Chapter 1, Section 5B
MIXING AND TRANSPORT OF POLLUTANTS IN SURFACE WATER

by

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a coefficient in Equation 5B-12 \([L^{2/3}/T]\)
a empirical coefficient in Equation 5B-15b
A cross-sectional area \([L^2]\)
\(A_{jk}\) cross-section area between boxes j and k in Equation 5B-8 \([L^2]\)
b empirical coefficient in Equation 5B-15b [dimensionless]
c speed of sound in water \([L/T]\)
C concentration of mass in the water \([M/L^3]\)
\(C_k\) concentration in box k in Equation 5B-8 \([M/L^3]\)
d depth of channel \([L]\)
D coefficient of proportionality known as the molecular diffusion coefficient \([L^2/T]\)
\(E_L\) longitudinal dispersion coefficient \([L^2/T]\)
\(E_t\) transverse dispersion coefficient \([L^2/T]\)
\( E_T \) Taylor dispersion coefficient \([L^2/T]\)

\( E_{jk} \) dispersion coefficient between boxes \( j \) and \( k \) in Equation 5B-8 \([L^2/T]\)

\( f \) friction factor \([\text{dimensionless}]\)

\( F \) rate of mass flux \([M/L^2/T]\)

\( F(l) \) neighbor diffusivity in Equation 5B-10 \([L^2/T]\)

\( g \) gravitational acceleration \([L/T^2]\)

\( k \) constant in Equation 5B-13 \([\text{dimensionless}]\)

\( K \) parameter in Equation 5B-26 \([L^2/T]\)

\( l_{jk} \) distance separating the centroids of boxes \( j \) and \( k \) in Equation 5B-8 \([L]\)

\( l \) neighbor separation in Equation 5B-10 \([L]\)

\( l \) cross-sectional mixing length in Equation 5B-20 \([L]\)

\( L \) distance downstream of discharge point in Equation 5B-20 \([L]\)

\( L \) length of estuary in Equation 5B-26 \([L]\)

\( m \) constant in Equation 5B-26 \([\text{dimensionless}]\)

\( N \) Brunt-Väisälä frequency \([1/T]\)

\( Q_{jk} \) flow between box \( k \) and neighboring box \( j \) in Equation 5B-8 \([L^3/T]\)

\( q(l) \) neighbor concentration function in Equation 5B-10 \([\text{dimensionless}]\)

\( r \) distance from the center of mass \([L]\)

\( R \) radius of curvature \([L]\)

\( R_H \) hydraulic radius \([L]\)

\( S \) cross-sectionally averaged salinity \([M/L^3]\)

\( S_{\text{crit}} \) critical water column stability \([1/L]\)

\( S_{\text{max}} \) maximum salinity at the ocean entrance to an estuary \([M/L^3]\)

\( t \) time \([T]\)

\( u,v,w \) velocity components in the \( x,y,z \) coordinate directions \([L/T]\)

\( U \) cross-sectional average velocity \([L/T]\)

\( \overline{U}, \overline{V} \) depth-averaged velocities \([L/T]\)

\( U_{\text{max}} \) maximum tidal velocity \([L/T]\)

\( u^* \) shear velocity \([L/T]\)

\( V_k \) volume of box \( k \) in Equation 5B-8 \([L^3]\)

\( w \) length scale in Equation 5B-12 \([L]\) in meters

\( W \) wind speed \([L/T]\)

\( \overline{W} \) mean channel width \([L]\)

\( z_t \) depth to the thermocline \([L]\)

\( x,y,z \) spatial coordinates \([L]\)

\( \alpha \) coefficient in Equation 5B-16 \([L]\)
α₀, α₂  empirical coefficients in Equation 5B-25
β coefficient in Equation 5B-16 and Equation 5B-22 [dimensionless]
εₓ, εᵧ, εᶻ  turbulent diffusion coefficients in the x, y, z coordinate directions [L²/T]
ε₉  horizontal turbulent diffusion coefficient [L²/T]
εᵥ  vertical diffusion coefficient [L²/T]
εₘᵢₙ  minimum value of εᵥ [L²/T]
εₘₐₓ  maximum value of εᵥ [L²/T]
ρ  water density [M/L³]
ρ₀  reference water density [M/L³]
\bar{\rho}  average water density [M/L³]
τ₀  bed shear stress [ML/T²]

GENERAL REFERENCES


INTRODUCTION

Most environmental engineers are familiar with this description of pollutant mixing: "Dilution is the solution to pollution." It is an irreverent but apt description: mixing indeed plays
a very important role in the environment's ability to assimilate pollutants and in protecting mankind from its own wastewater. This section describes the basic principles of mixing and transport in surface water, with an emphasis on developing quantitative estimates of dispersion or diffusion for use in predictive modeling of environmental water quality.

TRANSPORT PROCESSES

For the purposes of this section, transport may be broadly defined as the ability of moving water to convey mass or other properties (such as heat or momentum) from place to place. Mixing is one aspect of transport, and describes the dilution of mass when different parcels of water commingle. The following describes the transport processes significant in surface water.

Advection. The key to transport is movement of the water, and different types of movement create different types of transport. Conceptually simplest is advective motion, the organized motion of water over a large scale. The flow of a river from upstream to downstream is a good example of advection. Advective flow dominates the movement of pollutants in most situations in surface water—pollutants are carried by the flowing water in roughly the same direction and speed as the water itself.

Nonetheless, even a cursory look at a river reveals far more complicated motion than advection alone. The water is often highly turbulent, swirling behind rocks and along river banks, splashing through riffles, and rolling from bank to bank and top to bottom downstream of curves in the river. These deviations from simple advective flow create mixing processes known as diffusion and dispersion.

Molecular Diffusion. Diffusion in surface water bodies is most appropriately called turbulent diffusion, to differentiate it from its namesake, molecular diffusion. Molecular diffusion is a well-defined physical process in which matter is transported by random molecular motions. It is described by Fick's Law of Diffusion which states that the rate of mass movement due to molecular diffusion is inversely proportional to the gradient of mass concentration [Crank, *The Mathematics of Diffusion*, 2nd Ed., Oxford Univ. Press (1975)]:

\[ F = -D \frac{\partial C}{\partial x} \]  

(5B-1)
where, \( F \) is the rate of mass flux (the mass of concentrate crossing a unit area per unit time) \([\text{M}/\text{L}^2/\text{T}]\);

\( C \) is the concentration of mass in the water \([\text{M}/\text{L}^3]\);

\( D \) is a coefficient of proportionality known as the molecular diffusion coefficient \([\text{L}^2/\text{T}]\); and

\( x \) is length \([\text{L}]\).

(Within brackets are shown the units of terms in the equation using the convention of \( M \) for mass units, \( L \) for length units, and \( T \) for time units.) Equation 5B-1 states, in simple terms, that mass will naturally move from areas of high concentration to areas of low concentration and that the rate of that movement is greatest when the change in concentration occurs rapidly over the shortest distance.

Turbulent Diffusion. The process of molecular diffusion is generally unimportant to pollutant transport in water bodies. Nonetheless, the movement of mass by random molecular motion in molecular diffusion is an attractive analog for the movement of mass by random water movement in turbulent flow. This analogy is the basis for the concept of turbulent diffusion, the transport of mass by turbulent water motion in accordance with Fick's Law. The extension of Fick's Law to turbulent transport holds that mass is transported by turbulence in the same way as molecular diffusion: from areas of high concentration to areas of low concentration at a rate proportional to the concentration gradient. Though described using the same law as molecular diffusion, turbulent diffusion typically results in vastly greater rates of transport.

The representation of turbulent mass transport using Fick's Law is an imperfect—but highly successful—approximation. That it is an approximation can be understood by considering a microscale view of turbulent water movement. If the turbulent flow field was considered in sufficient detail in time and space, it would be entirely described as advection: there would be no random (turbulent) component and thus no diffusion. But in fact, it is impossible to consider a real flow field in such detail, and there always remains a random component. It can be shown mathematically [see for example Daily and Harleman, *Fluid Dynamics*, Addison-Wesley, p. 432 (1966)] that turbulent transport arises from the interaction of this random component in the flow velocity with the similar random component in the concentration distribution. Based on the mathematical derivation, turbulent transport is formally defined as the residual transport that remains after averaging the transient field of velocity and concentration in turbulent flow over a short, but finite, period of time. However, therein lies the inherent approximation in applying Fick's Law to turbulent transport: the averaging period is not intrinsically defined and the
coefficient of turbulent diffusion can vary depending on the time period of averaging. We return to this limitation and its implications in the sections below.

To summarize, turbulent diffusion is the mixing that results from random turbulent motion in flowing water; turbulent diffusion formally derives from averaging velocities and concentrations over time; and turbulent diffusion behaves similarly to molecular diffusion and is well described by Fick's Law.

Dispersion. A subtle distinction differentiates dispersive transport from diffusive transport. Consider once again a river and suppose a quantity of colored dye is dropped as a line across the river. In the central core of the river, where the flow is fastest, dye will move ahead, outpacing the dye placed in slower areas near the streambank. Thus, the dye spreads out—disperses—along the river much as if it had been mixed longitudinally. This apparent mixing is known as dispersion.

Just as turbulent diffusion can be shown mathematically to arise from averaging over time in a turbulent flow field, dispersion can be shown to arise from averaging over space in a spatially non-uniform flow field. And, just as with turbulent diffusion, dispersion is usually approximated using Fick's Law. The analogy with Fick's Law becomes still more imperfect when applied to dispersive transport, but a Fickian model remains a practical and widely used approximation for all manner of dispersive transport phenomena in a wide range of water bodies.

In summary, dispersion is the apparent mixing that results from spatial variations in advective velocity; dispersion formally derives from spatial averaging of the flow field and concentration distribution; and dispersion is usually represented using Fick's Law of Diffusion.

MATHEMATICAL REPRESENTATION

Diffusion Relations. The most fundamental mathematical relation of mixing is Fick's Law of Diffusion, which is stated as Equation 5B-1. Fick's Law is extended through mass balance considerations to the one-dimensional diffusion equation:

\[ \frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} \]  

(5B-2)

where, \( t \) is time [T].
The equation is further extended to the general case of three spatial dimensions in Equation 5B-3:

\[
\frac{\partial C}{\partial t} = D \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right)
\]  

(5B-3)

where, \(x, y, z\) are the three spatial coordinates [L].

Advection-Diffusion Equation. Equation 5B-3 gives the rate of change in mass concentration due to diffusion alone. A more general representation of mass transport considers the contribution of advection as well. (A still more general equation would consider biological, physical and chemical reactions also, topics not covered in this section.) The equation for mass transport due to advection and molecular diffusion is:

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = D \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right)
\]  

(5B-4)

where, \(u, v, w\) are velocity components in the \(x, y, z\) coordinate directions [L/T].

When applied to turbulent diffusion, the molecular diffusion coefficient \(D\) is replaced with the turbulent diffusion coefficient \(\varepsilon\). Turbulence may vary from one coordinate direction to another, and therefore, the turbulent diffusion coefficient is most generally represented as dimensionally dependent, as indicated by the subscripts in Equation 5B-5:

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = \varepsilon_x \frac{\partial^2 C}{\partial x^2} + \varepsilon_y \frac{\partial^2 C}{\partial y^2} + \varepsilon_z \frac{\partial^2 C}{\partial z^2}
\]  

(5B-5)

where, \(\varepsilon_x, \varepsilon_y, \varepsilon_z\) are turbulent diffusion coefficients in the \(x, y, z\) coordinate directions [L²/T].

Equation 5B-5 is the general expression for the transport of mass due to advection and turbulent diffusion in three dimensions.
Averaging Equation 5B-5 over one or more spatial dimensions gives rise to dispersion as described above. The most commonly used form is the one-dimensional equation applied to rivers and estuaries:

\[
\frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} = \frac{1}{A} \frac{\partial}{\partial x} \left( AE_L \frac{\partial C}{\partial x} \right) \tag{5B-6}
\]

where, \( U \) is the cross-sectional average velocity \([L/T]\); \( A \) is the cross-sectional area \([L^2]\); and \( E_L \) is the longitudinal dispersion coefficient \([L^2/T]\).

In wide rivers and estuaries, and in lakes, the spread of pollutants across, as well as along the direction of flow is important. For these problems, the two-dimensional horizontal dispersion equation is used:

\[
\frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} + V \frac{\partial C}{\partial y} = \frac{1}{A} \left[ \frac{\partial}{\partial x} \left( AE_L \frac{\partial C}{\partial x} \right) + \frac{\partial}{\partial y} \left( AE_t \frac{\partial C}{\partial y} \right) \right] \tag{5B-7}
\]

where, \( U, V \) are depth-averaged velocities \([L/T]\); and \( E_t \) is the transverse dispersion coefficient \([L^2/T]\).

In rivers and estuaries, \( V \) is usually taken as zero, but the transverse dispersion term is retained.

Finally, we consider an averaged form of Equation 5B-5 commonly used in modeling large lakes. This is the multiple-box model (or simply box model) in which the lake is subdivided into a number of fully-mixed volume elements or "boxes." Each box is represented by a three-dimensionally averaged form of the advection-diffusion equation:

\[
V_k \frac{dC_k}{dt} = \sum_j \left( Q_{jk} C_j - Q_{jk} C_k + \frac{E_{jk} A_{jk}}{l_{jk}} \left( C_j - C_k \right) \right) \tag{5B-8}
\]

where, \( C_k \) is the concentration in box k \([M/L^3]\); \( V_k \) is the volume of box k \([L^3]\); \( Q_{jk} \) is the flow from box j to downstream box k \([L^3/T]\);
\(E_{jk}\) is a dispersion coefficient between boxes \(j\) and \(k\) \([L^2/T]\);
\(A_{jk}\) is the cross-section area between boxes \(j\) and \(k\) \([L^2]\); and
\(l_{jk}\) is the distance separating the centroids of boxes \(j\) and \(k\) \([L]\).

HORIZONTAL MIXING IN OCEANS AND LARGE LAKES

THE NATURE OF TURBULENT DIFFUSION

Oceanic mixing is typically considered to be a diffusive rather than a dispersive process. There are conditions in which velocity varies with depth or horizontal distance, known as shear currents, and which cause dispersion. In most situations however, the turbulent diffusion of the ocean environment is presumed to dominate mixing.

Carl Eckart [J. Mar. Res., 7, p. 265 (1948)] has given a very intuitive explanation of the turbulent diffusion processes that cause oceanic mixing. He uses the analogy of the mixing of coffee and cream. He defines three stages after the cream has been poured into a cup of coffee. At first the two fluids are distinctly separated: sharp interfaces separate large areas of the coffee and cream. Although the concentration gradient of the cream is large at the interface, averaged over the entire cup surface the gradient is fairly small. This situation could persist a long time if stirring did not occur. In the second stage this pattern is disturbed when the coffee is stirred. The coffee and cream bodies are quickly distorted and the interface area consequently increases. Over the whole cup the average concentration gradient is increased. In the final stage molecular diffusion acts along the extensive interface areas to complete the mixing process quite abruptly, producing the liquid you wanted to drink.

This is the basic process at work in the oceans. Turbulent motions produce a mixing roughly equivalent to stage two in the coffee and cream mixture. The motion is not nearly as simple as that in a coffee cup, however. The one simple whirlpool motion in the coffee cup is replaced by a complicated series of eddy motions of all sizes in the ocean. Although an eddy is usually thought of as a small swirling motion, a more comprehensive definition is useful in the analysis of diffusion. Stommel [J. Mar. Res., 8, p. 199 (1949)] states that the essential concept of an eddy is the "area over which turbulent velocities are similar or correlated." This is quantified through a Fourier analysis of the motion that identifies Fourier components with various wave numbers. These wave numbers are then taken to indicate eddy sizes [Okubo, Impingement of Man on the Oceans, (Hood, ed.), Wiley-Interscience, p. 89 (1971)].
Analysis of the characteristics of the series of eddies leads to the concept of the eddy spectrum and energy transfer between eddies of different sizes. The largest eddies, the ocean-sized circulation patterns, constantly draw their energy from the wind. Since the kinetic energy of the ocean remains relatively constant, some form of energy dissipation must take place. What happens is that some of the energy of the general circulation is lost to form eddies of about 100 kilometer size. These in turn pass energy down to smaller eddies, and so on. Eventually, the smallest eddies dissipate into heat generated by viscous friction. Viscosity acts on all of the larger eddies in the spectrum also [Stommel, J. Mar. Res., 8, p. 199 (1949)]. The whole process has been very neatly summarized by L.F. Richardson [quoted in Batchelor, Proc. Cambridge Phil. Soc., 43, p. 33 (1947)]:

Big whorls have little whorls,
which feed on their velocity;
Little whorls have smaller whorls,
and so on into viscosity.

The eddy spectrum is quite significant in diffusion. If a bucket of dye is dumped in the open ocean, diffusion processes will enlarge it to produce a constantly spreading cloud. Figure 5B-1 gives a schematic view of what would happen to such a cloud in an ocean with only two scales of turbulence. At first only the smallest eddies would act to spread the cloud; the larger eddy appears only as advection, moving the entire cloud. With spreading, however, the cloud eventually reaches a size comparable to the large eddy, and then it too acts to spread the dye. In the ocean, with a continuous spectrum of motions, the dividing line between advection and turbulence constantly changes as the cloud enlarges. Eddies that happen to be about the same size as the cloud have a short period during which they produce a large change in the cloud. This process continues with larger and larger eddies until the dye is widely dispersed.

DESCRIPTIONS OF HORIZONTAL DIFFUSION

Fickian Diffusion. The process of dispersion can be cast in mathematical terms using the advection-diffusion equation based on Fick’s Law, Equation 5B-4. For now, we will simplify by considering the diffusion terms alone, separate from advection. This is done by transforming to a Lagrangian coordinate system in which the coordinate origin moves with center of mass of the advecting pollutant. This mathematical construct also has practical application. For example, Adams et al. [R.M. Parsons Lab. Report No. 205, MIT (1975)] develop an ocean diffusion model based on such a Lagrangian model.
In the transformed coordinate system, horizontal diffusion about the center of pollutant mass is given by:

\[
\frac{\partial C}{\partial t} = \varepsilon_H \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right)
\]  

(5B-9a)

or in radial coordinates:
\[
\frac{\partial C}{\partial t} = \varepsilon_H \frac{1}{r} \frac{\partial^2}{\partial r^2} \left( r \frac{\partial C}{\partial r} \right)
\]  
(5B-9b)

where, \( r \) is distance from the center of mass [L]; and
\( \varepsilon_H \) is the horizontal turbulent diffusion coefficient \([L^2/T]\).

\( \varepsilon_H \) is also called the horizontal turbulent diffusivity or eddy diffusivity. It is used in the presumption that horizontal diffusion is the same in the x and y directions (that is, \( \varepsilon_x = \varepsilon_y = \varepsilon_H \)). Equation 5B-9b is the model used in the earliest studies of oceanic diffusion, studies that pointed out some major deviations from Fick's Law.

When field experiments with dye clouds were carried out in the ocean, the diffusivity was found to increase with time. The reasons for this are related to the distribution of eddy sizes, as discussed above. Those eddies approximately equal in size to the cloud contribute the most to its mixing. As the cloud expands the dominant eddies shift to the larger portion of the eddy spectrum, and the mixing power consequently increases. Thus, \( \varepsilon_H \) is an increasing function of time and cloud size [Deacon, Inter. J. Air Water Poll., 2, p. 92 (1959)].

This is in direct conflict with the concepts underlying the Fickian approach. Fick's Law assumes that the velocity of a particle is independent of the velocities of nearby particles. With turbulence, however, the closer two particles are, the more similar their velocities will be. Thus, the basic predicate of the Fick equation is violated.

Statistical Description of Diffusion. Further insight to the limitations in applying Fick's Law to turbulent diffusion is gained by considering the statistics of the random movement of a single particle of pollutant. The mathematical derivation, which is complicated, is given by Fischer et al. [Mixing in Inland and Coastal Waters, Academic Press (1979)]. The derivation considers the statistical correlation between the particle's motion at one instant in time with motion at previous instants in time—the particle's "Lagrangian autocorrelation." An important related concept resulting from the statistical derivation is the Lagrangian time scale. The Lagrangian time scale can be thought of as the period of time for the autocorrelation to go to zero, in essence for the particle to forget its past motion. The statistical theory shows that for time periods shorter than the Lagrangian time scale the diffusion coefficient changes with time and a Fickian description is therefore invalid. For time periods that are long relative to the Lagrangian time scale, the diffusion coefficient becomes constant and Fick's Law can be used. Unfortunately, there is no a priori method to determine the Lagrangian time scale for a particular situation. Related derivations show, however, that if the scale of the diffusing plume is greater
than the scale of turbulent motion, the Fickian relation holds. This criterion has proven to be most useful in bounded water bodies, where the depth or width can be assumed to be an upper limit on the scale of turbulence. In the ocean, experiments with length scales as great as 40 kilometers fail to show a limit on the turbulent diffusion coefficient.

Richardson Neighbor Diffusion. Richardson [Proc. Royal Soc. London, 110A, p. 709 (1926)] addressed the apparent relation between the diffusion coefficient and length scale for time periods shorter than the Lagrangian time scale. Noting that the diffusivity of the atmosphere varied by a billion orders of magnitude, he sought an expression for the diffusion coefficient that would account for this variation, but still not be a function of position or time. The relevant parameter was, he said, the distance $l$ by which two particles are separated. He then went on to define a theory based on this "neighbor separation." The equation proposed was modeled after the Fick equation:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial l} \left( F(l) \frac{\partial q}{\partial l} \right)$$  \hspace{1cm} (5B-10)

where,
- $l$ is the neighbor separation [L], equal to the instantaneous distance between two particles;
- $q(l)$ is the neighbor concentration function [dimensionless], equal to the number of particle pairs separated by the distance $l$; and
- $F(l)$ is the neighbor diffusivity [L²/T].

$F(l)$ is of the same nature as $\varepsilon_H$ and has the same units. For diffusion of the Fickian type, when $F(l)$ is a constant, it can be shown that $F(l) = 2\varepsilon_H$. For oceanic turbulent diffusion, $F(l)$ is a function of $l$. From experimental data Richardson proposed:

$$F(l) = a l^{4/3}$$  \hspace{1cm} (5B-11)

Taylor [Advances in Geophysics, 6, p. 101 (1959)] attributes the choice of $4/3$ rather than 1.3 or 1.4 to a remarkable intuitive insight on Richardson's part. He states that Richardson felt that $F(l)$ was related to the energy transfer from larger to smaller eddies, and that a simple universal rule must therefore govern the process. Fifteen years later the eddy diffusivity was proven analytically to be proportional to the $4/3$ power of the eddy length by Kolmogoroff through a
Richardson's 4/3 Law may be used in practical problems to define a diffusivity based on the width of a diffusing plume or cloud. The following empirical formula is based on field data:

\[ \varepsilon_H = a w^{4/3} \]  

(5B-12)

where, \( a \) is a coefficient \([L^{2/3}/T]\) with a value typically between \(4 \times 10^{-6}\) and \(2 \times 10^{-5}\) \(m^{2/3}/\text{sec}\); and \( w \) is a pertinent length scale such as the plume width \([L]\).

The formula is based on field data collected and analyzed by A. Okubo as reported by Fischer et al. [Mixing in Inland and Coastal Waters, Academic Press (1979)]. In the ocean, the relation has been found to hold for length scales from 10 meters to over 10 kilometers. It should be applied with caution in bounded water bodies where free turbulence is impeded by shorelines and the bottom.

Murthy and Okubo [Symp. Modeling Transport Mechanisms in Oceans and Lakes, Manuscript Report Series No. 43, Mar. Science Directorate, Dept. Fisheries and Environ. Ottawa, Canada, p. 129 (1977)] compile field data from ocean and lake diffusion experiments and arrive at a slightly different dependency on the scale of diffusion. Their results, which are reproduced here in Figure 5B-2, show \( \varepsilon_H \) to be proportional to \( w^{1.1} \). The minor difference between the 4/3 power in Equation 5B-12 and the 1.1 power in Figure 5B-2 may arise from spatial intermittency in the turbulence field [Lawrence et al., Limnology and Oceanogr., 40, p. 1519 (1995)].

Application. A classic paper by Brooks [Proc. Inter. Conference on Waste Disposal Marine Environ. p. 246 (1960)], which is also described by Fischer et al. [Mixing in Inland and Coastal Waters, Academic Press, pp. 406-411 (1979)], considers the mixing of sewage effluent discharged to the ocean. Brooks' paper gives the mathematical solution for a sewage plume subject to turbulent diffusion that conforms to the 4/3 Law. The original paper or Fischer et al. should be consulted for a description of the solution and its application.
Figure 5B-2
Horizontal Turbulent Diffusivity in Oceans and Large Lakes as a Function of Length Scale
[based on Lawrence et al., Limnology and Oceanogr., 40, pp. 1519-1526 (1995), reproduced with permission]
HORIZONTAL DISPERSION IN LAKE MODELS

Transport and mixing in large lakes is often modeled using box models, previously defined using Equation 5B-8. A dispersion coefficient is defined in thes models to capture the effects of non-advective mixing between adjacent boxes. This is an essentially empirical formulation, however, and there are no direct means to compute the dispersion coefficient based on the geometric or physical characteristics of the water body. Shanahan and Harleman [J. Environ. Eng. ASCE, 110, p. 42 (1984)] describe a procedure to define the dispersion coefficient for box models.

VERTICAL MIXING IN THE OCEAN AND LARGE LAKES

The effects of vertical mixing are much less pronounced in the oceans and large lakes than are the effects of horizontal mixing. The reason is the existence of the thermocline, a depth interval over which temperature decreases rapidly with depth. (Formally, the thermocline is the interval of depth in which the change in temperature with depth exceeds 1°C per meter.) The thermocline is found below a warmer, mixed layer at the water's surface. The cool waters below the thermocline are denser than those above; thus, the thermocline acts as a physical barrier to mixing. In the oceans particularly, the thermocline is relatively shallow and horizontal mixing dominates vertical mixing as a dilution mechanism. Nonetheless, vertical mixing can be a very important water quality process, particularly in lakes and reservoirs where water quality varies greatly in the vertical dimension.

VERTICAL MIXING IN THE OCEAN

Vertical mixing in the ocean is reviewed by Okubo [Impingement of Man on the Oceans, (Hood, ed.), Wiley-Interscience, p.89 (1971)] and Murthy and Okubo [Syp. Modeling Transport Mechanisms in Oceans and Lakes, Manuscript Report Series No. 43, Mar. Science Directorate, Dept. Fisheries and Environ. Ottawa, Canada, p. 129 (1977)]. They report that most experimental values of the vertical turbulent diffusion coefficient found in ocean experiments range between 1 and 100 cm²/sec, with the smallest values representative of values below the thermocline. Kullenberg [Tellus, 23, p. 129 (1971)] relates the value of the vertical diffusion coefficient to the wind speed and water-column stability through this relation:

\[ \varepsilon_v = k \frac{W^2}{N^2} \left| \frac{du}{dz} \right| \]  (5B-13)
where, \( k \) is a constant [dimensionless] with a value of between \( 2 \times 10^{-8} \) and \( 8 \times 10^{-8} \);  
\( W \) is the mean wind speed [L/T];  
\( N \) is the Brunt-Väisälä frequency [1/T]; and  
\( u \) is the horizontal water velocity [L/T].

The variable \( N \) is related to the density stability of the water column. Physically, it represents the natural frequency of oscillation that occurs in the water column following a disturbance. It is defined as [Phillips, *The Dynamics of the Upper Ocean*, 2nd Ed., Cambridge Univ. Press (1977)]:

\[
N = \left( -\frac{g}{\rho_0} \frac{\partial \rho}{\partial z} - \frac{g^2}{c^2} \right)^{1/2}
\]  
(5B-14)

where,  
\( g \) is gravitational acceleration [L/T²];  
\( \rho \) is the water density as a function of \( z \) [M/L³];  
\( \rho_0 \) is the reference density at \( z = 0 \) [M/L³]; and  
\( c \) is the speed of sound in water [L/T], equal to approximately 1500 m/sec.

Figure 5B-3 is based on Murthy and Okubo's [Symp. Modeling Transport Mechanisms in Oceans and Lakes, Manuscript Report Series No. 43, Mar. Science Directorate, Dept. Fisheries and Environ. Ottawa, Canada, p. 129 (1977)] plot of Equation 5B-13 against data from the ocean and Lake Ontario.

**VERTICAL MIXING IN LAKES AND RESERVOIRS**

Vertical mixing in temperate lakes is strongly influenced by seasonal cycles in the lake's thermal structure. At the end of winter, a lake is typically mixed throughout its depth and shows a vertically isothermal temperature profile. As the lake surface is warmed by the sun and atmosphere through the spring, the shallowest water warms relative to the deeper water. Soon, a pronounced thermocline separates the warm surface water, the epilimnion, from the cold deeper water, the hypolimnion. Through the summer, the thermocline becomes stronger (that is, the difference in temperature over the vertical distance of the thermocline increases). Finally, with surface cooling in the fall, the temperature stratification weakens until the fall overturn, when the lake mixes throughout its depth.
Figure 5B-3

Empirical Relation for the Vertical Turbulent Diffusivity
[based on Murthy and Okubo, Symp. on Modeling of Transport Mechanisms in Oceans and Lakes, Manuscript Report Series No. 43, Mar. Sciences Directorate, Dept. of Fisheries and Environ. Ottawa, Canada, p. 129 (1977)]
<table>
<thead>
<tr>
<th>Lake</th>
<th>Vertical Diffusion Coefficient (cm$^2$/sec)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lake Erie</td>
<td>0.58</td>
<td></td>
</tr>
<tr>
<td>Lake Erie</td>
<td>102</td>
<td>Unstratified</td>
</tr>
<tr>
<td>Lake Erie</td>
<td>15</td>
<td>Stratified</td>
</tr>
<tr>
<td>Lake Huron</td>
<td>1.16</td>
<td></td>
</tr>
<tr>
<td>Lake Huron</td>
<td>1.16</td>
<td></td>
</tr>
<tr>
<td>Lake Ontario</td>
<td>3.47</td>
<td></td>
</tr>
<tr>
<td>Wellington Reservoir</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>White Lake, Michigan</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>Lake LBJ, Texas</td>
<td>0.18, 0.12, 0.01</td>
<td>Feb.-April, May-June, July-Jan.</td>
</tr>
<tr>
<td>Cayuga Lake, New York</td>
<td>2.31</td>
<td></td>
</tr>
<tr>
<td>Lake Greifensee, Switzerland</td>
<td>.02, 0.15, 0.05</td>
<td>April, May-Aug., Sept.-Nov.</td>
</tr>
</tbody>
</table>

**TABLE 5B-1**
SUMMARY OF VERTICAL DIFFUSION MEASUREMENTS IN LAKES

<table>
<thead>
<tr>
<th>Lake</th>
<th>Vertical Diffusion Coefficient (cm$^2$/sec)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lake Erie</td>
<td>0.21</td>
<td></td>
</tr>
<tr>
<td>Lake Ontario</td>
<td>0.125, 0.063</td>
<td></td>
</tr>
<tr>
<td>Cayuga Lake, New York</td>
<td>0.178, 0.25</td>
<td></td>
</tr>
<tr>
<td>Lake Greifensee, Switzerland</td>
<td>.25</td>
<td></td>
</tr>
<tr>
<td>Lake Zurich, Switzerland</td>
<td>.71, .14, .064, .039, .026, .074, .03</td>
<td>April, May, June, July, Aug., Sept., Oct.</td>
</tr>
<tr>
<td>Lake Mendota, Wisconsin</td>
<td>.03</td>
<td></td>
</tr>
<tr>
<td>Lake Tasmania, USSR</td>
<td>2.5 – 7.4</td>
<td></td>
</tr>
</tbody>
</table>

1 [Torgersen et al., Limnology and Oceanogr., 22, p. 181 (1977)]
2 [DiToro and Connolly, Report EPA-600/3-80-065, U.S. EPA (1980)]
3 [Heinrich et al., J. Great Lakes Res., 7, p. 284 (1981)]
5 [Imberger et al., J. Hydraulics Div., ASCE, 104, p. 725 (1978)]
8 [Bedford and Babajimopoulus, J. Environ. Eng., Div., ASCE, 103 p. 133 (1977)]
9 [Imboden and Emerson, Limnology and Oceanogr., 23, p. 77 (1978)]
10 [Snodgrass and O'Melia, Environ. Sci. Technol., 9, p. 937 (1975)]
Vertical mixing is of greatest interest during the summer stratification. The strong temperature and density gradient across the thermocline separates zones of distinctly different water quality. The epilimnion receives sunlight and atmospheric oxygen, and thus supports phytoplankton growth. Typically, dissolved oxygen is high and nutrient concentrations low in the epilimnion. Algal growth is limited in the cooler, darker hypolimnion, and dissolved oxygen is reduced or absent. Organic matter that settles into the hypolimnion and chemical constituents diffused from the lake bottom increase nutrient and other constituent concentrations, creating a dramatically different water quality than in the epilimnion. The distinct water qualities in these two layers make vertical mixing a very important water-quality process in lakes.

The vertical diffusion coefficient in lakes is typically on the order 0.1 to 0.01 cm$^2$/sec, but may range from as high as 1 cm$^2$/sec to as low as 0.001 cm$^2$/sec. Table 5B-1, which is drawn from Schnoor et al. [Report EPA/600/3-87/015, U.S. EPA (1987)], lists observed vertical diffusion coefficients from a number of published studies.

Deep Mixing. Mixing across a well-established thermocline or within the hypolimnion is very limited. Lake modeling studies by Wang and Harleman [R.M. Parsons Lab. Report No. 270, MIT (1982)] show that diffusion across and below the thermocline of stratified lakes is at or near the rate of molecular diffusion (1.6 x 10$^{-5}$ cm$^2$/sec for salt [NaCl] in water at 25°C). Hypolimnetic mixing may be higher if there is significant flow or other motion within the hypolimnion. For example, there may be flow in a reservoir from stream inflows to a deep dam outlet. Another source of motion is an internal seiche, the back-and-forth oscillation of the thermocline in a type of motion similar to sloshing in a bathtub.

Surface Mixing. The epilimnion of a lake or reservoir typically is well mixed owing to a nearly constant input of mixing energy from the wind. A more subtle and interesting question than mixing within the epilimnion is the rate at which mixing from the wind causes the surface layer to mix into the thermocline and become deeper. This is a far more important mechanism for transport from the hypolimnion to the epilimnion than diffusion across the thermocline.

(1969)] relied on specification of a single vertical diffusion coefficient to compute vertical mixing. The empiricism and lack of generality of this approach led to definitions of the diffusion coefficient in terms of the lake's thermal structure and/or input mixing energy.

Through the 1970s, descriptive formulae for the vertical diffusion coefficient in lakes were improved, but with what was still an essentially empirical approach. The widely-used WQRRS Model [Smith, “Water Quality for River-Reservoir Systems,” HEC, US Army Corps of Eng. (1978)] includes two diffusion coefficient formulae typical of this approach. For deep, well-stratified lakes, the diffusion coefficient is determined from the water column stability:

\[
\varepsilon_v = \varepsilon_{\text{max}} \quad \text{when} \quad \frac{1}{\bar{\rho}} \frac{\partial \rho}{\partial z} \leq S_{\text{crit}} \quad (5B-15a)
\]

\[
\varepsilon_v = a \left( \frac{1}{\bar{\rho}} \frac{\partial \rho}{\partial z} \right)^b \quad \text{when} \quad \frac{1}{\bar{\rho}} \frac{\partial \rho}{\partial z} > S_{\text{crit}} \quad (5B-15b)
\]

where, \( \varepsilon_v \) is the vertical diffusion coefficient \([L^2/T]\); \( \varepsilon_{\text{max}} \) is an empirically determined maximum value of \( \varepsilon_v \) \([L^2/T]\) with a recommended value ranging from 0.2 to 10.0 cm\(^2\)/sec and an average value of about 2.5 cm\(^2\)/sec; \( \rho \) is the density of the water as a function of \( z \) \([M/L^3]\); \( \bar{\rho} \) is the average density of the water \([M/L^3]\); \( S_{\text{crit}} \) is a critical water column stability \([1/L]\), typically between 1 \times 10^{-6} and 1 \times 10^{-5} m\(^{-1}\); \( a \) is an empirical coefficient with units that depend on the value of \( b \); and \( b \) an empirical coefficient [dimensionless] with a typical value of −0.7.

This equation predicts a diffusion coefficient that is constant and equal through the epilimnion and hypolimnion, but which decreases through the thermocline.

For lakes in which wind mixing is a dominant mechanism, WQRRS provides this relation:
\[ \varepsilon_v = \varepsilon_{\text{min}} + \alpha W \exp \left( -\frac{\beta z}{z_t} \right) \]  

(5B-16)

where, \( \varepsilon_{\text{min}} \) is the minimum value of \( \varepsilon_v \) [L²/T] with a recommended value of 0.1 to 0.5 cm²/sec in well-mixed reservoirs and 0.01 to 0.001 cm²/sec in stratified reservoirs;

\( W \) is the wind speed [L/T];

\( z_t \) is the depth to the thermocline [L];

\( \alpha \) is an empirical coefficient [L] with a recommended value of 0.01 to 0.02 cm in well-mixed reservoirs and 0.001 to 0.005 cm in stratified reservoirs; and

\( \beta \) is an empirical coefficient [dimensionless] with a recommended value of 4.6.

Equation 5B-16 predicts a diffusion coefficient that has its maximum value at the surface, where there is the greatest wind energy, decreasing exponentially with depth to a minimum value. The values recommended in the WQRRS manual for \( \varepsilon_{\text{max}} \) in Equation 5B-15 and \( \varepsilon_{\text{min}} \) in Equation 5B-16 are probably high based on more recent research that shows vertical diffusivities between 1 and 50 times molecular diffusivity at depth in lakes.

Equations 5B-15 and 5B-16 are just two widely-used examples of a large number of empirical equations for vertical diffusivity found in the literature. Good reviews of the many equations proposed are given by Henderson-Sellers [J. Environ. Eng. Div., ASCE, 102 p. 517 (1976)] and McCutcheon [Proc. Conference on Frontiers in Hydraulic Eng., ASCE, p.15 (1983)]. Table 5B-2 provides a summary of the literature addressing vertical dispersion in lakes.

Wind-Mixing Models. The most sophisticated and successful models of vertical mixing in lakes are those that simulate the effects of surface wind mixing, surface heat transfer, vertical density structure, and hydraulic forces through time. Harleman [J. Hydraulics Div. ASCE, 108, p. 302 (1982)] reviews these models with an emphasis on the first three mechanisms; the review by Fischer et al. [Mixing in Inland and Coastal Waters, Academic Press (1979)] places greater emphasis on the last mechanism.
TABLE 5B-2

SUMMARY OF REFERENCES ON VERTICAL DIFFUSION IN LAKES

<table>
<thead>
<tr>
<th>Reference</th>
<th>Epilimnion</th>
<th>Hypolimnion</th>
<th>Thermocline</th>
<th>Wind Mixing</th>
<th>Through Flow</th>
<th>Field Data</th>
<th>Formula for ( E_v )</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>DiGiano et al.(^1)</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
<td>Dispersion in shallow lakes</td>
</tr>
<tr>
<td>Henderson-Sellers(^2)</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td>Evaluation of literature formulae</td>
</tr>
<tr>
<td>Powell and Jassby(^3)</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Evaluation of formulae for hypolimnetic diffusion</td>
</tr>
<tr>
<td>Kullenberg(^4)</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td>( E_v ) from stability, wind</td>
</tr>
<tr>
<td>Imberger(^5)</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td>River underflow mixing</td>
</tr>
<tr>
<td>Adams et al.(^6)</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td>Diffusion in stratified cooling pond</td>
</tr>
<tr>
<td>Bloss and Harleman(^7)</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
<td>Wind-mixing model of thermocline</td>
</tr>
</tbody>
</table>

\(^1\)[DiGiano, Likiema and Van Straten, Ecol. Modelling, 4, p. 237 (1978)]
\(^4\)[Kullenberg, Tellus, 23, p. 129 (1971)]
\(^5\)[Imberger, J. Hydraulic Eng., ASCE, 113, p. 697 (1987)]
\(^6\)[Adams et al., J. Hydraulic Eng., ASCE, 113, p. 293 (1987)]
\(^7\)[Bloss and Harleman, R.M. Parsons Lab. Report No. 249, MIT (1979)]

DISPERSION IN RIVERS

RIVER DISPERSION PROCESSES

If a mass of material is introduced into a river, it will be mixed largely by two mechanisms. Firstly, and most importantly, the mass is dispersed due to variations in flow velocity across the river. Some portions of the river travel faster than others, and thus will carry some of the material further downstream. The net effect of this advective dispersion is to spread the material along the river. If no other mechanisms were important, this would produce quite a high rate of dispersion. However, this potentially high rate is lowered somewhat by the second process, turbulent diffusion. The effect of this process is to shift quantities of material between
zones of different velocity so that none remains in one zone indefinitely. This decreases the spread of pollutant by shifting the material in low velocity zones to flowing zones, and that in the maximum velocity zones into slower moving sectors [Glover, U.S. Geo. Survey Prof. Paper 433-B, (1964); Fischer, U.S. Geo. Survey Prof. Paper 582-A (1968)].

In practice, analysis of mass transport in rivers is frequently limited to a one-dimensional representation of longitudinal dispersion using Equation 5B-6. When necessary, the two-dimensional advective-dispersion equation (Equation 5B-7) can be used to explicitly consider transverse dispersion. Vertical transport is rarely modeled explicitly because rivers are usually well mixed vertically in the time periods of interest.

DESCRIPTIONS OF LONGITUDINAL DISPERSION IN RIVERS

Classical Descriptions. In order to understand the mathematical descriptions of longitudinal dispersion in rivers, we begin by reviewing Taylor's work concerning dispersion in pipes. Neglecting reactions and molecular diffusion, and assuming homogeneous turbulent diffusion in a uniform velocity field, longitudinal dispersion can be represented by the one-dimensional diffusion equation:

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = \epsilon_x \frac{\partial^2 C}{\partial x^2}
\]  

The solution of this equation for a mass of pollutant injected at \( t = 0 \) is an ever spreading Gaussian distribution moving downstream with velocity \( u \), with \( u \) a function of \( y \) and \( z \) (Figure 5B-4a).

Taylor [Proc. Royal Soc. London, 219A, p. 186 (1953); Proc. Royal Soc. London, 220A, p. 446 (1953)] realized that a formulation limited to turbulent diffusion alone would not describe the spread of a solute with much accuracy. His studies on flow in a pipe recognized that the variation of velocity from zero at the pipe walls to a maximum at the center would also cause a spreading of the pollutant. Figure 5B-4b shows the difference between this mechanism and diffusion as seen in Figure 5B-4a. The small amounts of concentrate that remain at the injection point and the greater spread of the solute "cloud" are noteworthy. Taylor concluded from his studies that dispersion could be represented by an expression identical to Equation 5B-17. The only difference is that \( u \) would be replaced by \( U \), the cross-sectionally averaged velocity, and \( \epsilon_x \)
would be replaced by a simple sum of the turbulent diffusion coefficient and an additional dispersive term: \( E_L = E_{\text{DISPERSION}} + E_{\text{TURBULENT DIFFUSION}} \). \( E_L \) is called the longitudinal dispersion coefficient and is approximately two orders of magnitude larger than the corresponding turbulent diffusivity. It includes the dispersive effects attributable to the non-uniform velocity field.

---

**Figure 5B-4**

Mixing Processes in Circular Pipe Flow

a. Turbulent Diffusion  b. Longitudinal Dispersion

[based on Daily and Harleman, *Fluid Dynamics*, Addison-Wesley (1966)]
Taylor's work was later extended by Elder [J. Fluid Mech., 5 p. 544(1959)] to an infinitely wide open channel. Here, the velocity variation was with depth, rather than with radius from the pipe centerline as in Taylor's experiments. Nevertheless, a similar behavior was predicted: a Gaussian concentration cloud advecting downstream at the average flow velocity. Elder was able to deduce an expression for $E_L$:

$$E_L = 0.23 \, d \, u^* \quad (5B-18)$$

Where, $d$ is the depth of the channel [L]; and $u^*$ is the shear velocity [L/T].

The shear velocity, $u^*$, is related empirically to the bed friction factor and the mean stream velocity, $U$, as:

$$u^* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{f} \frac{U^2}{8} \quad (5B-19)$$

where, $U$ is the cross-sectionally averaged stream velocity [L/T]; $\tau_0$ is the bed shear stress [ML/T²]; $\rho$ is the density of the fluid [M/L³]; and $f$ is a friction factor [dimensionless] equal to approximately 0.02 for natural, fully turbulent flow.

Dispersion in Natural Rivers. As would be expected, the theoretical developments of Taylor and Elder, based on straight pipes and channels with regular geometries and near ideal conditions can be translated to streams only imperfectly. The differences between the theoretical developments based on ideal systems and the behavior of natural streams are quite important.

There is a fundamental difference in the cross-sectional velocity distribution between the infinitely wide open channel and a natural stream. In the infinitely wide channel, the velocity is uniform across the channel and only varies with depth. In a natural stream there is usually a central zone of high velocity and slower velocity zones along one or both banks. These lateral variations are more important for dispersion than the vertical variations that control in Elder's experiments. In streams vertical mixing occurs fairly quickly. Velocity zones are further apart
horizontally than vertically however, and lateral mixing therefore takes a good deal longer [Fischer, J. Hydraulics Div., ASCE, 93, p. 187 (1967)].

Godfrey and Frederick [U.S.G.S. Prof. Paper 433-K (1970)] studied dispersion in natural streams and reported $E/\mu^*$ between 140 and 500 (versus 0.23 predicted by Elder). As shown in Table 5B-3, subsequent experiments have shown that $E/\mu^*$ varies widely for natural streams, but is always much greater than Elder's result. The larger values are caused by the variations in velocity across the stream, which are not accounted for in Elder's analysis of vertical velocity variations. The longitudinal dispersion coefficient is proportional to the square of the distance over which the shear profile extends. Since the width to depth ratio for natural streams is usually greater than 10, it is not surprising that the transverse velocity profile is much more important than the vertical profile in producing longitudinal dispersion.

Fischer [J. Hydraulics Div., ASCE, 93, p. 187 (1967)] extended Taylor's and Elder's classical studies and showed the limits of their applicability to natural streams. Fischer pointed out that pure dispersion (that is, the effect of differential advection alone) does not conform with Fick's Law. Fick's Law is met through the additional influence of vertical and lateral diffusion. These cross-sectional mixing processes create a more-or-less uniform cross-sectional concentration that behaves much like Fickian diffusion. However, cross-sectional mixing processes are not instantaneous—the pollutant must travel a certain distance (or an equivalent travel time) downstream of the discharge point before a uniform cross-sectional distribution is achieved. Based on this concept, Fischer derives the following criterion for use of Equation 5B-17, the one-dimensional advective-dispersion equation:

$$L > 1.8 \frac{l^2 U}{R_H \mu^*} \quad (5B-20)$$

where, $L$ is distance downstream of the discharge point [L]; $l$ is the cross-sectional mixing length [L], taken as the distance between the thread of maximum velocity and the furthest bank; and $R_H$ is the stream's hydraulic radius [L].

According to Equation 5B-20, a diffusion-type relation is an invalid description of stream dispersion for a distance less than the value of the equation's right hand side.
### TABLE 5B-3
SUMMARY OF LONGITUDINAL DISPERSION MEASUREMENTS IN RIVERS

<table>
<thead>
<tr>
<th>River</th>
<th>Longitudinal Dispersion Coefficient (m²/sec)</th>
<th>$E_L$ du²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Missouri River, Indiana-Nebraska¹</td>
<td>56,000</td>
<td></td>
</tr>
<tr>
<td>Missouri River, Indiana-Nebraska²</td>
<td>1500</td>
<td>7500</td>
</tr>
<tr>
<td>Missouri River, Indiana-Nebraska³</td>
<td>1487</td>
<td>5700</td>
</tr>
<tr>
<td>White River, Indiana¹</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Chattahoochee River, Georgia⁴</td>
<td>33</td>
<td></td>
</tr>
<tr>
<td>Susquehanna River, Pennsylvania⁴</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td>Elkhorn River, Nebraska⁴</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>John Day River, Oregon⁴</td>
<td>14 – 65</td>
<td></td>
</tr>
<tr>
<td>Comite River, Louisiana⁴</td>
<td>7 – 14</td>
<td></td>
</tr>
<tr>
<td>Amite River, Louisiana⁴</td>
<td>24 – 30</td>
<td></td>
</tr>
<tr>
<td>Yadkin River, North Carolina⁴</td>
<td>110 – 260</td>
<td></td>
</tr>
<tr>
<td>Muddy Creek, North Carolina⁶</td>
<td>14 – 33</td>
<td></td>
</tr>
<tr>
<td>Noosak River, Washington⁴</td>
<td>35 – 153</td>
<td></td>
</tr>
<tr>
<td>Monocracy River, Maryland⁴</td>
<td>5 – 40</td>
<td></td>
</tr>
<tr>
<td>Antietam Creek, Maryland⁴</td>
<td>9 – 26</td>
<td></td>
</tr>
<tr>
<td>Bayou Anacoco, Louisiana⁴</td>
<td>14 – 40</td>
<td></td>
</tr>
<tr>
<td>South Platte River, Nebraska⁵</td>
<td>16.2</td>
<td>510</td>
</tr>
<tr>
<td>Green-Duwanish River, Washington⁶</td>
<td>6.5 – 8.5</td>
<td>120 - 160</td>
</tr>
<tr>
<td>Clinch River, Virginia⁷</td>
<td>14</td>
<td>230</td>
</tr>
<tr>
<td></td>
<td>47</td>
<td>200</td>
</tr>
<tr>
<td></td>
<td>56</td>
<td>230</td>
</tr>
<tr>
<td></td>
<td>9.5 – 21</td>
<td>280</td>
</tr>
<tr>
<td>Sabine River, Texas⁴</td>
<td>320</td>
<td>2800</td>
</tr>
<tr>
<td></td>
<td>670</td>
<td>1700</td>
</tr>
<tr>
<td>Coachella Canal, California⁷</td>
<td>9.6</td>
<td>140</td>
</tr>
<tr>
<td>Powell River, Tennessee⁷</td>
<td>9.5</td>
<td>200</td>
</tr>
<tr>
<td>Nemadji River, Wisconsin⁵</td>
<td>16.9</td>
<td></td>
</tr>
<tr>
<td>Coraville Reservoir, Iowa River, Iowa⁸</td>
<td>0.5 – 9</td>
<td>50 – 900</td>
</tr>
<tr>
<td>Colorado River, Arizona¹⁰</td>
<td>55 – 243</td>
<td></td>
</tr>
</tbody>
</table>

6 [Fischer, U.S. Geo. Survey Prof. Paper 582-A (1968)]
8 [Hibbs et al., J. Environ. Eng., ASCE, 124, p. 752 (1998)]
A second major discrepancy between the idealized and natural systems arises with respect to geometry. Taylor and Elder dealt with straight channels of regular and constant cross-sectional geometry. This, of course, is not the case with natural streams. Among the nonuniformities that can be found in streams are bends, islands, dead zones, structures, reservoirs, water falls, expanding sections, navigation locks, density currents, diversions to industrial plants for process water, braided sections, and overbank flow during floods. These nonuniformities can all contribute to deviations from predicted behavior. For example, irregularities along the banks increase the length of time before the concentration distribution becomes Gaussian. A long "tail" of slightly elevated concentration is produced due to the initial detention and subsequent slow release of small amounts of the concentrate in dead zones. Bends, on the other hand, accentuate velocity differences across the stream, and therefore greatly increase the longitudinal dispersion coefficient. Sooky [J. Hydraulics Div., ASCE, 95, p. 1327 (1969)] gives a mathematical formulation for this effect.

Dispersion Coefficient Formulae. Many investigators have developed equations for estimating longitudinal dispersion in rivers. Table 5B-4 lists available technical references related to longitudinal dispersion. Two of the most frequently used formulae are by Fischer [J. Environ. Eng. Div., ASCE, 101, p. 453 (1975)] and Liu [J. Environ. Eng. Div., ASCE, 103, p. 59 (1977)]. Fischer develops the equation:

\[ E_L = \frac{0.011 U^2 \overline{W}^2}{d u^*} \]  

(5B-21)

where, \( U \) is the mean stream velocity [L/T]; \( \overline{W} \) is the mean channel width [L]; \( d \) is the mean depth [L]; and \( u^* \) is the shear velocity [L/T] defined in Equation 5B-19.

Liu's equation is similar:

\[ E_L = \frac{\beta U^2 W^2}{d u^*} \]  

(5B-22)

where, \( \beta = 0.5 \ u^*/U \) depends on the dimensionless bottom roughness.
These equations typically predict $E_L$ within a factor of 4 to 6 \cite{Schnoor:1987,Fischer:1979}.

\begin{table}[h]
\centering
\caption{Summary of references on longitudinal dispersion in rivers}
\begin{tabular}{|l|c|c|c|c|c|}
\hline
Reference & Factors Considered & Velocity & Channel Irreg. & Dead Zones & Mountain Streams & Other & Field Data & Formula for $E_L$ & Comments \\
\hline
Fischer\textsuperscript{1} & & X & & & X & & & & \text{Equation for $E_L$ from streamflow measurements} \\
Wu\textsuperscript{2} & & X & X & & & & & & \text{Effect of wind on $E_L$ in wide channels} \\
McQuivey and Keefer\textsuperscript{3} & & & X & & & & & & \text{Empirical function of streamflow, slope, width} \\
Fischer\textsuperscript{4} & & X & & X & X & & & & \text{Discussion of McQuivey and Keefer\textsuperscript{3}} \\
Liu, Liu & & X & & & & & & & \text{Widely-used formula} \\
& & & & & & & & & \text{Model of dispersion in pool-and-riffle streams} \\
Bencala & & & & & & & & & \text{Dead zone model} \\
& & & & & & & & & \text{Bed and side roughness} \\
Beltaos\textsuperscript{7} & & & & & & & & & \text{Steep streams} \\
Sobol and Nordin\textsuperscript{8} & & X & & & & & & & \text{Dead zone model} \\
Valentine & & & & & & & & & \text{Dead zone model, lab data} \\
Wood\textsuperscript{12,13} & & & & & & & & & \text{$E_L$ by time series analysis} \\
Beer and Young\textsuperscript{14} & & X & & & & & & & \text{Mountain stream storage model} \\
Bajraktarevic-Dobran\textsuperscript{15} & & & X & & & & & & \text{$E_L$ by numerical routing} \\
Jobson\textsuperscript{16} & & & & & & & & & \text{} \\
\hline
\end{tabular}
\end{table}

\textsuperscript{1}\cite{Fischer:1968} \textsuperscript{2}\cite{Wu:1969} \textsuperscript{3}\cite{McQuivey:1974} \textsuperscript{4}\cite{Fischer:1975} \textsuperscript{5}\cite{Liu:1977} \textsuperscript{6}\cite{Liu:1980} \textsuperscript{7}\cite{Bencala:1983} \textsuperscript{8}\cite{Yu:1988} \textsuperscript{9}\cite{Magazine:1988} \textsuperscript{10}\cite{Sobol:1978} \textsuperscript{11}\cite{Valentine:1977} \textsuperscript{12}\cite{Valentine:1979} \textsuperscript{13}\cite{Beer:1983} \textsuperscript{14}\cite{Bajraktarevic-Dobran:1982} \textsuperscript{15}\cite{Jobson:1987}
DESCRIPTIONS OF TRANSVERSE DISPERSION IN RIVERS

Transverse Dispersion in Straight Channels. In the absence of secondary currents, transverse dispersion in open channels is due to turbulent diffusion. Thus, the magnitude of the transverse dispersion coefficient is generally much lower than that of the longitudinal coefficient. Fischer et al. [Mixing in Inland and Coastal Waters, Academic Press (1979)] report over 75 separate experiments conducted by various investigators using straight rectangular laboratory channels, all yielding $E_{t}/u^*$ between 0.1 and 0.2. They suggest the following estimate of $E_{t}$:

$$E_{t} = 0.15 d u^*$$ (5B-23)

Detailed analyses by Okoye [Report KH-R-23, California Inst. Technol. (1970)] and Lau and Krishnappen [J. Hydraulics Div., ASCE, 103, p. 1173 (1977)] indicate that the width of the channel affects the transverse dispersion no matter how wide the channel. However, it is not yet clear exactly what role the width plays. Equation 5B-23 is usually correct within fifty percent for uniform straight rectangular channels [Fischer et al., Mixing in Inland and Coastal Waters, Academic Press (1979)].

Transverse Dispersion in Natural Rivers. Of course, natural rivers differ from uniform rectangular channels in several ways: the depth varies irregularly, the channel is likely to curve, and there may be irregularities along the banks such as man-made structures or points of land. Holley et al. [J. Hydraulic Res., 10, p. 27 (1972)] report the impact of regular cross-sectional depth variations on transverse dispersion, but generally this is not a major factor.

Sidewall irregularities of various types are very common in natural channels and have a major effect on transverse dispersion due to the presence of secondary lateral flows around the irregularities. Generally, the larger the irregularity, the faster transverse mixing occurs. Fischer et al. [Mixing in Inland and Coastal Waters, Academic Press (1979)] reports that sidewall irregularities increase $E_{t}/u^*$ to between 0.3 and 0.7.

Of the three characteristics affecting transverse dispersion in natural channels, channel curvature is the best understood. Bends increase $E_{t}/u^*$ with reported values ranging from 0.4 to 0.8 for slowly meandering rivers and even higher values in sharply curving channels [Fischer et al., Mixing in Inland and Coastal Waters, Academic Press (1979)]. This increase is due to lateral flows (often called secondary currents) that occur along the bend. Centrifugal forces induce flow toward the outside bank at the water surface with a compensating reverse flow near the channel...
bottom. Fischer [Water Resourc. Res., 5, p. 496 (1969)] used the velocity distribution around a curve to determine that:

\[
E_t \propto \left(\frac{U}{u^*}\right)^2 \left(\frac{d}{R}\right)^2
\]  \hspace{1cm} (5B-24)

where, \( R \) is the radius of curvature [L].

He reported that the constant of proportionality was approximately 25. However, Sayre and Yeh [Report No. 145, Iowa Inst. of Hydraulic Res., Univ. of Iowa (1973)] subsequently found that \( E_t \) varies along the bend, from a low of one-half its average value in the upstream portion of the bend, to twice the average in the downstream portion. Finally, Holley and Jirka [Tech. Report E-86-11, Environ. and Water Quality Operational Studies, U.S. Army Corps of Eng., pp. 301-315 (1986)] point out that stream bends must be sufficiently long and curved to produce a secondary current that affects dispersion. They give general guidelines on classifying meandering streams according to their potential influence on transverse mixing.

In summary, curves and sidewall irregularities increase transverse dispersion in natural streams, so that \( E_t/du^* \) is rarely less than 0.4. Table 5B-5 presents a summary of field and experimental values of \( E_t/du^* \) based on compilations by Bowie et al. [Report EPA/600/3-85/040, 2nd Ed., U.S. EPA (1985)] and Fischer et al. [Mixing in Inland and Coastal Waters, Academic Press (1979)]. Fischer et al. suggest using \( E_t/du^* = 0.6 \pm 50\% \) for slowly meandering streams with moderate sidewall irregularities, and higher values for sharply curving channels or rapid changes in channel geometry. A summary of references addressing transverse dispersion is provided in Table 5B-6.

RECOMMENDED PRACTICE

The complications of longitudinal and transverse dispersion in rivers are mitigated by an important fact: dispersion is not a practical concern in many problems. For example, the most common problem in riverine water quality is the development of a wasteload allocation, the determination of allowable rates of discharge for flow and oxygen-demanding wastes from a continuous wastewater discharge. Typically, wasteload allocations are developed using a one-dimensional steady-state model of the river. Fortunately, the predicted longitudinal distribution of pollutant is highly insensitive to the dispersion coefficient. In most cases, to predict the
steady-state pollutant distribution from a continuous discharge, dispersion can be neglected altogether!

**TABLE 5B-5**

SUMMARY OF TRANSVERSE DISPERSION MEASUREMENTS IN RIVERS

<table>
<thead>
<tr>
<th>River</th>
<th>Transverse Dispersion Coefficient (m²/sec)</th>
<th>( \frac{E_{t}}{du} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Missouri River, Nebraska¹</td>
<td>0.101</td>
<td>0.5</td>
</tr>
<tr>
<td>Missouri River, Nebraska²</td>
<td>1.11</td>
<td>0.26</td>
</tr>
<tr>
<td>South River, Virginia¹</td>
<td>0.0046</td>
<td>0.22</td>
</tr>
<tr>
<td>Aristo Feeder Canal, New Mexico¹</td>
<td>0.0093</td>
<td>0.3</td>
</tr>
<tr>
<td>Bernedo Conveyance Channel, New Mexico¹</td>
<td>0.013</td>
<td>0.75</td>
</tr>
<tr>
<td>Athabasca River, Alberta, Canada³</td>
<td>0.041, 0.093, 0.067, 0.010</td>
<td>1.16, 0.75, 0.41, 0.56</td>
</tr>
<tr>
<td>Beaver River, Alberta, Canada³</td>
<td>0.043, 0.020</td>
<td>1.01, 2.54</td>
</tr>
<tr>
<td>Grand River, Ontario, Canada⁴</td>
<td>0.009</td>
<td>0.26</td>
</tr>
<tr>
<td>Danube River, Hungary⁵</td>
<td>0.038</td>
<td>0.25</td>
</tr>
<tr>
<td>Mississippi River, Minnesota⁵</td>
<td>0.171</td>
<td>2.03</td>
</tr>
<tr>
<td>Coraville Reservoir, Iowa River, Iowa⁷</td>
<td>0.05</td>
<td>5</td>
</tr>
</tbody>
</table>

1 [Yotsukura and Cobb, U.S. Geo. Survey Prof. Paper 582-C (1972)]
3 [Beltaos, J. Hydraulics Div., ASCE, 106, p. 1607 (1980)]
4 [Lau and Krishnappan, J. Hydraulics Div., ASCE, 103, p. 1173 (1977)]
6 [Demetracopoulos and Stefan, J. Environ. Eng., ASCE, 109, p. 685 (1982)]
TABLE 5B-6

SUMMARY OF REFERENCES ON TRANSVERSE DIFFUSION IN RIVERS

<table>
<thead>
<tr>
<th>Reference</th>
<th>Turbulence</th>
<th>Channel Irreg.</th>
<th>Bends</th>
<th>Other Effects</th>
<th>Field Data</th>
<th>Formula for E.</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yotsukura and Sayre(^1)</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td>Streamtube model</td>
</tr>
<tr>
<td>Holley and Abraham(^2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td>Computation from dye dispersion data</td>
</tr>
<tr>
<td>Holley and Jirka(^3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Analysis of bends</td>
</tr>
<tr>
<td>Beltaos(^4)</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td>Effects of ice cover</td>
</tr>
<tr>
<td>Weber and Schatzmann(^5)</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td>Laboratory experiments for several parameters</td>
</tr>
<tr>
<td>Lau and Krishnappan(^6)</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Laboratory experiments for several parameters</td>
</tr>
</tbody>
</table>

\(^2\)[Holley and Abraham, J. Hydraulics Div., ASCE, 99, p. 2313 (1973)]
\(^4\)[Beltaos, J. Hydraulics Div., ASCE, 106, p. 1607 (1980)]
\(^5\)[Webel and Schatzmann, J. Hydraulic Eng., ASCE, 110, p. 423 (1984)]
\(^6\)[Lau and Krishnappan, J. Hydraulics Div., ASCE, 103, p. 1173 (1977)]

The exceptions to this general recommendation are several. The reason that the recommendation can be made is that pollutant concentration does not change rapidly along the river if the discharge is continuous. If the water at one location in the river is not drastically different from the water at a neighboring location, it makes little difference that the waters mix. The exceptions are those cases when concentration changes rapidly with position in the river. This occurs for sudden discharges such as spills. Here, the concentration distribution is a small area of high concentration in an otherwise unpolluted river. Dispersion may be the single most important mechanism in reducing the concentration in the river as the spill moves downstream. Another exception occurs for a pollutant that degrades rapidly as it flows downstream. Even with
a continuous discharge, the rapid loss of pollutant leads to significant concentration differences along the river and longitudinal mixing may be important.

A final important exception is the case of a shoreline discharge into a wide river. Here, the mixing of the pollutant across the river needs to be addressed and thus transverse mixing must be considered as well as longitudinal dispersion. A one-dimensional model could greatly underpredict concentrations in the river with the model's implicit assumption that the discharge has mixed throughout the river cross-section. In fact, the shoreline plume is diluted by a smaller portion of the river's flow and is therefore more concentrated.

In those cases where dispersion is an important process, and a dispersion coefficient is required, we recommend the following approach. First, the validity of the Fickian diffusion model should be checked using Equation 5B-20. Passing that criterion, a dispersion coefficient may be developed and applied. If there exist field measurements of mixing from the river in question, those data are generally preferred to any of the empirical formulae for predicting dispersion coefficients. Such data rarely exist, however, in which case the modeler should select and apply at least several pertinent relations from the references in Tables 5B-4 and 5B-6. These will yield a range of results, presenting the modeler with a judgement. If the results generally converge on a narrow range of results, an average of those would be a good choice. If there is scatter, a conservative approach would be to use the result giving the lowest dispersion coefficient, and thus the highest predicted concentrations.

DISPERSION IN ESTUARIES

ESTUARINE MIXING

The dispersive characteristics of a river become further complicated where a river nears the sea and becomes an estuary. In an estuary, the significant influence of tidal flows is added to the list of mechanisms that affect dispersion in rivers. Still another mechanism is added in stratified estuaries. In a stratified estuary the fresh river water flows atop the denser salt water that intrudes into the estuary from the sea. Not all estuaries are stratified however: in estuaries in which the back-and-forth tidal flow is significant compared to the freshwater river flow, the salt and fresh water mix more or less completely through the depth of the estuary. Such estuaries are known as well-mixed or partially-mixed depending on their degree of vertical stratification.
Fischer [Inter. Symposium on Discharge of Sewage from Sea Outfalls, London, 1974, Pergamon Press (1975)] and Fischer et al. [Mixing in Inland and Coastal Waters, Academic Press (1979)] describe the many physical phenomena that cause mixing in estuaries. The major mechanisms are described in the sections to follow.

Tidal Pumping. "Tidal pumping" is the residual circulation pattern created by the different patterns of flow that exist between the flood tide and the ebb tide. Different flow patterns are set up by the geometry and bathymetry of the estuary. For example, in a bay with a narrow inlet, flow on the flood tide creates a jet-like flow into the bay when it enters the inlet. On the ebb tide, however, flow in the bay tends to flow radially into the inlet. Averaged over a tidal cycle, the incoming jet flow and outgoing radial flow create a net circulation pattern as shown in Figure 5B-5.

![Figure 5B-5](Image)

(a) Flood tide  (b) Ebb Tide  (c) Net Circulation

Figure 5B-5
Illustration of Tidal Pumping in a Bay
Gravitational Circulation. "Gravitational circulation" refers to the net transport created by the flow of different density waters in a partially stratified estuary. In the surface layers of the estuary, there is a net seaward transport due to the river's freshwater flow. In the deeper layers, there is a net landward flow of saline water from the sea. At mid-depth, there is a vertically upwards flow associated with underlying salt water being entrained and mixed into the freshwater flow. The gravitational circulation also creates a net horizontal circulation with a landward flow in the deeper parts of the estuary and a seaward flow in the shallow sections.

Shear Flow Dispersion. Longitudinal dispersion caused by cross-sectional velocity variations occurs in an estuary in much the same fashion as it occurs in a river. It is modified, however, by the effects of the oscillating tidal flow. If the tidal period (the time period of one tidal oscillation) is shorter than the time period for cross-sectional mixing, then the dispersion coefficient is effectively reduced. Holley et al. [J. Hydraulics Div., ASCE, 96, p. 1691 (1970)] discuss this effect in detail and give a relation for determining the amount of decrease in the dispersion coefficient.

Dead Zones. Just as in rivers, dead zones increase longitudinal dispersion in estuaries. The effect may be magnified in estuaries by what Fischer et al. [Mixing in Inland and Coastal Waters, Academic Press (1979)] call "tidal trapping." Due to tidal oscillations, pollutants trapped in dead zones and tributary branches during the flood tide may rejoin the mainstem flow during the ebb tide in a different part of the flow than that from which they originated. The effect is an important dispersion mechanism in some estuaries.

Wind Mixing. Mixing induced by the wind may be important in some large estuaries and bays. It is most important in situations where a steady wind sets up current patterns that create large scale mixing.

DESCRIPTIONS OF ESTUARINE DISPERSION

Tidally-Averaged Dispersion. A confusing aspect of determining the dispersion coefficient in estuaries is the choice of the operating time period. The most common approach is the tidally-averaged model. In these models concentration predictions are intended to represent the average over the tidal cycle. Flow in the model is the freshwater flow only; the oscillatory flow of the tide is not explicitly considered. The dispersion coefficient for these models must capture all dispersive effects of the oscillatory tidal flow and is typically much larger than the dispersion coefficients seen in rivers.

Real-Time Dispersion. A second approach to predicting the dispersion equation is to use a time-varying, or "real-time" model. In this approach, the tidal flow is included in the advective term and the dispersion coefficient need capture only the effects of velocity shear and gravitational circulation. This approach is more deterministic than the tidally-averaged approach and some empirical formulae have been developed to predict the time-varying dispersion coefficient, $E_L$. This dispersion coefficient is appropriate for use in an equation such as Equation 5B-6, the one dimensional advection-dispersion equation. However, when Equation 5B-6 is used with the time-varying dispersion coefficient, the velocity term must include both the freshwater flow and a tidal flow that varies sinusoidally through the tidal cycle.

Prediction of the Dispersion Coefficient. There are few formulae in the literature to use in determining the longitudinal dispersion coefficient in an estuary. Chatwin and Allen [Ann. Rev. Flu. Mech., 17, p. 119 (1985)] give the following general equation for the longitudinal dispersion coefficient that accounts for both shear dispersion and gravitational circulation:

$$E_L = \alpha_0 + \alpha_2 \left( \frac{\partial S}{\partial x} \right)^2$$  \hspace{1cm} (5B-25)

where, $S$ is the cross-sectionally averaged salinity [M/L$^3$], used as an indicator of density; and

$\alpha_0$, $\alpha_2$ are empirical coefficients.

Chatwin and Allen do not give guidelines for determining $\alpha_0$ and $\alpha_2$, which are best found on a case-specific basis.

Thatcher and Harleman [J. Environ. Eng. Div., ASCE, 107, p. 11 (1981)] use a formula with a different dependence on the salinity (density) gradient. It is shown here in a modified form that is more directly comparable to Equation 5B-25:
\[ E_L = mE_T + K \frac{L}{S_{\text{max}}} \left( \frac{\partial S}{\partial x} \right) \]  

(5B-26)

where, 
- \( m \) is a constant [dimensionless] that accounts for the effects of modified shear flow dispersion and channel irregularities;
- \( E_T \) is the dispersion coefficient \([L^2/T]\) based on a Taylor-like dispersion coefficient, for example as computed by Equation 5B-21 or 5B-22;
- \( K \) is a parameter \([L^2/T]\) that depends upon the stratification in the estuary;
- \( S_{\text{max}} \) is the maximum salinity at the ocean entrance \([M/L^3]\); and
- \( L \) is the length of the estuary \([L]\) from the ocean entrance to the head of tide.

\( S_{\text{max}} \) and \( L \) are simply normalizing constants to preserve the equation's dimensionality. The stratification parameter varies between about \( 4 \times 10^{-4} U_{\text{max}} L \) for an unstratified estuary to \( 1.1 \times 10^{-3} U_{\text{max}} L \) for a strongly stratified estuary, where \( U_{\text{max}} \) is the maximum tidal velocity.


In the face of this uncertainty, a recommended procedure for developing a coefficient value for a particular study is to either 1) conduct field experiments to determine the coefficient by direct experimentation, or 2) calibrate the dispersion coefficient in a transport model of the salt distribution in the estuary. Salt is useful in this regard because it is a conservative substance and not subject to the complicating factors of degradation or decay. For general background, Table 5B-7 summarizes field observations of tidally-averaged dispersion coefficients.
TABLE 5B-7
SUMMARY OF LONGITUDINAL DISPERSION MEASUREMENTS IN ESTUARIES

<table>
<thead>
<tr>
<th>Estuary</th>
<th>Longitudinal Dispersion Coefficient (m²/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hudson River, New York¹</td>
<td>600</td>
</tr>
<tr>
<td>Hudson River, New York²</td>
<td>450 – 1500</td>
</tr>
<tr>
<td>Hudson River, New York³</td>
<td>160</td>
</tr>
<tr>
<td>Delaware River¹</td>
<td>150</td>
</tr>
<tr>
<td>Delaware River³</td>
<td>500 – 1500</td>
</tr>
<tr>
<td>East River, New York¹</td>
<td>300</td>
</tr>
<tr>
<td>Cooper River, South Carolina¹</td>
<td>900</td>
</tr>
<tr>
<td>Savannah River, Georgia, S.C.¹</td>
<td>300 – 600</td>
</tr>
<tr>
<td>Lower Raritan River, New Jersey¹</td>
<td>150</td>
</tr>
<tr>
<td>South River, New Jersey¹</td>
<td>150</td>
</tr>
<tr>
<td>Houston Ship Channel, Texas¹</td>
<td>810</td>
</tr>
<tr>
<td>Cape Fear River, North Carolina¹</td>
<td>60 – 300</td>
</tr>
<tr>
<td>Potomac River, Virginia¹</td>
<td>30 – 300</td>
</tr>
<tr>
<td>Potomac River, Virginia²</td>
<td>6 – 60</td>
</tr>
<tr>
<td>Potomac River, Virginia³</td>
<td>55</td>
</tr>
<tr>
<td>Potomac River, Virginia⁴</td>
<td>20 – 100</td>
</tr>
<tr>
<td>Compton Creek, New Jersey¹</td>
<td>30</td>
</tr>
<tr>
<td>Fishkill Creek, New Jersey¹</td>
<td>15 – 30</td>
</tr>
<tr>
<td>South San Francisco Bay²</td>
<td>20 – 200</td>
</tr>
<tr>
<td>North San Francisco Bay²</td>
<td>50 – 1900</td>
</tr>
<tr>
<td>Hudson River, New York⁵</td>
<td>47 ± 6</td>
</tr>
<tr>
<td></td>
<td>162 ± 22</td>
</tr>
<tr>
<td>Conwy Estuary, Wales⁵</td>
<td>0 – 35</td>
</tr>
<tr>
<td>Bremer River, Australia²</td>
<td>27.2</td>
</tr>
<tr>
<td>Oxley Creek, Australia²</td>
<td>7 – 12</td>
</tr>
</tbody>
</table>

2 [Officer, Physical Oceanography of Estuaries (and Associated Coastal Waters), Wiley and Sons (1976)]
3 [Thatcher and Harleman, R.M. Parsons Lab. Report No. 144, MIT (1972)]
5 [Clark et al., Environ. Sci. Technol., 30, p. 1527 (1996)]
6 [Guymer and West, J. Hydraulic Eng., ASCE, 118, p. 718 (1992)]