Spatially anisotropic systems greatly enrich our experience of collective phenomena, as exemplified by high-$T_c$ superconducting materials, in which the couplings along one direction are much weaker than those in the perpendicular plane. Anisotropic systems are also intriguing from a conceptual point of view, since vastly different critical phenomena are known to happen in different spatial dimensions, whereas, between $d$-dimensional systems stacked along a new direction, even the weakest coupling, while not affecting the critical temperature, induces $(d+1)$-dimensional critical behavior. Calculational results that yield the global phase diagram of anisotropic systems and thus provide a unified connected picture of the various anisotropic and isotropic behaviors at different subdimensions and at the full dimension have been rare and mostly confined to $d = 2$. In the present study, we obtain global phase diagrams for a variety of anisotropic $d = 3$ systems: Ising magnetic, XY magnetic, and percolation systems. Anisotropy along one direction (uniaxial) and full anisotropy, in which the couplings along each direction are different, are studied, yielding global phase diagrams. We use hierarchical models, which yield exact renormalization-group solutions [1, 2]. Thus, the construction of hierarchical lattices that incorporate correct dimensional reductions is an important step of the study. The exact solutions of hierarchical models can simultaneously be considered approximate position-space renormalization-group solutions of models on naturally occurring lattices [1]. The method developed in this study will be employed to extend, from isotropic to anisotropic systems, the renormalization-group solutions of the $tJ$ and Hubbard models of electronic conduction [4, 5, 6].

II. ANISOTROPIC HIERARCHICAL LATTICES

Hierarchical lattices are constructed by repeatedly self-imbedding a graph. These provide exactly solvable models, with which complex problems can be studied and understood. For example, frustrated [7], spin-glass [8], random-bond [9] and random-field [10], Schrödinger equation [11], lattice-vibration [12], dynamic scaling [13], aperiodic magnet [14], complex phase diagram [15], and directed-path [16] systems, etc., have been solved on hierarchical lattices.

In this study, we construct anisotropic hierarchical lattices by the parallel, mutual imbedding of several graphs. In each imbedding step, $b$ and $b^d$ respectively are the length and volume rescaling factors. We illustrate the method by the simplest case of the anisotropic $d = 2$ lattice, before moving on to the uniaxially or fully anisotropic $d = 3$ lattices. The parallel, mutual imbeddings of the two graphs shown in Fig.1 provide an anisotropic $d = 2$ hierarchical lattice. If either of the couplings $(K_x, K_y)$ is set to zero, the remaining coupling constitutes a one-dimensional lattice. When the couplings are of equal strength, $K_x = K_y$, the two directions, represented by the two imbedding sequences, are equivalent and the lattice is isotropic $d = 2$. This lattice will be referred to as $A_2$.

FIG. 1: The parallel, mutual imbeddings of these two graphs provide an anisotropic $d = 2$ hierarchical lattice. If either of the couplings $(K_x, K_y)$ is set to zero, the remaining coupling constitutes a one-dimensional lattice. When the couplings are of equal strength, $K_x = K_y$, the two directions, represented by the two imbedding sequences, are equivalent and the lattice is isotropic $d = 2$. This lattice will be referred to as $A_2$. 

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FIG. 2: The parallel, mutual imbeddings of these two graphs provide a uniaxially anisotropic $d = 3$ hierarchical lattice. If the coupling $K_z$ is set to zero, the coupling $K_x$ constitutes an isotropic two-dimensional lattice. If the coupling $K_{xy}$ is set to zero, the coupling $K_z$ constitutes a one-dimensional lattice. When the couplings are of equal strength, $K_{xy} = K_z$, the $z$ direction, represented by the first imbedding sequence, and the $x, y$ directions, represented by the second imbedding sequence, are all equivalent and the lattice is isotropic $d = 3$. This lattice will be referred to as $U_3$.

will be referred to as $A_2$.

It is thus seen that generally our requirements in the construction of anisotropic hierarchical lattices are (1) the proper reduction to the lower dimension when one (or more, see below) of the couplings is set to zero and (2) the restitution of an isotropic lattice when the couplings are of equal strength.

The parallel, mutual imbeddings of the two graphs shown in Fig.2 provide a uniaxially anisotropic $d = 3$ hierarchical lattice. If the coupling $K_z$ is set to zero, the coupling $K_{xy}$ constitutes an isotropic two-dimensional lattice. If the coupling $K_{xy}$ is set to zero, the coupling $K_z$ constitutes a one-dimensional lattice. When the couplings are of equal strength, $K_{xy} = K_z$, the $z$ direction, represented by the first imbedding sequence, and the $x, y$ directions, represented by the second imbedding sequence, are all equivalent and the lattice is isotropic $d = 3$. This lattice will be referred to as $U_3$.

FIG. 3: The parallel, mutual imbeddings of the top graph with either one of the following graphs provide uniaxially anisotropic $d = 3$ hierarchical lattices. If the last graph is used, isotropy is not restored when $K_{xy} = K_z$. These lattices, differentiated by the choice of the second imbedding graph, will be respectively referred to as $U_{3a}, U_{3b}, U_{3c}$.

couplings are set to zero, the remaining coupling constitutes a one-dimensional lattice. When the couplings are of equal strength, $K_x = K_y = K_z$, the three directions, represented by the three imbedding sequences, are equivalent and the lattice is isotropic $d = 3$. These lattices will be referred to as $A_{3a}$ and $A_{3b}$.

The anisotropic systems that we study are located on the anisotropic lattices constructed above. These hierarchical models admit exact renormalization-group solutions, with recursion relations obtained by decimations in direction opposite to their construction direction. The exact solutions of hierarchical models can simultaneously be considered approximate position-space renormalization-group solutions of models on naturally occurring lattices. In fact, the recursion relations obtained for the models below correspond to Migdal-Kadanoff approximate recursion relations, which are hereby generalized to anisotropic systems.

A fully anisotropic $d = 3$ hierarchical lattice is provided in Fig.4 by each shown imbedding in parallel with the two imbeddings obtained by permuting $K_x$ (full line), $K_y$ (dashed), and $K_z$ (dotted). If any one the couplings $K_u$ is set to zero, the remaining two couplings constitute an anisotropic two-dimensional lattice. If any two of the
FIG. 4: Each imbedding shown in this figure, in parallel with the two imbeddings obtained by permuting $K_x$ (full line), $K_y$ (dashed), and $K_z$ (dotted), yields a fully anisotropic $d=3$ hierarchical lattice. If any one the couplings $K_u$ is set to zero, the remaining two couplings constitute an anisotropic two-dimensional lattice. If any two of the couplings are set to zero, the remaining coupling constitutes a one-dimensional lattice. When the couplings are of equal strength, $K_x = K_y = K_z$, the three directions, represented by the three mutual imbedding sequences, are equivalent and the lattice is isotropic $d=3$. These lattices will be respectively referred to as $A_{3a}$ and $A_{3b}$.

III. ANISOTROPIC ISING MAGNETS

The Ising model is defined by the Hamiltonian

$$-\beta H = \sum_u K_u \sum_{\langle ij \rangle_u} s_is_j,$$  \hspace{1cm} (1)

where, at each lattice site $i$, $s_i = \pm 1$, and $\langle ij \rangle_u$ denotes summation over bonds of type $u$. The various decimations in the models are composed of two elementary steps, $K = K_u + K_v$ for bonds in parallel and $K = \tanh^{-1}(\tanh K_u + \tanh K_v)$ for bonds in series, where $K$ is the effective coupling of the combined bonds.

The phase boundaries for the Ising model on the $d=2$ anisotropic hierarchical lattices $A_2$, $A_{3a}$, and $A_{3b}$ (setting $K_z = 0$ in the latter two) are given in Fig.5, along with the exact result for the anisotropic square lattice \cite{19}. The renormalization-group flows are indicated on the phase boundary of the hierarchical models. The fixed point occurs at isotropy, $K_x = K_y$, to which the $d=2$ anisotropic critical points flow, thereby sharing the same critical exponents.

The phase boundaries for the Ising model on the $d=3$ uniaxially anisotropic hierarchical lattices $U_3$, $U_{3a}$, $U_{3b}$, $U_{3c}$, $A_{3a}$, and $A_{3b}$ (setting $K_x = K_y$) are given in Fig.6. The exact phase transition points for the square \cite{19} and cubic \cite{21} lattices are also shown. For each model, the phase transitions at $d=1$ (at infinite coupling) and $d=2$ cross over to $d=3$ criticality, which is thus universal for all $d=3$ anisotropic and the $d=3$ isotropic cases.

The phase boundary surface for the Ising model on the $d=3$ fully anisotropic hierarchical lattice $A_{3b}$ is given in Fig.7. The dashed lines on the planes are the exact $d=2$ solutions for the square lattice \cite{19}. Again, all points on the critical surface of the $d=3$ fully anisotropic model flow onto the fixed point located at isotropy, thereby sharing its critical exponents. The critical exponents found for this model are $\nu_T = 0.69$, $\nu_H = 1.68$ for $d=2$ (for the square lattice $\nu_T = 1$, $\nu_H = 1.875$ \cite{19}) and $\nu_T = 0.92$, $\nu_H = 2.20$ for $d=3$ (for the cubic lattice $\nu_T = 1.59$, $\nu_H = 2.50$ \cite{21,22}).
FIG. 6: Phase boundaries for the Ising model on the $d = 3$ uniaxially anisotropic hierarchical lattices $U_3, U_{3a}, U_{3b}, U_{3c}, A_{3a}$, and $A_{3b}$ (setting $K_x = K_y$). The exact phase transition points for the square [19] and cubic [20] lattices are shown by the black circles. For each model, the phase transitions at $d = 1$ (at infinite coupling) and $d = 2$ cross over (as shown for $A_{3a}$) to $d = 3$ criticality, which is thus universal for all $d = 3$ anisotropic and the $d = 3$ isotropic cases. The $d = 2$ fixed point of $A_{3b}$ is not marked by a star, since it coincides with the square lattice exact transition point, marked by the black circle on the horizontal axis.

IV. ANISOTROPIC XY MAGNETS

The XY model is defined by the Hamiltonian

$$-\beta H = \sum_u J_u \sum_{ij} s_i \cdot s_j = \sum_u J_u \sum_{ij} \cos(\theta_i - \theta_j), \quad (2)$$

where at each lattice site $i$, $s_i$ is a unit vector confined to the $xy$ plane at angle $\theta_i$ to the $x$ axis and $<ij>_u$ denotes summation over bonds of type $u$. Under renormalization-group transformations, the coupling between nearest-neighbor sites takes the general form of a function $V_u(\theta_i - \theta_j)$. The various decimations in the models are composed of two elementary steps,

$$V = V_u + V_v \quad \text{and} \quad (3)$$

$$V(\theta_i - \theta_k) = \ln \int_0^{2\pi} d\theta_j \exp[V_u(\theta_i - \theta_j)V_v(\theta_j - \theta_k)],$$

respectively for bonds in parallel and in series, where $V$ is the effective coupling of the combined bonds. In terms of Fourier components,

$$f_u(s) = \int_0^{2\pi} d\theta e^{is\theta} \exp[V_u(\theta) - V_u(0)], \quad (4)$$

$$\exp[V_u(\theta) - V_u(0)] = \sum_s e^{-is\theta} f_u(s),$$

Eqs.(3) respectively are

$$f(s) = \sum_s f_u(p)f_v(s-p) \quad \text{and} \quad (5)$$

$$f(s) = f_u(s)f_v(s),$$

in a form that is more conveniently followed in our calculations. The phase boundaries for the XY model on the $d = 3$ uniaxially anisotropic hierarchical lattices $U_3, U_{3a}, U_{3b}$, and $U_{3c}$ are given in Fig.8. The exact phase transition points for the square [23] and cubic [24] lattices are also shown.

In $d = 2$, namely along the horizontal axis, above a critical interaction strength marked by the squares on the figure, the systems exhibit algebraic order [25, 26, 27]: The starting Hamiltonian [Eq.(2)] flows to a Villain potential [28],

$$f_v(s) = A \exp(-s^2/2J_v), \quad (6)$$

exhibiting a fixed-line behavior parametrized by $J_v$. This corresponds to a system without true long-range order, namely with zero magnetization, but infinite correlation length and algebraic order in which the correlations decay as a power law. In $d = 3$, true long-range order occurs: points in the ferromagnetic phase renormalize to a
FIG. 8: Phase boundaries for the XY model on the d=3 uniaxially anisotropic hierarchical lattices $U_3, U_{3a}, U_{3b}$, and $U_{3c}$. The exact phase transition points for the square [23] and cubic [24] lattices are shown by the black circles. For each model, the phase transitions at $d=1$ (at infinite coupling) and $d=2$ (onset of algebraic order) cross over to $d=3$ criticality, which is thus universal for all $d=3$ anisotropic and the $d=3$ isotropic cases. The onsets of effective algebraic order in $d=2$ are marked, for models $U_3, U_{3a}$ with the open square and for models $U_{3b}, U_{3c}$ with the full square.

delta function potential; points on the phase boundaries renormalize to single true fixed potential, shown in Fig.9, differing from the Villain potential as also seen on the figure. The behavior here for $d=2$ is not true fixed-line behavior. After tens of thousands of renormalization-group iterations (corresponding to a scale change factor of 210,000), the Villain potential decays [27] to a disordered sink with $V(\theta)_{\text{max}} - V(\theta)_{\text{min}} < 10^{-4}$. The sharp change in the necessary number of iterations, as seen in Fig.10, indicates the onset of effective algebraic order.

As seen in Fig.8, for each XY model, the phase transitions at $d=1$ (at infinite coupling) and $d=2$ (onset of algebraic order) cross over to $d=3$ criticality, which is thus universal for all $d=3$ anisotropic and the $d=3$ isotropic cases.

V. ANISOTROPIC PERCOLATION

Anisotropic percolation is defined such that on each connection of direction $u$, a bond exist with probability $p_u$. The various decimations in the models are composed of two elementary steps, $p = p_u p_v + p_a (1 - p_u) + p_v (1 - p_a)$ for connections in parallel and $p = p_u p_v$ for connections in series, where $p$ is the effective connectedness probability of the combined connections. The phase diagram for percolation on the $d=2$ anisotropic hierarchical lattice $A_2$ is given in Fig.11. The percolation fixed point occurs at isotropy, $p_x = p_y$, to which the $d=2$ anisotropic percolation onsets flow, thereby sharing the same critical exponents. The phase boundaries for percolation on the $d=3$ uniaxially anisotropic hierarchical lattices $U_3, U_{3a}, U_{3b}$, and $U_{3c}$ are given in Fig.12. The perco-
FIG. 11: The phase boundaries for percolation on the $d = 2$ anisotropic hierarchical lattice ($A_2$). The renormalization-group flows are indicated on the phase boundary of the hierarchical model. The onset of percolation for the square lattice is shown by the black circle. For each model, percolation onset at $d = 1$ (at $p_z = 1$) and $d = 2$ cross over to $d = 3$ percolation onset, which is thus universal for all $d = 3$ anisotropic and the $d = 3$ isotropic cases.

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