LECTURE 4 SUPPLEMENTAL MATERIAL

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1. Introduction

Due to the complexity of the analysis in class I have included these notes.

2. Direct Adaptive Control of Scalar Plant.

Let us begin with a simple scalar adaptive system,

$$\dot{x}_p(t) = a_p x_p(t) + k_p u(t) \tag{1}$$

where $x_p(t) \in \mathbb{R}$ is the plant state, $u(t) \in \mathbb{R}$ is the control input, $a_p \in \mathbb{R}$ is an unknown scalar and only the sign of $k_p \in \mathbb{R}$ is known. For the direct adaptive controller we choose a reference model as

$$\dot{x}_m(t) = a_m x_m(t) + k_m r(t) - \ell(x(t) - x_m(t)). \tag{2}$$

All of the parameter above are known and scalar, $x_m(t)$ is the reference model state, $r(t)$ is a bounded reference input. We choose $a_m, \ell < 0$ so that the reference model and the subsequent error dynamics will be stable. Compared to the direct adaptive controller we presented in Lecture 3, we now need to distinguish between *Open-loop Reference Models* (ORM) and *Closed-loop Reference Models* (CRM):

$$ORM : \dot{x}^o_m(t) = a_m x_m(t) + k_m r(t) \tag{3}$$

$$CRM : \dot{x}_m(t) = a_m x_m(t) + k_m r(t) - \ell(x(t) - x_m(t)). \tag{4}$$

The control law is chosen as

$$\begin{aligned} u(t) =& \theta(t) x_p(t) + k(t) r(t) \\ =& \bar{\theta}^T(t)\phi(t) \end{aligned} \tag{5}$$

where we have defined

$$\bar{\theta}(t) = \begin{bmatrix} \theta(t) \\ k(t) \end{bmatrix} \text{ and } \phi(t) = \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}.$$

From this point forward we will suppress the explicit time dependance of parameters accept for emphasis.

We now define the two errors in the system, the model following error

$$e(t) = x_p(t) - x_m(t) \tag{6}$$

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and the parameter errors

$$\tilde{\bar{\theta}}(t) = \bar{\theta}(t) - \bar{\theta}^*.$$

The term $\bar{\theta}^* \in \mathbb{R}^2$ is a constant matched term which satisfies

$$\bar{\theta}^* = \begin{bmatrix} \frac{a_m - a_p}{k_p} \\ \frac{k_m}{k_p} \end{bmatrix}. \tag{7}$$

This allows us to rewrite the plant dynamics as

$$\begin{aligned} \dot{x}_p(t) &= a_m x_p(t) + k_p(\theta(t)x_p + k(t)r) - k_p(\theta^* x_p + k^* r) \\ &= a_m x_p(t) + k_p(\tilde{\bar{\theta}}^T(t)\phi) \end{aligned} \tag{8}$$

Using the definition of the error term in (6), the error dynamics are now of the following form:

$$\dot{e}(t) = (a_m + \ell)e + k_p \tilde{\bar{\theta}}^T \phi. \tag{9}$$

The error dynamics above suggest an adaptive update law of the form

$$\dot{\tilde{\bar{\theta}}} = \dot{\bar{\theta}} = \begin{bmatrix} \dot{\theta} \\ \dot{k} \end{bmatrix} = -\gamma \mathrm{sgn}(k_p) e \phi = \begin{bmatrix} -\gamma \mathrm{sgn}(k_p) e x_p \\ -\gamma \mathrm{sgn}(k_p) e r \end{bmatrix} \tag{10}$$

where $\gamma > 0$ is a free design parameter commonly referred to as the *adaptive tuning gain.*

Theorem 1. *The plant in* (1)*, with the controller defined by* (5)*, the update law in* (10) *with the reference model as in* (2)*, has the following properties*

- (a) $e, x_p, \tilde{\bar{\theta}}, \bar{\theta}$ *are all bounded.*
- (b) $e \in \mathcal{L}_2$
- (c) $\lim_{t\to\infty} e(t) = 0$

Proof. Consider the lyapunov candidate function

$$V(e(t), \tilde{\theta}(t)) = \frac{1}{2}e^2 + \frac{1}{2\gamma}|k_p|\, \tilde{\bar{\theta}}^T \tilde{\bar{\theta}}$$

Taking the time derivative of V along the system directions we have

$$\dot{V} = (a_m + \ell)e^2 \leq 0.$$

Given that V is positive definite and $\dot{V}$ is negative semi-definite we have that

$$V(e(t), \tilde{\theta}(t) \leq V(e(0), \tilde{\theta}(0)) < \infty. \tag{11}$$

Thus V is bounded and this means in tern that e and $\tilde{\bar{\theta}}$ are bounded. For an explicit bound on e we expand (11) giving us

$$\frac{1}{2}e(t)^2 + \frac{1}{2\gamma}|k_p|\, \tilde{\bar{\theta}}^T(t)\tilde{\bar{\theta}}(t) \leq \frac{1}{2}e(0)^2 + \frac{1}{2\gamma}|k_p|\, \tilde{\bar{\theta}}^T(0)\tilde{\bar{\theta}}(0)$$

Given that $\frac{1}{2\gamma}|k_p|\, \tilde{\bar{\theta}}^T(t)\tilde{\bar{\theta}}(t) \geq 0$ the following also holds

$$\frac{1}{2}e(t)^2 \leq \frac{1}{2}e(0)^2 + \frac{1}{2\gamma}|k_p|\, \tilde{\bar{\theta}}^T(0)\tilde{\bar{\theta}}(0) \quad \forall t \geq 0$$

or in compact form

$$\|e(t)\|^2_{L_\infty} \leq 2V(0). \tag{12}$$

Given that r and e are bounded and the fact that $a_m < 0$, the reference model is stable. Therefore, we can conclude x_m is bounded and along with the boundedness of e we can conclude that x_p is bounded. Given that $\bar{\theta}^*$ is a constant we can conclude that $\bar{\theta}$ is bounded from the boundedness of $\tilde{\bar{\theta}}$. This can be compactly stated as $e, x_p, \tilde{\bar{\theta}}, \bar{\theta} \in \mathcal{L}_\infty$, and we have proved (a).

In order to prove (b) we note that

$$-\int_0^t \dot{V} = V(e(0), \tilde{\theta}(0)) - V(e(t), \tilde{\theta}(t)) \leq V(e(0), \tilde{\theta}(0))$$

Substitution of $\dot{V}$ and $V(e(0), \tilde{\theta}(0))$ we have

$$|a_m + \ell| \int_0^t e(t)^2 \leq \frac{1}{2} e(0)^2 + \frac{1}{2\gamma} |k_p| \, \tilde{\bar{\theta}}^T(0) \tilde{\bar{\theta}}(0) \quad \forall \ t \geq 0$$

Dividing by $|a_m + \ell|$ and taking the limit as $t \to \infty$ we have

$$\|e\|^2_{L_2} \leq \frac{1}{|a_m + \ell|} \left(\frac{1}{2} e(0)^2 + \frac{1}{2\gamma} |k_p| \, \tilde{\bar{\theta}}^T(0) \tilde{\bar{\theta}}(0) \right) < \infty \tag{13}$$

or equivalently

$$\|e\|^2_{L_2} \leq \frac{V(0)}{|a_m + \ell|} < \infty \tag{14}$$

Thus we have proved (b).

In order to prove (c) we will use Barbalat Lemma as presented in Corollary 4. e is bounded from (a) and from (b) e is also in $\mathcal{L}_2$. Therefore, $e \in \mathcal{L}_2 \cap L_\infty$. We also have that $\dot{e}$ is bounded because all of the terms on the right hand side of (9) are bounded as given in (a). This allows us to conclude that e is uniformly continuous and thus all of the conditions of Corollary 4 are satisfied. □

2.1. **L-2 norm of $\dot{k}$.** From (9) we can deduce that

$$\|\dot{k}(t)\|^2 = \gamma^2 e(t)^2 r(t)^2 \tag{15}$$

Inetgrating both sides we have

$$\begin{aligned}
\int_0^t \|\dot{k}(\tau)\|^2 d\tau &= \gamma^2 \int_0^t e(\tau)^2 r(\tau)^2 d\tau \\
&\leq \gamma^2 \int_0^t \|r(\tau)\|^2_{L_\infty} e(\tau)^2 d\tau \\
&= \gamma^2 \|r(\tau)\|^2_{L_\infty} \int_0^t e(\tau)^2 d\tau \\
&\leq \gamma^2 \|r(t)\|^2_{L_\infty} \|e(t)\|^2_{L_2}
\end{aligned} \tag{16}$$

Using the definition of $\|e\|_{L_2}$ from (14) we have that

$$\boxed{\|\dot{k}(t)\|_{L_2}^2 \leq \frac{2\gamma^2\|r(t)\|_{L_\infty}^2 V(0)}{|a_m+\ell|}} \tag{17}$$

2.2. **L-2 norm of $\dot{\theta}$.** Before beginning directly with the norm of $\dot{\theta}$ we first find the upper bound on $x_m(t)$.

2.2.1. *L-∞ norm of x_m.* The solution to the ODE in (4) is

$$x_m(t) = \exp{(a_m t)} x_m(0) + \int_0^t \exp{(a_m(t-\tau))} r(\tau) d\tau - \ell \int_0^t \exp{(a_m(t-\tau))} e(\tau) d\tau \tag{18}$$

The solution to the ODE in (3) is

$$x_m^o(t) = \exp{(a_m t)} x_m^o(0) + \int_0^t \exp{(a_m(t-\tau))} r(\tau) d\tau \tag{19}$$

Also, noting that regardless of whether we use the ORM or CRM, they will both have the same initial conditions, we have that $x_m(0) = x_m^o(0)$ and thus

$$\|\exp{(a_m t)} x_m(0) + \int_0^t \exp{(a_m(t-\tau))} r(\tau) d\tau\| \leq \|x_m^o(t)\|_{L_\infty} \tag{20}$$

We note that $\|x_m^o(t)\|_{L_\infty}$ is only a function of the reference model and the initial condition of the reference model, and is not affected by γ or ℓ. Using (20), (18) can be bounded as

$$\|x_m(t)\| \leq \|x_m^o(t)\|_{L_\infty} + |\ell| \int_0^t \exp{(a_m(t-\tau))} \|e(\tau)\| d\tau \tag{21}$$

Using Cauchy Schwartz Ineqality on $\int_0^t \exp{(a_m(t-\tau))}\|e(\tau)\|d\tau$ we have that

$$\int_0^t \exp{(a_m(t-\tau))} \|e(\tau)\| d\tau \leq \sqrt{\int_0^t (\exp{(a_m(t-\tau))})^2 d\tau} \sqrt{\int_0^t \|e(\tau)\|^2 d\tau} \tag{22}$$

Noting the rule for the powers of exponents, we have

$$\begin{aligned} \sqrt{\int_0^t (\exp{(a_m(t-\tau))})^2 d\tau} &= \sqrt{\int_0^t (\exp{(2a_m(t-\tau))}) d\tau} \\ &= \sqrt{-\frac{1}{2a_m}\left[1-\exp(2a_m(t))\right]} \\ &\leq \sqrt{\frac{1}{|2a_m|}} \end{aligned} \tag{23}$$

where the last step arises do to the fact that $a_m < 0$. Using the bound on $\sqrt{\int_0^t (\exp(a_m(t-\tau)))^2 d\tau}$ from (23) and the bound for $\|e(t)\|_{L_2}$ from (14), (22) can be bounded as

$$\int_0^t \exp(a_m(t-\tau))\|e(\tau)\|d\tau \leq \sqrt{\frac{1}{|2a_m|}}\sqrt{\frac{V(0)}{|a_m+\ell|}} \tag{24}$$

Using the above inequality in (21) we finally have that

$$\|x_m(t)\| \leq \|x_m^o(t)\|_{L_\infty} + |\ell|\sqrt{\frac{1}{|2a_m|}}\sqrt{\frac{V(0)}{|a_m+\ell|}} \tag{25}$$

Finally, using the fact that $(x+y)^2 \leq 2x^2 + 2y^2$, and the fact that the above inequality is true for all time, the above inequality can be expressed in terms of $\|x_m(t)\|_{L_\infty}^2$ as

$$\|x_m(t)\|_{L_\infty}^2 \leq 2\|x_m^o(t)\|_{L_\infty}^2 + \frac{|\ell|^2}{|a_m|}\frac{V(0)}{|a_m+\ell|} \tag{26}$$

2.2.2. *L-2 norm of $\dot{\theta}$.* From the update law in (10) we have that

$$\|\dot{\theta}\|^2 = \gamma^2 e^2 x_p^2.$$

Using the fact that $e = x_p - x_m$ we have

$$\|\dot{\theta}\|^2 = \gamma^2 e^2 (e + x_m)^2.$$

Using the the same inequality that $(x+y)^2 \leq 2x^2 + 2y^2$, we have

$$\|\dot{\theta}\|^2 \leq 2\gamma^2 e^2 e^2 + 2\gamma^2 e^2 x_m^2.$$

Integrating we have

$$\int_0^t \|\dot{\theta}(\tau)\|^2 d\tau \leq 2\gamma^2 \int_0^t e(\tau)^2 e(\tau)^2 d\tau + 2\gamma^2 \int_0^t e(\tau)^2 x_m(\tau)^2 d\tau.$$

Taking out the supremum norms for one of the e^2 terms in the first integral and the supremum norm of x_m^2 in the second integral we have

$$\int_0^t \|\dot{\theta}(\tau)\|^2 d\tau \leq 2\gamma^2 \|e(t)\|_{L_\infty}^2 \int_0^t e(\tau)^2 d\tau + 2\gamma^2 \|x_m(\tau)\|_{L_\infty}^2 \int_0^t e(\tau)^2 d\tau. \tag{27}$$

Using the bounds for $\|e(t)\|_{L_\infty}$ in (12), the bounded for $\|e(t)\|_{L_2}$ in (14) and the bound for $\|x_m^o(t)\|_{L_\infty}$ in (26), the bound in (27) can be simplified as

$$\boxed{\|\dot{\theta}(t)\|_{L_2}^2 \leq 4\gamma^2 \frac{V(0)\|x_m^o(t)\|_{L_\infty}^2}{|a_m+\ell|} + 4\gamma^2 \frac{V(0)^2}{|a_m+\ell|} + 2\gamma^2 \frac{|\ell|^2}{|a_m|}\frac{V(0)^2}{|a_m+\ell|^2}} \tag{28}$$

3. Discussion

This analysis shows how we can rigorously prove that the use of a CRM can reduce the amount of chattering in the adaptive parameters k and θ. This is realized through the feedback gain ℓ in the CRM. For the L-2 norm of $\dot{k}$ in (17) we see that increasing $|\ell|$ will decrease the norm to be as small as one would like. For the L-2 norm of $\dot{\theta}$ we do not have the same uniform decrease in the norm. While the middle term in (28) can be arbitrarily decreased the last term has ℓ^2 in both the numerator and the denominator. Therefore the affect of the initial condition of the Lyapunov function $V(0)$ can not be completely removed. We can however decrease the affect of excitation due to the reference input $r(t)$ which is realized through the first expression which is proportional to $\|x_m^o(t)\|_{L_\infty}$. Without the ℓ we can not decrease the chattering that arises from the excitation from the reference model input. If you look back at the ppt file with the simulation results, we see that the step command is the point of excitation in the system.

References

1. K. S. Narendra and A. M. Annaswamy, *Stable adaptive systems*, Dover, 2005.

Appendix A. barbalat

Lemma 2 (Gronwall-Bellman Lemma 2.1 in [1]). *If $u, v \geq 0$, c_1 is a positive constant and if*

$$u \leq c_1 + \int_0^t uv d\tau$$

then

$$u \leq c_1 \exp\left(\int_0^t v d\tau\right).$$

Lemma 3 (Barbalat in pure analysis context, Lemma 2.12 in [1]). *If $f : \mathbb{R}^+ \to \mathbb{R}$ is uniformly continuous for $t \geq 0$ and if*

$$\lim_{t\to\infty} \int_0^t |f(\tau)|\, d\tau < \infty$$

thus $f(t) \in \mathcal{L}_1$, then

$$\lim_{t\to\infty} f(t) = 0.$$

Corollary 4 (Corollary 2.9 in [1]). *If $g \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{g}$ is bounded, then* $\lim_{t\to\infty} g(t) = 0$.

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