperbolic case. We then have the following theorems analogous to Theorems 5.3 and 5.6.

Theorem 6.2 (Existence of Optimal Control)

There exists a unique $u \in U_{ad}$ such that $J(u) = \underset{v\in U_{ad}}{\text{Inf.}}\, J(v)$.

Theorem 6.3 (Necessary and Sufficient Conditions)

Under the above assumptions, $u \in U_{ad}$ is an optimal control if and only if,

(6.10) $(\Lambda^{-1}B^*p(u) + Nu,v-u)_U \geq 0 \quad \forall\, v \in U_{ad}$,

$u \in U_{ad}$,

(1) In section 5.1.2 we could take $f \in L^2(0,T;V)$ and $B \in L(U; L^2(0,T;V'))$

where p(u) is defined by ,

$$(6.11)\quad \begin{cases} \dfrac{d^2y(u)}{dt^2} + A(t)y(u) = f + Bu \\ y(o;u) = y_o \ , \ y'(o;u) = y_1,^{(1)} \end{cases}$$

$$(6.12)\quad \begin{cases} \dfrac{d^2p(u)}{dt^2} + A^*(t)p(u) = C^*\Lambda(Cy(u)-z_d), \\ p(T;u) = 0, \quad p'(T;u) = 0 \ , \end{cases}$$

and further,

$$(6.13)\quad \begin{cases} y(u) \ , \ p(u) \in L^2(0,T;V) \\ y'(u), \ p'(u) \in L^2(0,T;H). \end{cases}$$

Examples of control problems similar to those considered in Section 5 can now be considered in this setting.

7. Remarks on Existence Theorems for Optimal Controls

The existence theorems for optimal control problems which we have presented in this paper are relatively simple--once an appropriate existence theorem for the linear partial differential equation was obtained, the problem of existence of an optimal control was reduced to the problem of existence of a minimum of a convex functional on a closed, convex set. The situation is, however, far more complicated when we are dealing with partial differential equations which are non-linear. For very general results in this direction, see Lions [44]. For a different approach to the problem, see Cesari [45],[46].

8. Controllability and Observability

Consider a system whose state y(v) is given by,

$$(8.1)\quad \begin{cases} \dfrac{d}{dt}\, y(v) + A(t)y(v) = B(t)v(t) \quad t \in\,]0,T[\\ y(o) = 0. \end{cases}$$

Let us suppose that A(t) satisfies

$$(8.2)\quad (A(t)\phi,\phi) \geq \alpha \, \|\phi\|_V^2 \quad \forall\, \phi \in V \ , \ \forall\, t \in [0,T] \, , \alpha > 0 .$$

(1) ' denotes differentiation w.r. to t.

The control u is assumed to be an element of $L^2(0,T;E)$ (see notation of section 5.7) and $B(t) \in L(E;H)$ with $\sup_{t\in[o,T]} \|B(t)\|_{L(E;H)} < \infty$.

Let us introduce the Green's operator $G(t,\tau)$ associated with $A(t)$ (see Lions [47]). We have that $G(t,\tau) \in L(H;H)$ and the function $t \to G(t,\tau)g$ is a continuous function of $[\tau,T]$ into H, $\forall$ $g \in H$. Further $G(t,\tau)$ satisfies

$$\frac{d}{dt} G(t,\tau) + A(t)G(t,\tau) =$$

Definition 8.1

The system (8.1) is said to be controllable at time T if

$$(8.3) \qquad A(T) = \{y(T,0;v) \mid v \in L^2(o,T;E)\}$$

is a dense subspace of H.

Remark 8.2

This is the natural extension of the definition of controllability in the finite dimensional case. The definition says that for every $z \in H$, there exists a control u which steers the system arbitrarily close to z.▌

Theorem 8.3

The system (8.1) is controllable if and only if

$$(8.4) \qquad C(T) = \int_o^T G(T,t)B(t)B^*(t)G^*(T,t)dt > 0 .$$

Proof

a) Sufficiency : We have,

$$y(T,0;v) = \int_o^T G(T,t)B(t)v(t)dt .$$

Consider a control

$$v(t) = B^*(t)G^*(T,t)\xi , \text{ where } \xi \in H.$$

Hence ,

$$y(T,o;v) = C(T)\xi .$$

But the operator $C(T)$ is a self-adjoint positive operator and hence its range $R[C(T)]$ is dense in H. This proves the sufficiency part.

b) Necessity: Suppose that the system is controllable and (8.4) is not true. Then there exists a $z \in H$, $z \neq 0$ such that

$$\left(\int_o^T G(T,t)B(t)B^*(t)G^*(T,t)dtz , z \right) = 0 , \text{ which implies}$$

$$(8.5) \qquad B^*(t)G^*(T,t)z = 0 \text{ almost everywhere.}$$

Let $p \in W(0,T)$ be a solution of

(8.6) $$\begin{cases} -\frac{dp}{dt} + A^*(t)p = 0 & t \in]0,T[, \\ p(T) = z . \end{cases}$$

Then, $p(t) = G^*(T,t)z.$

Multiplying the first equation in (8.5) by $y(t;v)$ and integrating by parts, we obtain,

$$(z,\ y(T,o;v)) = \int_0^T (v(t), B^*(t)G^*(T,t)z) = 0$$

and hence from (8.5) $(z,\ y(T,o;v)) = 0 \quad \forall\ v \in L^2(0,T;E).$

Therefore $A(T)$ is not a dense subspace of H. ∎
Theorem 8.3 was formally derived by Wang [28].

For examples of partial differential equations which are controllable in the sense of Definition 8.1, see Lions [7 , section 10] where the question of controllability is reduced to a study of the uniqueness properties of solutions. ∎

Consider the system,

(8.7) $$\begin{cases} \frac{dy}{dt} + A(t)y = f , & t \in]0,T[, \ f \text{ given in } L^2(0,T;H) \\ y(o) = \xi , \end{cases}$$

with the same hypotheses as for (8.1).
Let the observation equation be,

(8.8) $z = Cy(\xi)$, where $C \in \mathcal{L}(W(0,T);K)$, K being the space of observations.

Definition 8.4

The system is said to be observable at time T if

$$O(T) = \{Cy(\xi) \mid \xi \in H\}$$

is a dense subspace of K.

A theorem analogous to Theorem 8.3 can now be proved for observability.

For a study of controllability and observability for abstract control systems, see Jurdjevic [48]. For studies on strict controllability for partial differential equations, see Russell [23],[24].

9. Miscellaneous Comments

All the problems we have considered in this paper (excepting time-optimal problems) are free-end point problems. Problems in which the end-point is prescribed to lie in a fixed set are, of course, more difficult. For an extension of some of the results presented in this paper to the fixed end-point problem with a cost function

$$\int_0^T \|x(t)\|^2 dt$$

see the recent paper by Friedman [19].

Another interesting development is a duality theory (in the mathematical programming sense) for control of systems governed by partial differential equation. It can be shown that some of the abstract results of Rockefellar [49],[50] can be specialized to obtain results of the maximum principle type for partial differential equations [51],[52]. In this way a duality theory between estimation and control can also be developed (for the ordinary differential equation case, see Pearson [53]).

10. Numerical Solution of Optimal Control Problems

To the author's knowledge two main schemes for the numerical solution of optimal control problems have been proposed. A complete description of these schemes is beyond the scope of this paper. In both schemes the problem is reformulated as a mathematical programming problem in an infinite dimensional space. In one scheme the constraints (including the differential equation) is adjoined to the cost function by means of a penalty function and the resulting cost function minimized using iterative techniques. Details of this may be found in [7],[54]. In the second scheme, a Galerkin technique is used to solve the state and adjoint equations and the gradient of the cost function is calculated making use of the state and adjoint trajectories (similar to the Kelley-Bryson technique for the ordinary differential equation case). Iterative techniques such as conjugate gradient techniques can then be used to minimize the performance functional. For details of this technique, see [21], [22].

REFERENCES

1. E.B. Lee and L. Markus: Foundations of Optimal Control Theory, Wiley, 1967.
2. H. Halkin and L.W. Neustadt: General Necessary Conditions for Optimization Problems, Proc. National Acad. Sci., U.S.A. 56(1966), pp. 1066-1071.
3. R.V. Gamkrelidze and G.L. Kheratishrili: Extremal Problems in Linear Topological Spaces, Mathematical Systems Theory 1 (1967), pp. 229-256.
4. L.W. Neustadt: A General Theory of Extremals, Journal of Computer and System Sciences, 3 (1969), pp. 57-92.
5. H.T. Banks: Necessary Conditions for Control Problems with Variable Time Lags, SIAM J. on Control, 6 (1968), pp. 9-47.
6. H.T. Banks: A Maximum Principle for Optimal Control Problems with Functional Differential Equations, Bull. Am. Math. Soc. 75 (1969), pp. 158-161.
7. J.L. Lions: Contrôle Optimal de Systèmes Gouvernes par des Équations Aux Dérivées Partielles, Dunsd-Gauthier-Villass, 1968. English translation by S.K. Mitter, to be published by Springer-Verlag, 1969.
8. A. Bensoussan: Doctoral Thesis, University of Paris, 1969.
9. A.V. Balakrishnan and J.L. Lions: State Estimation for Infinite Dimensional Systems, Journal of Computer and System Sciences, (1967), pp. 391-403.
10. A.V. Balakrishnan: On a New Computing Technique in Optimal Control, SIAM J. on Control, 6 (1968), pp. 149-173.
11. A.V. Balakrishnan: Optimal Control Problems in Banach Spaces, SIAM J. on Control, 3 (1965), pp. 109-127.
12. H.O. Fattorini: Control in Finite Time of Differential Equations in Banach Space, Commun. Pure and Applied Math., 19 (1966), pp. 17-34.
13. H.O. Fattorini: On Complete Controllability of Linear Systems, J. of Diff. Engns., 3 (1967), pp. 391-402.
14. H.O. Fattorini: Time Optimal Control of Solutions of Operational Differential Equations, SIAM J. on Control, 2 (1964), pp. 54-59.
15. H.O. Fattorini: Some Remarks on Complete Controllability, SIAM J. on Control, 4 (1966), pp. 686-693.
16. H.O. Fattorini: A Remark on the "Bang-Bang" Principle for Linear Control Systems in Infinite-Dimensional Space, SIAM J. on Control, 6 (1968), pp. 109-113.
17. A. Friedman: Optimal Control for Parabolic Equations, J. of Math. Analysis and Applns. 18 (1967), pp. 479-491.
18. A. Friedman: Optimal Control in Banach Spaces, J. of Math. Analysis and Applns, 19 (1967), pp. 35-55.
19. A. Friedman: Optimal Control in Banach Space with Fixed End Points, J. of Math. Analysis and Applns., 24 (1968), pp. 161-181.
20. G.A. Phillipson and S.K. Mitter: State Identification of a Class of Linear Distributed Systems, Proc. Fourth IFAC Congress, Warsaw, Poland, June 1969.
21. G.A. Phillipson and S.K. Mitter: Numerical Solution of a Distributed Identification Problem via a Direct Method, Computing Methods in Optimization Problems, eds. A.V. Balakrishnan and L.W. Neustadt, Academic Press, 1969.
22. G.A. Phillipson: Identification of Distributed Systems, monograph to be published by American Elsevier, 1969.

23. D.L. Russell: On Boundary Value Control of Linear Symmetric Hyperbolic Systems, in Mathematical Theory of Control, eds. A.V. Balakrishnan and L.W. Neustadt, Academic Press, 1967.
24. D.L. Russell: Optimal Regulation of Linear Symmetric Hyperbolic Systems with Finite Dimensional Controls, SIAM J. on Control 4 (1966), pp. 276-294.
25. D.L. Russell: Nonharmonic Fourier Series in the Control Theory of Distributed Parameter Systems, J. of Math. Anal. and Applns., 18 (1967), pp. 542-560.
26. Yu V. Egorov: Sufficient Conditions for Optimal Control in a Banach Space, Mat. Sbornik, 64 (1964), pp. 79-101.
27. A.G. Butkovsky, A.I. Egorov, K.A. Lurie: Optimal Control of Distributed Systems (A Survey of Soviet Publications), SIAM J. on Control, 6 (1968), pp. 437-476.
28. P.K.C. Wang: Control of Distributed Parameter Systems, in Advances in Control Systems, ed. C.T. Leondes, Academic Press, 1 (1964), pp. 75-172.
29. P.K.C. Wang: Optimum Control of Distributed Parameter Systems with Time Delays, IEEE Trans. on Automatic Control 9, (1964), pp. 13-22.
30. H. Erzberger and M. Kim: Optimum Boundary Control of Distributed Parameter Systems, Information and Control, 9 (1966), pp. 265-278.
31. M. Kim and H. Erzberger: On the Design of Optimum Distributed Parameter Systems with Boundary Control Functions, IEEE Trans. on Automatic Control, Vol. AC-12 (1967), pp. 22-28.
32. E.A. Coddington and N. Levinson: Theory of Ordinary Differential Equations, McGraw Hill, 1955.
33. J. Dieudonné: Foundations of Modern Analysis, Academic Press, 1960.
34. J.L. Lions and G. Stampacchia: Variational Inequalities, Comm. Pure and Applied Maths., XX (1967), pp. 493-519.
35. J.L. Lions and E. Magenes: Problèmes aux Limites Non Homogènes et Applications, vol. 1,2, Durod, Paris, 1968.
36. J. Horvath: Topological Vector Spaces and Distributions, vol. 1, Addison-Wesley, 1966.
37. L. Schwartz: Théorie des Noyaux, Proceedings of the International Congress of Mathematicians, 1950, 1, pp. 220-230.
38. S.K. Mitter and G.A. Phillipson: State Identification of Distributed Parameter Systems, to appear.
39. C. Baiocchi: Sulle Equazioni Differenziale astratte lineari del Primo e del Seconde Ordine negli Spazi di Hilbert, Annali Mat. Pure ed Applic., 76 (1967), pp. 233-304.
40. S.K. Mitter: Synthesis of Optimal Feedback Control for Differential-Delay Systems, to appear.
41. D.L. Lukes and D.L. Russell: The Quadratic Criterion for Distributed Systems, MRC Technical Report #860, University of Wisconsin, 1968.
42. D.L. Russell: Linear Stabilization of the Linear Oscillator in Hilbert Space, J. of Math. Analysis and Applns., 3 (1969), pp. 663-675.
43. R. Conti: Time Optimal Solution of a Linear Evolution Equation in Banach Spaces, J. of Opt. Theory and Applns., 2 (1968), pp. 277-284.
44. J.L. Lions: Optimisation pour Certaines Classes d'Équations d'Évolution Non Linéaire, Annali di Mat. Pure ed Appliceta, LXXII (1966), pp. 275-294.

45. L. Cesari: Existence Theorems for Multidimensional Problems of Optimal Control, in Differential Equations and Dynamical Systems, Acad. Press, 1967, pp. 115-132.
46. L. Cesari: Existence Theorems for Multidimensional Legrange Problem, J. of Optimization Theory and Applns., 1 (1967).
47. J.L. Lions: Équations Différentielles Opérationnelles et Problèmes Aux Limites, Springer, 1961.
48. V. Jurdjevic: Controllability and Observability of Abstract Control Systems, Ph.D. Thesis, Case Western Reserve University, 1969.
49. R.T. Rockafellar: Duality and Stability in Extremum Problems Involving Convex Functions, Pacific J. of Math, 21 (1967), pp. 167-187.
50. R.T. Rockafellar: Integrals which are Convex Functionals,Pacific J. of Math., 24 (1968), pp. 525-539.
51. W. Heins and S.K. Mitter: Conjugate Convex Functions, Duality and Optimal Control Problems I: Ordinary Differential Equations, Information Sciences, 1969.
52. S.K. Mitter and W. Heins: Conjugate Convex Functions, Duality, and Optimal Control Problems II: Functional Differential and Partial Differential Equations, Information Sciences, 1969.
53. J.D. Pearson: On the Duality between Estimation and Control, SIAM J. on Control 4 (1966).
54. P. Kenneth, M. Sibony, J.P. Yvon: Penalization Techniques for Optimal Control Problems Governed by Partial Differential Equations, Computing Methods in Optimization Problems, eds. A.V. Balakrishnan and L.W. Neustadt, Academic Press, 1969.