MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.334 Statistical Mechanics II — Spring 1997

			Off Hr*	Office	Phone
Lectures:	MWF2-3:30 in Rm. 12-132	Prof. Nihat Berker	M4	12-135	3-2176
Recitations:	Tue9,10 in Rm. 12-132	Prof. Mehran Kardar	Tue2	12-108	3-3259
Teaching Assistants:		Mr. Alkan Kabakçıoğlu	M4	12-109	3-2287
		Mr. Gabriele Migliorini	M4	12-109	3-2287
Course Administration:		Ms. Anne Conklin		4-352	3-4855

*Office consultation can also be arranged by appointment. Do call us!

Problem Set and Examination Information

Physics is a collaborative science! You are strongly encouraged to discuss the problem set problems with other students in the class. However, each student is expected to compose her/his own written problem set that is turned in. Do not seek or consult solution sets for these problems that may be available from previous years.

The problem sets are given and due weekly on Friday 2pm, in Rm. 12-132. Solution sets will be available on the same day as you turn in the problem set, also in Rm. 12-132. Please check the solutions while the problems are still fresh in your mind. Late problem sets cannot be accepted.

The outstanding problem set will be discussed in recitation. Therefore, it is important that you familiarize yourself with the outstanding problem set before attending the recitation. Otherwise, it will be unlikely for you to fully complete and benefit from the problem set.

The graded problem sets will be returned to you, in lecture, on the Monday immediately after you turn them in. Again, please check your graded problem set while the problems are still fresh in your mind.

We are grateful to the Teaching Assistants for providing such remarkably prompt service, from which all the students will benefit.

Any questions you may have on the lecture material, outstanding problem set, and returned problem sets should be addressed, respectively, to the lecturer, recitation professor, and teaching assistants.

The examinations are closed book. The midterm quiz, given during lecture time (1.5 hours), will cover the first part of the course. The final, given during finals period (3 hours), will target the second part of the course, although knowledge of the first part will be necessary as a basis.

For corrections on your grades, either in the problem sets or in the examinations, please see the teaching staff member who has graded that problem. Please do not hesitate to do so. However, all grades will become final after three weeks from being returned to you. The overall grades are computed as follows: problem sets 1/3, quiz 1/3, final 1/3.

It will be our sincere pleasure to teach this course, from which we hope you can benefit as much as possible. Best wishes for the entire semester!

Nihat Berker and Mehran Kardar,

February 7, 1997

8.334 Phase Transitions and Critical Phenomena (Spring 1982)

Berker, John Negele, Susan McKay

(1) One sentence summary: Qualitatively new phenomena due to a very large number of interacting degrees of freedom.

(2) Expanding field.

(3) Close connection between theory and experiment.

(4) Application outside condensed matter physics:

high energy, biological, turbulence.

Even within condensed matter: universality → variety of system

(5) Captivating method: renormalization group

Each step has physical content.

A mapping, a trajectory: cf. mountain climbing or electric trains.

Problem Sets handed out and due on Thursdays.

Quiz qualitative understanding

calculational expertise tested in problem sets, not quiz.

Final could be short, same as quiz

Computing/library project

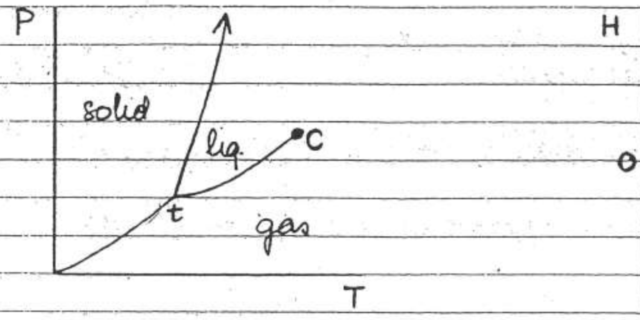
Companion courses:

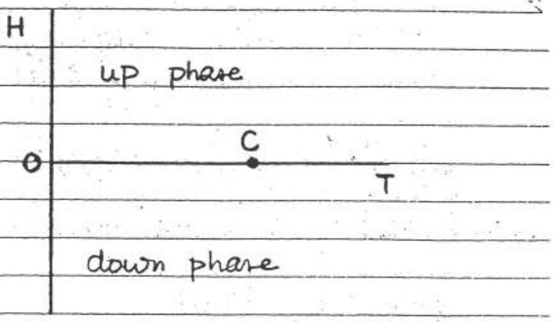
8.382 Intro to the Physics of Disordered Media, Henri Orland

8.872 Lattice Gauge Theories, Julius Kuti

I. Introduction

1. Phase diagrams and types of phase transitions

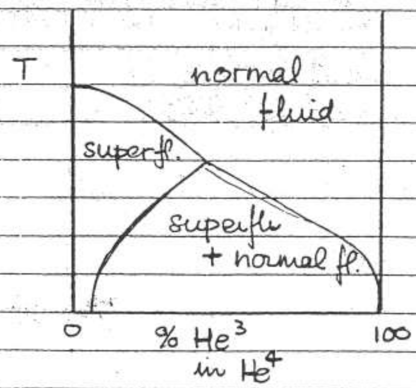




First-order phase transitions: discontinuous density or magnetizatu
two-phase coexistence

Second-order phase transitions:
where first-order transition disappears:
termination of two-phase coexistence
statistically uniform, but
large, long-lived fluctuations
infinite correlation length.
singular (divergent) thermodynamic properties
such as infinite response functions, i.e. compressibility, susceptib

Another phase diagram:



2. Necessity of thermodynamic limit

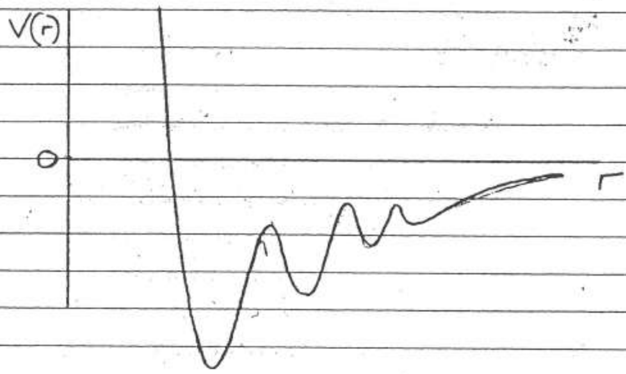
$$\mathcal{F} = -kT \ln Z = -kT \ln \sum_{\{r,s\}} e^{-\beta\mathcal{H}\{r,s\}}$$

$\lim$ $N \to \infty$
$V \to \infty$
N/V constant.

Otherwise, free energy is perfectly analytic function

Good things happen:
various ensembles become equivalent,
theoretical treatment, whether exact or approximate, easier,
experiments independent of container shape.

Existence of thermodynamic limit
For a gas, $V(r \to 0) \to \infty$ strong repulsion close-in
$V(r \to \infty) \to 0$ decaying interaction far-out



This can usually be made more precise.

Griffiths in DG 1.

3. Critical phenomena

(A) Absence of physical length scale

Fluctuations occur at all length scales:

Characteristic size $\xi = \infty$.

Ocean/bathtub example. or dunes/sandbox
Demonstration next lecture.

Correlation length infinite: $\left[\langle \rho_r \rangle = Z^{-1} \sum_{\{r\}} \rho e^{-\beta \mathcal{H}} \right]$

$$\langle (\rho_r - \langle \rho_r \rangle)(\rho_{r'} - \langle \rho_{r'} \rangle) \rangle \sim \frac{e^{-r/\xi}}{r^u}, \quad \xi \to \infty.$$

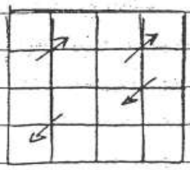
$$= \langle \rho_r \rho_{r'} \rangle - \langle \rho_r \langle \rho_{r'} \rangle \rangle - \langle \langle \rho_r \rangle \rho_{r'} \rangle + \langle \langle \rho_r \rangle \langle \rho_{r'} \rangle \rangle =$$

$$= \langle \rho_r \rho_{r'} \rangle - \langle \rho_r \rangle \langle \rho_{r'} \rangle$$ Correlation function
"uncorrelated part subtracted"

(B) Thermodynamic singularities

These two phenomena are related

This is best seen in the <u>Ising model</u>, which will be quite useful.



$$-\beta \mathcal{H} = J \sum_{\langle ij \rangle} s_i s_j + H \sum_i s_i$$

$$s_i = \pm 1$$

note: $J^{-1} \sim$ temperature

$$Z = \sum_{\{s\}} e^{-\beta \mathcal{H}\{s\}} = \mathrm{Tr}\, e^{-\beta \mathcal{H}}$$

$$M = \langle s_i \rangle = \frac{\mathrm{Tr}\, s_i e^{-\beta \mathcal{H}}}{\mathrm{Tr}\, e^{-\beta \mathcal{H}}} = \frac{1}{N} \frac{\partial}{\partial H} \ln Z$$

$$\frac{1}{N} \frac{\partial}{\partial H} \ln Z = \frac{1}{N} \frac{1}{Z} \frac{\partial}{\partial H} \mathrm{Tr}\, e^{J \sum s_i s_j + H \sum s_i} = \frac{1}{\cancel{N}} \frac{1}{Z} \mathrm{Tr} \left(\sum_i s_i \right) e^{-\beta \mathcal{H}} = M$$

$$\chi = \frac{\partial M}{\partial H} = \frac{\mathrm{Tr}\,(\sum_i s_i)\, e^{-\beta\mathcal{H}} (\sum_j s_j)}{N Z} - \frac{[\mathrm{Tr}(\sum_i s_i)\, e^{-\beta\mathcal{H}}][\mathrm{Tr}(\sum_j s_j)\, e^{-\beta\mathcal{H}}]}{N Z^2} =$$

$$= \frac{1}{N}\sum_{i,j}\left\{ \frac{\mathrm{Tr}\, s_i s_j e^{-\beta\mathcal{H}}}{Z} - \frac{\mathrm{Tr}\, s_i e^{-\beta\mathcal{H}}}{Z}\,\frac{\mathrm{Tr}\, s_j e^{-\beta\mathcal{H}}}{Z} \right\} =$$

$$= \frac{1}{N}\sum_{ij}\left\{ \langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle \right\} = \frac{1}{N}\sum_{ij} \langle (s_i - \langle s_i\rangle)(s_j - \langle s_j\rangle)\rangle =$$

$$= \frac{1}{N}\sum_{ij} \Gamma(i,j)$$

Susceptibility sum rule.

Translationally invariant system: $\Gamma(i,j) = \Gamma(i-j)$

$$\chi = \sum_j \Gamma(i-j) = \sum_r \Gamma(r) = \Gamma(\vec{k}=0).$$

$$\sum_r \Gamma(r) \sim \int_a^L r^{d-1} dr\ \Gamma(r)$$

① At criticality, $\chi \sim \int_a^L r^{d-1} dr \frac{1}{r^{d-2+\eta}} = \int_0^L dr\, r^{1-\eta} = L^{2-\eta}\int_{a/L\sim 0}^{1} dx\, x^{1-\eta}$

$\to \infty$ in thermo limit for $\eta < 2$.
Typically, η in $[0, .25]$

② Away from criticality

$$\chi \sim \int_a^L dr\, r^{1-\eta} e^{-r/\xi} \sim \int_a^{\xi} dr\, r^{1-\eta} = \xi^{2-\eta}$$

If the correlation length ξ diverges as $|T-T_c|^{-\nu}$,
the susceptibility χ diverges as $|T-T_c|^{-\gamma}$, $\gamma = (2-\eta)\nu$
ν, γ are called critical exponents (indices).

#2 $\quad \chi = \sum_r \Gamma(r)$

Divergence of χ tells us something about $\Gamma(r)$.

$\Gamma(r)$ is bounded between $[-1, 1]$.

So, divergence of $\sum_r \Gamma(r)$ can only come from

#1 the thermo limit $N \to \infty$ and from the long-range part o

Critical phenomena: Truely many-body problem
Effective-medium theories fail.

Many details of the hamiltonian do not affect many quantitative properties (e.g. critical exponents).

What is impatant? Things which affect large fluctuati

① Space dimension d

② Symmetry of the fluctuation degree of freedom $\updownarrow$

③ Long-range interactions

Thermodynamic Singularities at a critical point

		analogies
Susceptibility	$\chi = \frac{\partial M}{\partial H} \sim \lvert T - T_c \rvert^{-\gamma} \to \infty$	$\partial V / \partial P$ compressib
Specific heat	$C = \frac{\partial U}{\partial T} \sim \lvert T - T_c \rvert^{-\alpha}$	
Magnetization	$M \sim (T_c - T)^{\beta} \to 0$	$\rho_L - \rho_G$
Correlation length	$\xi \sim \lvert T - T_c \rvert^{-\nu} \to \infty$.	

<u>Universality</u> - same numerical values for exponents of systems as diverse as

$YFeO_3$ a ferromagnet
CO_2, Xe fluids
FeF_2 an antiferromagnet
β-brass an alloy
NH_4Cl a molecular crystal
3d Ising model a highly idealized model (cubic lattice)

<u>or</u> superfluid helium
magnetic systems
coupled rigid rotators

<u>universality classes</u>
exponents: classification indices

L.P. Kadanoff et al. RMP <u>39</u>, 395 (1967).

P. Heller, Rept. Prog. Phys. (1967)

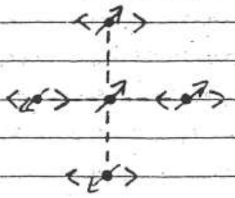
<u>Theory</u>	1960	1970 $\rightarrow t$
Mean field theory (molecular field, Curie-Weiss, van der Waals, Hartree, Landau, Gibbs)	high-order perturbative expansions, Monte Carlo, Molecular dynamics) computer experiments, droplet theories, scaling (1966)	Renormalization-group theory.

L.D. Landau, Z. Phys. Sowjetunion <u>11</u>, 26 (1937)
(reprinted in <u>Collected Papers of L.D. Landau</u>, ed. Ter Haar)

Landau and Lifshitz, <u>Statistical Mech.</u>

II. Classical Theory

1. Naive Mean-Field Theory



Each spin sees the average value of its neighbors.
All correlated fluctuations are ignored.

One-spin Hamiltonian:

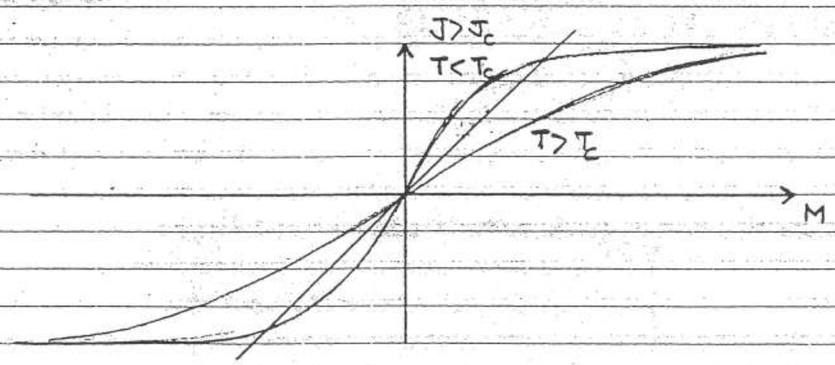
$$-\beta\mathcal{H}_1 = s_1 (qJM + H)$$

Require self-consistency:

$$M = \langle s_1 \rangle = \frac{\sum_{s_1} e^{-\beta\mathcal{H}_1} s_1}{\sum_{s_1} e^{-\beta\mathcal{H}_1}} = \tanh(qJM + H)$$

$$M = \tanh(qJM + H)$$

@ $H = 0$



$$1 = \frac{d}{dM} \tanh(qJM)\Big]_{M=0} = qJ \qquad \text{determines } T_c.$$

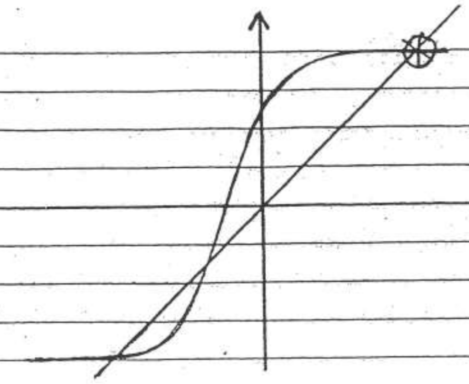
$$T_c = J_c^{-1} = q$$ overestimate: ignored fluctuations
12% in 3d, 50% in 2d, serious trouble in 1

How to choose a solution for $T < T_c$?

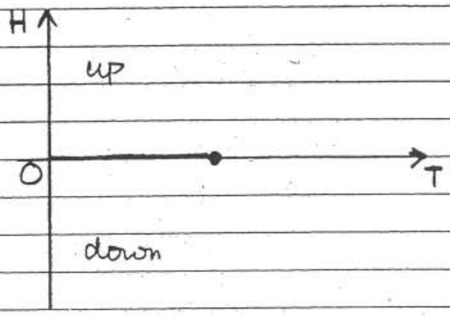
$$F_1 = -kT \ln \sum_{s_1} e^{s_1(qJM+H)} = -kT \ln\{2\cosh(qJM+H)\}.$$

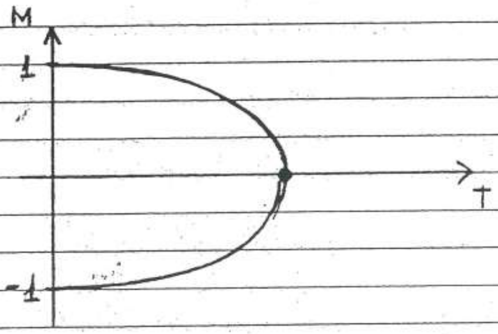
$M \neq 0$ minimizes: coexistence of two solutions.

(b) $H > 0$



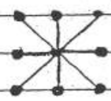
Phase diagram:





When is naive mean-field theory "more justified"?

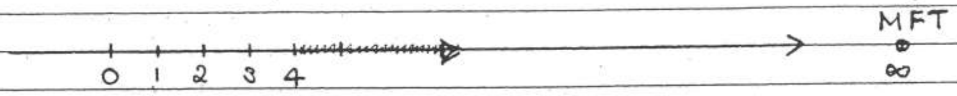
Equivalent neighbor model



$\sum_j s_j = NM \simeq qM$ (central limit theorem)

$J s_i \sum_j s_j \simeq J s_i q M$

Watch for (i) long-range interactions
(ii) high dimensionalities $q \sim 2d$ (simple cubic)



2. Constructive Mean-Field Theory

A. Gibbs variational principle

[H. Falk, Am. J. Phys. 38, 858 (1970)]

$$F = -kT \ln Z \leq \mathrm{Tr}\,\rho\mathcal{H} + kT\,\mathrm{Tr}\,\rho\ln\rho \equiv \Psi(\rho)$$

where ρ is any density matrix (non-neg, Hermitian, $\mathrm{Tr}\,\rho = 1$).

Equality holds when $\rho = \dfrac{e^{-\beta\mathcal{H}}}{\mathrm{Tr}\,e^{-\beta\mathcal{H}}} = \rho_{canonical}$

Proof of Gibbs variational principle

I. When $\rho = \rho_{can}$,

$\rho = \mathcal{P}/\mathcal{N}$

$\mathrm{Tr}\,\rho\ln\rho = \mathrm{Tr}\,\dfrac{\mathcal{P}}{\mathcal{N}}\ln\mathcal{P} - \mathrm{Tr}\,\dfrac{\mathcal{P}}{\mathcal{N}}\ln\mathcal{N} = -\ln\mathcal{N}$

$\mathrm{Tr}\,\rho\mathcal{H} = \langle\mathcal{H}\rangle = U$, $\quad S = -k\,\mathrm{Tr}\,\rho\ln\rho$

$\Psi(\rho_{can}) = U - TS = F$

II. $\Psi(\rho) = \sum_n \rho_n \left(\langle n|\mathcal{H}|n\rangle + kT\ln\rho_n\right)$

Vary the elements ρ_n:

$0 = \dfrac{\partial}{\partial\rho_n}\left(\Psi + \Lambda\,\mathrm{Tr}\,\rho\right) =$ $\quad$ Note: $\mathrm{Tr}\,\rho = 1$.

$= \langle n|\mathcal{H}|n\rangle + kT\ln\rho_n^* + \text{const}$

$\rho_n^* = \text{const}\; e^{-\beta\langle n|\mathcal{H}|n\rangle} = \dfrac{e^{-\beta\langle n|\mathcal{H}|n\rangle}}{\sum_{n'} e^{-\beta\langle n'|\mathcal{H}|n'\rangle}}$, normalized.

Take another derivative to verify minimum.

Quantum version: T.D. Schultz, Nuovo Cim. 8, 943 (1958).

B. Mean-field approximation

Use (most general) single-point density matrix, $\rho = \prod_i^N \rho_i$

Minimize $\Psi(\rho_i)$.

#2 [Bethe-Peierl approximation: $\rho = \prod_{\langle ij\rangle} \rho_{ij}$]

#3

C. Example: Ising model.

(a) $\rho_i = A e^{Bs_i}$

Normalization: $A^{-1} = 2\cosh B$, B minimization parameter

(b) $-\beta \operatorname{Tr} \rho \mathcal{H} = \operatorname{Tr} \prod_i \rho_i \left\{ J \sum_{\langle ij \rangle} s_i s_j + H \sum_i s_i \right\} =$

$= J \sum_{\langle ij \rangle} (\operatorname{tr} \rho_i s_i)(\operatorname{tr} \rho_j s_j) + H \sum_i (\operatorname{tr} \rho_i s_i) =$

$= J \frac{Nq}{2} m^2 + HNm.$

$m = \operatorname{tr} \rho_i s_i = \sum_{s_i} A e^{Bs_i} s_i = A\, 2 \sinh B = \tanh B$

note: not real magnetizat

$\operatorname{Tr} \rho \ln \rho = \operatorname{Tr} \prod_i \rho_i \left(\sum_j \ln \rho_j \right) =$

$= \sum_j \operatorname{tr} \rho_j \ln \rho_j = N \operatorname{tr} \rho_j (Bs_j + \ln A) =$

$= N(Bm + \ln A).$

Combining,

$$-\frac{\beta \Psi}{N} = \frac{Jq}{2} m^2 + Hm - Bm - \ln A$$

(c) $\cosh^2 B = (2A)^{-2}$ $\sinh^2 B = (2A)^{-2} - 1$

$m^2 = \tanh^2 B = 1 - 4A^2$

$4A^2 = 1 - m^2$

$A = \frac{1}{2}(1-m^2)^{1/2}$ $B = \tanh^{-1} m = \frac{1}{2} \ln \frac{1+m}{1-m}$

Now m can be the minimization parameter.

$$-\frac{\beta \Psi}{N} = \frac{Jq}{2} m^2 + Hm - m \tanh^{-1} m - \frac{1}{2} \ln(1-m^2) + \cancel{\ln 2}$$

$$\Rightarrow \frac{Jq}{2} m^2 + Hm - \frac{m}{2} \ln \frac{1+m}{1-m} - \frac{1}{2} \ln (1+m)(1-m) =$$

$$= \frac{Jq}{2} m^2 + Hm - \frac{(1+m)}{2} \ln(1+m) - \frac{(1-m)}{2} \ln(1-m)$$

Now minimize:

$$0 = \frac{d}{dm}\left(-\frac{\beta\Psi}{N}\right) = Jqm + H - \tanh^{-1}m - \cancel{\frac{m}{1-m^2}} - \cancel{\frac{1}{2}\frac{(-2m)}{(1-m^2)}}$$

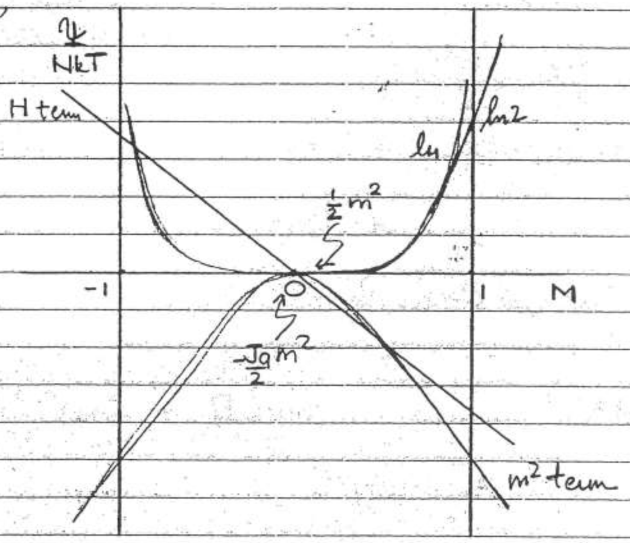
$$Jqm + H - \tanh^{-1}m = 0$$

<u>Self-consistent condition recovered.</u>

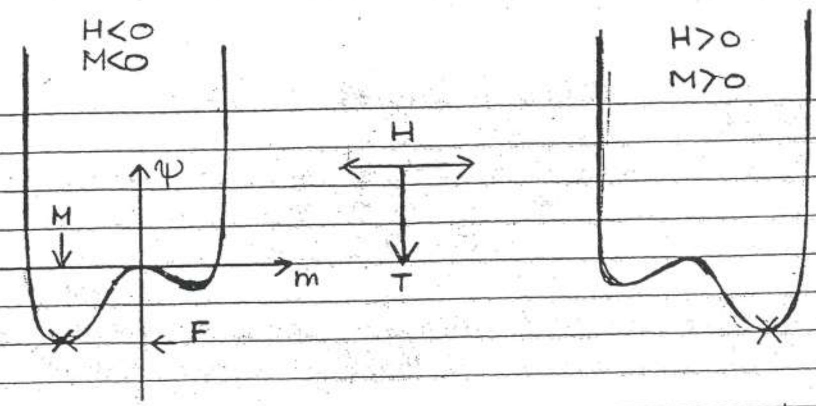
$$\frac{\Psi}{N} = -kT\left\{\frac{Jq}{2}m^2 + Hm - \frac{1}{2}(1+m)\ln(1+m) - \frac{1}{2}(1-m)\ln(1-m)\right\}$$

"<u>Bragg-Williams form</u>"

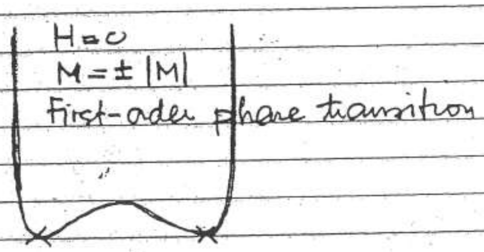
Pictorially,



① T low
J high

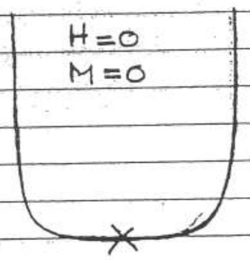


H=0 coexistence condition of two phases, $M = \pm |M|$



(Critical point)

② T high
J low



Only one phase at M=0

Looking back at free energy functional

$$\frac{\beta \Psi}{N} = -\frac{Jq}{2} m^2 - Hm + \frac{1}{2}(1+m)\ln(1+m) + \frac{1}{2}(1-m)\ln(1-m)$$

① Ordinary point M : $\frac{d\Psi}{dm} = 0$

② Critical point : $\frac{d^2\Psi}{dm^2} = 0$ in addition $M_c = 0$ cu at dis

③ First-order (coexistence) $\Psi(M_1) = \Psi(M_2)$ Jq=
In this specific case, $H = 0$, $M_1 = -M_2$ by examining Ψ

④ x

④ Stability. Expand around $m=0$.

$$\frac{\beta\Psi(m)}{N} = -Hm - \frac{Jq}{2}m^2 + \left(-\frac{m^2}{2} - \frac{m^4}{4} - \dots\right) + m\left(m + \frac{m^3}{3} + \dots\right) =$$

$$= -Hm + \frac{1}{2}(1-Jq)\,m^2 + \frac{1}{12}m^4 + \dots$$

positive coefficients

ⓐ In this case, no real danger, because $|M| \leq 1$ physically. If global minimum at $|m| > 1$, check $|m| = 1$.

ⓑ Unrestrained order-parameter problems. Then stability essential.

3. Landau Theory

Assume that there exists a free energy function $\Psi(m; H, T, L)$ such that

e.g. $L \sum_{\langle ijk \rangle} s_i s_j s_k$

① Ψ(minimized wrt m) $\simeq F$

m(minimizing) $\simeq M$

② Ψ can be expanded in m.

$$\left.\frac{\partial\Psi}{\partial m}\right)_M = 0, \quad \frac{\partial^2\Psi}{\partial m^2} > 0$$

ordinary point (global min), i.e. equation of state

$$\left.\frac{\partial\Psi}{\partial m}, \frac{\partial^2\Psi}{\partial m^2}, \frac{\partial^3\Psi}{\partial m^3}\right)_M = 0, \quad \frac{\partial^4\Psi}{\partial m^4} > 0$$

critical point

$$\left.\frac{\partial\Psi}{\partial m}\right)_{M_1, M_2} = 0, \quad \Psi(M_1) = \Psi(M_2)$$

first-order phase transit[ion]

Consider the neighborhood of critical point $T=T_c$

$H=H_c=0$, $L=L_c=0$, $M_c=0$

$$\Psi = \Psi_c + a_0 + a_1 m + a_2 m^2 + a_3 m^3 + a_4 m^4$$

For $n = 0,1,2,3$:

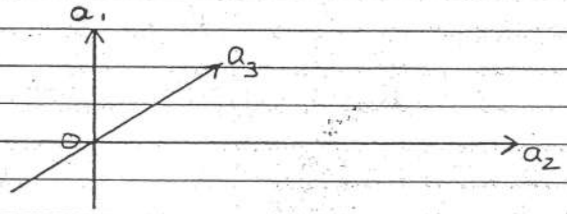
$$a_n = a_{n1}(H-H_c) + a_{n2}(T-T_c) + a_{n3}(L-L_c) + \text{higher order}$$

$$a_4 = \underset{\text{? positive}}{c} + \text{small quantities}$$

Note: symmetry $\Psi(m;H,T,L) = \Psi(-m;-H,T,-L)$

Thus, for example $a_{21}=0$, $a_{12}=0$, etc.

Axes:



$$\left.\frac{\partial\Psi}{\partial m}\right)_M = 0 = a_1 + 2a_2 M + 3a_3 M^2 + 4cM^3 \qquad \text{Eq. of state.}$$

① Other nearby critical points?

$$\left.\frac{\partial^2\Psi}{\partial m^2}\right)_M = 0 = 2a_2 + 6a_3 M + 12cM^2$$

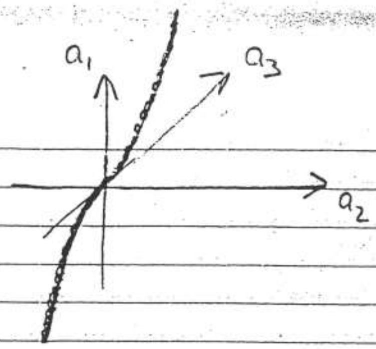
$$\left.\frac{\partial^3\Psi}{\partial m^3}\right)_M = 0 = 6a_3 + 24cM$$

$$\left.\begin{aligned} M_c &= -\frac{a_3}{4c} \\ a_2 &= \frac{3}{8}\frac{a_3^2}{c} \\ a_1 &= \frac{a_3^3}{16c^2} \end{aligned}\right\} \text{line of critical points parametrized by } a_3. \text{ ?}$$

$$0 = 2a_2 - 12cM_c^2$$

$$a_2 = 6cM_c^2 = 6c\frac{a_3^2}{16c^2} = \frac{3}{8}\frac{a_3^2}{c}$$

$$a_1 = \left(-2a_2 + 3a_3\frac{a_3}{4c} - 4c\frac{a_3^2}{16c^2}\right)M_c = \frac{a_3^3}{16c^2}$$



ⓒ <u>First-order transition?</u> Specialise to $a_3 = 0$:

Eq. of state $\quad a_1 + 2a_2 M + 4c M^3 = 0$

$\Big[\; q = \frac{1}{3}\frac{a_2}{2c} \qquad r = -\frac{1}{2}\frac{a_1}{4c}$

$q^3 + r^2 = \frac{a_2^3}{6^3 c^3} + \frac{1}{8^2}\frac{a_1^2}{c^2} \quad \ldots \;\Big]$

also,

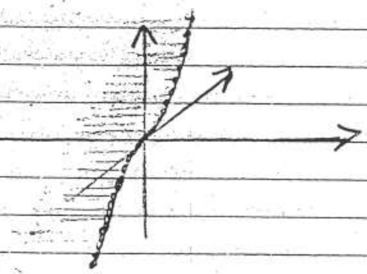
$\psi(M) = \psi_c + a_0 + a_1 M + a_2 M^2 + a_4 M^4 =$

$= \psi_c + a_0 - \cancel{2}a_2 M^2 - \cancel{4}^{3} a_4 M^4 + \cancel{a_2 M^2} + \cancel{a_4 M^4} =$

$= \psi_c + a_0 - a_2 M^2 - 3a_4 M^4$

$\psi(M_1) = \psi(M_2)$ satisfied for $a_1 = 0$, $a_2 < 0$

$M_{1,2} = \pm |M|$



a_3 = irrelevant field
a_2 = weak relevant field (thermal, even)
a_1 = strong relevant field (magnetic, odd)

<u>#3</u>

#4

Critical Behavior and Critical Exponents

(A) Thermal direction : $a_1 = 0$, $a_3 = 0$

Eq. of state $2a_2 M + 4cM^3 = 0$

$a_2 > 0$: $M = 0$

$a_2 < 0$: $M = \pm\left(\frac{-a_2}{2c}\right)^{1/2} \sim \left(\frac{T_c - T}{T_c}\right)^{1/2}$ $\beta = 1/2$

(B) Odd (symmetry-breaking) direction: $a_2 = 0$, $a_3 = 0$

Eq. of state: $a_1 + 4cM^3 = 0$

$M = \left(\frac{-a_1}{4c}\right)^{1/3} \sim H^{1/3}$

critical exponent $H \sim M^\delta$ $\delta = 3$

(C) Susceptibility $\chi = \left.\frac{\partial M}{\partial H}\right)_{H=0} \sim -\left.\frac{\partial M}{\partial a_1}\right)_{a_1=0}$

specialize

Eq. of state: $\left.\frac{\partial \Psi}{\partial m}\right)_M = 0 = a_1 + 2a_2 M + \cancel{3a_3 M^2} + 4cM^3$

Take one derivative wrt a_1: $0 = -1 + (2a_2 + \cancel{6a_3 M} + 12cM^2)\chi$

$$\chi = \frac{+1}{2a_2 + \cancel{6a_3 M} + 12cM^2}$$

(1) $a_2 \to 0^+$: $M = 0$, $\chi = 1/2a_2 \to +\infty$

$\sim \left[\frac{T - T_c}{T_c}\right]^{-1}$ $\gamma = 1$

(2) $a_2 \to 0^-$: $M^2 = -\frac{a_2}{2c}$, $\chi = -1/4a_2 = 1/4|a_2| \to \infty$

$\sim \left[\frac{T_c - T}{T}\right]^{-1}$ $\gamma = 1$

Exponents symmetrical, amplitudes not. Explain: order parameter saturation eff

Ⓓ Specific heat: a_0 contributes to non-singular background

$$\frac{F}{N} = \psi(M) = \psi_c + a_0 + a_2 M^2 + c M^4 \qquad (\text{taken } a_1 = a_3 = 0).$$

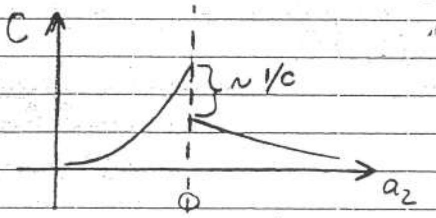
$$S = -\frac{1}{N}\frac{\partial F}{\partial T}$$

$$C = T\frac{\partial S}{\partial T} = -\frac{T}{N}\frac{\partial^2 F}{\partial T^2} \sim \frac{\partial^2 \psi(M)}{\partial a_2^2}\left[a_{22}\right]^2 + \text{background}$$

$a_2 \to 0^+$: $\psi(M) = \psi_c + a_0$

$a_2 \to 0^-$: $\psi(M) = \psi_c + a_0 + a_2\left(-\frac{a_2}{2c}\right) + c\left(\frac{a_2}{2c}\right)^2 =$

$$= \psi_c + a_0 - \frac{a_2^2}{4c}$$



Discontinuity.

Scaling Forms

Useful when both a_1, a_2 present

Ⓐ Equation of state: $0 = a_1 + 2a_2 M + 4c M^3$

$1/a_2^{3/2}$: $$0 = \frac{a_1}{a_2^{3/2}} + 2\left(\frac{M}{a_2^{1/2}}\right) + 4c\left(\frac{M}{a_2^{1/2}}\right)^3$$

$$\frac{M}{a_2^{1/2}} = f\left(\frac{a_1}{a_2^{3/2}}\right)$$

$$M = a_2^{1/2} f\left(\frac{a_1}{a_2^{3/2}}\right)$$

Generally, $M = a_2^{\beta} f\left(\frac{a_1}{a_2^{\Delta}}\right)$, homogeneous fn of degree β of variables a_1, a_2^{Δ}.

Importance. data collapsing.

$$\beta_{MF} = 1/2 \qquad \Delta_{MF} = 3/2$$

① $a_1 = 0 \qquad M = a_2^{\beta} f(0)$

② $a_2 = 0 \qquad f(x) = x^{u} g(x^{-1}) \quad$ as $x \to \infty$.

$$M = a_2^{\beta - \Delta u} a_1^{u} g\left(\frac{a_2^{\Delta}}{a_1}\right) = a_2^{\beta - \Delta u} a_1^{u} g(0).$$

For M to be finite (which we know it is), $u = \beta/\Delta$.

$$M = a_1^{\beta/\Delta} g(0) \sim a_1^{1/\delta}$$

Conclude $\quad \Delta = \beta\delta$.

MF: $\quad \frac{3}{2} = \frac{1}{2} 3 \quad \checkmark$

③ Free energy $\qquad F = \Psi_c'' + a_0 + F_s$

$$F_s = a_1 M + a_2 M^2 + c M^4 =$$

$$= a_2^2 \left\{ \frac{a_1}{a_2^{3/2}} \left(\frac{M}{a_2^{1/2}}\right) + \left(\frac{M}{a_2^{1/2}}\right)^2 + c\left(\frac{M}{a_2^{1/2}}\right)^4 \right\} =$$

$$= a_2^2 \left\{ \frac{a_1}{a_2^{3/2}} f\left(\frac{a_1}{a_2^{3/2}}\right) + \left[f\left(\frac{a_1}{a_2^{3/2}}\right)\right]^2 + c\left[f\left(\frac{a_1}{a_2^{3/2}}\right)\right]^4 \right\} =$$

$$= a_2^2 \, \mathrm{func}\left(\frac{a_1}{a_2^{3/2}}\right) =$$

$$= a_2^{2-\alpha} \, \mathrm{func}\left(\frac{a_1}{a_2^{\Delta}}\right) \qquad \alpha_{MF} = 0, \quad \Delta_{MF} = 3/2$$

$$= a_2^{2-\alpha-\Delta v} a_1^{v} G(0), \qquad v = \frac{2-\alpha}{\Delta} = \frac{4}{3}$$

<u>Exponent equalities</u>

$$M \sim \left.\frac{\partial F}{\partial a_1}\right)_{a_1=0} = a_2^{2-\alpha-\Delta} \, \mathrm{func}'(0) \qquad \beta = 2 - \alpha - \Delta$$

$$\chi \sim \left.\frac{\partial^2 F}{\partial a_1^2}\right)_{a_1=0} = a_2^{2-\alpha-2\Delta} \, \mathrm{func}''(0) \qquad \gamma = 2 - \alpha - 2\Delta \quad \text{"gap ex}$$

$$\alpha' = \alpha$$
$$\gamma' = \gamma$$

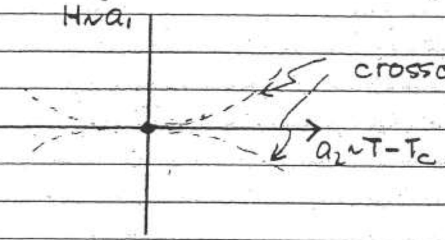
Only two independent exponents.

Crossover phenomena

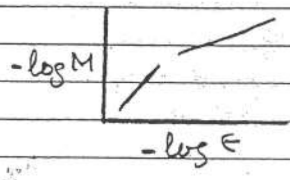
$$M \sim \underset{\substack{\text{temp direction}\\ x \to 0}}{a_2^{\beta}} \quad \leftarrow \quad a_2^{\beta} f\left(\frac{a_1}{a_2^{\beta\delta}}\right) \quad \rightarrow \quad \underset{\substack{\text{mag field direction}\\ x \to \infty}}{a_1^{1/\delta}}$$

Possible because, as we have seen, $f(x) \sim x^{1/\delta}$ as $x \to \infty$

Boundary between two regimes: $a_1/a_2^{\Delta} = x = 1$ (or const)



$M \sim \epsilon^{1/2}$ crosses over to $\epsilon^{1/3}$



~~#4~~

Overview of Classical Theory - thus far

① Global or local analysis

variational principle — Landau theory

phase diagrams — critical exponents

② Power laws for singular dependences
(MF quantitatively wrong below $d=4$)

$A_{\pm} t^{\alpha}$ becomes discontinuity for $\alpha = 0$.

or

$$A_{\pm} \frac{1}{\alpha} t^{\alpha} = A_{\pm}\frac{1}{\alpha} e^{\alpha \ln t} \underset{\alpha\to 0}{\Rightarrow} A_{\pm} \ln t \qquad \text{logarithm}$$

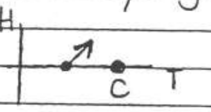
③ Scaling fields, axes.

④ Scaling forms.

⑤ Exponents equalities

Rigorous thermodynamics: bounds, e.g. $\alpha + 2\beta + \gamma \geq 2$

$$C_M = T\left(\frac{\partial S}{\partial T}\right)_M$$



$$C_H \equiv T\left(\frac{\partial S}{\partial T}\right)_H = T\left(\frac{\partial S}{\partial T}\right)_M + T\left(\frac{\partial S}{\partial M}\right)_T\left(\frac{\partial M}{\partial T}\right)_H$$

$$S = -\left[\frac{\partial A(T,M)}{\partial T}\right]_M \quad \text{Helmholtz potential} \quad dA = -S\,dT + H\,dM$$

$$\left(\frac{\partial S}{\partial M}\right)_T = -\left(\frac{\partial}{\partial T}\left(\frac{\partial A}{\partial M}\right)_T\right)_M = -\left(\frac{\partial H}{\partial T}\right)_M$$

$$C_H - C_M = T(-)\left(\frac{\partial H}{\partial T}\right)_M\left(\frac{\partial M}{\partial T}\right)_H =$$

$$= T(+)\left[\left(\frac{\partial T}{\partial M}\right)_H\left(\frac{\partial M}{\partial H}\right)_T\right]^{-1}\left(\frac{\partial M}{\partial T}\right)_H =$$

$$= T\left(\frac{\partial M}{\partial T}\right)_H^2 \chi^{-1}$$

(for $\alpha_H > 0$:

$$C_H \geq T\left(\frac{\partial M}{\partial T}\right)_H^2 \chi^{-1}$$

$$(T_c - T)^{-\alpha} \geq A\,(T_c - T)^{2(\beta-1)+\gamma}$$

$$-\alpha \ln \tau \geq (2\beta + \gamma - 2)\ln \tau$$

$$-\alpha \leq 2\beta + \gamma - 2$$

Refs. Stanley; Buckingham

RG: Equalities

DG2

Can prove equality by thermodynamics?

Ferromagnet χ positive.

If $\alpha > 0$: C_M either as singular or less singular than C_H. Otherwise

Then equality follows. pos

$$C_H - C_M = T\left(\frac{\partial M}{\partial T}\right)^2 \chi^{-1} = |B|\, t^{2\beta-2+\gamma}$$

$$C_x \approx A_x t^{-\alpha_x} + f_0 + f_1 t + \ldots \qquad \alpha \equiv \alpha_H$$

$$A \equiv A_H$$

① $\alpha_H > \alpha_M, 0$

$$A t^{-\alpha} = |B|\, t^{2\beta-2+\gamma}$$

$$-\alpha = 2\beta - 2 + \gamma$$
$$\alpha + 2\beta + \gamma = 2$$

② $\alpha_H = \alpha_M > 0, \qquad A_H > A_M, 0$

Same result as ①.

③ $\alpha_H = \alpha_M > 0, \qquad A_H = A_M$

next power $\bar{A} t^{-\bar{\alpha}} = |B|\, t^{2\beta-2+\gamma}$, $\bar{\alpha} < \alpha$ by definition

$$-\alpha < -\bar{\alpha} = 2\beta - 2 + \gamma$$
$$\alpha + 2\beta + \gamma > 2$$

④ $\alpha_H \leqslant 0 \quad \Rightarrow \quad \alpha_M \leqslant 0$

Analytic parts cancel

$$A_H t^{-\alpha_H} - A_M t^{-\alpha_M} = |B|\, t^{2\beta-2+\gamma}$$

A_x can be of either sign.

④a $-\alpha_M > -\alpha_H$ proves equality

④b $-\alpha_M < -\alpha_H$

$-\alpha_H > -\alpha_M = 2\beta - 2 + \gamma$ $\quad \boxed{\alpha + 2\beta + \gamma < 2.}$

④c $\alpha_M = \alpha_H, \quad A_M \neq A_H$ proves equality

④d $\alpha_M = \alpha_H, \quad A_M = A_H$

$\alpha + 2\beta + \gamma > \bar{\alpha} + 2\beta + \gamma = 2 \qquad -\bar{\alpha} > -\alpha \qquad \bar{\alpha} < \alpha$

$(\alpha + 2\beta + \gamma > 2)$

(5) <u>Universality</u> too large in MF, independent of d, n.

① $H(r)$: use $\rho = \prod_r \rho(r) \rightarrow m(r)$

4. <u>Non-uniform fluctuations</u>

② $m(r+a) = m(r) + a\underbrace{\frac{dm(r)}{dr}}_{a\frac{1}{\lambda}} + \frac{a^2}{2}\underbrace{\frac{d^2m}{dr^2}}_{a^2\frac{1}{\lambda^2}}$

$$\frac{J}{2}\sum_{\vec{r}}\sum_{\hat{a}} m(\vec{r})\, m(\vec{r}+a)$$

$$m(r)\,m(r+a) = \frac{1}{2}\left\{m^2(r) + m^2(r+a) - a^2\frac{[m(r+a)-m(r)]^2}{a^2}\right\}$$

This term becomes, in the continuum limit,

$$\frac{Jq}{2}\sum_r m^2(r) - \frac{J}{2}a^2\sum_r |\nabla m(r)|^2 \quad + \text{higher order}$$

geometric factor
for non-cubic lattices

λ: length over which $m(r)$ varies appreciably.
a/λ expansion parameter
$\Theta\,(a/\lambda)^2$ above.

Supposed that $m(r)$ both slowly varying and small }
From variational principle:

③ $$\beta\Psi = \beta\Psi_c + \beta a_0 + \underbrace{\sum_r}_{\frac{N}{V}\int(dr)}\Big\{-H(r)m(r) + \underbrace{\tfrac{1}{2}(1-Jq)m^2(r) + \tfrac{1}{12}m^4(r)}_{\beta a_2(T)\, m^2(r) + \beta c\, m^4(r)} + \frac{Ja^2}{2}|\nabla m(r)|$$

$$g_0[m(r)]$$

Minimize: $\left.\frac{\partial \Psi}{\partial m(r)}\right]_{M(r)} = 0$

$m(r) \to m(r) + \delta m(r)$

$2\nabla \delta m(r) \cdot \nabla m(r)$

$\int$ by parts: $-2\delta m(r) \nabla^2 m(r)$

$$-H(r) + \left.\frac{\partial g_0}{\partial m(r)}\right]_{M(r)} - Ja^2 \nabla^2 M(r) = 0 \qquad \text{Eq. of state}$$

Consider near-uniformity: $H(r) = H + \delta H(r)$
$M(r) = M + \delta M(r)$

$$\left.\frac{\partial g_0}{\partial m(r)}\right]_M = H \qquad \underline{\text{Eq of state of uniform case}}$$

$$\frac{\partial}{\partial M}\left[\left.\frac{\partial g_0}{\partial m(r)}\right]_M\right] = \frac{\partial H}{\partial M} = \chi^{-1}; \qquad \text{to be used in expansion:}$$

$$\left.\frac{\partial g_0}{\partial m(r)}\right]_{M(r)} = \left.\frac{\partial g_0}{\partial m(r)}\right]_M + \frac{\partial}{\partial M}\left[\left.\frac{\partial g_0}{\partial m(r)}\right]_M\right] \delta M(r) = H + \chi^{-1}\delta M(r)$$

Substituting in eq. of state

$$-\delta H(r) + \chi^{-1}\delta M(r) - Ja^2 \nabla^2 M(r) = 0.$$

$$\nabla^2 [M + \delta M(r)] = \nabla^2 \delta M(r)$$

$$\{\chi^{-1} - Ja^2 \nabla^2\}\, \delta M(r) = \delta H(r)$$

$$\{(\chi J a^2)^{-1} - \nabla^2\}\, \delta M(r) = \delta H(r)/Ja^2$$

units of inverse length square. Define length ξ

$$\{\xi^{-2} - \nabla^2\}\, \delta M(r) = \delta H(r)/Ja^2$$

~~#5~~ 5

#6

Operate with $\frac{\partial}{\partial[\delta H(r')]}$: in $-\beta\mathcal{H}$, $\sum_{r'} H(r') s(r')$

$+M$

$$\frac{\partial[\delta M(r)]}{\partial[\delta H(r')]} = \frac{\partial}{\partial[\delta H]} \frac{\mathrm{Tr}\, s(r) e^{-\beta\mathcal{H}}}{\mathrm{Tr}\, e^{-\beta\mathcal{H}}} = \{\langle s(r)s(r')\rangle - \langle s(r)\rangle\langle s(r')\rangle\} = \Gamma$$

$$\{\xi^{-2} - \nabla^2\}\, \Gamma(r,r') = \frac{v_0}{Ja^2}\, \delta(r - r') \qquad \underline{\text{Eq. for correlation fn}}$$

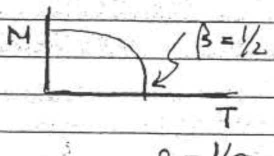
Characteristic (physical) length $\xi = (Ja^2/\chi)^{1/2}$

high-temp $a_2 \to 0^+$ $\quad \xi = (Ja^2)^{1/2} \frac{1}{\sqrt{2}} \frac{1}{(a_2)^{1/2}} \sim \left(\frac{T-T_c}{T_c}\right)^{-1/2} \to \infty$

low-temp $a_2 \to 0^-$ $\quad \xi = (Ja^2)^{1/2} \frac{1}{2} \cdot \frac{1}{(-a_2)^{1/2}} \sim \left(\frac{T_c-T}{T_c}\right)^{-1/2} \to \infty$

Again asymmetry in amplitudes, due to order parameter saturation.

$$\xi \simeq \xi_0^{\pm} \left|\frac{T-T_c}{T_c}\right|^{-\nu}$$



MFT: $\nu = 1/2$ $\quad \xi_0^+/\xi_0^- = \sqrt{2}$

2d Ising: $\nu = 1$ $\quad \xi_0^+/\xi_0^- = 2$ $\quad \beta = 1/8$

Solution of the eq

$$\{\xi^{-2} - \nabla^2\}\,\Gamma(r-r') = \frac{V_0}{Ja^2}\,\delta(r-r')$$

Define $\rho \equiv (r-r')/\xi$ $\quad \delta^d(r-r') = \xi^{-d}\delta(\rho)$

$$\{1-\nabla_\rho^2\}\,\Gamma(\rho) = \frac{V_0}{Ja^2\xi^{d-2}}\,\delta(\rho)$$

Define $G(\rho) = \dfrac{\xi^{d-2}Ja^2}{V_0}\,\Gamma(\rho)$

$$\{1-\nabla_\rho^2\}\,G(\rho) = \delta(\rho)$$

Fourier transform $\quad G(\rho) = \dfrac{1}{(2\pi)^{d/2}}\displaystyle\int (dk)\, G(k)\, e^{ik\rho}$

Solve $\quad G(k) = \dfrac{1}{(2\pi)^{d/2}}\dfrac{1}{(1+k^2)}$

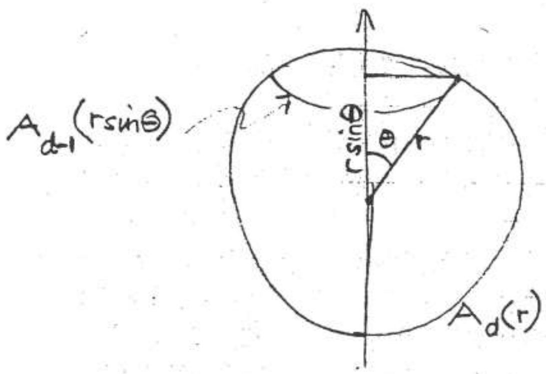
Invert transform

$$G(\rho) = \frac{1}{(2\pi)^d}\int (dk)\,\frac{e^{ik\rho}}{1+k^2} = \frac{1}{(2\pi)^d}\int \frac{k^{d-1}dk}{1+k^2}\int_0^\pi d\theta\,(\sin\theta)^{d-2}\,e^{ik\rho\cos\theta}\,\overset{A_{d-1}}{C_d}$$

$$C_d = \frac{(2\pi)^d}{(2\pi)}\,\frac{1}{2^{d-2}\,\pi^{(d-1)/2}\,\Gamma[(d-1)/2]} = \frac{2\pi^{(d-1)/2}}{\Gamma[(d-1)/2]}$$

Check: $\int_0^\pi d\theta\,(\sin\theta)^{d-2}\,C_d =$
$= \pi\; 2\pi^{1/2}/\Gamma(1/2) = 2\pi$ $\quad d=2$
$= 2\; 2\pi/\Gamma(1) = 4\pi$ $\quad d=3$

d-dimensional phase space: Vilenkin, Special Fns and the Theory of Group Repres. Ch. IX.



$$\int (dx)\, e^{-r^2} = \left[\int_{-\infty}^{\infty} dx\, e^{-x^2}\right]^d = \pi^{d/2}$$

$$= A_d \int_0^{\infty} r^{d-1} dr\, e^{-r^2} = A_d\, \frac{1}{2}\, \Gamma\left(\frac{d}{2}\right)$$

$$A_d = 2\,\frac{\pi^{d/2}}{\Gamma(d/2)}$$

d-dimensional phase space.

Vilenkin, Special Fns. and the Theory of Group Repres. Ch. IX

$$x_1 = r\cos\theta_1$$
$$x_2 = r\sin\theta_1 \cos\theta_2$$
$$x_3 = r\sin\theta_1 \sin\theta_2 \cos\theta_3$$
$$\vdots$$
$$x_{d-2} = r\sin\theta_1 \quad \cdots \quad \sin\theta_{d-3}\cos\theta_{d-2}$$
$$x_{d-1} = r\sin\theta_1 \quad \cdots \quad \sin\theta_{d-3}\sin\theta_{d-2}\cos\varphi$$
$$x_d = r\sin\theta_1 \quad \cdots \quad \sin\theta_{d-3}\sin\theta_{d-2}\sin\varphi$$

$$\int (dx)\,\delta(r-1) = A\,\frac{\Gamma(d/2)}{2\pi^{d/2}}\int \sin^{d-2}\theta_1 \cdots \sin\theta_{d-2}\, d\theta_1 \ldots d\theta_{d-2}\, d\varphi$$

using $\displaystyle\int_0^\pi \sin^n\theta\, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n+2}{2}\right)}$,

$$= A\,\frac{\Gamma(d/2)}{\cancel{2}\pi^{d/2}}\,\pi^{(d-3)/2}\,\frac{\Gamma(1)}{\Gamma\left(\frac{d-1}{2}\right)}\,\cancel{2}\pi\int \sin^{d-2}\theta_1\, d\theta_1 =$$

$$= A\,\pi^{-1/2}\,\frac{\Gamma(d/2)}{\Gamma\left(\frac{d-1}{2}\right)}\int \sin^{d-2}\theta_1\, d\theta_1 \qquad = A$$

$$A = A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \int_0^\pi \sin^{d-2} d\theta\; A_{d-1} = \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\sqrt{\pi}\; A_{d-1}$$

$$= \frac{2\pi^{d/2}}{\cancel{\Gamma(d/2)}}\,\pi^{-1/2}\,\frac{\cancel{\Gamma(d/2)}}{\Gamma\left(\frac{d-1}{2}\right)}\int \sin^{d-2}\theta_1\, d\theta_1 = \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}\int \sin^{d-2}\theta_1\, d\theta_1$$

$$\int (dx)\,\delta(r-\bar r) = \int dx_1 \ldots dx_{d-1}\int dx_d\,\delta(r-\bar r) = \int dx_1\ldots dx_{d-1}\int dx_d\,\delta\left(x_d \pm \sqrt{\bar r^2 - \textstyle\sum^{d-1} x_i^2}\right)\left|\frac{\partial x_d}{\partial r}\right| =$$

$$= \int dx_1\ldots dx_{d-1}\int dx_d\,\delta(x_d \pm \sqrt{\ })\,\frac{\bar r}{|x_d|} = \int \bar r^{d-2}\sin^{d-2}\theta_1 \cdots \sin^2\theta_{d-3}\sin\theta_{d-2}\,|x_d|\,\frac{\bar r}{|x_d|\,\delta(x_d \pm \sqrt{\ })}\,(d\theta)\,d\varphi\, dx_d =$$

$$= \int \bar r^{d-1}\sin^{d-2}\theta_1 \ldots \sin\theta_{d-2}\,(d\theta)\,d\varphi\,.$$

$$G(\rho) = \frac{C_d}{(2\pi)^d}\int \frac{k^{d-1}dk}{1+k^2}\int d\theta (\sin\theta)^{d-2} e^{ik\rho\cos\theta} =$$

Gradshteyn and Ryzhik; p. 482 # 3.915.5:

$$= \frac{C_d}{(2\pi)^d}\int \frac{k^{d-1}dk}{1+k^2}\, \pi^{1/2}\left(\frac{2}{k\rho}\right)^{\frac{d}{2}-1}\Gamma[(d-1)/2]\, J_{\frac{d}{2}-1}(k\rho) = \qquad d>1$$

eaning up, $$= \frac{\cancel{C_d}}{\cancel{(2\pi)^d}_{\,2^{d/2}\pi^{d/2}}}\, \frac{\pi^{1/2}2^{\frac{d}{2}-1}}{\rho^{d/2-1}}\, \frac{2\pi^{(d-1)/2}}{\cancel{C_d}}\int \frac{k^{d/2}dk}{1+k^2} J_{\frac{d}{2}-1}(k\rho) =$$

$$= \frac{1}{(2\pi)^{d/2}\rho^{d/2-1}}\int \frac{k^{d/2}dk}{1+k^2} J_{\frac{d}{2}-1}(k\rho) =$$

GR, p. 686, # 6.565.4:

$$= \frac{1}{(2\pi)^{d/2}\rho^{d/2-1}} K_{\frac{d}{2}-1}(\rho) \qquad \text{condition: } -1 < \frac{d}{2}-1 < \frac{3}{2}$$

$d<5$?

GR, p. 967, # 8.468: Asymptotic series for $\rho \gg 1$:

$$\sim \frac{1}{(2\pi)^{d/2}\rho^{d/2-1}}\left(\frac{\pi}{2\rho}\right)^{1/2}e^{-\rho} = \frac{1}{2(2\pi)^{(d-1)/2}}\frac{e^{-\rho}}{\rho^{(d-1)/2}}$$

$$\Gamma(r) \sim \frac{1}{2(2\pi)^{(d-1)/2}}\frac{v_0}{\xi^{d-2}Ja^2}\,\xi^{(d-1)/2}\,\frac{e^{-r/\xi}}{r^{(d-1)/2}}$$

Exact results: violated $T<T_c$ for $d=2$ Ising: $\frac{e}{r^2}$
O.k. $T>T_c$: $\frac{e}{r^{1/2}}$

① $r/\xi \to \infty$ long-distance (asymptotic) behavior

$$\Gamma \sim f(T)\frac{e^{-r/\xi}}{r^{(d-1)/2}} \qquad \text{dominant exponential decay}$$

#6

#7

② $r/\xi \to 0$ critical point

$$\Gamma \simeq \frac{v_0}{\cancel{\xi^{d-2}}Ja^2}\frac{1}{(2\pi)^{d/2}}\left(\frac{\xi}{r}\right)^{d/2-1}\frac{1}{2}\Gamma\left(\frac{d}{2}-1\right)\left(\frac{1}{2}\right)^{-\frac{d}{2}+1}\left(\frac{\xi}{r}\right)^{\frac{d}{2}-1}$$

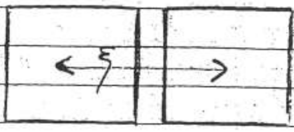
Cancellation!

$$= \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}}\frac{v_0}{Ja^2}\frac{1}{r^{d-2}} \qquad \text{algebraic decay (slower)}$$

dimensional argument

$$\text{cf } \frac{1}{r^{d-2+\eta}} \rightarrow \eta)_{MF} = 0$$

Ginzburg Criterion V.L. Ginsburg, Sov. Phys. Sol. St. 2, 1824 (1960).



$$\Gamma(\xi) \ll M^2$$

$$\langle S_0 S_\xi \rangle - \langle S_0 \rangle \langle S_\xi \rangle \ll \langle S_0 \rangle \langle S_\xi \rangle$$

$$\Gamma(r) = \frac{V_0}{\xi^{d-2} J a^2} \frac{1}{(2\pi)^{d/2}} \left(\frac{\xi}{r}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\left(\frac{r}{\xi}\right), \quad \xi = (\chi J a^2)^{1/2}$$

$$\frac{V_0}{\xi^{d-2} J a^2} \frac{1}{(2\pi)^{d/2}} K_{\frac{d}{2}-1}(1) \overset{?}{\ll} M^2$$

$$1 \overset{?}{\gg} M^{-2} \frac{V_0}{J a^2 (\chi J a^2)^{d/2-1}} \frac{K(1)}{(2\pi)^{d/2}} =$$

$$= \frac{1}{M^2 \chi^{d/2-1}} \frac{K(1)}{(2\pi)^{d/2}} \frac{V_0}{(J a^2)^{d/2}} =$$

$$= \frac{2c}{|a_2|} (4|a_2|)^{d/2-1} \frac{K(1)}{(2\pi J)^{d/2}} =$$

$$= |a_2|^{\frac{d}{2}-2} \left(\frac{2}{\pi J}\right)^{d/2} K(1) \frac{c}{2} \overset{?}{\ll} 1$$

Conclusions:

① Always satisfied, mean-field theory O.K. for $d > 4$.

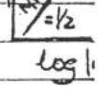
② $d = 2$: $K_0(1) = 1.1 e^{-1} = 0.42$, take $c/2 = 1$.
$d = 3$: $K_{1/2}(1) = (2/\pi)^{1/2} (0.58) = 0.46$
$d = 4$: $K_1(1) = 1.6 e^{-1} = 0.60$

Crossover from mean-field theory to something else at

$$|a_2| \sim \left(\frac{\pi J}{2}\right)^{\frac{d}{2}/(\frac{d}{2}-2)} K^{1/(2-d/2)} \qquad J_c \sim 1/2d$$

$$\sim \left(\frac{\pi}{4d}\right)^{d/(d-4)} K^{2/(4-d)} = \left[\left(\frac{4d}{\pi}\right)^d K^2\right]^{1/(4-d)} = \begin{array}{ll} 1.07 & d=2 \\ 11.79 & d=3 \end{array}$$

power law?



[For $\frac{K_c}{2} = 1$, there are 0.39 and 0.02] ?

III. Ising Models and Exact Results

1. Preliminary Comments

Ⓐ Ising model

$$-\beta\mathcal{H} = +\beta\hat{J}\sum_{\langle ij\rangle} s_i s_j + \beta\hat{H}\sum_i s_i \equiv J\sum_{\langle ij\rangle} s_i s_j + H\sum_i s_i$$

inverse temperature absorbed.

Ⓑ s_i "degree of freedom"

We shall be putting fluctuating degrees of freedom on a network (connectivity) of interactions - the lattice

$\hat{J}$ wavefunction overlap
exchange integral " qm effect

Also, s_i not necessarily spin, any angular momentum vector $\vec{l}, \vec{j}$.

Magnetism in Solids - Martin

Ⓒ $\langle ij\rangle$ nn sum.

Generalization $J\sum_{\langle ij\rangle} s_i s_j \to \frac{1}{2}\sum_{i,j} J_{ij} s_i s_j$ will be necessary in RG.

$J_{ij} = J\delta_{i,j+\hat{a}}$ nn recovered.

Ⓓ Already absorbed $\mu\hat{\hat{H}} \equiv \hat{H}$. Now:

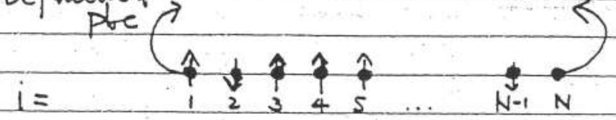
$\beta\hat{H} \equiv H \qquad \beta\hat{J} \equiv J$

Ⓔ Equivalence to lattice gas of PS

Decrease mesh size, increase range: arbitrarily realistic.

2. One-Dimensional Ising Model (Thompson, DG 1)

~~(A) Definition~~



(B) Exactly soluble, outcome:

1. No phase transition at $T>0$.
   Phase transition at $T=0$.

2. Interesting (illustrative) and manageable critical phenomena at $T=0$ in one-dimensional models.

   Example: tricritical point.

3. Methodology important
   (a) Exact solution in $d=2$ (Onsager)
   (b) Approximate RG solution (Migdal-Kadanoff)

(C) Heuristic argument

$$Z = \sum_{\{s\}} e^{-\beta\mathcal{H}} \sim \omega_m e^{-\beta E_m} = e^{-\beta(E_m - kT\ln\omega_m)}$$ ← minimum Ψ

$\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$ $\quad E_m - kT\ln\omega_m = -N\hat{J} - kT\ln 1 = -N\hat{J}$

$\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow$ $\quad -N\hat{J} + 2\hat{J} - kT\ln N$

Difference $\Delta\Psi = 2\hat{J} - kT\ln N$, favorable for large N.

R. Peierls (1936), LL, Thouless, PRB (1969).

(D) Solution: Transfer matrix method

$$Z = \sum_{\{s\}} e^{-\beta\mathcal{H}} = \sum_{\{s\}} \left(e^{\frac{H}{2}s_1 + Js_1s_2 + \frac{H}{2}s_2}\right)\left(e^{\frac{H}{2}s_2 + Js_2s_3 + \frac{H}{2}s_3}\right)\cdots$$

$$\cdots\left(e^{\frac{H}{2}s_N + Js_Ns_1 + \frac{H}{2}s_1}\right)$$

$$\langle s_i|T|s_j\rangle \equiv e^{\frac{H}{2}s_i + Js_is_j + \frac{H}{2}s_j} = \begin{array}{cc} \begin{bmatrix} e^{J+H} & e^{-J} \\ e^{-J} & e^{J-H} \end{bmatrix} & \begin{array}{l} (s_i=+1) \\ (s_i=-1) \end{array} \\ \begin{array}{cc} (s_j=+1) & (s_j=-1) \end{array} & \end{array}$$

$$Z = \sum_{\{s\}} \langle s_1|T|s_2\rangle\langle s_2|T|s_3\rangle\cdots\langle s_N|T|s_1\rangle = \mathop{\mathrm{Tr}}_{2\times2} T^N$$

$\uparrow$ matrix multiplication

T real symmetric $\rightarrow$ diagonalize by orthogonal transformation

$$S^{-1}TS = \tilde{T} = \begin{bmatrix} \lambda_> & 0 \\ 0 & \lambda_< \end{bmatrix}$$

$$Z = \mathrm{Tr}\,\tilde{T}^N = \lambda_>^N + \lambda_<^N = \lambda_>^N\left\{1 + \left(\frac{\lambda_<}{\lambda_>}\right)^N\right\}$$

$$f = -\frac{kT}{N}\ln Z = -kT\ln\lambda_> - \frac{kT}{N}\ln\left\{1 + \left(\frac{\lambda_<}{\lambda_>}\right)^N\right\}$$

$$= -kT\ln\lambda_> - \frac{kT}{N}\left(\frac{\lambda_<}{\lambda_>}\right)^N$$

finite-size effect.

Diagonalization: $x \equiv e^J, \quad y \equiv e^H$

$$0 = (xy-\lambda)(xy^{-1}-\lambda) - x^{-2} = \lambda^2 - x(y+y^{-1})\lambda + (x^2 - x^{-2})$$

$$\lambda_{\gtrless} = \frac{1}{2}\left\{x(y+y^{-1}) \pm \left[x^2(y \cancel{+} - y^{-1})^2 \cancel{- 4x^2} + 4x^{-2}\right]^{1/2}\right\}$$

$N \to \infty$ Thermodynamic limit

$$f = -kT\ln\lambda_>$$

$\lambda_>$ is analytic for $J < \infty$. $(T > 0)$

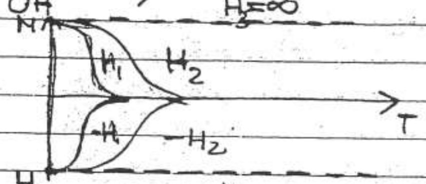
No phase transition.

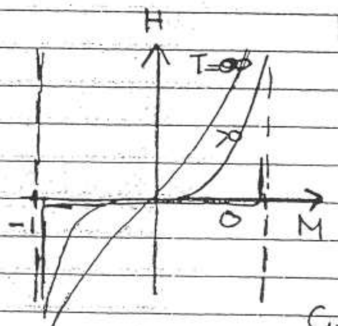
(E) Thermodynamic properties

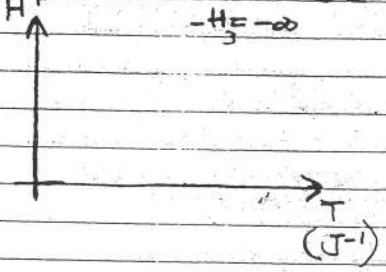
$$M = \langle s_i \rangle = \frac{1}{N}\frac{\partial}{\partial H} \ln Z = \frac{\partial}{\partial H} \ln \lambda_>$$

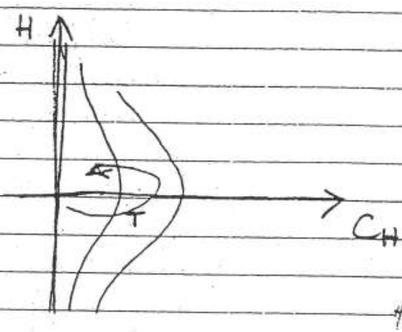
$$U = -\hat{J}\langle s_i s_j \rangle = -\frac{\hat{J}}{N}\frac{\partial}{\partial J} \ln Z = -\hat{J}\frac{\partial}{\partial J} \ln \lambda_>$$

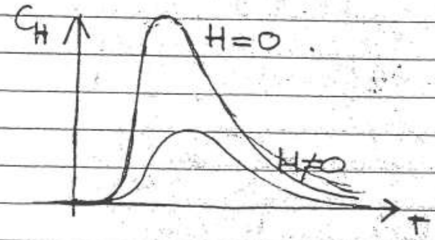
$$C_H = \left.\frac{\partial U}{\partial T}\right)_H$$











#7

Generalizations

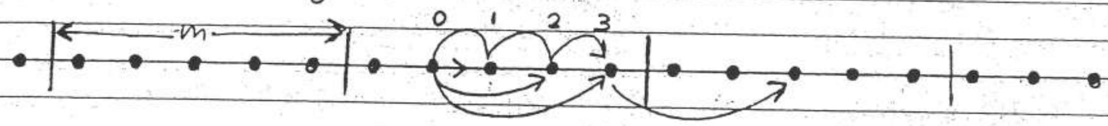
(A) Non-existence of finite-temperature phase transition.

General feature for $d=1$, for
finite number q of states per site (spin-½ Ising has $q=2$)
bounded Hamiltonian ($\frac{|E|}{N}$ finite as $N\to\infty$)
(finite-ranged interaction)

Why?

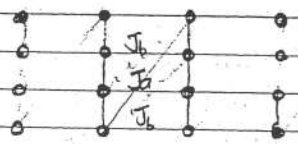
① Heuristic argument still works

② Consider range $m<\infty$.



$J_a S_0 S_1 + J_b S_0 S_2 + J_c S_0 S_3$ $\quad\quad H S_0$

$M S_0 S_1 S_2 S_3$ $\quad\quad L S_0 S_1 S_2$



Block-block interactions nn
Transfer matrix $q^m \times q^m$
finite-dimensional
All elements are finite ($T>0$)

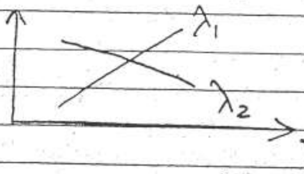
And, again, real symmetric:

$$Z = \sum_{n=1}^{q^m} (\lambda_n)^{N/m}, \quad f = \lim_{N\to\infty}\left[-\frac{kT}{N}\ln Z\right] = -\frac{kT}{m}\ln\lambda_{max}$$

<u>How might f develop non-analyticity?</u>

① λ_{max} non-analytic function of H, T.

② Eigenvalue crossing.



S.B. Frobenius, Pruss. Akad. Wiss., 471 (1908):

① Maximum eigenvalue of finite matrix ($q^m < \infty$) of positive elements ($T_{ij} > 0$) is non-degenerate
(rules out level crossing)

② λ_{max} analytic function of matrix elements, which in turn are $e^{-\beta\mathcal{H}(s_i s_j)}$ analytic fn of H, -

(B) Minimum Magnetization of d=1 Systems

Recall heuristic argument $\Delta\Psi = 2m\hat{J} - kT \ln N$

Breaks down as m (range) algebraically grows.

Rigorous results (Griffiths DG1):

① No phase transition ($M=0$ for $T>0$, $H=0$) if
$\sum_{r=1}^{\infty (N)} r|\hat{J}(r)| < \infty$ $(\log\log N)$, i.e. $\hat{J}(r) \sim \frac{1}{r^{2+\epsilon}}$ Ruelle

② Phase transition ($M \neq 0$ for some $T>0$, $H=0$) if
$\sum_{r=1}^{\infty} \frac{1}{r^3 \hat{J}(r)} < \infty$, $\log\log^r$, i.e. $\hat{J}(r) \sim \frac{1}{r^{2-\epsilon}}$ Dyson
$J(r) \geq 0$

Heuristic argument:

③ Borderline case [D.J. Thouless, PR B 187, 732 (1969)]
Σ heuristic argument.

Rigorous proof: M. Aizenman and C. Newman, Comm. Math. Phys. 107, 611 (1986)
(analogous percolation phenomenon).

M. Aizenman, J. Chase, L. Chase, and C. Newman, J. Stat. Phys. 50, 1 (1988)
(application to spin problem).

$$Z \simeq e^{-\beta(E_0 - kT \ln \omega_0)} \equiv e^{-\beta \Psi_0}$$

$$\Psi_0 = E_0 - TS_0$$
$$S_0 = k \ln \omega_0$$

Correspondingly $M_0 = \langle s_i \rangle_0 \quad > 0$ hypothesized.

* Reverse spins to the right of site l.

$$\Delta E_0 = \sum_{i \le l} \sum_{j > l} 2\hat{J}(i-j) \langle s_i s_j \rangle_0 \qquad \frac{\langle ss \rangle}{\langle s \rangle^2} = 1 + \frac{\langle ss \rangle - \langle s \rangle^2}{\langle s \rangle^2}$$

$$\underset{M_0 \quad M_0}{\langle s_i \rangle_0 \langle s_j \rangle_0}$$ Ignored correlations, MFA.

$$= 2M_0^2 \sum_{r=1}^{N} r\, J(r)$$

l can be in one of N places, $\Delta S_0 = k \log N$.

$$\Delta \Psi_0 = 2M_0^2 \sum_{r=1}^{N} r\, J(r) - kT \ln N$$

① $\sum_r^N r\, J(r)$ diverges slower than $\log N$: $\quad \Delta \Psi_0 \ll 0$.

Systems breaks up into domains, $M_0 = 0$ (cf. Ruelle)

Argued that $M_0 > 0$ hypothesis not self-consistent.

② $\sum_r^N r\, J(r)$ diverges faster than $\log N$: $\quad \Delta \Psi_0 \gg 0$.

Single domain stable with $M_0 > 0$. (cf. Dyson)

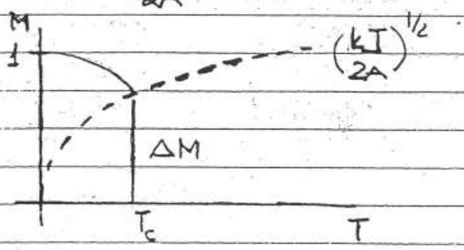
Hypothesis self-consistent.

③ Interesting borderline case:

$$\sum_r^N r\hat{J} \underset{N\to\infty}{\simeq} A \ln N \quad \text{(external specification)}$$

$$\Delta\Psi_0 = (2M_0^2A - kT)\ln N = \pm\infty.$$

$$M_0 > \pm\left(\frac{kT}{2A}\right)^{1/2} \quad \text{or} \quad M_0 = 0.$$



minimum magnetization curve.

$\frac{\Delta M}{T_c^{1/2}} = \left(\frac{k}{2A}\right)^{1/2}$ universal in the sense that it depends only on long-range properties of Hamiltonian.

* Real life analog: Superfluidity in helium layers $\left(\frac{\Delta\rho_s}{T_c}\right)$

Predicted: Nelson + Kosterlitz, PRL 39, 1201 (1977),

Confirmed: Bishop + Reppy, PRL 40, 1727 (1978).

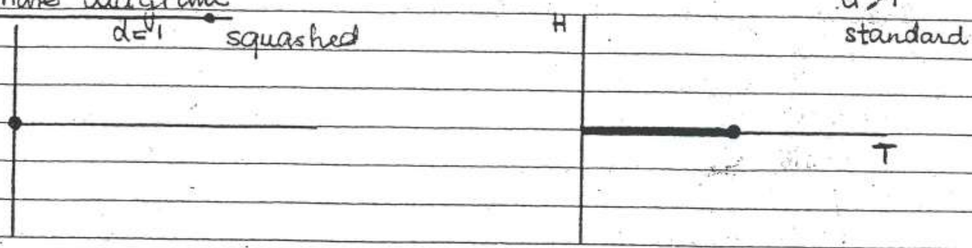
Back to the nn Ising Chain:

A Zero-Temperature Phase Transition

$$f = -kT\ln\lambda_> = -kT\ln\left\{e^{\hat{J}/kT}\cosh H + \left[e^{2\hat{J}/kT}\sinh^2 H + e^{-2\hat{J}/kT}\right]^{1/2}\right\}$$

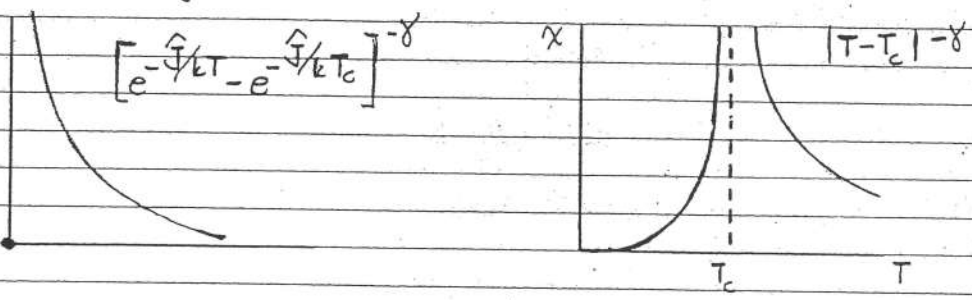
As $T\to 0$, $e^{\hat{J}/kT}$ gives singularity.

① Phase diagram



$d = 1+\epsilon$ expansion

Susceptibility $\quad \chi = \partial M / \partial \hat{H}$

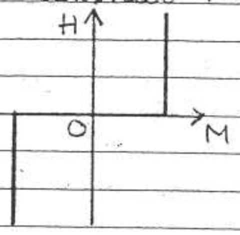


By differentiating $\ln \lambda_>$ twice wrt $H, \hat{H}$: $\quad \gamma = 2$

Note: must specify scaling field when $T_c = 0$.

② Frobenius?

Consider $T=0$: $\quad \lambda_\pm = e^J(\cosh H \pm \sinh H) = e^{J \pm H}$

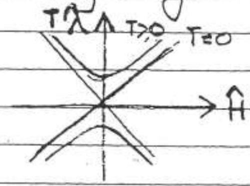


$H>0$: $\quad \ln \lambda_> = J+H \qquad M = \dfrac{\partial}{\partial H} \ln \lambda_> = +1$

$H<0$: $\quad \ln \lambda_< = J-H \qquad M \qquad = -1$

First-order transition by eigenvalue crossing non-analyticity

Frobenius does not hold, zero entries in $T_{ij} = \begin{bmatrix} e^{J+H} & 0 \\ 0 & e^{J-H} \end{bmatrix}$



② Correlation function by the transfer matrix method.

$$\langle s_1\rangle = \frac{1}{Z}\sum_{\{s\}} s_1\, e^{\frac{H}{2}s_1 + Js_1s_2 + \frac{H}{2}s_2} \cdots =$$

$$= \frac{\mathrm{Tr}\,\sigma_z T^N}{\mathrm{Tr}\,T^N} =$$ Pauli spin matrix.

$$= \frac{\mathrm{Tr}(S^{-1}\sigma_z S)\,\tilde{T}^N}{\mathrm{Tr}\,\tilde{T}^N}$$ $\tilde{T} = S^{-1}TS = \begin{bmatrix}\lambda_> & 0\\ 0 & \lambda_<\end{bmatrix}$

$S^{-1}\sigma^z S = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ easy to calculate

$$\langle s_1\rangle = \frac{a\lambda_>^N + d\lambda_<^N}{\lambda_>^N + \lambda_<^N} \underset{N\to\infty}{=} a(H,T) \underset{H=0}{=} 0.$$

$$\langle s_1 s_{1+r}\rangle = Z^{-1}\,\mathrm{Tr}\,(S^{-1}\sigma_z S)\,T'^{r}\,(S^{-1}\sigma_z S)\,T^{N-r} =$$

$$= Z^{-1}\,\mathrm{Tr}\begin{pmatrix} a\lambda_>^r & b\lambda_<^r\\ c\lambda_>^r & d\lambda_<^r\end{pmatrix}\begin{pmatrix} a\lambda_>^{N-r} & b\lambda_<^{N-r}\\ c\lambda_>^{N-r} & d\lambda_<^{N-r}\end{pmatrix} =$$

$$= \frac{a^2\lambda_>^N + d^2\lambda_<^N + bc\left(\lambda_<^r\lambda_>^{N-r} + \lambda_>^r\lambda_<^{N-r}\right)}{\lambda_>^N + \lambda_<^N} = a^2 + bc\left(\frac{\lambda_<}{\lambda_>}\right)^r$$

$$\Gamma(r) = \langle s_1 s_{1+r}\rangle - \langle s_1\rangle\langle s_r\rangle = bc\left(\frac{\lambda_<}{\lambda_>}\right)^r$$

$$= bc\, e^{-r\ln \lambda_>/\lambda_<}$$

No inverse power in this case.

$$\xi = \frac{1}{\ln \lambda_>/\lambda_<} \to \infty \quad \text{as } \frac{\lambda_<}{\lambda_>} \to 1$$

Infinite ranged correlations $\leftrightarrow$ eigenvalue degeneracy

$H=0$: $\quad \lambda_{\gtrless} = \begin{matrix}\cosh J \\ \sinh J\end{matrix}$

$$\xi = [\ln \coth J]^{-1} \simeq [\ln(1+2e^{-2J})]^{-1} \simeq \frac{1}{2}\left(e^{-\hat{J}/kT}\right)^{-2}$$

cf $|T-T_c|^{-\nu}$ $\qquad \nu = 2$

Two-Dimensional Ising Model

A. Peierls argument for the existence of a phase transition

R Peierls, Proc. Cambr. Philos. soc. 32, 477 (1936)
Griffiths, DG1

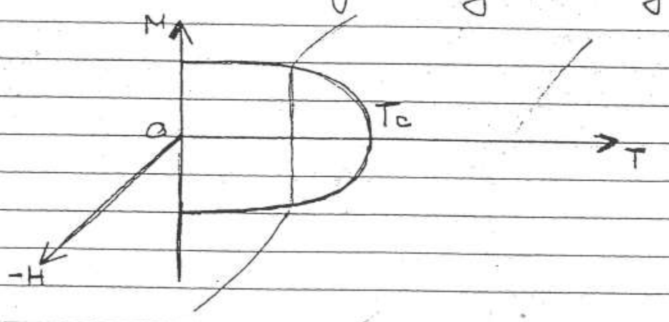
(1) $-\beta\mathcal{H} = J\sum_{\langle ij\rangle} s_i s_j + H\sum_i s_i$

$H=0$: $\quad -\beta\mathcal{H} = \sum_{\langle ij\rangle} s_i s_j \qquad$ up-down symmetry

Consider $M(H=0, T)$.

The existence of non-zero $M(H=0, T<T_c)$

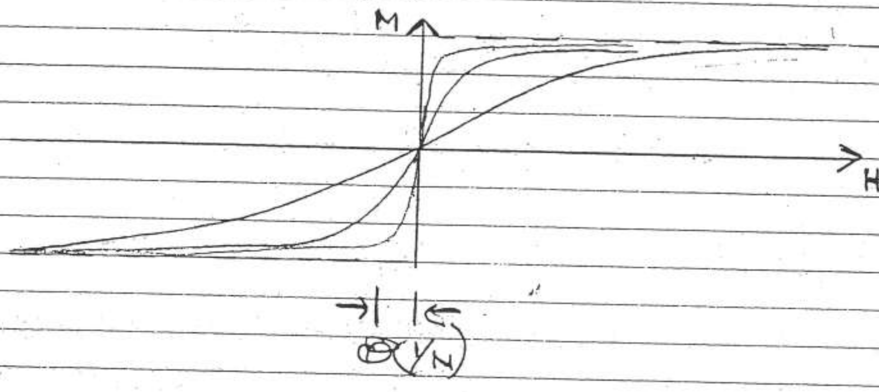
spontaneous symmetry breaking



Consider any finite system:

$$M(N) = \frac{\sum_{\{s\}} \left(\frac{1}{N}\sum_i s_i\right) e^{J\Sigma s_i s_j}}{\sum_{\{s\}} e^{J\Sigma s_i s_j}} \overset{s'=-s}{=} \frac{\sum_{\{s'\}} \left(\frac{1}{N}\sum_i [-s_i']\right) e^{J\Sigma s_i' s_j'}}{\sum_{\{s'\}} e^{J\Sigma s_i' s_j'}} = -M(N)$$

$\therefore \; M(N) = 0$ Fixed $T < T_c$

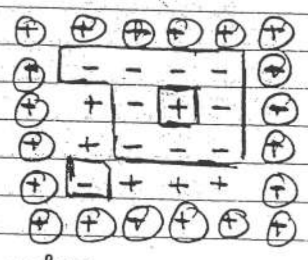


$\lim\limits_{N\to\infty} \lim\limits_{H\to 0} M(N,H,T) = 0$

$\lim\limits_{H\to 0} \lim\limits_{N\to\infty} M(N,H,T) = M_0(T)$

} non-interchangeable limits

Proof.



islands of outer boundary b. "outer" means can be reached from boundary via ⊕ spins.

"operator" $P_b(I) = 1$ if I occurs, zero otherwise.

Maximum number of $-$ spin in I is $\left(\frac{b}{4}\right)^2$:

$\tilde{N}_- \leq \left(\frac{b}{2} - x\right) x$

$x)_{max} = \frac{b}{4}$

$N_- \leq \tilde{N}_- \leq \left(\frac{b}{4}\right)^2$

For given global configuration:

$$\langle N_- \rangle \leq \sum_{b=4,6,8,\dots} \left(\frac{b}{4}\right)^2 \sum_{I=1}^{n_b} \langle P_b(I) \rangle$$

only configurations containing I.

$$\langle P_b(I) \rangle = \frac{\sum e^{-\beta\mathcal{H}} P_b(I)}{\sum e^{-\beta\mathcal{H}}} = \frac{\sum' e^{-\beta\mathcal{H}}}{\sum e^{-\beta\mathcal{H}}} \leq \frac{\sum' e^{-\beta\mathcal{H}}}{\sum'' e^{-\beta\mathcal{H}}} = e^{-2bJ}$$

$\sum''$: only configurations in numerator but with spins inside outer boundary flipped.

n_b: # translations $\leq N$
shapes $\leq 4\,3^{b-2}$

$$\langle N_- \rangle \leq \sum_b \left(\frac{b}{4}\right)^2 \left(N4\,3^{b-2}\right) e^{-2bJ}$$

$$= \frac{N}{36} \sum_b b^2 \left(3e^{-2\beta\hat{J}}\right)^b < \infty \text{ for } y = 2\beta\hat{J} - \ln 3 > 0$$

$$\frac{\langle N_- \rangle}{N} \leq \frac{1}{36} \sum_b b^2 \left(3e^{-2\beta\hat{J}}\right)^b < \frac{1}{2} - \frac{|\alpha|}{2}$$

for small enough $T < T_0$

$$S(y) = \sum_{b=0}^{\infty} \left(e^{-y}\right)^b = \frac{1}{1-e^{-y}}$$

$$S'(y) = -\frac{e^{-y}}{(1-e^{-y})^2} = -\left(e^{y/2} - e^{-y/2}\right)^{-2}$$

$$S''(y) = \frac{+2\left(e^{y/2}+e^{-y/2}\right)}{\left(e^{y/2}-e^{-y/2}\right)^3}\frac{1}{2} \to 0^+ \text{ as } y \to \infty.$$

$$\sum_b b^2 \left(3e^{-2\beta\hat{J}}\right)^b < S''(y) \xrightarrow{y\to\infty} 0^+$$

$$M(N, H=0, T<T_0) = \frac{\langle N_+ \rangle - \langle N_- \rangle}{N} = 1 - 2\frac{\langle N_- \rangle}{N} > |\alpha|$$

<u>Problem</u>: For given T, find maximum possible lower bound on M(T). Graph.

① Shows that fluctuation are suppressed at low T.

② Why boundary condition?

Operationally, necessary for $\langle P_b(I) \rangle$ bound.

Conceptually, equivalent to a small H going to zero with $N \to \infty$.

③ Proved that $M > 0$ for $N \to \infty$. Could repeat for $M < 0$. Spontaneous symmetry breaking.

Actually, proved the stronger condition that the − spins do not percolate, which in general is sufficient but necessary for M. Interesting comparison of T_{Perc} in $d = 2$ and 3, based on this.

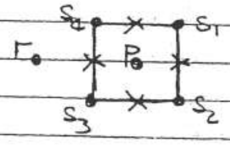
④ Generalized to hypercubic lattice $d \geq 2$.

See why it does not work in $d = 1$.

Some quantum (small transverse), more decorated classical systems. (antiferromagnet higher spin, further neighbors)

B. Dual transformation, self-duality Syozi DG 1

$$Z = \sum_{\{S\}} e^{J \sum_{\langle ij \rangle} S_i S_j}$$



$t_{12} = S_1 S_2 = \pm 1$

N variables S_i

2N variables t_{ij}

N conditions.

$t_{12} t_{23} t_{34} t_{41} = S_1 S_2 S_2 S_3 S_3 S_4 S_4 S_1 = 1$

$$Z = \sum_{\{t\}} e^{J\sum_{\langle ij\rangle} t_{ij}} \left\{ \prod_{P} \left(1 + t_{12}t_{23}t_{34}t_{41}\right)_P / 2 \right\} =$$

$$= \frac{1}{2^N} \sum_{\{t\}} e^{J\sum_{\langle ij\rangle} t_{ij}} \left\{ \prod_{P} \sum_{n_p=0,1} \left(t_{12}t_{23}t_{34}t_{41}\right)^{n_p} \right\} =$$

$$= \frac{1}{2^N} \sum_{\{n_p=0,1\}} \sum_{\{t\}} \left\{ \prod_{\langle ij\rangle} e^{Jt_{ij}} \right\} \left\{ \prod_{P} \left(t_{12}t_{23}t_{34}t_{41}\right)^{n_p} \right\} =$$

$$= \frac{1}{2^N} \sum_{\{n_p\}} \sum_{\{t\}} \prod_{\langle ij\rangle} e^{Jt_{ij}}\, t_{ij}^{\,n_p+n_r} =$$

$$= \frac{1}{2^N} \sum_{\{n_p\}} \prod_{\langle ij\rangle} \sum_{t_{ij}=\pm 1} e^{Jt_{ij}}\, t_{ij}^{\,n_p+n_r} =$$

$$= \frac{1}{2^N} \sum_{\{n_p\}} \prod_{\langle ij\rangle} \left[e^{J} + (-1)^{n_p+n_r} e^{-J} \right] =$$

$\sigma_p = 2n_p - 1$

$n_p = \frac{1}{2}(\sigma_p + 1)$

$$= \frac{1}{2^N} \sum_{\{n_p\}} \prod_{\langle pr\rangle} \left[e^{J} - (-1)^{(\sigma_p+\sigma_r)/2} e^{-J} \right]$$

$$e^{J} - (-1)^{(\sigma_p+\sigma_r)/2} e^{-J} = e^{\tilde{J}\sigma_p\sigma_r + \tilde{G}}$$

$(+1,+1)$: $\quad e^{J} + e^{-J} = e^{\tilde{J}+\tilde{G}}$

$(+1,-1)$: $\quad e^{J} - e^{-J} = e^{-\tilde{J}+G}$

$$e^{2\tilde{J}} = \coth J \qquad\qquad e^{2\tilde{G}} = 2\sinh 2J$$

$$Z(J) = \frac{1}{2^N} \sum_{\{\sigma_p\}} \prod_{\langle pr\rangle} (2\sinh 2J)^{1/2} e^{\tilde{J}\sigma_p\sigma_r} =$$

$$= (\sinh 2J)^N \sum_{\{\sigma_p\}} e^{\sum_{\langle pr\rangle} \tilde{J}\sigma_p\sigma_r} = (\sinh 2J)^N\, Z(\tilde{J})$$

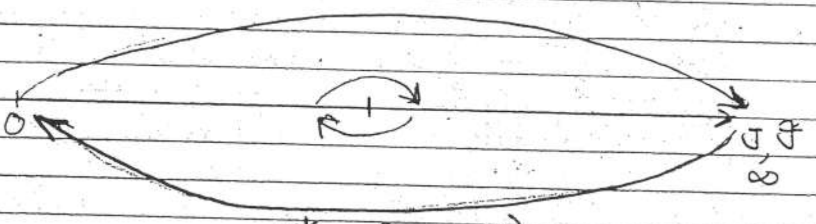
$$-\beta f(J) = \frac{1}{N} \ln Z(J) = \ln(\sinh 2J) + \frac{1}{N} \ln Z(\tilde{J}) =$$

$$= \ln(\sinh 2J) - \beta f(\tilde{J}).$$

$$\begin{pmatrix} J \to \infty & -\beta f(J) \propto J \\ J = 0 & -\beta f(J) = \ln 2 \end{pmatrix}$$

$$e^{-2\tilde{J}} = \tanh J$$

$$\tilde{J} = 0 \leftrightarrow J = \infty$$
$$\tilde{J} = \infty \leftrightarrow J = 0$$



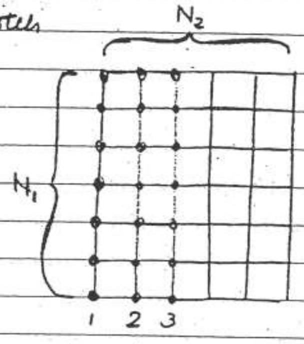
$$J^* = \frac{\ln(1+\sqrt{2})}{2}$$

#5

C. Exact solution

Onsager, PR 65, 117 (1944)
B.M. McCoy and T.T. Wu, The Two-Dimensional Ising Model Harvard UP
Huang, Stat Mech Appendix
Temperley in DG 1 *M.E. Fisher, Rep. Prog. Phys. 30, 615 (1967).

Sketch



Set up transfer matrix between columns, e.g. $\langle col\,1|T|col\,2\rangle$ $\quad 2^{N_1}\times 2^{N_1}$

$$Z = \sum_{n=1}^{2^{N_1}} \left[\lambda_n(N_1)\right]^{N_2}$$

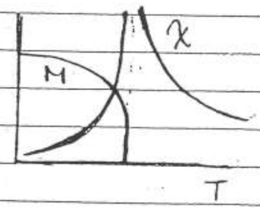
$$f = \lim_{N_1\to\infty} \frac{kT}{N_1} \ln \lambda_{max}(N_1)$$

Solved only at $H=0$. But Yang-Lee theorem: all phase transitions occur at $H=0$ for ferro system.

$$C_H = -C_0 \ln\left|\frac{T-T_c}{T_c}\right| + C_1 + \ldots \qquad \alpha = 0 \text{ log singularity}$$

$$M = B\left(\frac{T_c - T}{T_c}\right)^{1/8} + \ldots \qquad \beta = 1/8$$

$$\chi = A^{(\pm)}\left|\frac{T-T_c}{T_c}\right|^{-7/4} + \ldots \qquad \gamma = \gamma' = 7/4$$



$$\Gamma(r \gg a, T_c) = \frac{D}{r^{1/4}} \quad \text{usually } r^{d-2+\eta}$$

$\eta = 1/4$

$$\Gamma(r\gg a, T \gtrsim T_c) = E \frac{e^{-r/\xi(t)}}{?(t^{1/4}) r^{1/2}} \qquad \xi = \xi_0 t^{-1} \qquad \nu = \nu' = 1$$

$$t = |T-T_c|/T_c$$

$$T \lesssim T_c \qquad E' \frac{e^{-2r/\xi(t)}}{?(t^{7/4}) r^2}$$

Derivations along axis and diagonal directions give identical results, strongly suggesting cyl. sym.

usually $r^{\frac{d-1}{2}}$
Here, violation of classical theory away from T_c!

$$\Gamma(r \gg a, \overset{t \sim 0}{T \sim T_c}) \simeq t^{1/4} f_{\pm}\left(\frac{\xi}{r}\right) e^{-r/\xi} \quad , \quad \xi = A_{\pm} t^{-1}$$

$$f_{+}(x \to 0) \sim x^{1/2} \qquad \Gamma(r \gg a, T \gtrsim T_c) \sim \frac{e^{-r/\xi}}{t^{1/4} r^{1/2}} \qquad \text{for } rt \to \infty$$

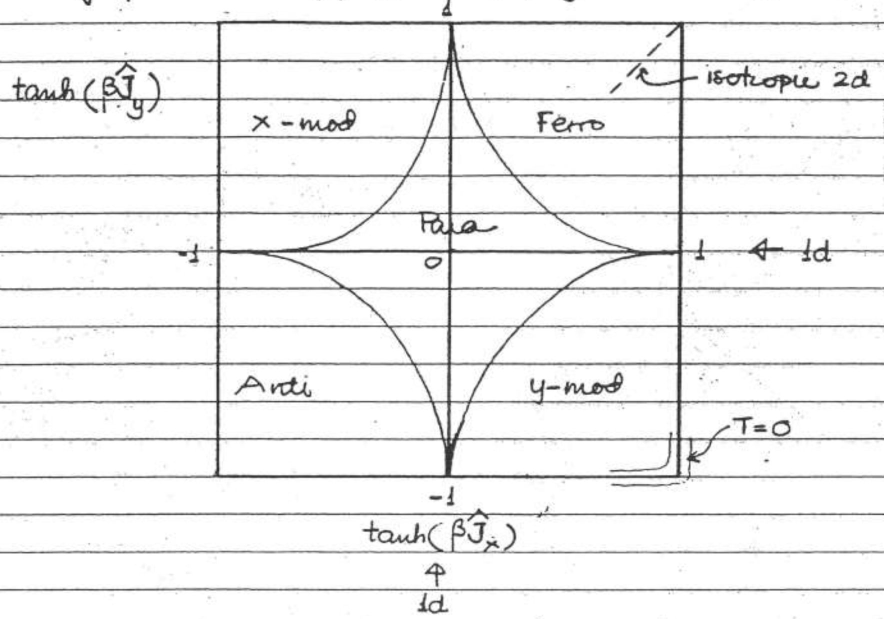
$$f_{-}(x \to 0) \sim x^{2} \qquad \Gamma(r \gg a, T \lesssim T_c) \sim \frac{e^{-r/\xi}}{t^{7/4} r^{2}} \qquad \text{for } rt \to \infty$$

$$f_{\pm}(x \to \infty) \sim x^{1/4} \qquad \Gamma(r \gg a, T = T_c) \sim \frac{1}{r^{1/4}} \qquad \text{for } \begin{matrix} t = 0 \\ r \to \infty . \end{matrix}$$

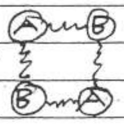
Global Phase Diagrams

A. Anisotropic Ising Model (d=2)

Duality gives $\sinh(2\beta\hat{J}_x)\sinh(2\beta\hat{J}_y) = 1$ as criticality condition.

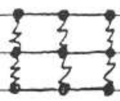


① Antiferromagnetism $\mathcal{H} = |\hat{J}| \sum_{\langle ij \rangle} s_i s_j$

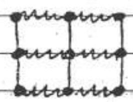


sublattices $\quad s_i'^A \equiv -s_i^A \quad s_i'^B \equiv s_i^B, \quad \mathcal{H} = -|\hat{J}| \sum_{\langle ij \rangle} s_i' s_j'$

② Also for

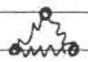


y-modulated



x-modulated

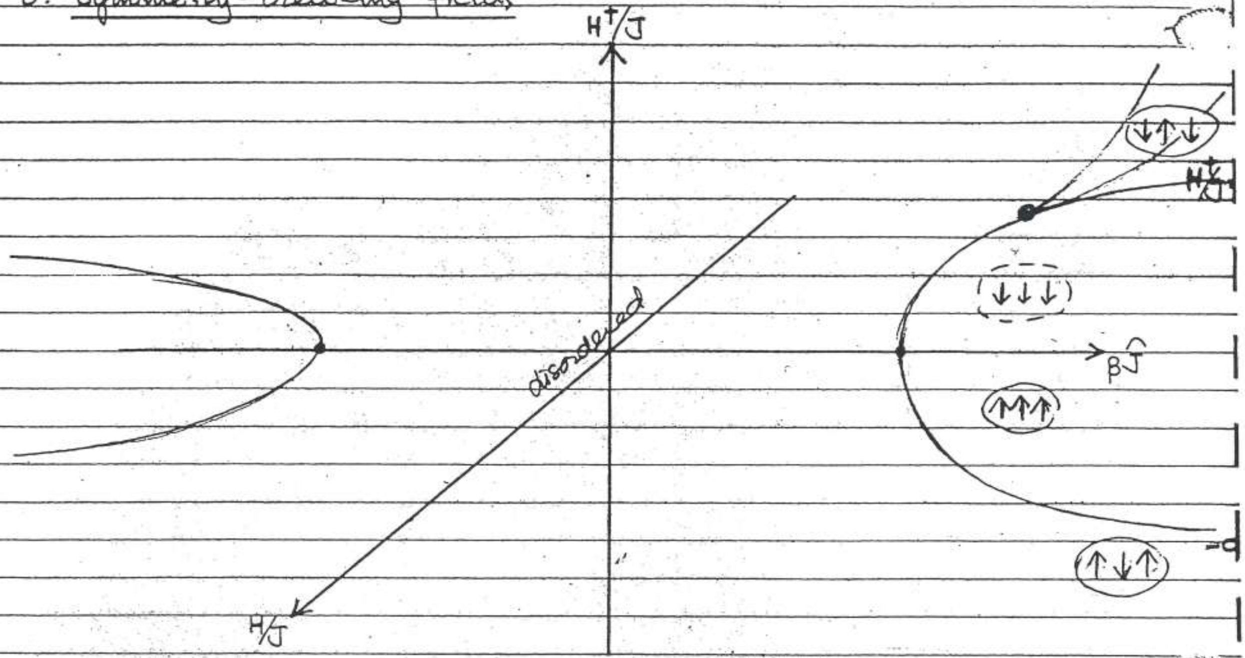
③ Can also do for hexagonal lattice.
Not for triangular antiferromagnet
Close packing for d=2 → Adsorption problems



Frustration.
Competing interactions, cannot be removed.

Question: Sketch same phase diagram for anisotropic d=3 Ising.
Note T_c (isotropic) as function of d.

B. Symmetry-breaking fields



$$-\beta\mathcal{H} = \beta\hat{J}\sum_{\langle ij\rangle_1} s_i s_j \quad + \beta\hat{H}\sum_i s_i \quad + \beta\hat{H}^\dagger \sum_i (-1)^{l_1+l_2+\dots+l_d} s_i \quad + \beta\hat{K}\sum_{\langle ij\rangle_2} s_i s_j$$

$$i \leftrightarrow l_1, l_2, \dots, l_d$$

$$-\beta\mathcal{H}(\uparrow\uparrow\uparrow) = N\beta\left(\tfrac{q}{2}\hat{J} + \hat{H}\right)$$

$$-\beta\mathcal{H}(\downarrow\downarrow\downarrow) = N\beta\left(\tfrac{q}{2}\hat{J} - \hat{H}\right)$$

$$-\beta\mathcal{H}(\downarrow\uparrow\downarrow) = N\beta\left(-\tfrac{q}{2}\hat{J} + \hat{H}^\dagger\right)$$

$$-\beta\mathcal{H}(\uparrow\downarrow\uparrow) = N\beta\left(-\tfrac{q}{2}\hat{J} - \hat{H}^\dagger\right)$$

Is the zero-temperature transition first-order? $H = 0$.

$$\begin{matrix} + & + & + \\ + & + & + \\ + & + & + \end{matrix} \quad \rightarrow \quad \begin{matrix} + & + & + \\ + & - & + \\ + & + & + \end{matrix} \qquad -\beta\Delta E = -2qJ + 2H^\dagger - 2q_2 K$$

$$\text{at } H^\dagger/J = q, \quad = -2q_2 K$$

$$\langle \Theta \rangle = \sum_{\{s\}} e^{-\beta\mathcal{H}\{s\}}\Theta\{s\} \Big/ \sum_{\{s\}} e^{-\beta\mathcal{H}\{s\}} =$$

$$= \Theta\{\text{ground state}\} + \left[C e^{-\beta\Delta E} \;\dots \right]$$

$K=0$ no gap infinite degeneracy difficult problem
second-order transition

$K>0$ gap just ground-state degeneracy
three phase coexistence – easy
first-order transition
wings and tricritical point

Question: What are the three coexisting phases in Problem 3 of Set 2?

Other quadrants of global phase diagram:

$s_A \to -s_A$

$s_B \to s_B$

$J \to -J$

$K \to K$

$H \to H^{\dagger}$

$H^{\dagger} \to H$

B. Quantum Statistical Mechanics

Entropy: $S = k \ln \Omega$ — k: Boltzmann constant; Ω: number of microscopic states corresponding to macroscopic specification.

- Completely random ensemble: $w_i = 1/N$ for all $|a_i\rangle$ of any basis set:

$$\rho_R = \frac{1}{N}\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

- Pure quantum-state (polarized) ensemble: $w_i = 1$ for one $|a_i\rangle$, 0 for all others, using basis set that includes.

$$\rho_P = \begin{bmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{bmatrix}$$

can be evaluated in any basis.

Consider $\boxed{\sigma = -\mathrm{Tr}\,\rho \ln \rho} = \ln N$ for ρ_R,
$= 0$ for ρ_P.

In fact, $\boxed{S = k\sigma}$, measures disorder.

Thermal equilibrium: S is maximized subject to fixed $[\mathcal{H}] = U$
(result of a steepest descent integration, canonical ensemble)

$\frac{d\rho}{dt} = 0$, $[\rho, \mathcal{H}] = 0$, ρ and $\mathcal{H}$ can be simultaneously diagonalized. Use energy eigenstates as basis.

$$\delta\sigma = -\sum_i (\ln \rho_{ii} + 1)\,\delta\rho_{ii}$$

$\mathrm{Tr}\,\rho\mathcal{H} = U$, $\quad \delta[\mathcal{H}] = \sum_i \delta\rho_{ii} E_i = 0$.

$\mathrm{Tr}\,\rho = 1$, $\quad \delta[1] = \sum_i \delta\rho_{ii} = 0$.

With Lagrange multipliers, $\sum_i \delta\rho_{ii}\left[(\ln\rho_{ii} + 1) + \beta E_i + \gamma\right] = 0$.

$$\rho_{ii} = e^{-\beta E_i - \gamma - 1} \overset{\mathrm{Tr}\,\rho = 1}{=} \frac{e^{-\beta E_i}}{\sum_j e^{-\beta E_j}} = \frac{e^{-\beta E_i}}{Z}$$

Z ← partition function.

$\frac{1}{kT} \equiv \beta$. For $\beta \to 0$ (infinite temperature), the completely random case is recovered.

Z, the partition function, is also a generating function: $U = -\frac{\partial}{\partial\beta}\ln Z$.

Example: Two $s = 1/2$ spins coupled by an exchange constant J: $\mathcal{H} = -J\vec{S}_1 \cdot \vec{S}_2 =$

basis: $|+\rangle|+\rangle = |++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$

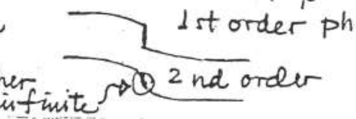
$$= -\frac{\hbar^2 J}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{\hbar^2 J}{4}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \frac{\hbar^2 J}{4}\begin{bmatrix} 0 & 0 & 0 & -i(-i) \\ 0 & 0 & -i \cdot i & 0 \\ 0 & i(-i) & 0 & 0 \\ i \cdot i & 0 & 0 & 0 \end{bmatrix} = -\frac{\hbar^2 J}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Eigenstates:
$|++\rangle$, $E_1 = -\hbar^2 J/4$,
$|--\rangle$, $E_2 = -\hbar^2 J/4$,
$\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$, $E_3 = -\hbar^2 J/4$,
$\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$, $E_4 = 3\hbar^2 J/4$.

$$Z = 3e^{\beta\hbar^2 J/4} + e^{-\beta 3\hbar^2 J/4}.$$

$$U = -\frac{\hbar^2 J}{4}\left(3e^{\beta\hbar^2 J/4} - 3e^{-\beta 3\hbar^2 J/4}\right)/Z.$$

spins mutually aligned.



Infinitely many coupled spins, $d > 1$:

1st order phase transition

2nd order

slope or higher derivative infinite

Aside: Recall propagator:

$$\langle \vec{x},t|\vec{x}',t'\rangle \equiv K(\vec{x},t;\vec{x}',t') = \sum_a \langle \vec{x}|a\rangle e^{-iE_a(t-t')/\hbar}\langle a|\vec{x}'\rangle .$$

$$\left.\int d\vec{x}\, K(\vec{x},t;\vec{x},0)\right]_{t=-i\beta\hbar} = Z(\beta).$$

For spin systems, $K = K(S_z,t;S_z',t') = \langle S_z,t|S_z',t'\rangle$

Can be generalized to many particles.

① "Easy" quantum statistical mechanics: $\mathcal{H}$ obviously diagonalizable.

e.g., $\mathcal{H} = \sum_{ij} J_{ij} S_i^z S_j^z$.

Work in diagonal basis. Becomes a classical problem.

e.g., $d=1$: $\mathcal{H} = -J\sum S_i^z S_{i+1}^z$

$$Z = \sum_{\{S_i^z=\pm\frac{\hbar}{2}\}} e^{\beta J\sum_i S_i^z S_{i+1}^z} = \sum_{\{S_i^z=\pm\frac{\hbar}{2}\}} \prod_i e^{\beta J \overbrace{S_i^z S_{i+1}^z}^{\equiv t_i}} = \left(2\cosh\frac{\beta J\hbar^2}{4}\right)^N .$$

② Quantum effects occur when the Hamiltonian is composed of non-commuting term.

e.g., $\mathcal{H} = -J\sum_i S_i^z S_{i+1}^z - \Gamma\sum_i S_i^x$. This $\mathcal{H}$ is not obviously diagonalizable. Procedure:

$e^{-\beta\mathcal{H}} = (e^{-\beta\mathcal{H}/n})^n$

$e^{-\beta\mathcal{H}/n} = e^{-\beta\mathcal{H}_1/n-\beta\mathcal{H}_2/n} = e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n}(1+$ terms involving $[\mathcal{H}_1,\mathcal{H}_2]$, $[\mathcal{H}_1,[\mathcal{H}_1,\mathcal{H}_2]]$, $[\mathcal{H}_2,[\mathcal{H}_1,\mathcal{H}_2]]$, ...$)$

e.g., term of order m_1, m_2 on each side $\displaystyle \frac{1}{(m_1+m_2)!}\sum_{\text{all permutations}}^{\frac{(m_1+m_2)!}{m_1!\,m_2!}} \mathcal{H}_1^{m_1}\mathcal{H}_2^{m_2} \Leftrightarrow \frac{\mathcal{H}_1^{m_1}}{m_1!}\frac{\mathcal{H}_2^{m_2}}{m_2!}$

$= e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n}\left(1+\Theta(\tfrac{1}{n})^2 \text{ and higher}\right)$

Take limit $n\to\infty$: $(e^{-\beta\mathcal{H}_1/n-\beta\mathcal{H}_2/n})^n = (e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n})^n$ Trotter formula

$$Z = \sum_{b_i}\langle b_i|e^{-\beta(\mathcal{H}_1+\mathcal{H}_2)}|b_i\rangle = \sum_{b_i}\langle b_i|e^{-\frac{\beta\mathcal{H}_1}{n}}e^{-\frac{\beta\mathcal{H}_2}{n}}e^{-\frac{\beta\mathcal{H}_1}{n}}e^{-\frac{\beta\mathcal{H}_2}{n}}\ldots|b_i\rangle =$$

insert complete set of states

Use eigenstates of $\mathcal{H}_1$

$$= \sum_{b_i^{(1)},\ldots,b_i^{(n)}} \underbrace{\langle b_i^{(1)}|e^{-\frac{\beta\mathcal{H}_1}{n}}|b_i^{(1)}\rangle\langle b_i^{(1)}|e^{-\frac{\beta\mathcal{H}_2}{n}}|b_i^{(2)}\rangle}_{\text{first replica}}\underbrace{\langle b_i^{(2)}|e^{-\frac{\beta\mathcal{H}_1}{n}}|b_i^{(2)}\rangle\ldots}_{\text{second replica}}\ldots\langle b_i^{(n)}|e^{-\frac{\beta\mathcal{H}_2}{n}}|b_i^{(1)}\rangle .$$

Each $\mathcal{H}_1$ operates in a given replica. Each $\mathcal{H}_2$ operates between adjoining replicas.

Define $|b_i^{(1)}\rangle|b_i^{(2)}\rangle\ldots|b_i^{(n)}\rangle \equiv |B_i\rangle$, basis of $d+1$ dimensional system.

$\langle b_i^{(m)}|e^{-\frac{\beta\mathcal{H}_1}{n}}|b_i^{(m)}\rangle \overset{\checkmark}{\equiv} \langle B_i|e^{-\frac{\beta\mathcal{H}_1^{(m)}}{n}}|B_i\rangle$, $\langle b_i^{(m)}|e^{-\frac{\beta\mathcal{H}_2}{n}}|b_i^{(m+1)}\rangle \overset{?}{\equiv} \langle B_i|e^{-\frac{\beta\mathcal{H}_2^{(m,m+1)}}{n}}|B_i\rangle$.

Aside:

Similar to path integral summation of propagator:

$\langle\vec{x}_n,t|\vec{x}_1,t\rangle = \int d\vec{x}_{N-1}\ldots d\vec{x}_2\langle\vec{x}_N t_N|\vec{x}_{N-1}t_{N-1}\rangle\langle\vec{x}_{N-1}t_{N-1}|\vec{x}_{N-2}t_{N-2}\rangle\ldots\langle\vec{x}_2 t_2|\vec{x}_1 t_1\rangle$.

$\langle\vec{x}_2|e^{-\frac{i\mathcal{H}(t_2-t_1)}{\hbar}}|\vec{x}_1\rangle = \langle\vec{x}_2|e^{\frac{-ip^2(t_2-t_1)}{2m\hbar}}|\vec{x}_1\rangle\langle\vec{x}_1|e^{\frac{-iV(\vec{x})(t_2-t_1)}{\hbar}}|\vec{x}$

When is this useful? When $\mathcal{H}_1$ and $\mathcal{H}_2$ are each composed of mutually commuting (i.e., simultaneously diagonalizable) local terms. Example above:

$|b_i^{(m)}\rangle$ = kets of spins in mth replica. Use $|\pm z\rangle$ basis for each spin.

$= |S_1^z,S_2^z,\ldots,S_N^z\rangle^{(m)} = |\{S_j^z\}\rangle^{(m)} = |S_1^z\rangle^{(m)}|S_2^z\rangle^{(m)}\ldots|S_N^z\rangle^{(m)}$, outer product.

$${}^{(m)}\langle\{S_j^z\}|e^{-\frac{\beta\mathcal{H}_1}{n}}|\{S_j^z\}\rangle^{(m)} = e^{\frac{\beta J}{n}\sum_i^{(m)} S_i^z S_{i+1}^z} \equiv e^{\frac{\beta J}{n}\sum_i S_{i,m}^z S_{i+1,m}^z} .$$

$${}^{(m)}\langle\{S_j^z\}|e^{-\frac{\beta\mathcal{H}_2}{n}}|\{S_j^z\}\rangle^{(m+1)} = \prod_j \langle S_{j,m}^z|e^{\frac{\beta\Gamma}{n}\frac{\hbar}{2}\sigma_j^x}|S_{j,m+1}^z\rangle =$$

$$= \prod_j \langle S_{j,m}^z|\left(\cosh\frac{\beta\Gamma}{n}\frac{\hbar}{2} + \sigma_j^x \sinh\frac{\beta\Gamma}{n}\frac{\hbar}{2}\right)|S_{j,m+1}^z\rangle =$$

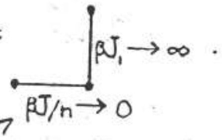
$$= \prod_j \begin{bmatrix} \cosh\frac{\beta\Gamma}{n}\frac{\hbar}{2} & \sinh\frac{\beta\Gamma}{n}\frac{\hbar}{2} \\ \sinh\frac{\beta\Gamma}{n}\frac{\hbar}{2} & \cosh\frac{\beta\Gamma}{n}\frac{\hbar}{2} \end{bmatrix} = \prod_j e^{\beta J_0 + \beta J_1 S_{j,m}^z S_{j,m+1}^z} .$$

$$e^{\beta J_0 + \beta J_1 \hbar^2/4} = \cosh\frac{\beta\Gamma}{n}\frac{\hbar}{2}$$
$$e^{\beta J_0 - \beta J_1 \hbar^2/4} = \sinh\frac{\beta\Gamma}{n}\frac{\hbar}{2}$$
$$e^{-\beta J_1 \hbar^2/2} = \tanh\frac{\beta\Gamma}{n}\frac{\hbar}{2} \longrightarrow 0$$
$$\beta J_1 = -\frac{2}{\hbar^2}\ln\tanh\frac{\beta\Gamma}{n}\frac{\hbar}{2} \longrightarrow \infty$$

βJ_0 just an additive constant, does not couple to the degrees of freedom.

Classical $d=2$ anisotropic Ising magnet is obtained:

bonds:



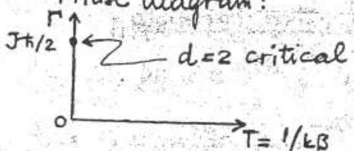
Criticality condition: $\sinh 2\frac{\beta_c J}{n}\frac{\hbar^2}{4} \sinh 2\beta_c J_1 \frac{\hbar^2}{4} = 1$.

Anisotropic limit: $\cancel{2}\frac{\beta_c J}{n}\frac{\hbar^2}{4}\left(e^{2\beta_c J_1 \frac{\hbar^4}{4}}\right)/\cancel{2} = \frac{\beta_c J}{\cancel{n}}\frac{\hbar^2}{4}\left(\tanh\frac{\beta_c\Gamma}{\cancel{n}}\frac{\hbar}{2}\right)^{-1} = \frac{J(\hbar/2)}{\Gamma} = 1$.

For the phase transition to occur, $\frac{\beta J}{n}$ has to be infinitesimal, but non-zero. Otherwise, system decomposes to decoupled strands.

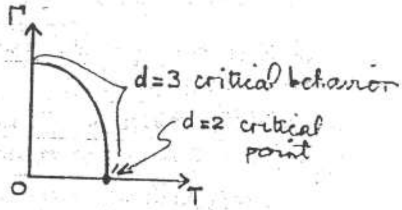
Thus, $n \to \infty \gg \beta J \to \infty \gg 1$.

Phase diagram:



$d=1$ quantum Ising model in a transverse field.

Similarly,



$d=2$ quantum Ising model in a transverse field.

M. Suzuki, Progr. Theor. Phys. 56, 1454 (1976).

$\boxed{\text{d dimensional quantum statistical mechanics} \to \to \text{d+1 dimensional classical statistical mechanics}}$.

Other applications: • Quantum Heisenberg magnet: $\mathcal{H} = -J\sum_i \vec{S}_i \cdot \vec{S}_{i+1}$.
$\mathcal{H}_1, \mathcal{H}_2$ taken as every other bond. Also in higher d.
• Electrons hopping in a lattice.

Gibbs Inequality: A Variational Principle Underlying Approximate Treatmen

$$\log Z = \log \mathrm{Tr}\, e^{-\beta\mathcal{H}} \equiv -\beta F .$$

$$\rho = \frac{e^{-\beta\mathcal{H}}}{Z} = e^{\beta(F-\mathcal{H})} , \quad \text{"canonical density matrix"}$$

$$\beta F = \beta\mathcal{H} + \log\rho = \mathrm{Tr}\,\rho\beta\mathcal{H} + \mathrm{Tr}\,\rho\log\rho .$$

Also consider an arbitrary density matrix (Hermitian, unit trace, non-negative):
$\rho_0 = e^{\beta(F-\mathcal{H})+\Delta}$. Δ must be Hermitian, but need not commute with $\mathcal{H}$.

$$u(\lambda) \equiv \mathrm{Tr}\, e^{\beta(F-\mathcal{H})+\lambda\Delta} \neq \sum_i e^{\beta(F-E_i)} \langle E_i | e^{\lambda\Delta} | E_i \rangle .$$ see page 56

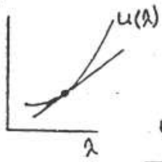
$$\frac{du}{d\lambda} = \sum_i e^{\beta(F-E_i)} \langle E_i | \Delta e^{\lambda\Delta} | E_i \rangle .$$

$$\frac{d^2u}{d\lambda^2} = \sum_i e^{\beta(F-E_i)} \langle E_i | \Delta^2 e^{\lambda\Delta} | E_i \rangle = \sum_{i,j} e^{\beta(F-E_i)} \langle E_i | \Delta_j \rangle \Delta_j^2 e^{\lambda\Delta_j} \langle \Delta_j | E_i \rangle \geq 0$$

$$\sum_{i,j} e^{\beta(F-E_i)} \langle E_i | \Delta_j \rangle \Delta_j^2 e^{\lambda\Delta_j} \langle \Delta_j | E_i \rangle \overset{?}{=} 0$$

$$0 \overset{?}{=} \sum_{i,j} \langle E_i | \Delta_j \rangle \Delta_j^2 \langle \Delta_j | E_i \rangle = \sum_j \langle \Delta_j | \Delta_j \rangle \Delta_j^2 = \sum_j \Delta_j^2 .$$

Equality holds iff $\{\Delta_\ell\} = 0$, i.e. $\Delta = 0$.



$$u(\bar\lambda) + (\lambda - \bar\lambda) \frac{du}{d\lambda}(\bar\lambda) \leq u(\lambda)$$

Take $\lambda = 0$, $\bar\lambda = 1$: $\cancel{\mathrm{Tr}\rho_0} - [\Delta]_0 \leq \cancel{\mathrm{Tr}\rho}$

$$0 \leq [\Delta]_0 = [(\log\rho_0 - \beta F + \beta\mathcal{H})]_0 = [(\log\rho_0 + \beta\mathcal{H})]_0 - \beta F$$

$$\mathrm{Tr}\,\rho_0 \beta\mathcal{H} + \mathrm{Tr}\,\rho_0 \log\rho_0 \geq \beta F = \mathrm{Tr}\,\rho\beta\mathcal{H} + \mathrm{Tr}\,\rho\log\rho$$

$$[\mathcal{H}]_0 + kT\, \mathrm{Tr}\,\rho_0 \log\rho_0 \geq F = [\mathcal{H}] + kT\, \mathrm{Tr}\,\rho\log\rho$$

$$\boxed{F_0 = U_0 - TS_0 \geq F = U - TS}$$

The free energy F_0 evaluated with an arbitrary density matrix ρ_0 ("test density matrix") is bounded below by the true free energy F, evaluated with the canonical density matrix $\rho = e^{-\beta\mathcal{H}}/\mathrm{Tr}\, e^{-\beta\mathcal{H}}$.

$$\boxed{F_0 = F \text{ if and only if } \rho_0 = \rho .}$$

H. Falk, Am. J. Phys. 38, 858 (1970).

ρ_0 = most general single-site (single-particle) factorizable density matrix $\Rightarrow$
$\Rightarrow$ mean-field (effective-field) theories.

ρ_0 = most general two-site factorizable density matrix $\Rightarrow$ Bethe-Peierls approximation.

$u(\lambda) \equiv \mathrm{Tr}\, e^{A+\lambda B}$, where A and B are arbitrary Hermitian operators.

$\dfrac{du}{d\lambda} = \mathrm{Tr}\, e^{A+\lambda B}$ by cyclic permutation inside trace.

$$\frac{d^2u}{d\lambda^2} = \frac{d}{d\lambda}\,\mathrm{Tr}\sum_{n=0}^{\infty}\frac{1}{n!}(A+\lambda B)^n B = \mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=1}^{n}(A+\lambda B)^{m-1}B(A+\lambda B)^{n-m}B =$$

using eigenstates of $A+\lambda B$ as basis, with eigenvalues G_i which are real,

$$= \sum_{ij}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=1}^{n} G_i^{m-1}\langle i|B|j\rangle G_j^{n-m}\langle j|B|i\rangle = \sum_{i,j}|B_{ij}|^2\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=1}^{n} G_i^{m-1}G_j^{n-m}.$$

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{m=0}^{n} G_i^{m}G_j^{n-m} \overset{?}{=} \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,\frac{G_j^{n+1}-G_i^{n+1}}{G_j-G_i} \quad \text{for } G_j \neq G_i .$$

$$(G_j-G_i)\sum_{m=0}^{n}G_i^{m}G_j^{n-m} = \sum_{m=0}^{n}G_j^{n+1-m}G_i^{m} - \sum_{m=0}^{n}G_j^{n-m}G_i^{m+1} = \sum_{m=0}^{n}G_j^{n+1-m}G_i^{m} - \sum_{m=1}^{n+1}G_j^{n+1-m}G_i^{m}$$

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,\frac{G_j^{n+1}-G_i^{n+1}}{G_j-G_i} = \sum_{n=1}^{\infty}\frac{1}{n!}\,\frac{G_j^{n}-G_i^{n}}{G_j-G_i} \doteq \frac{e^{G_j}-e^{G_i}}{G_j-G_i} > 0 .$$

$$\text{For } G_i = G_j, \quad \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{m=0}^{n}G_i^{m}G_j^{n-m} = \sum_{n=0}^{\infty}\frac{1}{n!}G_i^{n} = e^{G_i} > 0 .$$

$\therefore\ \dfrac{d^2u}{d\lambda^2} \geqslant 0$, with equality when all $B_{ij} = 0$, i.e., $B = 0$.

Use $A = \beta(F-\mathcal{H})$ and $B = \Delta$ for Gibbs inequality.

IV. Scaling Theory of Kadanoff

P. Kadanoff, Physics $\underline{2}$, 263 (1966).

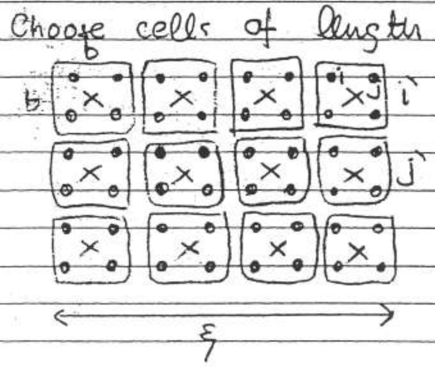
$$-\beta\mathcal{H} = J\sum_{\langle ij\rangle} s_i s_j + H\sum_i s_i$$

critical point $\quad J = J_c \,, \quad H = H_c = 0$

small deviations $\quad t = \dfrac{J_c - J}{J_c} \,, \quad H$

free energy $\quad -\beta F = \ln Z(t, H)$

Choose cells of length b:



$s_{i'}$ is a cell variable which reflects an "important" collective property of spins inside cell.

For example, ferromagnetism:
$s_{i'} = b^{-d} \sum_i^{i'} s_i$ (local magnetization)

If s_i is a spin-half Ising variable, take $s_{i'} = \mathrm{sign}\left(\sum_i^{i'} s_i\right)$, to conserve form.

Hypothesis.

s-problem:	s'-problem
N spins	$N' = N/b^d$
lattice spacing a	$a' = ba$
$-\beta\mathcal{H} = J\sum_{\langle ij\rangle} s_i s_j + H\sum_i s_i$	① $-\beta'\mathcal{H}' = J'\sum_{\langle i'j'\rangle} s_{i'} s_{j'} + H'\sum_{i'} s_{j'}$
t, H	t', H'

Can be made thermodynamically equivalent by the proper choice of (J', H'), or equivalently (t', H')

Microscope analogy

$-\beta \hat{F}(t, H) = -\beta \hat{F}(t', H')$ Since fam conserved.

$Z(t, H) = Z(t', H')$

$t' = t'(t, H)$ Recursion relations

$H' = H'(t, H)$ Expect $t'(t) > t \neq 0$, $H'(H) > H > 0$.

At critical point, system is scale invariant

$t'(t = H = 0) = 0$ $H'(t = H = 0) = 0$.

or, alternatively

$J'(J = J_c ; H = 0) = J_c$ Fixed point

③ Crucial sub-hypothesis

Recursion relations analytic

One can linearize $t'(t, H)$ and $H'(t, H)$ around $t = H = 0$.

$t' = At + BH$

$H' = Ct + DH$

$$\left.\begin{array}{l} s \to -s \\ s' \to -s' \end{array}\right\} \qquad \left.\begin{array}{l} H \to -H \\ H' \to -H' \\ t \to t \\ t' \to t' \end{array}\right\} \qquad \begin{array}{l} B = 0 \\ C = 0 \end{array}$$

$t' = b^{y_T} t$ since $t' = t$ for $b = 1$.

$H' = b^{y_H} H$ $H' = H$

Also takes care of multiplicative property: $b_{12} = b_1 b_2$

Ⓐ Dimensionless free energy per particle: $f = \frac{1}{N} \ln Z$) for successive transformations

$\ln Z = \ln Z'$

$N f(t, H) = N' f(t', H')$

$f(t, H) = b^{-d} f(b^{y_T} t, b^{y_H} H)$, b arbitrary

Generalized homogeneous function

Choose $b = t^{-1/y_T}$:

$$f(t,H) = t^{d/y_T} f\left(1, \frac{H}{t^{y_H/y_T}}\right) =$$

$$= t^{d/y_T} \text{ funct.}\left(\frac{H}{t^{y_H/y_T}}\right)$$ <u>scaling form</u>

Similarly, for choice $b = H^{-1/y_H}$
$f = H^{d/y_H} \text{ funct}\left(\frac{t}{H^{y_T/y_H}}\right)$
$\frac{\partial f}{\partial t}, \frac{\partial f}{\partial H}$ bounded
$\rightarrow d \geq y_T, y_H$

Identifies: $\boxed{2-\alpha = d/y_T}$ $\boxed{\Delta = y_H/y_T}$

#12

#13 (B) <u>Correlation length</u> $(H=0)$

$$\xi = \xi_0 t^{-\nu} \qquad \xi' = \xi_0 t'^{-\nu} = \xi_0 b^{-y_T \nu} t^{-\nu}$$

$$= \frac{1}{b}\xi = b^{-1} \xi_0 t^{-\nu}$$

$\boxed{\nu = 1/y_T}$

Could repeat this for any physical length.
Single diverging correlation length.
But recent MIT le experiments (anisotropic)

(A) $\rightarrow d\nu = 2-\alpha$
True experimental test.

$$(t=0) \qquad \nu_H = 1/y_H$$

(C) <u>Magnetization</u>

$$M(t,H) = \frac{1}{N}\frac{\partial}{\partial H} \ln Z(t,H) = \frac{1}{b^d N'} \frac{\partial}{b^{-y_H}\partial H'} \ln Z(t',H')$$

$$= b^{y_H-d} M(t',H').$$

$$M' = b^{d-y_H} M \qquad \text{cf} \qquad H' = b^{y_H} H$$

$$\left(U' = b^{d-y_T} U \qquad\qquad t' = b^{y_T} t \right)$$

$$M(t,H) = b^{y_H-d} M(b^{y_T}t, b^{y_H}H)$$

Choose $b = t^{-1/y_T}$: $$M(t,H) = t^{(d-y_H)/y_T} M\left(1, \frac{H}{t^{y_H/y_T}}\right) =$$

$$= t^{(d-y_H)/y_T} \text{ funct.}\left(\frac{H}{t^{y_H/y_T}}\right)$$

Scaling form

[redacted]

Take $H=0$: $\quad M(t) = t^{(d-y_H)/y_T}\, \text{funct}(0)$

$$\boxed{\beta = (d-y_H)/y_T}$$

Similarly, ~~$b = H^{-1/y_H}$~~ gives $M = H^{(d-y_H)/y_H}\, \text{funct}\left(\frac{t}{H^{y_T/y_H}}\right)$

$$\frac{1}{\delta} = \frac{d-y_H}{y_H} = \beta/\Delta$$

Generalize:

$$M_u = \frac{1}{N}\frac{\partial}{\partial K_u} \ln Z$$

$$M_v(K_w = K_w^c + t_w;\ \text{all other } K_u = K_u^c) = A\, t_w^{\beta_{vw}},$$

$$\beta_{vw} = (d-y_v)/y_w$$

All other exponents can be derived and satisfy exponent equalities.

[Question: determine susceptibility expo γ in terms of y.]

Ⓓ Correlation function $\quad H_i^{\prime}$

Uniform field $\quad H^{\prime} = b^{y_H} H \quad$, in $H^{\prime}\sum_{i^{\prime}} s_{i^{\prime}}$ and $H\sum_i s_i$

Local field $\quad H_{i^{\prime}}^{\prime} = b^{y_H}\frac{1}{b^d} H_i \quad$, in $\sum_{i^{\prime}} H_{i^{\prime}}^{\prime} s_{i^{\prime}}$ and $\sum_i H_i s_i$

$$\Gamma(r,t,H) = \langle s_0 s_r\rangle - \langle s_0\rangle\langle s_r\rangle = \frac{\partial^2}{\partial H_0 \partial H_r}\ln Z =$$

$$= b^{2y_H-2d}\frac{\partial^2}{\partial H_{0^{\prime}}^{\prime}\partial H_{r^{\prime}}^{\prime}}\ln Z^{\prime} = b^{2y_H-2d}\,\Gamma(r^{\prime},t^{\prime},H^{\prime})$$

$$\Gamma(r,t,H) = b^{2y_H-2d}\,\Gamma(b^{-1}r,\ b^{y_T}t,\ b^{y_H}H)$$

Choose $b=r$: $\quad \Gamma(r,t,H) = \dfrac{\text{funct.}(r^{y_T}t,\ r^{y_H}H)}{r^{2d-2y_H}}$

Identify $\quad 2d - 2y_H = d-2+\eta$

$$\eta = d+2-2y_H$$

Expect $y_T, y_H > 0$: Physical consideration of critical fluctuations: $b\uparrow$ away from criticality

All critical exponents in terms of two parameters y_T, y_H

$\rightarrow$ exponent equalities.

y_T, y_H better be independent of b. $\quad$ Also, $0 < y_T, y_H \leq d$.

Summary: Close to Critical Point

Ⓐ Technical $\quad r \to r' = r/b \quad$ length rescaling

$t, H \to t' = b^{y_T} t, \; H' = b^{y_H} H \quad$ Thermo. fields

$w \qquad w' = b^{y_w} w$

$y > 0 \quad$ amplified under rescaling – relevant field.

$y < 0 \quad$ suppressed, disappears – irrelevant field.

line of critical points collapse to single fixed point, universality

$y = 0 \quad$ marginal.

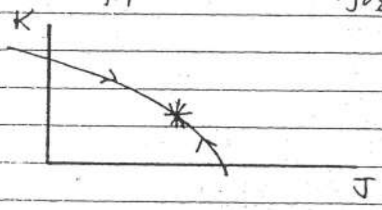
$s_i, \; s_i s_j \to s_i' = b^{d-y_H} s_i, \quad s_i' s_j' = b^{d-y_T} s_i s_j \quad$ operators $\mathcal{O}_u$.

$\langle \mathcal{O}_u \mathcal{O}_v \rangle - \langle \mathcal{O}_u \rangle \langle \mathcal{O}_v \rangle \sim 1/r^{(d-y_u)+(d-y_v)} \quad$ for large r.

Operator algebra – L.P. Kadanoff, PRL 23, 1430 (1969).

Ⓑ Generalize: e.g two types of interaction.

$$\beta \hat{J} \sum_{\langle ij \rangle_1} s_i s_j + \beta \hat{K} \sum_{\langle ij \rangle_2} s_i s_j = J \sum_{\langle ij \rangle_1} s_i s_j + K \sum_{\langle ij \rangle_2} s_i s_j$$



$$J' = J'(J,K)$$
$$K' = K'(J,K)$$

Consider eigenvalues and eigendirection of

$$\begin{bmatrix} \partial J'/\partial J & \partial J'/\partial K \\ \partial K'/\partial J & \partial K'/\partial K \end{bmatrix}^*$$

(C) Philosophy

Try to eliminate all behavior on the scale $\tilde{x}$, $a < \tilde{x} < ba$.

Renormalize parameters such as temperature, maintain thermodynamic equivalence.

Each cell has b^d degrees of freedom $\{s\}_{b^d}$.
Replace by one degree of freedom s', judiciously chosen.
Deal in some systematic way with $b^d - 1$ degrees of freedom.

(D) Shortcomings: pre-RG:

1. Ad hoc: Why should cell spins s' act like nn $s = 1/2$ Ising?
2. $t' = t'(t,H)$, $H' = H'(t,H)$, but no method for calculating recursion relations.
3. Quantities other than exponents (amplitudes), away from criticality.

(E) Future Success:

$t'(t,H)$ indeed analytic form, can be approximated well.

K.G. Wilson, PRB $\underline{4}$, 3174, 3184 (1971).

#14

V Exact Renormalization-Group Treatment in One Dimension

D.R. Nelson and M.E. Fisher, Ann. Phys. (N.Y.) 91, 226 (1975).

spin-1/2 Ising chain

$$Z = \sum_{\{s\}} e^{-\beta \mathcal{H}\{s\}} = \sum_{\{s'\}} \left(\sum_{\{\sigma\}} e^{-\beta \mathcal{H}\{s',\sigma\}} \right) \overset{RGT}{=} \sum_{\{s'\}} e^{-\beta' \mathcal{H}'\{s'\}}$$

$i =$ 1 (2) 3 (4) 5 (6)

$i' =$ 1 2 3

$b = 2$

$s'_{i'} = s_{i=2i'}$ $\qquad \sigma_{i'} = s_{2i'-1}$

Choice of RG transformation arbitrary, up to solvability consideration

Derive recursion relation:

$$\sum_{s_3} e^{J s_2 s_3 + H s_3 + J s_3 s_4} = e^{J' s_2 s_4 + \tilde{H}(s_2+s_4) + \tilde{G}} = R(s_2 s_4).$$

$$x \equiv e^{J}, \quad y \equiv e^{H}, \quad \tilde{g} \equiv e^{\tilde{G}}$$

$$R(+1,+1) = x^2 y + x^{-2} y^{-1} = x' \tilde{y}^2 \tilde{g}$$

$$R(+1,-1) = y + y^{-1} = x'^{-1} \tilde{g} = R(-1,+1)$$

$$R(-1,-1) = x^2 y^{-1} + x^{-2} y = x' \tilde{y}^{-2} \tilde{g}$$

$$x'^4 = \frac{R(+1,+1)\,R(-1,-1)}{R(+1,-1)\,R(-1,+1)}$$

$$\tilde{y}^4 = \frac{R(+1,+1)}{R(-1,-1)}$$

$$\tilde{g}^4 = R(+1,+1)\,R(-1,-1)\,R(+1,-1)\,R(-1,+1)$$

$$x'^4 = \frac{(x^2y + x^{-2}y^{-1})(x^2y^{-1} + x^{-2}y)}{(y+y^{-1})^2}$$

Recursi relatio

$H' = H + 2\tilde{H}$

$$y'^2 = y^2 \frac{x^2y + x^{-2}y^{-1}}{x^2y^{-1} + x^{-2}y}$$

additive constant
NG in Hamiltonian

$$g'^4 = g^8 (x^2y + x^{-2}y^{-1})(x^2y^{-1} + x^{-2}y)(y+y^{-1})^2$$

Operators: s_i, s_is_j, 1

$G\sum 1 = NG$

$G' = 2G + \tilde{G}$

$$J' = J'(J,H)$$
$$H' = H'(J,H)$$
$$G' = b^d G + \tilde{G}(J,H) \quad \leftarrow \text{captive variabl}$$

Importance of additive constant

$$Z = \sum_{\{s\}} e^{-\beta\mathcal{H} + NG} = \sum_{\{s'\}} e^{-\beta'\mathcal{H}' + N'G'}$$

$$f + G = \frac{1}{N}\ln Z = \frac{1}{b^d N'}\ln Z = b^{-d}(f' + G')$$

$$f' + G' = \frac{1}{N'}\ln Z$$

Notation $\quad f \to f^{(0)} \qquad f' \to f^{(1)}$

$$f^{(0)} + G^{(0)} = b^{-dn}(f^{(n)} + G^{(n)})$$

large n: soluble $\uparrow$ singularity accumulates here.
(some derivative infinite)

$$G^{(n)} = b^{dn}G^{(0)} + b^{d(n-1)}\tilde{G}(J^{(0)}, H^{(0)}) + b^{d(n-2)}\tilde{G}(J^{(1)}, H^{(1)}) + \dots$$

This leads to total free energy. What one really computes is $b^{-dn}G^{(n)}$, converge

Flow space: (J,H) or, equivalently (x,y)
$(\tanh J, \tanh H)$

Ⓐ. $H'(H=0) = 0$

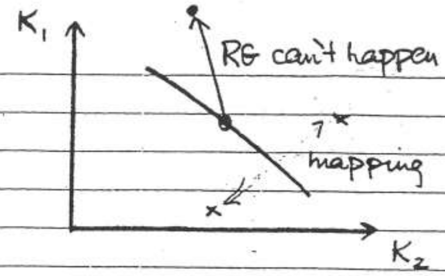
$H=0$ is a subspace closed under this RG.

Result of up-down symmetry: $s, s' \to -s, -s'$ $\quad$ redefining dummy varia

$H, H' \to -H, -H'$

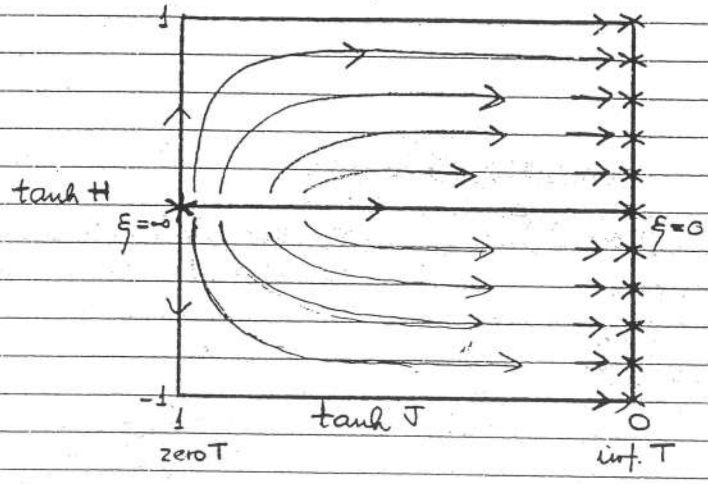
$J, J' \to J, J'$

(Also, remember $F \Leftrightarrow AF$ mapping)



Chosen RG commutes with up-down symmetry.

At $H=0$ subspace, $\tanh J' = (\tanh J)^2$



Fixed points: $H^* = 0$, $J^* = 0$ stable; decoupled.
$H^* = 0$, $J^* = \infty$ unstable; strongly coupled

Correlation length at a fixed point

$\xi = \text{func}(J, H)$ $\qquad\qquad$ $\xi' = \text{same func}(J', H')$

$$\xi = b\xi'$$

At fp, $\quad J' = J \equiv J^*$, $\qquad H' = H \equiv H^*$

$\therefore \quad \xi' = \xi = 0$ or ∞.
trivial $\qquad$ critical

[redacted]

(B) $J'(J=0) = 0$ closed subspace
(decoupled spins)

$H' = H$ fixed line, trivially, $\xi^* = 0$.
stable

(C) Global flows: up-down symmetry.

One critical point with an unstable fp.
Disordered phase with totally stable, trivial fl. (sink)

(D) Linearize recursion relation around critical fp.

(a) $\tanh J' = (\tanh J)^2$ " Note slow flow away from fp.
→ accumulate divergences

$2e^{-2J'} = 4e^{-2J}$

$e^{-J'} = \underbrace{2^{1/2}}_{b^{y_T}} e^{-J}$

$b=2$, $y_T = 1/2$, $\nu = 1/y_T = 2$

$\xi \sim \left(e^{-\hat{J}/kT} - e^{-\hat{J}/kT_c}\right)^{-2}$

Zero T fp: must specify scaling field for expo to have meaning.

(b) $H' = \underbrace{2}_{b^{y_H}}H$

$y_H = 1$

$\gamma = (2y_H - d)/y_T = 2$ $\chi \sim (e^{-\hat{J}/kT})^{-2}$

$\eta = d+2-2y_H = 1$

Recall direct solution: $\Gamma(r) = bc\, e^{-r\ln \lambda_>/\lambda_<}$
at cp, $\lambda_> = \lambda_<$ $\Gamma(r) = bc = \dfrac{bc}{r^{d-2+\eta}}$, $\eta = 1$.

RG using transfer matrix

$$Z = \mathrm{Tr}\ \underbrace{T\cdot T\cdot T\cdot T}_{N \text{ factors}} \qquad T \equiv \langle s_i | T | s_{i+1} \rangle$$

$$Z' = \mathrm{Tr}\ \underbrace{T'\cdot T'}_{N' = N/b \text{ factors}} \qquad T' \equiv \langle s'_i | T' | s'_{i+1} \rangle$$

decimation: $= \langle s_{bi} | T' | s_{b(i+1)} \rangle$

$$T' = T^b$$

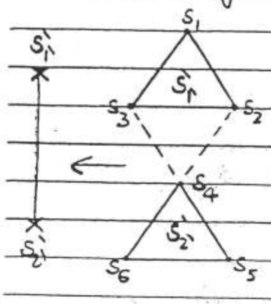
$$\lambda'_\xi = \lambda_\xi^b ,$$

which explicitly shows that $\xi' = \xi / b$.

[redacted]

VI. Approximate Renormalization-Group Treatment in Two Dimensions

Niemeijer and van Leeuwen, DG 6



(A) Cell lattice triangular, $b = \sqrt{3}$

$s_1' = \text{sign}(s_1+s_2+s_3) = \pm 1$ Majority rule.

Structure conserved.

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(B) Quantitative equivalence of thermodynamics

$$Z = \sum_{\{s\}} e^{-\beta \mathcal{H}\{s\}} = \sum_{\{s'\}} \left(\sum_{\{\sigma\}} e^{-\beta \mathcal{H}\{s',\sigma\}} \right) =$$

$$= \sum_{\{s'\}} e^{-\beta' \mathcal{H}'\{s'\}}$$

$$e^{-\beta' \mathcal{H}'\{s'\}} = \sum_{\{\sigma\}} e^{-\beta \mathcal{H}\{s',\sigma\}}$$

Less degrees of freedom.

By construction, $Z' = \sum_{\{s'\}} e^{-\beta' \mathcal{H}'\{s'\}} = \sum_{\{s'\}} \sum_{\{\sigma\}} e^{-\beta \mathcal{H}\{s',\sigma\}} = \sum_{\{s\}} e^{-\beta \mathcal{H}\{s\}} = Z$

Free energy conserved.

We want to study $-\beta\mathcal{H} = J \sum_{\langle ij \rangle} s_i s_j$

No guarantee that form of Hamiltonian will be conserved - in general, it will not.

However, as a simple approximation, use finite cluster shown above

$-\beta\mathcal{H}$ has eight bonds, $-\beta'\mathcal{H}'$ has one bond.

<u>Intuitive local nature of transformation.</u>

$$R(s_1', s_2') = e^{J' s_1' s_2' + \tilde{G}} = \sum_{\{J\}} e^{J(s_1 s_2 + s_2 s_3 + \dots)}$$

$$e^{2\tilde{G}} = R(++)\, R(+-)$$

$$e^{2J'} = R(++) / R(+-)$$

$R(++)$ = polynomial of order 8 in $x = e^J$

$R(+-)$ = order ≤ 6.

$R(++)$	$s_1' = +1, \sigma_1' = 1$	$\sigma = 2$	$\sigma = 3$	$\sigma = 4$	$R(+-)$
	$3J$	$-J$	$-J$	$-J$	
$=1, \sigma_2' = 1$ $3J$	$+2J = 8J$ $-2J = 4J$	$+2J = 4J$ $-2J = 0$	$+0 = 2J$ $+0 = 2J$	$+0 = 2J$ $+0 = 2J$	$s_2' = -1, \ \sigma_2' = 1$ $3J$
$\sigma_2' = 2$ $-J$	$-2J = 0$ $+2J = 4J$	$-2J = -4J$ $+2J = 0$	$+0 = -2J$ $+0 = -2J$	$+0 = -2J$ $+0 = -2J$	$\sigma_2' = 2$ $-J$
$\sigma_2' = 3$ $-J$	$+2J = 4J$ $-2J = 0$	$+2J = 0$ $-2J = -4J$	$+0 = -2J$ $+0 = -2J$	$+0 = -2J$ $+0 = -2J$	$\sigma_2' = 3$ $-J$
$\sigma_3' = 4$ $-J$	$+2J = 4J$ $-2J = 0$	$+2J = 0$ $-2J = -4J$	$+0 = -2J$ $+0 = -2J$	$+0 = -2J$ $+0 = -2J$	$\sigma_2' = 4$ $-J$

$R(++) = x^8 + 3x^4 + 2x^2 + 3 + 6x^{-2} + x^{-4}$, sum. coeff $= 16$ ✓

$R(+-) = 2x^4 + 2x^2 + 4 + 6x^{-2} + 2x^{-4}$

Recursion relation: $x'^2 = R(++) / R(+-)$

$\tilde{q}^2 = R(++)\, R(+-)$ captive

① $x \to \infty$ $\quad$ $x'^2 = x^8/2x^4 = \frac{1}{2}x^4$

$J \to \infty$

$$J' = 2J > J \qquad J^* = \infty \quad \text{stable}$$

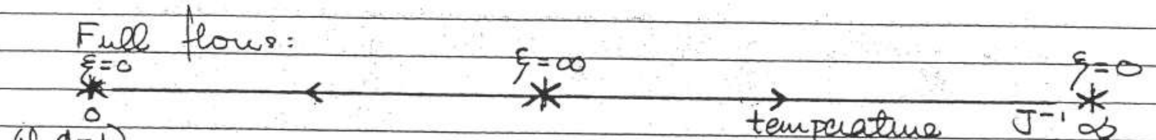
② $x \to 1$ $\quad$ $x \simeq 1 + J$

$J \to 0$

$$1 + 2J' \simeq x'^2 \simeq \frac{16 + 8J}{16 - 8J} \simeq 1 + J$$

$$J' = J/2 < J \qquad J^* = 0 \quad \text{stable}$$

Full flows:



(cf. $d=1$)

$J^* = 0.365$ $\quad$ (exact 0.274) $\quad$ (MFT 0.167)

Close to J^*, $\quad J'(J) = J^* + \left.\frac{\partial J'}{\partial J}\right)_{J^*} (J - J^*)$

$$J' - J^* = \left.\frac{\partial J'}{\partial J}\right)_{J^*} (J - J^*) = b^{y_T}(J - J^*)$$

$$t' = b^{y_T}\, t$$

$b^{y_T} = 1.544 > 1$

$(b = \sqrt{3})$ $\quad$ unstable

$y_T = 0.79 > 0$ relevant

(exact 1) $\quad$ (MFT 2)

<u>Better than MFT</u>, even at this level.

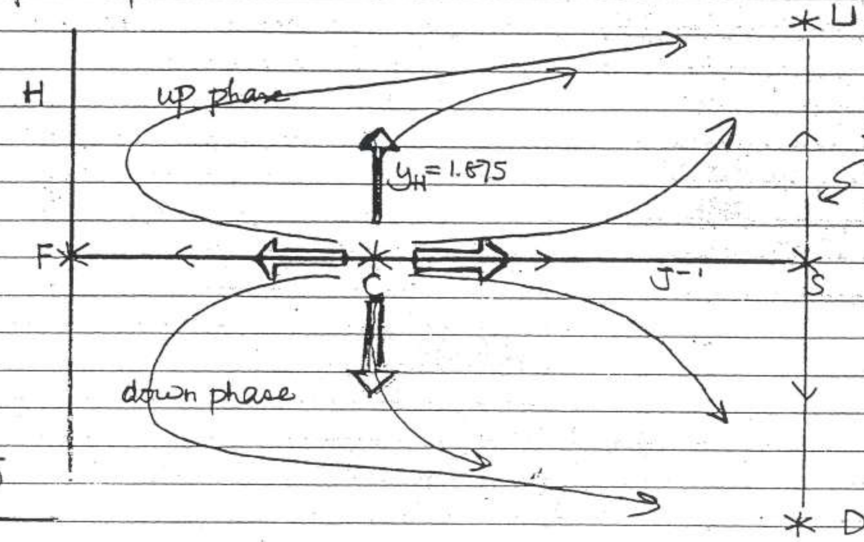
Can carry to 10^{-7} without much technical sophistication.

<u>Question</u>: Would this RG transformation work for <u>antiferro</u>? Why?

Now include magnetic field in Hamiltonian, $H \sum_i s_i$

If want to carry out above approximation, $\frac{H}{q} \sum_{\langle ij \rangle} (s_i + s_j)$. Advisable for most truncations.

#16 5 fixed points:



cf. fixed line in 1-d decimation

Can this happen for non-trivial fixed points?

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Each special locus (phase, boundary of given type) has its own fixed point[s]. [redundancy].

Fixed point classification:

critical	C	$\xi = \infty$	
first-order	F	$0, \infty$	(Goldstone modes, or ground-state entropy)
sink	U, D	0	
null	S	0	

- We have learned how to calculate phase boundaries (map out connectivity of flows), critical exponents (linearise RGT at non-trivial fp).

- How about entire thermodynamic fns? Amplitude, analytic background, away from cp, etc. Eqs of state? See below.

– Algebraic interlude – Projection operator

$$e^{-\beta \mathcal{H}'\{s'\}} = \sum_{\{s\}}' e^{-\beta \mathcal{H}\{s\}} \qquad \text{restricted sum}$$

$$= \sum_{\{s\}} P(\{s'\},\{s\})\, e^{-\beta \mathcal{H}\{s\}} \qquad \text{unrestricted sum}$$

Sufficient condition for Z-conservation:

$$\sum_{\{s'\}} P(\{s'\},\{s\}) = 1$$

Usually, factorized form chosen:

$$P = \prod_i^{\text{cells}} p(s'_i, \{s\}_i).$$

Example ① $d=1$ decimation:

$$p = \tfrac{1}{2}\left(1 + s'_i s_{2i}\right)$$

Example ② $d=2$ generalized NvL:

$$p = \tfrac{1}{2}\left\{ 1 + s'_i \left[u(s_1+s_2+s_3) + v s_1 s_2 s_3 \right] \right\}$$

Z-conservation

no even terms, up-down symmetry conserved

triangular symmetry

$u = 1/2,\ v = -1/2$ give "majority (signum) rule" as special case.

Question: Which symmetry of the original system is violated?

Varying u, v: exact treatment: shouldn't make diff.
approximate treatment: move around f.p., useful (e.g., ϵ expansion)

But be sure to project important (critical) degrees of freedom.

Approximations: On analytic recursion relation.
(Singularities come from infinite repetition)

systematic vs. one-shot
truncation vs. controlled (convergent, asymptotic)
interpolation, symmetries used
variational
Monte Carlo RG

No cook-book. Have to consider case by case.

How about thermodynamic functions?

General form: $-\beta\mathcal{H} = \sum_a K_a s_a$, $\quad s_a = \prod_i^a s_i$

inversion: $K_a = \frac{1}{2^N} \sum_{\{s\}} s_a(-\beta\mathcal{H})$.

We have already used this to extract renormalized interaction constants (primed system):

$$K'_a = \frac{1}{2^{N'}} \sum_{\{s'\}} s'_a(-\beta'\mathcal{H}') = \frac{1}{2^{N'}} \sum_{\{s'\}} s'_a \ln \sum_{\{s\}}' e^{-\beta\mathcal{H}} = \frac{1}{2^{N'}} \sum_{\{s'\}} s'_a \ln R\{s'\}$$

RG flows in $\{K_a\}$.
Somehow avoid an infinite-dimensional space.

Generalized densities: $M_a = \langle s_a \rangle = \frac{\partial}{\partial K_a} \ln Z$.

This is a local density, M_a, conjugate to the local interaction K_a

Generic interactions and densities:

$$-\beta\mathcal{H}\{s\} = \sum_a K_a s_a = \sum_\alpha K_\alpha \sum_a^\alpha s_a ,$$

such that M_a, by some symmetry, same for all a in α,
$= M_\alpha$.

Ⓐ $M_\alpha = \frac{1}{N_\alpha} \frac{\partial}{\partial K_\alpha} \ln Z = \frac{1}{N_\alpha} \sum_a^\alpha M_a$

Examples:

$M_\alpha = M_a = \langle s_i \rangle$ $\quad K_a = H_i \quad K_\alpha = H \quad N_\alpha = N$

$\langle s_i s_j \rangle \quad J_{ij} \quad J \quad \frac{q}{2} N$

$\langle 1 \rangle \quad G_i \quad G \quad N$

in general, $N_\alpha = N q_\alpha$ ← translations; ← rotations + reflections (shapes?)

$$= \frac{1}{b^d N'} \frac{1}{q_\alpha} \frac{\partial}{\partial K'_\beta} \ln Z \underbrace{\frac{\partial K'_\beta}{\partial K_\alpha}}_{\equiv T_{\beta\alpha}}$$ chain rule

$\equiv T_{\beta\alpha}$ recursion matrix

Ⓑ $M'_\alpha = \frac{1}{N'} \frac{1}{q_\alpha} \frac{\partial}{\partial K'_\alpha} \ln Z^{(1)}$

$$\boxed{M_\alpha = b^{-d} M'_\beta \left(\frac{q_\beta}{q_\alpha}\right) T_{\beta\alpha}}$$ (absorb)

recursion relation for densities

① At fixed point, $M'_\alpha = M_\alpha = M^*_\alpha$

Ⓐ M^*_α left eigenvector of $T_{\alpha\beta}$ with eigenvalue exponent $y=d$

(Recall $y>d$ not allowed, $\frac{\partial f}{\partial t} \sim t^{(d-y_T)/y_T}$ non-divergent.)

$\vec{M} \neq \vec{0}$ because of $M_\alpha = \langle 1 \rangle$

Ⓑ There must always be at least one eigenvalue exponent $y=d$.

How does this check with previous example? →

In previous example (NVL, $s=\frac{1}{2}$ triangular, 2-cell cluster approx.):

$T_{\alpha\beta}$ is a 2×2 matrix.

$M_\alpha = \langle 1 \rangle$ $\quad K_\alpha = G$ $\quad$ additive constant per site.

$M_\beta = \langle s_i s_{i+1} \rangle$ $\quad K_\beta = J$ $\quad$ nn coupling

$$J' = f_\beta(J)$$

$$G' = b^d G + \tilde{G}(J) \qquad \text{captive}$$

$$T = \begin{bmatrix} \partial G'/\partial G = b^d & (3/4)\,\partial G'/\partial J \\ \partial J'/\partial G = 0 & \partial J'/\partial J = b^{y_T} \end{bmatrix} = \begin{bmatrix} b^d & (3/4)\,\partial G'/\partial J \\ 0 & b^{y_T} \end{bmatrix}$$

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#17 Now include magnetic field: $\quad K_\gamma = H, \quad M_\gamma = \langle s_i \rangle$

up-down symmetry: $Z(J,H) = \sum_{\{s\}} e^{J\sum s_i s_j + H\sum s_i} \underset{\tilde{s}=-s}{=} \sum_{\{\tilde{s}\}} e^{J\sum \tilde{s}_i \tilde{s}_j - H\sum \tilde{s}_i} = Z(J,-H)$.

Similarly, $e^{-\beta\mathcal{H}'(\{s'\};J',H')} = \sum'_{\{s\}} e^{-\beta\mathcal{H}(\{s\};J,H)}$

$(s,s') \to (-s,-s')$ $\quad$ find that

RG chosen not to violate symmetry (Σ').

$$H' = f_\gamma(J,H) = -f_\gamma(J,-H)$$

$$J' = f_\beta(J,H) = f_\beta(J,-H)$$

$$G' = b^d G + \tilde{G}(J,H) = b^d G + \tilde{G}(J,-H).$$

Derivatives calculated at $H^* = 0$:

$$\left.\frac{\partial J'}{\partial H}\right)^* = -\left.\frac{\partial J'}{\partial H}\right)^* = 0 = \frac{\partial G'}{\partial H} = \frac{\partial H'}{\partial J}$$

$$T_{3\times 3} = \begin{bmatrix} b^d & \times & 0 \\ 0 & b^{y_T} & 0 \\ 0 & 0 & b^{y_H} \end{bmatrix}$$

First-order fixed-point: $y_H = d$ at F^*

Two density eigenvectors: (use variable e^{-J}).

$$\vec{M} = \begin{bmatrix} 1 \\ \langle s_i s_j \rangle \\ \langle s_i \rangle \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ +1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

At $H^* = 0$, $T^* = 0$ ($J^* = \infty$), two phases coexist $\begin{matrix}\uparrow\uparrow\uparrow \\ \uparrow\uparrow\uparrow\end{matrix}$ $\begin{matrix}\downarrow\downarrow\downarrow \\ \downarrow\downarrow\downarrow\end{matrix}$

In general, at $H^* = 0$ fixed point — generically, odd interactions

$$T = \begin{bmatrix} b^d & \boxed{\times} & \\ \boxed{0} & \boxed{\begin{matrix}\frac{\partial J'_\alpha}{\partial J_\beta} \\ y_T\end{matrix}} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{\begin{matrix}\frac{\partial H'_\alpha}{\partial H_\beta} \\ y_H\end{matrix}} \end{bmatrix}$$

① $M_\alpha = b^{-d} M'_\beta \, T_{\beta\alpha}$ even and odd pieces separate.

② $\vec{M}^*_{\substack{\text{odd} \\ \text{eigen}}} = b^{d-y_H} \vec{M}^*_{\substack{\text{odd} \\ \text{eigen}}}$ $\neq 0$ only if $y_H = d$.

E.g., not at Ising C^* with $y_H = 1.875$.

If non-zero, in combining with even ($y=d$) eigenvector,

$$\vec{M}^*_{y=d} = \begin{bmatrix} \vec{M}^*_{\text{even}} \\ \vec{M}^*_{\text{odd}} \end{bmatrix} \text{ or } \begin{bmatrix} \vec{M}^*_{\text{even}} \\ -\vec{M}^*_{\text{odd}} \end{bmatrix}$$ double-valued, up-down symmetry

③ (# of $y=d$) = (# of coexisting phases at fp)

Question: How would one see a latent heat?

② Calculation of densities of a non-fixed point
(includes ordinary points, as well as almost all phase boundary points)

Iterate recursion relation:

$$\vec{M}^{(0)} = b^{-dn}\,\vec{M}^{(n)} \cdot \underbrace{\hat{T} \cdot \hat{T} \cdot \hat{T}}_{n \text{ different factors}} \cdots$$

$\vec{M}^{(n)}$ calculated or, more often, guessed.

$U^* \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad D^* \Rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $\quad F^* \Rightarrow \begin{bmatrix} 1 \\ 1 \\ \pm 1 \end{bmatrix}$

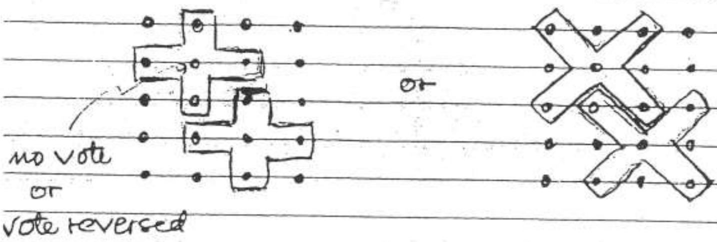
C^* must calculated $\quad N^* \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

In principle, infinitely many iterations to reach stable fp.
In practice, this is no problem.

Antiferromagnet

⁘ does not make sense (note: ties for ferromagnet no problem)
does not distinguish antiferro from para.

Use:



#13

VII. Continuous Spin Models

will lead us to **Gaussian Model**

Momentum-Space RG

$s_i = \pm 1$, or $s_i^z = \pm 1/2$ spin-1/2 system.

Generalize to spin-s: $s_i^z = -s, -s+1, \dots, s-1, s$.

The coupling is $J \frac{s_i^z}{s} \frac{s_j^z}{s}$

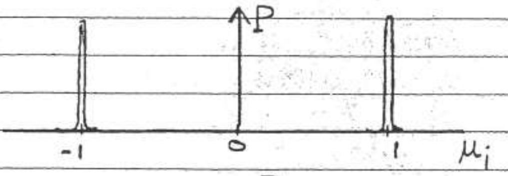
Define normalized spin variable $\mu_i = \frac{s_i^z}{s}$

$$J \mu_i \mu_j$$

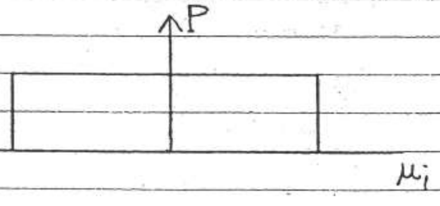
Partition function

$$\sum_{\{\mu_i\}} e^{-\beta \mathcal{H}} = \int_{-\infty}^{\infty} \left(\prod_i d\mu_i\right) \underbrace{\prod_i \left\{ \delta\left(\mu_i - \frac{s}{s}\right) + \delta\left(\mu_i - \frac{s-1}{s}\right) + \dots + \delta\left(\mu_i + \frac{s}{s}\right) \right\}}_{\equiv P\{\mu_i\} \equiv \prod_i P(\mu_i)} e^{-\beta \mathcal{H}}$$

Examples: spin-1/2 Ising



$s \to \infty$ Ising

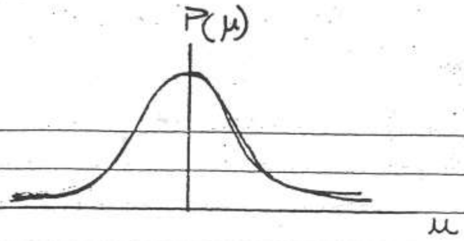


#12

We shall eventually introduce other vector components, e.g. s_i^x

Commutation rule $[\mu_i^\alpha, \mu_i^\beta] = i\hbar \epsilon_{\alpha\beta\gamma} \frac{\mu_i^\gamma}{s} \underset{s \to \infty}{=} 0$.

Everything commutes, classical vectors.



Gaussian model.

$$P(\mu) = \sqrt{\frac{\alpha}{2\pi}}\, e^{-\frac{\alpha}{2}\mu^2}$$

$$Z = \left(\frac{\alpha}{2\pi}\right)^{N/2} \int_{-\infty}^{\infty} (d\mu)\; e^{-\frac{1}{2}\sum_{ij} K_{ij}\mu_i\mu_j + \sum_i H_i \mu_i}$$

$$K_{ij} \equiv \alpha\,\delta_{ij} - J_{ij}$$

Exactly soluble because Gaussian integrals can be performed.

Standard Gaussian integral:

$$\vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \end{pmatrix} \qquad \mu^T = (\mu_1\ \mu_2\ \dots)$$

$$\left[\, Z = \left(\frac{\alpha}{2\pi}\right)^{N/2} \int (d\mu)\; e^{-\frac{1}{2}\mu^T K \mu + H^T\mu} = \frac{(\alpha)^{N/2}}{|\det K|^{1/2}}\, e^{\frac{1}{2} H^T K^{-1} H} \,\right]$$

as long as K is real symmetric and positive definite.

This is essentially "completing the square". Show this:

K can be diagonalized by a real orthogonal transformation.

$$\exists\, S, \qquad S^{-1} = S^T$$

$$1 = |S^T S| = |S^T|\,|S| = |S|^2$$

$$S^{-1} K S = \begin{bmatrix} K_1 & & \\ 0 & K_2 & \\ 0 & 0 & K_3 \\ \dots & & \end{bmatrix} \equiv \tilde{K}$$

$$\frac{1}{2}\mu^T \overset{SS^{-1}}{\vee} K \overset{SS^{-1}}{\vee} \mu = \frac{1}{2}\mu^T S \tilde{K} S^{-1}\mu = \frac{1}{2}\sigma^T \tilde{K}\sigma$$

$$H^T\mu = H^T S \sigma \qquad \begin{aligned}\mu &\equiv S\sigma \\ \sigma &\equiv S^{-1}\mu\end{aligned} \qquad \left|\frac{\partial\mu}{\partial\sigma}\right| = 1 \quad \text{Jacobian is unity}$$

$$Z = \left(\frac{\alpha}{2\pi}\right)^{N/2} \int_{-\infty}^{\infty} (d\sigma)\; e^{-\frac{1}{2}\sum_i K_i \sigma_i^2 + \sum_i (H^T S)_i \sigma_i}$$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}K x^2} = \sqrt{\frac{2\pi}{K}}$$

$$Z = \left(\frac{\alpha}{2\pi}\right)^{N/2} \prod_i \sqrt{\frac{2\pi}{K_i}}\; e^{\frac{1}{2} H_j^T S_{ji} \frac{1}{K_i} S^T_{ik} H_k}$$

$$= \frac{(\alpha)^{N/2}}{|K|^{1/2}}\, e^{\frac{1}{2} H_j^T (K^{-1})_{jk} H_k}$$

$$S_{ji}\frac{1}{K_i} S^T_{ik}\; K_{kl} =$$

$$= S_{ji}\frac{1}{K_i}\; K_i S^T_{il} = \delta_{jl}$$

$$S^T_{ik} K_{kl} S_{lj} = \delta_{ij} K_j$$

$$S^T_{ik} K_{kl} = \delta_{ij} K_j S^T_{jl}$$

$$= K_i S^T_{il}$$

$$\boxed{F = -kT\left[\frac{N}{2}\ln\alpha + \frac{1}{2} H^T K^{-1} H - \frac{1}{2}\ln \prod_i K_i\right]}$$

Phase transition of gaussian model

$K_i \to 0$ $\quad$ Z diverges

No need for thermodynamic limit!

$$K_{ij} = \alpha\delta_{ij} - J_{ij} = \alpha\delta_{ij} - \beta\hat{J}_{ij}$$

1) high T: β small

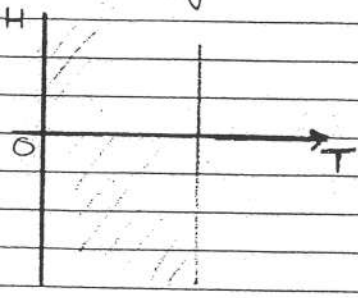
$-\beta\hat{J}_{ij}$ neglected compared to $\alpha\delta_{ij}$: $\quad$ $K_i \simeq \alpha$.

2) low T:

$\beta\hat{J}_{ij}$ dominated

negative eigenvalue: divergence.

Phase diagram:



Now specialize and study eigenvalues K_i :

$K_{ij} = K(\vec{r}_i - \vec{r}_j)$

Assume translational invariance, pbc.

$$\mu(\vec{r}) = \frac{1}{\sqrt{N}} \sum_k^{BZ} e^{i\vec{k}\vec{r}} \mu(\vec{k})$$

$$\mu(\vec{k}) = \frac{1}{\sqrt{N}} \sum_r e^{-i\vec{k}\vec{r}} \mu(r) \quad = \mu^*(-\vec{k})$$

$$K(\vec{r}) = \frac{1}{N} \sum_k e^{i\vec{k}\vec{r}} K(\vec{k})$$

$$K(\vec{k}) = \sum_r e^{-i\vec{k}\cdot\vec{r}} K(\vec{r}) \quad = K^*(-\vec{k}) \quad = K(-\vec{k})$$

$$\frac{1}{N} \sum_k e^{i\vec{k}\vec{r}} = \delta_{\vec{r},\vec{0}} \qquad \frac{1}{N} \sum_r e^{i\vec{k}\vec{r}} = \delta_{\vec{k},\vec{0}}$$

$$\begin{cases} e^{iq(r-r')+ikr+ik'r'} \\ \delta(q+k)\,\delta(q-k') \end{cases}$$

$$-\beta\mathcal{H}_{Gauss} = -\frac{1}{2} \sum_k K(k) \underbrace{\mu(k)\mu(-k)}_{|\mu(k)|^2} + \frac{1}{2} \sum_k \left[H(-k)\mu(k) + H(k)\mu(-k) \right]$$

$$\mu(k) = \frac{1}{\sqrt{2}} \left[\tau_1(k) - i\,\tau_2(k) \right]$$

$$\mu(-k) = \frac{1}{\sqrt{2}} \left[\tau_1(k) + i\,\tau_2(k) \right]$$

$$\sqrt{2}\,\tau_1(k) = \mathrm{Re}\,\mu(k) \quad = \mathrm{Re}\,\mu(-k)$$

$$\sqrt{2}\,\tau_2(k) = -\mathrm{Im}\,\mu(k) \quad = +\mathrm{Im}\,\mu(-k)$$

$$\tau_1(k) = \tau_1(-k)$$

$$\tau_2(k) = -\tau_2(-k)$$

$$-\beta\mathcal{H}_{Gauss} = -\frac{1}{2} \sum_k^{\frac{1}{2}BZ} K(k) \left[\tau_1^2(k) + \tau_2^2(k) \right] + \frac{1}{2} \sum_k^{\frac{1}{2}BZ} \left[h_1(k)\,\tau_1(k) + h_2(k)\,\tau_2(k) \right]$$

$$\mu(r) = \frac{1}{\sqrt{2N}} \sum_k e^{i\vec{k}\vec{r}} \left[T_1(k) - i T_2(k)\right] =$$

$$= \frac{1}{\sqrt{2N}} \sum_k e^{-i\vec{k}\vec{r}} \left[T_1(k) + i T_2(k)\right] =$$

$$= \sqrt{\frac{2}{N}} \sum_k^{\frac{1}{2}BZ} \left[\cos(kr)\, T_1(k) + \sin(kr)\, T_2(k)\right]$$

Matrix, $S: \langle nk|r\rangle = \begin{cases} \sqrt{\frac{2}{N}} \cos(kr), & n=1 \\ \sqrt{\frac{2}{N}} \sin(kr), & n=2 \end{cases}$ Real matrix

Orthogonal?

$$\langle r'|nk\rangle\langle nk|r\rangle = \frac{2}{N} \sum_k^{\frac{1}{2}BZ} \left\{\cos(kr')\cos(kr) + \sin(kr')\sin(kr)\right\}$$

$ka = \pi$

$$\frac{1}{N}\sum_k = \frac{a}{2\pi}\int_{-\pi/a}^{\pi/a} dk$$

$$= \frac{1}{N}\sum_k \cos\left[k(r-r')\right] =$$

$$= \frac{a}{2\pi}\int_{-\pi/a}^{\pi/a} dk \cos\left[k(r-r')\right] = \delta_{\frac{r}{a},\frac{r'}{a}} \quad ✓$$

Jacobian is unity

$$Z_{Gaussian} = \int_{-\infty}^{\infty} \left\{\prod^{\frac{1}{2}BZ} dT_1(k)\, dT_2(k)\right\} e^{-\beta \mathcal{H}_{Gauss}} \leftarrow \underline{\text{diagonal form}}$$

$\sigma(k) \equiv T_1(k)$ This half of BZ

$\sigma(k) \equiv T_2(-k)$ Other half of BZ

$$Z_{Gauss} = \left(\frac{\alpha}{2\pi}\right)^{N/2} \int_{-\infty}^{\infty} \left\{\prod^{BZ} d\sigma(k)\right\} e^{-\frac{1}{2}\sum_k^{BZ} K(k)\sigma^2(k) + \sum_k h(k)\sigma(k)}$$

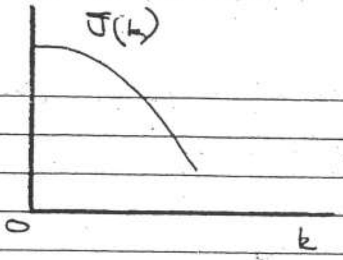
$$= (\alpha)^{N/2} \left\{\prod_k \frac{1}{K(k)^{1/2}}\right\} e^{h^2(k)/2K(k)}$$

Agrees with previous derivation. Now, we have also characterized $\sigma(k)$ variables.

especially $K(k) = K_i$

Form of $K(k)$:

$K(k) = \alpha - J(k)$



$J(k) = J(0) - ck^2 + \ldots$

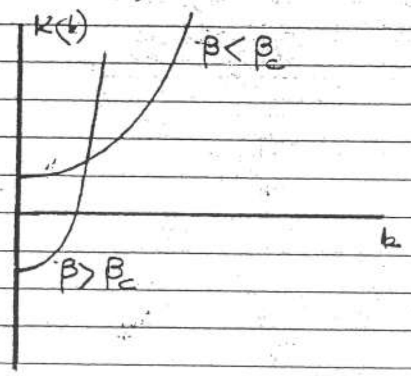
1) Example: nn interaction J, hypercubic lattice.

$$J(k) = J \sum_{\hat{a}} e^{i\vec{k}\cdot\hat{a}} = 2J \sum_{i=1}^{d} \cos(k_i a)$$

$$= qJ - Ja^2k^2 + \ldots$$

$$K(k) = (\alpha - q\beta\hat{J}) + \beta\hat{J}a^2k^2 + \mathcal{O}(k^4)$$

$\beta_c = \dfrac{\alpha}{q\hat{J}}$



2) Continuum spin model: $\mu(r)$, r continuously variable, not on lattice.

$$-\nabla^2 \leftrightarrow k^2$$

Critical Exponents

$$F_{sing} \sim \ln \prod_k K(k) = \sum_k \ln K(k) \overset{N\to\infty}{=} \frac{N}{(2\pi)^d}\int (d^dk) \ln(t + \cancel{c}k^2)$$

$$\frac{C_{sing}}{N} \sim \frac{\partial^2}{\partial t^2}\int k^{d-1}dk \ln(t + \cancel{c}k^2)$$

$$\sim \int^{\pi/a} \frac{k^{d-1}dk}{(t+k^2)^2} \qquad x = k/t^{1/2}$$

($H \neq 0$ different critical behavior, not interesting.)

$$= t^{-2}\int^{\pi/a}\frac{k^{d-1}dk}{(1+k^2/t)^2} = t^{-2+d/2}\int^{\pi/at^{1/2}\to\infty}\frac{x^{d-1}dx}{(1+x^2)^2}.$$

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① $d<4$ Integral converges.

$$\frac{C_{sing}}{N} \sim t^{-2+d/2} \qquad \alpha = 2 - d/2$$

② $d>4$ Integral diverges

$$\frac{C_{sing}}{N} \sim t^{-2+d/2} \int^{t^{-1/2}} x^{d-5}\,dx \sim t^{-2+d/2}\left(t^{-1/2}\right)^{d-4} = t^{0}$$

Note: new mechanism Singularity comes from zone boundary, not zone cent

③ $d=4$ $\qquad \frac{C_{sing}}{N} \sim \ln t$

Correlation function

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$$\Gamma(k) \sim \frac{\int (d\sigma)\, \sigma^2(k)\, e^{-\beta H}}{Z} = \frac{\int d\sigma(k)\, \sigma^2(k)\, e^{-\frac{1}{2}K(k)\sigma^2(k)}}{\int d\sigma(k) \quad e^{-\frac{1}{2}K(k)\sigma^2(k)}}$$

$$= \frac{-2\frac{d}{dK(k)}\sqrt{\frac{2\pi}{K(k)}}}{\sqrt{\frac{2\pi}{K(k)}}} = \frac{1}{K(k)} \sim \frac{1}{t+k^2}$$

At critical point, $t=0$, $\qquad \Gamma(k) \sim \frac{1}{k^2} = \frac{1}{k^{2-\eta}} \qquad \eta = 0$.

Close to critical point

$$G(r) \sim \int \frac{e^{ikr}\,dk}{t+k^2} \sim e^{-r t^{1/2}(\text{const})} \cong e^{-r/\xi}$$

pole at $k = i t^{1/2}$ $\qquad \xi \sim t^{-1/2} \qquad \nu = 1/2$

cf mean field theory lecture.

Exponent relation: $\alpha = 2 - d\nu$

hyperscaling violated at $d>4$.

RG treatment of the Gaussian model

Wilson + Kogut Phys. Repts. C 12, 75 (1974)

$$Z = \left(\frac{\alpha}{2\pi}\right)^{N/2} \int_{\mathcal{C}} (d\mu)\, e^{-\frac{1}{2}\sum_k K(k)\overbrace{\mu(k)\mu(-k)}^{[\tau_1^2(k)+\tau_2^2(k)]} + \frac{1}{\sqrt{N}}\sum_k \tilde{H}(k)\mu(-k)}$$

$$= \left(\frac{\alpha}{2\pi}\right)^{N/2} \int (d\sigma)\, e^{-\frac{1}{2}\frac{N}{(2\pi)^d}\int (dk)\, K(k)\sigma(k)\sigma(-k) + \frac{\sqrt{N}}{(2\pi)^d}\int (dk) H(k)\sigma(-k)}$$

$$= \left(\frac{\alpha}{2\pi}\right)^{N/2}\left(\frac{1}{N}\right)^{N/2} \int (d\sigma)\, e^{-\frac{1}{2}\int \frac{(dk)}{(2\pi)^d} K(k)\sigma(k)\sigma(-k) + \int \frac{(dk)}{(2\pi)^d} H(-k)\sigma(k)}$$

$$\mu(k) = \frac{1}{\sqrt{2}}\left[\tau_1(k) - i\tau_2(k)\right]$$

$$\mu(-k) = \frac{1}{\sqrt{2}}\left[\tau_1(k) + i\tau_2(k)\right]$$

Deform contour of integration $\mathcal{C}$

Traverse finite region along real axis.

Completing arcs at infinity vanish.

$$Z = \left(\frac{\alpha}{2\pi}\right)^{N/2} \int_{-\infty}^{\infty} (d\sigma)\, e^{-\frac{1}{2}\sum_k K(k)\sigma(k)\sigma(-k) + \frac{1}{\sqrt{N}}\sum_k H(k)\sigma(k)}, \qquad H(k) = \mathrm{Re}\,\tilde{H}(k)$$

$$K(k) = (\alpha - q\beta\hat{J}) + \beta\hat{J}a^2k^2$$

$$K(k) = r + ck^2$$

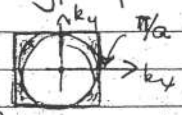
Transformation:

① Separate degrees of freedom:

$$\sigma(k) \begin{cases} 0 < k < b^{-1} & \text{(long wavelength keep)} \\ b^{-1} < k < 1 & \text{(short wavelength } \int \text{out)} \end{cases}$$

write as p.

Approximate Brillouin zone by a hypersphere.



[What happens at corners should not affect $k \to 0$ phenomena.]

Choose length scale so that radius is 1.

[PSRG: Choosing cells, $a' = ba$, and summing out internal degrees of freedom]

For Gaussian model, this is easily done, because $\vec{k}$ and $\vec{p}$ are not coupled.

$$Z = \left(\frac{\alpha}{2\pi N}\right)^{N/2} \left\{ \prod_p \sqrt{\frac{2\pi}{K(p)}}\; e^{\frac{1}{2} H(p)\frac{1}{K(p)} H(-p)} \right\} \int_{-\infty}^{\infty} \left(\prod_{k \neq p} d\sigma\right) e^{-\frac{1}{2}\int K(k)\sigma(k)\sigma(-k) + \int H(k)\sigma(-k)}$$

One apparent effect: additive constant in Hamiltonian changed.

(2) Define a new length scale, in relation to previous one:

$k' = bk$

We want to make renormalized system structurally identical to original system.

Above choice guarantees that Brillouin zone has radius 1.

[PSRG: $r' = r/b$]

$$-\beta'\mathcal{H}' = -\frac{1}{2}\int\frac{(dk')}{(2\pi)^d} b^{-d}(r + c b^{-2} k'^2)\,\sigma(k')\,\sigma(-k') + \int\frac{(dk')}{(2\pi)^d} b^{-d} H(k')\,\sigma(-k')$$

(3) Rescale spin variable, i.e. fix definition of renormalized spin.

$\sigma'(k) = \zeta^{-1}\sigma(k)$ zeta!

[PSRG: choice of projection operator]

$$-\beta'\mathcal{H}' = -\frac{1}{2}\int\frac{(dk')}{(2\pi)^d} \zeta^2 b^{-d}(r + c\, b^{-2} k'^2)\,\sigma'(k')\,\sigma'(-k') + \int\frac{(dk')}{(2\pi)^d} \zeta b^{-d} H(k')\,\sigma'$$

$$= -\frac{1}{2}\int\frac{(dk')}{(2\pi)^d} (r' + c' k'^2)\,\sigma'(k')\,\sigma'(-k') + \int\frac{(dk')}{(2\pi)^d} H'(k')\,\sigma'(-k').$$

Recursion relation: $r' = \zeta^2 b^{-d} r$

$c' = \zeta^2 b^{-d-2} c$

$H'(k') = \zeta b^{-d} H(bk)$

What have we done?

① Spherical Brillouin zone.

② Eliminate Outside Shell (onion)

③ Dilate

④ However, now thinned out.
Factor introduced in continuum limit changes:

$$\frac{N}{(2\pi)^d} \rightarrow \frac{N'}{(2\pi)^d} = \frac{N}{(2\pi)^d}\, b^{-d},$$

Thus far, ζ quite arbitrary.

) If both r^* and c^* were either zero or infinite,

the recursion relation, which we need in linearized form

around the fixed point,

would be indeterminate and useless.

) $r^* \neq 0, \infty$. $\rightarrow$ $\zeta = b^{d/2}$

$c' = b^{-2} c$, $c^* = 0$ stable fp, not a phase transition, will have to look at k^4 term.

③. $c^* \neq 0, \infty$. $\rightarrow$ $\zeta = b^{d/2+1}$.

$r' = b^2 r$ $r^* = 0$ unstable fp, phase transition.

$y_T = 2$, $\nu = 1/2$

Look at $H'(k=0) = b^{1-d/2} H(k=0)$ irrelevant for $d>2$ - artificial aspect of model - cf. phase diagram

[redacted]

Fixed point: $r^* = 0$
c^* finite, arbitrary (usually taken as 1)

Checks with previous solution: Transition point does not depend on c, neither do the exponents

Correlation function:

$$\Gamma(r) = \frac{\int (d\sigma)\, \sigma(0)\sigma(r)\, e^{-\beta\mathcal{H}}}{Z} \underset{\text{limit}}{\overset{\text{continuum}}{\Rightarrow}} \frac{\int (d\hat{\sigma})\, a^d\hat{\sigma}(0)\, a^d\hat{\sigma}(r)\, e^{-\beta\mathcal{H}}}{Z} =$$

$$= a^{2d}\,\hat{\Gamma}(r) \qquad (\wedge \text{ means density})$$

At fixed point,

$$\hat{\Gamma}(r_1 - r_2)\, d^d r_1\, d^d r_2 = \hat{\Gamma}'(r_1' - r_2')\, d^d r_1'\, d^d r_2'$$

$$\hat{\Gamma}(r_1 - r_2) = \hat{\Gamma}'(r_1' - r_2')\, b^{-2d}$$

$$\frac{\text{const}}{|r_1 - r_2|^{d-2+\eta}} = \frac{\zeta^2\, \text{const}}{|(r_1 - r_2)/b|^{d-2+\eta}}\, b^{-2d}$$

$$1 = \zeta^2\, b^{d-2+\eta}\, b^{-2d}$$
$$= b^{d+2}\, b^{-2+\eta}\, b^{-d} = b^{\eta}$$

$$\eta = 0.$$

Careless application of RG gives
specific heat singularity $\alpha = 2 - d/y_T = 2 - d/2$, $C \sim t^{-\alpha}$

Additive constant generates dominant term t^0 for $d > 4$.

i.e.
what we do is $\quad \frac{1}{N}\ln Z = \frac{1}{b^d N'}\ln Z' + \tilde{G}$

Differentiate both sides.
Hypothesize power law behavior of $C \sim At^{-\alpha}$
Obtain α.
In this case, $\tilde{G}$ will dominate RHS at small t $(d > 4)$. #

At this stage, impossible to tell whether analytic background or discont.
Careful treatment of low-T phase (Landau-Wilson model) shows it is discont. (log at $d=$)

Landau-Wilson Model

We went from spin-½ Ising,
to $s = \infty$ Ising,
to Gaussian weight.

Found an exactly soluble,
somewhat artificial model,
mean-field critical behavior.

Something has evidently been lost along the way

Return to spin-½ model. **Hubbard transformation**

$$s_i = \pm 1$$

$$e^{\frac{1}{2}s^T J s} = \frac{|J^{-1}|^{1/2}}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} (\prod_i d\mu)\, e^{-\frac{1}{2}\mu^T J^{-1}\mu + s^T\mu}$$

μ in $[-\infty, \infty]$.

$$Z = \sum_{\{s\}} e^{\frac{1}{2}s^T J s + H^T s} = \sum_{\{s\}} \frac{1}{(2\pi)^{N/2}|J|^{1/2}} \int_{-\infty}^{\infty} (d\mu)\, e^{-\frac{1}{2}\mu^T J^{-1}\mu + \mu^T s + H^T s} =$$

$$= \left(\frac{2}{\pi}\right)^{N/2} \frac{1}{|J|^{1/2}} \int (d\mu)\, e^{-\frac{1}{2}\mu^T J^{-1}\mu} \prod_i \cosh(\mu_i + H) =$$

$$= \left(\frac{2}{\pi}\right)^{N/2} \frac{1}{|J|^{1/2}} \int (d\mu)\, e^{-\frac{1}{2}\mu^T J^{-1}\mu + \sum_i \ln\cosh(\mu_i + H_i)}$$

$$\ln\cosh(\mu_i + H) \simeq \ln\left[1 + \frac{(\mu_i + H_i)^2}{2} + \frac{(\mu_i + H_i)^4}{4!}\right] \simeq$$

$$\simeq \ln\left[1 + \frac{\mu_i^2}{2} + \frac{\mu_i^4}{4!} + \mu_i H_i\right] \simeq$$

$$\simeq \frac{\mu_i^2}{2} + \frac{\mu_i^4}{4!} + \mu_i H_i - \frac{1}{2}\left(\frac{\mu_i^2}{2}\right)^2 =$$

$$= \frac{\mu_i^2}{2} - \frac{\mu_i^4}{12} + \mu_i H_i$$

1+2 terms: $-\frac{1}{2}\sum \mu(k)\left[\left(\frac{1}{q\beta\hat{J}} - 1\right) + \frac{a^2k^2}{q^2\beta\hat{J}}\right]\mu(-k)$

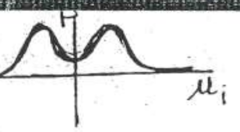
As before.

1: $-\frac{1}{2}\sum_k \mu(k)(q\beta\hat{J} - \beta\hat{J}a^2k^2)^{-1}\mu(-k)$

$= -\frac{(q\beta\hat{J})^{-1}}{2}\sum_k \mu(k)(1 + q^{-1}a^2k^2)\mu(-k)$

$$Z \simeq \left(\frac{2}{\pi}\right)^{N/2} \frac{1}{|J|^{1/2}} \int (d\mu)\, e^{-\frac{1}{2}\sum_{ij}\mu_i (J^{-1})_{ij}\mu_j + \frac{1}{2}\sum_i \mu_i^2 - \frac{1}{12}\sum_i \mu_i^4 + \sum_i H_i\mu_i}$$

$$P(\mu_i) = e^{\frac{1}{2}\mu_i^2 - \frac{1}{12}\mu_i^4}$$



Thus, we are motivated to study:

$$Z = \int (d\mu)\, e^{-\frac{1}{2}\sum_{ij} \mu_i K_{ij} \mu_j - u \sum_i \mu_i^4 + \sum_i H_i \mu_i}$$

Connection: ① $K_{ij} = \alpha\,\delta_{ij} - J_{ij}$, $u \hat{=} 0$ Gaussian model

② $K_{ij} = (J^{-1})_{ij} - \delta_{ij}$, $u = \frac{1}{12}$ leading terms of Hubbard-transformed $s = 1/2$ Ising

③ $K_{ij} = -\alpha\,\delta_{ij} - J_{ij}$, $u = \frac{\alpha}{4} \to \infty$

$$P(\mu) \sim e^{-\frac{\alpha}{4}(\mu_i^2 - 1)^2}$$

Again, $s = 1/2$ Ising.

③ Not very plausible because we'll treat u perturbatively

② More satisfactory

$$Z = \int (d\mu)\, e^{-\frac{1}{2}\int \frac{(dk)}{(2\pi)^d}(r + k^2)\mu(k)\mu(-k) - u\int \frac{(dk_1)}{(2\pi)^d}\frac{(dk_2)}{(2\pi)^d}\frac{(dk_3)}{(2\pi)^d}\mu(k_1)\mu(k_2)\mu(k_3)\mu(-k_1-k_2-k_3) + \int \frac{(dk)}{(2\pi)^d} H(-k)\mu(k)}$$

<u>Renormalization - Group</u> $(H = 0)$.

Integrate $\sigma(p)$, $b^{-1} < p < 1$

leave $\sigma(q)$, $0 < q < b^{-1}$

$$Z = \int (d\mu)\, e^{\mathcal{H}_G(q) - \frac{1}{2}\int \frac{(dp)}{(2\pi)^d}(r + p^2)\mu(p)\mu(-p)} \Big\{ 1 - u \int \frac{(dk_1)}{(2\pi)^d}\frac{(dk_2)}{(2\pi)^d}\frac{(dk_3)}{(2\pi)^d}\mu(k_1)\mu(k_2)\mu(k_3)\mu(-k_1-k_2-k_3) + \frac{1}{2}u^2 \iint \cdots$$

Do $\int \left[\prod d\mu(p)\right]$, reduces to calculating Gaussian averages.

$$\int \left[\prod^{p} d\mu(p)\right] e^{-\frac{1}{2}\int \frac{(dp)}{(2\pi)^d}(r + p^2)\mu(p)\mu(-p)}\, \mu(k_1)\mu(k_2)\mu(k_3)\mu(-k_1-k_2-k_3) \equiv$$

$$\equiv \langle \mu(k_1)\mu(k_2)\mu(k_3)\mu(-k_1-k_2-k_3)\rangle_G \cdot Z_G$$

Gaussian Integrals

$$Z_G = \int \left[\prod_p d\mu(p)\right] e^{-\frac{1}{2}\frac{1}{N}\sum_p K(p)\mu(p)\mu(-p) + \frac{1}{N}\sum_p H(-p)\mu(p)} =$$

$$= \left[\prod_p \frac{2\pi}{K(p)}\right]^{1/2} e^{\frac{1}{2N}\sum_p [K(p)]^{-1} H(p)H(-p)}$$

Used discrete sums - otherwise functional integration - identical results.

$$\langle \mu(p_1)\rangle Z_G = N\frac{\partial}{\partial H(-p_1)} Z_G = \frac{H(p_1)}{K(p_1)} Z_G \xrightarrow{H=0} 0.$$

$$\langle \mu(p_1)\mu(p_2)\rangle Z_G = \frac{N^2\partial^2}{\partial H(-p_1)\partial H(-p_2)} Z_G =$$

$$= N\frac{\partial}{\partial H(-p_1)} \frac{H(p_2)}{K(p_2)} Z_G \xrightarrow{H=0} N\delta_{-p_1,p_2}\frac{1}{K(p_2)} Z_G$$

$$\xrightarrow{\text{cont}} (2\pi)^d \frac{\delta(p_1+p_2)}{K(p_2)} Z_G$$

$$\sum_k \to \frac{N}{(2\pi)^d}\int (dk) \quad \text{because} \quad \sum_k \to \left(\frac{a}{N^{1/d}}\right)^d \int_0^{2\pi/a} (dk) = \left(\frac{a^d}{N}\frac{(2\pi)^d}{a^d}\right)\int_0^1 (dk)$$

$$\sum_k \delta_k \to \frac{N}{(2\pi)^d}\int (dk)\,\frac{(2\pi)^d}{N}\delta(k)$$

$$\left\langle \prod_{i=1}^n \mu(p_i)\right\rangle = \begin{cases} 0 & \text{for } n \text{ odd} \\ \sum_{\text{contraction}} \prod_{\substack{\text{pair} \\ ij}} \frac{(2\pi)^d \delta(p_i+p_j)}{K(p_j)} & \text{for } n \text{ even.}\end{cases}$$

For example,

$$\langle \mu(p_1)\mu(p_2)\mu(p_3)\mu(p_4)\rangle = \frac{(2\pi)^d\delta(p_1+p_2)}{K(p_2)}\,\frac{(2\pi)^d\delta(p_3+p_4)}{K(p_4)} + 2 \text{ pairings}$$

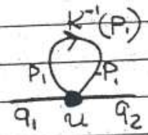
Returning to partition function, $\frac{1}{(2\pi)^d}$ absorbed

$$Z = \int \left[\prod_k d\mu(k)\right] e^{\mathcal{H}(q) + \mathcal{H}_Q(p)} \Big\{ 1 - u \int (dk_1)(dk_2)(dk_3)\, \mu(k_1)\mu(k_2)\mu(k_3)\mu(-k_1-k_2-k_3) +$$
$$+ \frac{u^2}{2} \int (dk_1)\cdots(dk_6)\, \mu(k_1)\cdots\mu(k_6)\mu(-k_1-k_2-k_3)\mu(-k_4-k_5-k_6) + \ldots$$

$$= \int \left[\prod_q d\mu(q)\right] e^{\mathcal{H}_1(q) + \hat{\mathcal{H}}(q)}$$

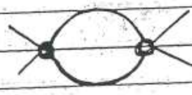
$e^{\hat{\mathcal{H}}(q)}$ will have

① $\mu(q_1)\mu(q_2)$ term:



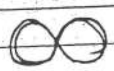
multiplicity $\frac{4\cdot 3}{2} = 6$

② $\mu(q_1)\mu(q_2)\mu(q_3)\mu(q_4)$ term:



$\frac{4\cdot 4}{2}\,\frac{3\cdot 3}{2} = 36$

$\mu(q)$-independent term not interesting, e.g.



<u>Diagram rules for $e^{\hat{\mathcal{H}}(q)}$</u>

1. Enumerate topologically distinct diagrams and determine multiplicity factor: number of contractions $\times \left(\frac{1}{n!}\right)$
2. At each vertex internal lines p_i, external lines q_i. Associate factor
$$-u(2\pi)^d \delta(k_1+k_2+k_3+k_4)$$
3. With each internal line associate factor
$$(2\pi)^d \delta(p_1+p_2) / (r^2+p_1^2)$$
∴ propagator
4. Integrate over internal momenta $\int \frac{(dp)}{(2\pi)^d}$

<u>Rule for $\hat{\mathcal{H}}(q)$:</u>

Remove all disconnected diagrams.

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6  $= -6u(2\pi)^d \int \frac{(dp_1)(dp_2)}{(2\pi)^d(2\pi)^d} \quad \frac{\delta(q_1+q_2+p_1+p_2)(2\pi)^d\delta(p_1+p_2)}{r+p_1^2} =$

$$= -6u(2\pi)^d \int \frac{(dp_1)}{(2\pi)^d} \quad \delta(q_1+q_2) \quad \frac{1}{r+p_1^2} =$$

$$= -6u\,\delta(q_1+q_2) \int \frac{(dp_1)}{\cancel{(2\pi)^d}} \frac{\cancel{(2\pi)^d}}{r+p_1^2}$$

#15

Spherical coordinates in d dimensions:

$x_1 = r\cos\theta_1$

$x_2 = r\sin\theta_1\cos\theta_2$

$x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3$

$\cdots$

$x_{d-1} = r\sin\theta_1 \cdots \sin\theta_{d-2}\cos\varphi$

$x_d = r\sin\theta_1 \cdots \sin\theta_{d-2}\sin\varphi$

$$\int (dp)\, f(p^2) = S_d \int dp\, p^{d-1} f(p^2), \qquad S_d = 2\pi^{d/2}/\Gamma(d/2)$$

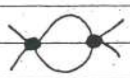
$\Gamma(n) = (n-1)!$

$\Gamma(n+\tfrac{1}{2}) = 1\cdot 3\cdot 5 \cdots (2n-1)\sqrt{\pi}/2^n =$

$= (2n)!\sqrt{\pi}/4^n n!$

6  $= -6u\,\delta(q_1+q_2)\, S_d \int_{b^{-1}}^{1} \frac{dp\, p^{d-1}}{r+p^2} \simeq -6u\,\delta(q_1+q_2)\, S_d \int_{b^{-1}}^{1} dp\, p^{d-3} =$

$$= -6u\,\delta(q_1+q_2)\, S_d \frac{(1-b^{2-d})}{d-2}$$

$)$ 36  $=$

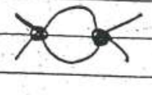
$$= 36\, u^2 \cancel{(2\pi)^{2d}}\cancel{(2\pi)^{2d}} \int \frac{(dp_1)(dp_2)(dp_3)(dp_4)}{\cancel{(2\pi)^d(2\pi)^d(2\pi)^d(2\pi)^d}} \delta(q_1+q_2+p_1+p_2)\,\delta(p_3+p_4+q_3+q_4) \frac{\delta(p_1+p_3)}{r+p_3^2}\frac{\delta(p_2+p_4)}{r+p_4^2} =$$

$$= 36u^2 \int (dp_1)(dp_2)\,\delta(q_1+q_2+p_1+p_2)\,\delta(p_1+p_2-q_3-q_4) \frac{1}{(r+p_1^2)(r+p_2^2)} =$$

$$= 36u^2\,\delta(q_1+q_2+q_3+q_4) \int (dp_1)(dp_2)\,\delta(q_1+q_2+p_1+p_2) \frac{1}{(r+p_1^2)(r+p_2^2)} =$$

$$= 36u^2\,\delta(q_1+q_2+q_3+q_4) \int (dp_1) \frac{1}{(r+p_1^2)[r+(p_1+q_1+q_2)^2]}$$

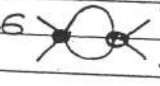
leading term: r^0, q^0:

36  $= 36u^2\,\delta(q_1+q_2+q_3+q_4)\,S_d \int \frac{dp\,p^{d-1}}{p^4} =$

$= 36u^2\,\delta(q_1+q_2+q_3+q_4)\,S_d\left(\frac{1-b^{4-d}}{d-4}\right).$

$\mathcal{H}(q)+\hat{\mathcal{H}}(q) = -u\int\frac{(dq_1)(dq_2)(dq_3)}{(2\pi)^d(2\pi)^d(2\pi)^d}\,\mu(q_1)\mu(q_2)\mu(q_3)\mu(-q_1-q_2-q_3)$

$= -\frac{1}{2}\int\frac{(dq)}{(2\pi)^d}(r+q^2)\mu(q)\mu(-q)$ ✓ $+\int\frac{(dq_1)(dq_2)}{(2\pi)^d(2\pi)^d}\Big[6$  $\Big]\mu(q_1)\mu(q_2)\;+$

$+\int\frac{(dq_1)(dq_2)(dq_3)(dq_4)}{(2\pi)^d(2\pi)^d(2\pi)^d(2\pi)^d}\Big[36$  $\Big]\mu(q_1)\mu(q_2)\mu(q_3)\mu(q_4)\;+\dots=$

$= -\frac{1}{2}\int\frac{(dq)}{(2\pi)^d}(r+q^2)\mu(q)\mu(-q)$ ✓ $-\frac{6u}{(2\pi)^d}\frac{S_d(1-b^{2-d})}{d-2}\int\frac{(dq)}{(2\pi)^d}\mu(q)\mu(-q)\;+$

$+\frac{36u^2 S_d}{(2\pi)^d}\frac{(1-b^{4-d})}{d-4}\int\frac{(dq_1)(dq_2)(dq_3)}{(2\pi)^d(2\pi)^d(2\pi)^d}\mu(q_1)\mu(q_2)\mu(q_3)\mu(-q_1-q_2-q_3).$

Step ① of the RG completed: <u>high-k shell thinned away</u>

② $k' = bq$ $\qquad 0<|k'|<1$ $\qquad$ <u>length rescaling</u>

③ $\mu'(k') = \zeta^{-1}\mu(q=\frac{k'}{b})$ $\qquad$ <u>spin rescaling</u> $\quad$ zeta!

$\mathcal{H}'(k') = -\frac{1}{2}\int\frac{(dk')}{(2\pi)^d}(r+b^{-2}k'^2)\zeta^2 b^{-d}\mu'(k')\mu'(-k') - u\int\frac{(dk_1')(dk_2')(dk_3')}{(2\pi)^d(2\pi)^d(2\pi)^d}\zeta^4 b^{-3d}\mu\mu\mu\mu$

$-\frac{6u}{(2\pi)^d}\frac{S_d(1-b^{2-d})}{d-2}\int\frac{(dk')}{(2\pi)^d}\zeta^2 b^{-d}\mu'(k')\mu'(-k')\;+$

$+\frac{36u^2}{(2\pi)^d}S_d\frac{(1-b^{4-d})}{d-4}\int\frac{(dk_1')(dk_2')(dk_3')}{(2\pi)^d(2\pi)^d(2\pi)^d}\zeta^4 b^{-3d}\mu'(k_1')\mu'(k_2')\mu'(k_3')\mu'(-k_1'-k_2'-k_3')$

$+\dots$

④ Fix ζ. Coefficient of k^2 term invariant. (Search for f.p.)

$$\zeta = b^{1+d/2}$$

Recursion relation.

$$r' = r\zeta^2 b^{-d} + \frac{12u}{(2\pi)^d} \frac{S_d(1-b^{2-d})}{d-2} \zeta^2 b^{-d} =$$

$$= \zeta^2 b^{-d}\left[r + \frac{12u}{(2\pi)^d} \frac{S_d(1-b^{2-d})}{d-2}\right] = b^2\left[r + \frac{12u}{(2\pi)^d} S_d \frac{(1-b^{2-d})}{d-2}\right]$$

$$u' = \zeta^4 b^{-3d}\left[u + \frac{36u^2}{(2\pi)^d} S_d \frac{(1-b^{4-d})}{4-d}\right] = b^{4-d}\left[u + \frac{36u^2}{(2\pi)^d} S_d \frac{(1-b^{4-d})}{4-d}\right]$$

Search Fixed point: $1 - b^{4-d} = u^* \left[\frac{36}{(2\pi)^d} S_d \frac{(1-b^{4-d})}{4-d}\right] b^{4-d}$

u is small and perturbation theory justified for $d \sim 4$
$d = 4 - \epsilon$

$$r' = b^2\left[r + \frac{12u}{(2\pi)^4} S_4 \frac{(1-b^{-2})}{2}\right] =$$

$$= b^2\left[r + \frac{6u}{(2\pi)^4} \frac{2\pi^2}{\Gamma(2)} (1-b^{-2})\right] =$$

$$r' = b^2\left[r + \frac{3}{4\pi^2} u (1-b^{-2})\right]$$

$$u' = b^{\epsilon}\left[u + \frac{36u^2}{(2\pi)^4} 2\pi^2 \left(\frac{1-e^{(4-d)\ln b}}{4-d}\right)\right] =$$

$$u' = b^{\epsilon}\left[u - \frac{9}{2\pi^2} u^2 \ln b\right]$$

Fixed points: ① $u^* = 0 \rightarrow r^* = 0$ Gaussian

$u^* \neq 0$. $b^{-\epsilon} - 1 = -\frac{9}{2\pi^2} u^* \ln b$ $\qquad \epsilon = \frac{9}{2\pi^2} u^*$

$e^{-\epsilon \ln b} - 1 = -\frac{9}{2\pi^2} u^* \ln b$ $\qquad u^* = \frac{2\pi^2 \epsilon}{9}$

$$(1-b^2)\, r^* = b^2 \frac{3u^*}{4\pi^2}(1-b^{-2})$$

$$(b^{-2}-1)\, r^* = \frac{3u^*}{4\pi^2}(1-b^{-2})$$

$$r^* = -\frac{3u^*}{4\pi^2} = -\frac{\cancel{3}}{\cancel{4\pi^2}_{2}}\,\frac{\cancel{2\pi^2}}{\cancel{9}_{3}}\epsilon = \boxed{-\frac{\epsilon}{6} = r^*}$$

$$\boxed{u^* = \frac{2\pi^2}{9}\epsilon}$$

Non-Gaussian

#17 Stability of fixed points

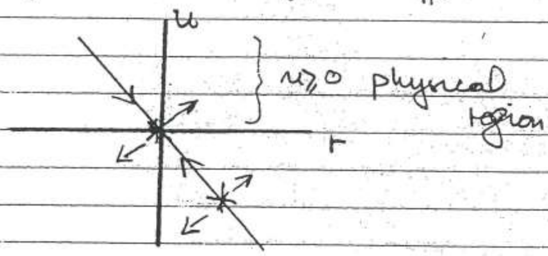
$$T = \begin{bmatrix} b^2 & \frac{3}{4\pi^2}(b^2-1) \\ 0 & b^{\epsilon}\left(1-\frac{9}{\pi^2}u^*\ln b\right) = b^{\epsilon}e^{-\frac{9}{\pi^2}u^*\ln b} = b^{\epsilon-\frac{9}{\pi^2}u^*} \end{bmatrix}$$

Gaussian: $y_1 = 2$ $\quad y_2 = \epsilon$ $\quad$ Stable for $d>4$

Non-gaussian $y_1 = 2$ $\quad y_2 = \epsilon - \frac{9}{\pi^2}u^* = \epsilon - \frac{\cancel{9}}{\cancel{\pi^2}}\frac{\cancel{2\pi^2}}{\cancel{9}}\epsilon = -\epsilon$

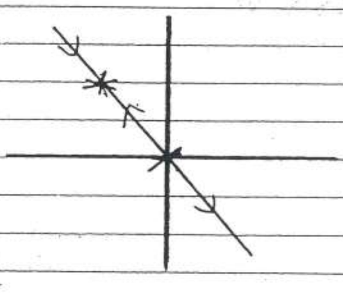
Stable for $d<4$

$d>4$



Phase boundary - universality

$d<4$



#16

Eigenvectors

$$T = \begin{bmatrix} b^2 & \frac{3}{4\pi^2}(b^2-1) \\ 0 & b^4 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad V_2 = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

$$x b^2 + \frac{3}{4\pi^2}(b^2-1) = x b^4$$

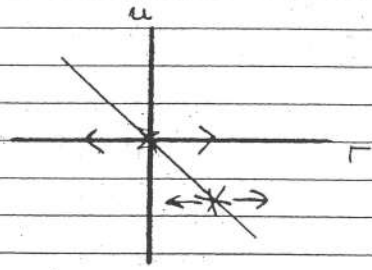
$$x(b^2 - b^4) = -\frac{3}{4\pi^2}(b^2-1)$$

to leading order in ϵ

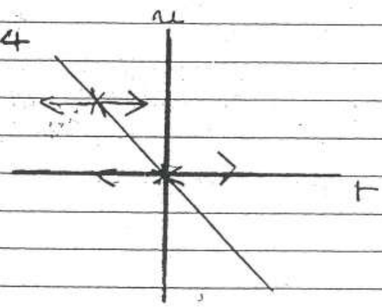
$$x(b^2-1) = -\frac{3}{4\pi^2}(b^2-1)$$

$$x = -\frac{3}{4\pi^2}$$

$a>4$



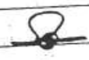
$a<4$



Next correction to strongest eigenvalue:

$$r' = b^2\left[r + \frac{3}{4\pi^2}u(1-b^{-2})\right] + ur \text{ term} + u^2 \text{ term}$$

$\uparrow$ calculate this

In , $$\int \frac{dp\, p^{d-1}}{r+p^2} = \int \frac{dp\, p^{d-1}}{p^2} - r\int\frac{dp\,p^{d-1}}{p^4} =$$

$$= \frac{(1-b^{-d+2})}{d-2} - r\,\frac{(1-b^{-d+4})}{d-4} =$$

$$= \frac{1-b^{-2}}{2} - r\ln b$$

$$r' = b^2\left[r + \frac{3u}{4\pi^2}(1-b^{-2})\left(1 - \frac{2}{1-b^{-2}}r\ln b\right)\right] =$$

$$= b^2\left[r + \frac{3u}{4\pi^2}(1-b^{-2}) - \frac{3u}{{}_2\cancel{4}\pi^2}\cancel{2}r\ln b\right] =$$

$$= b^2\left[r\left(1 - \frac{3u}{2\pi^2}\ln b\right) + \frac{3u}{4\pi^2}(1-b^{-2})\right] =$$

$$= b^{2-\frac{3u}{2\pi^2}}\, r \qquad + \frac{3u}{4\pi^2}(b^2-1)$$

$$\frac{\partial r'}{\partial r} = b^{2-\frac{3u}{2\pi^2}} \quad \text{at NG}$$

$$= b^{2-\frac{3}{\cancel{2\pi^2}}\frac{\cancel{2\pi^2}}{9}\epsilon} = b^{2-\frac{1}{3}\epsilon}$$

$$y_1 = 2 - \frac{\epsilon}{3} \qquad \nu = y_1^{-1} = \frac{1}{2}\left(1 + \frac{1}{6}\epsilon\right)$$

Set $d=3$, $\epsilon=1$: $\quad \nu = \frac{1}{2}\frac{7}{6} = 0.56 \qquad$ (series: 0.64; MF: 0.50)

Actually, asymptotic series:

$$2\nu = 1 + 0.167\epsilon + 0.04\epsilon^2 - 0.016\epsilon^3 + 0.077\epsilon^4 - 0.2\epsilon^5 + 0.67\epsilon^6 - 2.5\epsilon^7 + 10.3\epsilon^8 + \ldots$$

$1.26 \overset{?}{=} 1,\ 1.167,\ 1.207,\ 1.191,\ 1.268,\ 1.068,\ 1.738,\ -0.762,\ 9.538$

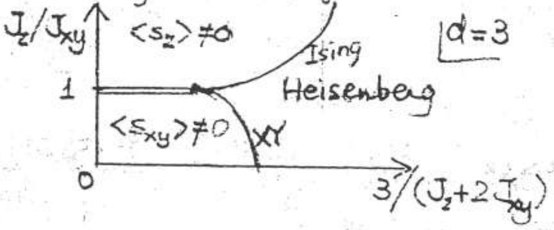
IX. The XY Model in Two Dimensions
$(n=2, \quad d=2)$

The Hamiltonian: $-\beta \mathcal{H} = J \sum_{\langle ij \rangle} \hat{s}_i \cdot \hat{s}_j$, $\quad \hat{s}_i = (\cos\theta_i, \sin\theta_i)$

$$= J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

Realizations: Anisotropic magnets: $-\beta\mathcal{H} = J_z \sum_{\langle ij \rangle} s_i^z s_j^z + J_{xy} \sum_{\langle ij \rangle} (s_i^x s_j^x + s_i^y s_j^y)$

$J_z > J_{xy}$ Ising
$J_z = J_{xy}$ Heisenberg
$J_z < J_{xy}$ XY



Superfluidity $\quad \psi = A(r)\, e^{i\theta(r)}$
Liquid crystals

Spin Waves: Fourier transforming, $\hat{s}(\vec{r}) = \frac{1}{\sqrt{N}} \sum_k e^{i\vec{k}\cdot\vec{r}} \vec{s}(\vec{k})$.

$$1 = \left(\frac{1}{N}\sum_r\right) \hat{s}(\vec{r})\cdot\hat{s}(\vec{r}) = \left(\frac{1}{N}\sum_r\right) \frac{1}{N}\sum_{kk'} e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}\, \vec{s}(\vec{k})\vec{s}(\vec{k}') =$$

$$\frac{1}{N}\sum_k \vec{s}(\vec{k})\vec{s}(-\vec{k}) . \quad \text{Normalization.}$$

Also, without $\sum_r$, consider $\vec{r}=0$: $\left[\sum_k \vec{s}(\vec{k})\right]^2 = N$.

Reality: $\hat{s}(\vec{r})^* = s(\vec{r})$

$$\sum_k e^{-i\vec{k}\cdot\vec{r}} \vec{s}(\vec{k})^* = \sum_k e^{i\vec{k}\cdot\vec{r}} \vec{s}(\vec{k}), \quad \text{so} \quad \vec{s}(-\vec{k})^* = \vec{s}(\vec{k}).$$

Hamiltonian: $-\beta\mathcal{H} = J \sum_r \hat{s}(\vec{r})\cdot\hat{s}(\vec{r}+\hat{x}) \;+ y$ part $=$

$$= \frac{J}{N} \sum_r \sum_{kk'} e^{i\vec{k}\cdot\vec{r} + i\vec{k}'\cdot(\vec{r}+\hat{x})} \vec{s}(\vec{k})\cdot\vec{s}(\vec{k}') \;+ y \text{ part} =$$

$$= J \sum_k \left(e^{-ik_x a} + e^{-ik_y a}\right) \vec{s}(\vec{k})\cdot\vec{s}(-\vec{k}) =$$

$$= J \sum_k \left(2 - \frac{k^2 a^2}{2} + \Theta(k^4)\right) \vec{s}(\vec{k})\cdot\vec{s}(-\vec{k}) =$$

$$= \underset{\substack{\uparrow \\ \text{ground state energy}}}{2NJ} - \sum_k \epsilon_k \underbrace{\vec{s}(\vec{k})\vec{s}(-\vec{k})}_{\text{spin wave (magnon) density}} , \qquad \underset{\substack{\uparrow \\ \text{excitation energies}}}{\epsilon_k} = \frac{J}{2} k^2 a^2 + \Theta(k^4).$$

excitation:

Two Dimensions: Peierls Argument

R. E. Peierls, Helv. Phys. Acta $\underline{7}$, Suppl. II, 81 (1934); Ann. Inst. Henri Poincaré $\underline{5}$, 177 (1935).

Assume magnetization along x-direction.

$$\frac{1}{N}\sum_{r}\langle s_y^2(\vec r)\rangle = \frac{1}{N}\sum_{r}\frac{1}{N}\sum_{kk'} e^{i(\vec k+\vec k')\cdot\vec r}\langle s_y(\vec k)\, s_y(\vec k')\rangle = \frac{1}{N}\sum_{k\neq 0}\langle s_y(\vec k)s_y(-\vec k)\rangle$$

$$s_y(\pm\vec k) = \tau_1(k) \pm i\,\tau_2(k), \quad \tau_i \text{ real}.$$

$s_y(\vec k)s_y(-\vec k)\Big]_{k=0} = s$

$\frac{1}{\sqrt N}s_y(0) = \frac{1}{N}\sum_r s_y($

$= 0$ by d

of x-

$$\langle s_y(\vec k)s_y(-\vec k)\rangle = \frac{\int\{d\tau_i\}\,[\tau_1^2(\vec k)+\tau_2^2(\vec k)]\,e^{-\sum_{k'}[\tau_1^2(\vec k')+\tau_2^2(\vec k')]\epsilon_{k'}}}{\int\{d\tau_i\}\,e^{-\sum_{k'}[\tau_1^2(\vec k')+\tau_2^2(\vec k')]\epsilon_{k'}}} =$$

Note: normalization condition ignored.

$$= \frac{\int d\tau_1 d\tau_2\,[\tau_1^2(\vec k)+\tau_2^2(\vec k)]\,e^{-[\tau_1^2(\vec k)+\tau_2^2(\vec k)]\epsilon_k}}{\int d\tau_1 d\tau_2\,e^{-[\tau_1^2(\vec k)+\tau_2^2(\vec k)]\epsilon_k}} =$$

$$= -\frac{d}{d\epsilon_k}\ln\int_{-\infty}^{\infty} d\tau_1 d\tau_2\, e^{-[\tau_1^2+\tau_2^2]\epsilon_k} = -\frac{d}{d\epsilon_k}\ln\frac{\pi}{\epsilon_k} = \frac{1}{\epsilon_k}$$

$$\frac{1}{N}\sum_r\langle s_y^2(\vec r)\rangle = \frac{1}{N}\sum_{k\neq 0}\frac{1}{\epsilon_k} = \frac{1}{N}\,N\left(\frac{a}{2\pi}\right)^d\int\frac{d\vec k}{\epsilon_k} = \left(\frac{a}{2\pi}\right)^d A_d\int_{\pi/L}^{\pi/a} k^{d-1}dk\,\frac{2}{Jk^2a^2}$$

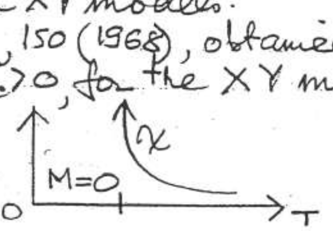
$$\sim \ln L \text{ in } d=2.$$

Divergent fluctuations in the y direction, coming from the long-wavelength spin-waves, not consistent with magnetizat along the x direction.
Suggests that long-wavelength fluctuations destroy magnetiza

Mec

High-Temperature Series for Susceptibility χ

In 1966, H. E. Stanley and T. A. Kaplan saw indication of χ diverging at $T_c > 0$, and questioned the (admittedly approximate) Peierls argument (for Heisenberg model).
In response, N. D. Mermin and H. Wagner, Phys. Rev. Lett. $\underline{17}$, 1133 (1
proved $M=0$ rigorously for Heisenberg and XY models.
Nevertheless, H. E. Stanley, Phys. Rev. Lett. 20, 150 (1968), obtained even stronger evidence for $\chi\to\infty$ at $T_c > 0$, for the XY model
He suggested a new type of low-T phase, characterized by zero magnetization and infinite susceptibility.

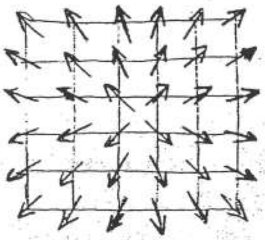


3'

Another type of elementary excitation: Vortices
J.M. Kosterlitz and D.J. Thouless, J. Phys. C $\underline{5}$, L124 (1972);
J. Phys. C $\underline{6}$, 1181 (1973).

Spin waves describe local alignments that vary smoothly in space, with a non-singular continuum limit $(a\to 0)$.

On the other hand, a vortex:



A point singularity upon taking the continuum limit.

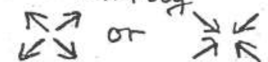
Let's calculate it energy in the continuum limit:

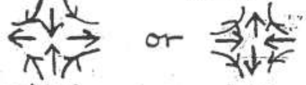
$$-\beta\mathcal{H} = J\sum_{\langle ij\rangle}\cos\theta_{ij} \underset{a\to 0}{\simeq} 2NJ - \frac{J}{2}\sum_{\langle ij\rangle}\theta_{ij}^2 = \cancel{2N}J - \frac{J}{2}\frac{1}{\cancel{a^2}}\int d^2r\,\vec\nabla\theta\cdot\vec\nabla\theta\,\cancel{a^2}$$

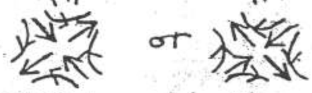
$$-\beta\mathcal{H} = -\frac{J}{2}\int d^2r\,\vec\nabla\theta\cdot\vec\nabla\theta$$

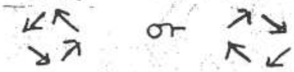
additive constant ignored.

Vortex configurations: $\theta = \tan^{-1}\frac{ax+by}{cx+dy}$.

$\tan\theta = \frac{y}{x}$: $a=d=0$, $b=c$ gives above:  Other vortices:

$\tan\theta = -\frac{y}{x}$: $a=d=0$, $b=-c$: 

$\tan\theta = \frac{x}{y}$: $b=c=0$, $a=d$ 

$\tan\theta = -\frac{x}{y}$: $b=c=0$, $a=-d$ 

$|b|\neq|c|$ or $|a|\neq|d|$ distorts from fourfold symmetry.

$$\vec\nabla\theta = \frac{(cx+dy)^2}{(ax+by)^2+(cx+dy)^2}\left\{\hat x\,\frac{(cx+dy)a-(ax+by)c}{(cx+dy)^2} + \hat y\,\frac{(cx+dy)b-(ax+by)d}{(cx+dy)^2}\right\} =$$

$$= \frac{(ad-bc)}{(ax+by)^2+(cx+dy)^2}\left\{\hat x y - \hat y x\right\}$$

$$\vec\nabla\theta\cdot\vec\nabla\theta = \frac{(ad-bc)^2(x^2+y^2)}{[(ax+by)^2+(cx+dy)^2]^2} = \frac{D^2(x^2+y^2)}{[Ax^2+2Bxy+Cy^2]^2}$$

$A = a^2+c^2$
$B = ab+cd$
$C = b^2+d^2$
$D = ad-bc$

$$I(A,B,C) \equiv \int dx\,dy\,\frac{x^2}{[Ax^2+2Bxy+Cy^2]^2} \overset{\text{Dwight 160.22}}{=} \int dy\,\frac{2Cy^2}{4ACy^2-4B^2y^2}\int dx\,\frac{1}{Ax^2+2Bxy+Cy^2}$$

$$= \frac{C}{2(AC-B^2)}\int\frac{dx\,dy}{Ax^2+2Bxy+Cy^2} = \frac{C\int dx\,dy/[Ax^2+2Bxy+Cy^2]}{2(a^2b^2+a^2d^2+c^2b^2+c^2d^2-a^2b^2-c^2d^2-2abcd)}$$

Dwight 160.01

$$I(A,B,C) = \frac{C}{2D^2}\int \frac{dx\,dy}{Ax^2+2Bxy+Cy^2} = \frac{C}{2D^2}\int dy\, \frac{\cancel{2}}{(\cancel{4}ACy^2-\cancel{4}B^2y^2)^{1/2}}\, \underbrace{\tan^{-1}\frac{2Ax+2By}{(\quad)^{1/2}}}_{\text{same}}\Big|_{x=}$$

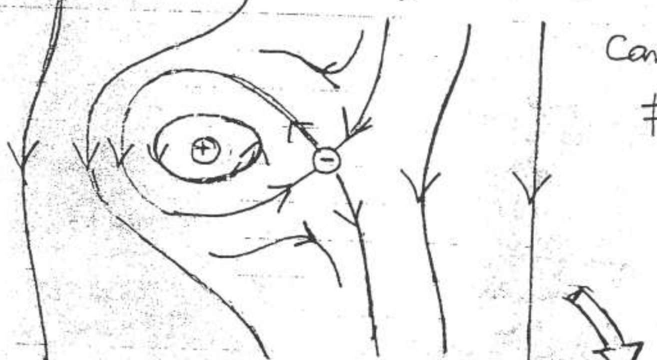
$$= \frac{C}{2D^2}\int \frac{dy}{|Dy|}\,\pi = \frac{\pi C}{2|D|^3}\int\frac{dy}{|y|} = \frac{\cancel{2}\pi C}{\cancel{2}|D|^3}\ln\frac{L/2}{a/2}$$

$$-\beta\mathcal{H} = -\frac{JD^2}{2}\left[I(A,B,C)+I(C,B,A)\right] = -\frac{J\pi(A+C)}{2|D|}\ln\frac{L/2}{a/2} \to -\infty.$$

$$-\beta\mathcal{H} = -\frac{\pi J}{2}\frac{(a^2+b^2+c^2+d^2)}{|ad-bc|}\ln\frac{L/2}{a/2} \to -\infty.$$

Highly energetic, due to an accumulation of infinitesimal displacements.
Pictorially: a single vortex cannot be locally imbedded into the ground state: causes misalignments around the infinite perimeter.

On the other hand, a vortex pair:

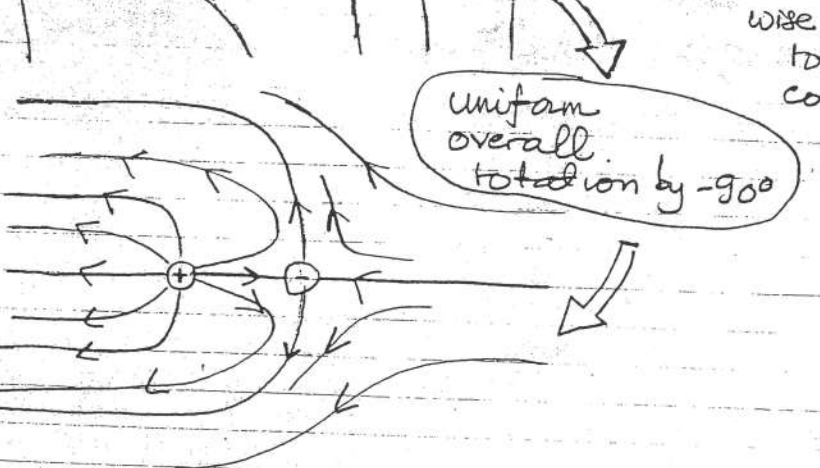


Can be imbedded locally.

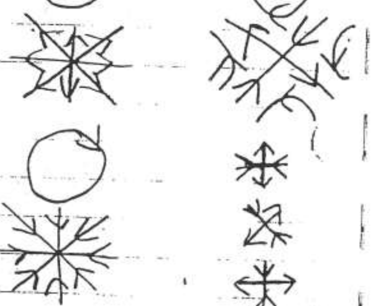
Finite energy.

The charge q of a vortex: convention:
Along a counterclockwise closed countour $\hat{s}$ rotates by $2\pi q$ in the counterclockwise direction.

Or:



$q=+1$: $q=-1$:



Recall: $\mathcal{J} = \frac{J}{kT}$

Two Ways of Obtaining a Vortex Phase Transition:

① A <u>single-vortex configuration</u>: Energy: $+\pi\hat{J}\ln L/a = \mathcal{H}$

Entropy: $k\ln L^2/a^2 = 2k\ln L/a = S$

Free energy: $F = \mathcal{H} - TS = (\pi\hat{J} - 2kT)\ln L/a \to \pm\infty$.

Single vortices spontaneously appear at $T_c = \frac{\pi\hat{J}}{2k}$

or $\mathcal{J}_c = 2/\pi$.

② A <u>vortex pair</u> separated by r:

The misalignment heals at $\sim r/2$ from each center.
Assuming each vortex contributes to its own circle, for an estimate:

Energy $\simeq 2\left\{\pi\hat{J}\ln\frac{r/2}{a/2}\right\} = 2\pi\hat{J}\ln r/a$.

Crudest variant:

$$\left\langle\frac{r^2}{a^2}\right\rangle = \frac{\int^L d\vec{r}\,\frac{r^2}{a^2}e^{-2\pi\mathcal{J}\ln r/a}}{\int_a d\vec{r}\;e^{-2\pi\mathcal{J}\ln r/a}} = \frac{\int^{L/a} dx\; x^{3-2\pi\mathcal{J}}}{\int_1^{L/a} dx\; x^{1-2\pi\mathcal{J}}} =$$

This ln potential makes all technical progress possible: $e^{\ln r} = r$.

$$= \left(\frac{2-2\pi\mathcal{J}}{4-2\pi\mathcal{J}}\right)\frac{\left(\frac{L}{a}\right)^{4-2\pi\mathcal{J}}-1}{\left(\frac{L}{a}\right)^{2-2\pi\mathcal{J}}-1} = \frac{\pi\mathcal{J}-1}{\pi\mathcal{J}-2}\;\frac{\left(\frac{L}{a}\right)^{4(1-\frac{\pi\mathcal{J}}{2})}-1}{\left(\frac{L}{a}\right)^{2(1-\pi\mathcal{J})}-1}$$

$\mathcal{J} > 2/\pi$: $\quad \left\langle\frac{r^2}{a^2}\right\rangle = \frac{\pi\mathcal{J}-1}{\pi\mathcal{J}-2}$ $\quad$ $a\to 0$ limit: $\langle r^2\rangle = 0$ then thermo limit

$\frac{2}{\pi} > \mathcal{J} > 1/\pi$: $\quad \left\langle\frac{r^2}{a^2}\right\rangle = \frac{\pi\mathcal{J}-1}{2-\pi\mathcal{J}}\left(\frac{L}{a}\right)^{4(1-\frac{\pi\mathcal{J}}{2})} \to +\infty$. $\quad \langle r^2\rangle = 0$

$1/\pi > \mathcal{J}$: $\quad \left\langle\frac{r^2}{a^2}\right\rangle = \frac{1-\pi\mathcal{J}}{2-\pi\mathcal{J}}\left(\frac{L}{a}\right)^2 \to +\infty$. $\quad \langle r^2\rangle \to \infty$

Hauge + Hemmer Helv. Physica Acta, 1972

Identify: $\mathcal{J}_c = 2/\pi$, $\quad T_c = \frac{\pi\hat{J}}{2k}$ $\quad$ (same as in ① !)

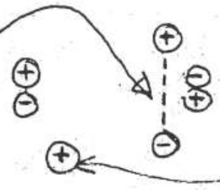
Vortex pair unbinds!

③ <u>Many-vortex interactions</u>: Actually:

⊖⊕

It will be easier to unbind anyone

#4



It will be easier energywise to appear

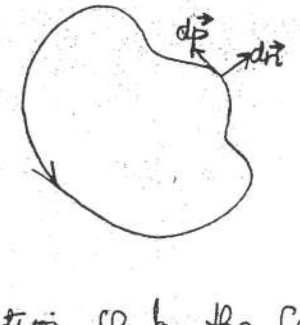
Conclude: $\mathcal{J}_c > 2/\pi$ and need many-body calculation.

MF: Kost. + Thou.
RG: Kost.

[redacted]

Derivation of the Vortex-Vortex Interaction:

$$\oint d\Theta = 2\pi q$$



$$d\Theta = \vec\nabla\Theta\cdot d\vec p = \frac{\partial\Theta}{\partial x}dp_x + \frac{\partial\Theta}{\partial y}dp_y$$

$$dn_x = dp_y, \quad dn_y = -dp_x$$

$$d\Theta = -\frac{\partial\Theta}{\partial x}dn_y + \frac{\partial\Theta}{\partial y}dn_x$$

Introduce a conjugate function φ by the Cauchy-Riemann conditions:

$$\frac{\partial\varphi}{\partial x} = -\frac{\partial\Theta}{\partial y} \quad \text{and} \quad \frac{\partial\varphi}{\partial y} = +\frac{\partial\Theta}{\partial x}$$

applies to the real and imaginary parts of an analytic funct on the complex plane

$$d\Theta = -\frac{\partial\varphi}{\partial x}dn_x - \frac{\partial\varphi}{\partial y}dn_y = -\vec\nabla\varphi\cdot d\vec n$$

$$\oint d\Theta = -\oint\vec\nabla\varphi\cdot d\vec n = -\iint\nabla^2\varphi\,dx\,dy = 2\pi q.$$

$\boxed{\nabla^2\varphi = -2\pi\rho}$ where ρ is the vortex charge distribution occurring on the dual lattice : Poisson's eq.

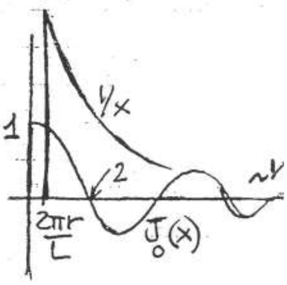
Solution: $\varphi(r) = -2\pi\int d^2r'\rho(\vec r')g(\vec r'-\vec r)$

$g(\vec r)$ is the Green's function of a square lattice.

$\nabla^2 g(\vec r) = \delta(\vec r)$ Fourier transform:

$$-k^2 g(\vec k) = 1/2\pi$$

$$g(\vec r) = -\frac{1}{4\pi^2}\int d^2k\,\frac{e^{i\vec k\vec r}}{k^2}$$



For $\vec r=0$: $g(0) = -\frac{1}{4\pi^2}2\pi\ln k\Big]_{2\pi/L}^{2\pi/a} = -\frac{1}{2\pi}\ln\frac{L}{a}$.

Ignore lattice and use continuum for

$$g(\vec r\gg a) = -\frac{1}{4\pi^2}\int_0^{2\pi}d\theta\int_{2\pi/L}^{2\pi/a\to\infty}\frac{dk}{k}e^{ikr\cos\theta} = -\frac{1}{4\pi^2}\int_0^{2\pi}d\theta\int_{2\pi/L}^{2\pi/a\to\infty}\frac{dk}{k}\cos(kr\cos\theta) =$$

$$= -\frac{1}{2\pi}\int_{2\pi/L}^{\infty}\frac{dk}{k}J_0(kr) = -\frac{1}{2\pi}\int_{2\pi r/L}^{\infty}\frac{dx}{x}J_0(x) \simeq -\frac{1}{2\pi}\int_{2\pi r/L}^{1}\frac{dx}{x} = \frac{1}{2\pi}\ln\frac{r}{L} \ (+\text{small corr.})$$

$\simeq g(a)$ (since $a \to \rho$) smooth function

Combining $g(0) = -\frac{1}{2\pi}\ln\frac{L}{a}$ and $g(\vec{r} \gg a) \simeq \frac{1}{2\pi}\ln\frac{r}{L}$ + small const,

$$\boxed{g(\vec{r}) \simeq \frac{1}{2\pi}\ln\frac{r}{a} - \frac{1}{2\pi}\ln\frac{L}{a} \quad \text{for } r \gg a.}$$

and

$$\boxed{g(0) = 0 - \frac{1}{2\pi}\ln\frac{L}{a}}$$

is supposed to be a good approximation [F. Spitzer, Principles of Random Walk (Princeton: Van Nostrand, 1964) pp 148-151].

$$-\beta\mathcal{H} = -\frac{J}{2}\int d\vec{r}\,\vec{\nabla}\theta\cdot\vec{\nabla}\theta = -\frac{J}{2}\int d\vec{r}\,\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi \overset{\text{by parts}}{=} \frac{J}{2}\int d\vec{r}\,\varphi\nabla^2\varphi$$

another way to surface term → surface terms: $-\frac{J}{2}\int_S d\vec{n}\,\varphi\cdot\vec{\nabla}\varphi$ spin wave part $1\theta\cdot\vec{\nabla}\theta$ $\frac{L}{a}\cdot Lk$

from Poisson:

$$= \frac{J}{2}\int d\vec{r}\,(-2\pi)\int d\vec{r}'\rho(\vec{r}')\,g(\vec{r}'-\vec{r})\,(-2\pi)\rho(\vec{r}) =$$

$$= 2\pi^2 J\int d\vec{r}\,d\vec{r}'\,\rho(\vec{r}')\,g(\vec{r}'-\vec{r})\,\rho(\vec{r}) =$$

$$= \pi J\sum_{i\neq j} q_i q_j \ln\frac{|\vec{r}_i-\vec{r}_j|}{a} + \Delta\sum_i q_i^2 - \pi J\Big(\sum_i q_i\Big)^2 \ln\frac{L}{a} \to \infty$$

vortex chem. pot. $\Delta \sim -4J$

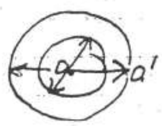
must be zero for configuration to contribute: overall charge neutrality.

Thus, one is to study a two-dimension Coulomb gas ($\equiv$ charged parallel wires).

<u>Renormalization-Group Solution</u>

J. M. Kosterlitz, J. Phys. C $\underline{7}$, 1046 (1974).

Treat a continuum, dilute vortex gas, with core radius a. Eliminate positional degrees of freedom by integrating over possible positions inside a shell around the core: new effective core radius $a' > a$ and renormalized temperature, chemical potential.



$$\int_{D_1} d^2r_1 \int_{D_2} d^2r_2 \cdots \int_{D_{2n}} d^2r_{2n} \simeq \int_{D_1'} d^2r_1 \int_{D_2'} d^2r_2 \cdots \int_{D_{2n}'} d^2r_{2n} +$$

$$+ \frac{1}{2} \sum_{i \neq j} \int_{D_1'} d^2r_1 \cdots \int_{D_{i-1}'} d^2r_{i-1} \int_{D_{i+1}'} d^2r_{i+1} \cdots \int_{D_{j-1}'} d^2r_{j-1} \int_{D_{j+1}'} d^2r_{j+1} \cdots \int_{D_{2n}'} d^2r_{2n} \int_{\tilde{D}_{ji}} d^2r_j \int_{\delta_{ij}} d^2r_i$$

$D_i \equiv \vec{r}_i$ such that $|\vec{r}_i - \vec{r}_{k \neq i}| > a$

$D_i' \equiv \vec{r}_i$ such that $|\vec{r}_i - \vec{r}_{k \neq i}| > a' = a + \delta a$

$\tilde{D}_{ji} \equiv \vec{r}_j$ such that $|\vec{r}_j - \vec{r}_{k \neq i,j}| > a'$

$\delta_{ij} \equiv \vec{r}_i$ such that $a < r_{ij} < a'$

i and j are restricted to oppositely charged pairs: $q_j = -q_i$

#6

$$\int_{\delta_{ij}} d^2r_i \exp\left\{ \pi J \sum_k q_i q_k \ln \frac{r_{ik}}{a} + \pi J \sum_k q_j q_k \ln \frac{r_{jk}}{a} \right\} =$$

$$= \int_{\delta_{ij}} d^2r_i \exp\left\{ \frac{\pi}{2} J q_i \sum_k q_k \ln \frac{r_{ik}^2}{r_{jk}^2} \right\} = \quad \#5$$

$$= \int_{\delta_{ij}} d^2r_i \exp\left\{ \frac{\pi J}{2} q_i \sum_k q_k \ln \frac{r_{ij}^2 + r_{jk}^2 - 2\vec{r}_{ij}\cdot\vec{r}_{jk}}{r_{jk}^2} \right\} =$$

$$= a\,da \int d\theta \prod_k \left\{ 1 + \frac{2\vec{a}\cdot\vec{r}_{jk}}{r_{jk}^2} + \frac{a^2}{r_{jk}^2} \right\}^{\frac{\pi J}{2} q_i q_k} \simeq$$

$$\simeq a\,da \int d\theta \prod_k \left\{ 1 + \pi J q_i q_k \frac{\vec{a}\cdot\vec{r}_{jk}}{r_{jk}^2} + \frac{\pi J}{2} q_i q_k \frac{a^2}{r_{jk}^2} + \frac{1}{\cancel{2}} \frac{\pi J}{\cancel{2}} q_i q_k \left(\frac{\pi J}{2} q_i q_k - 1\right) \cancel{4} \frac{(\vec{a}\cdot\vec{r}_{jk})^2}{r_{jk}^4} \right\} \simeq$$

$$\simeq a\,da \int d\theta \left\{ 1 + \sum_k \pi J q_i q_k \frac{\vec{a}\cdot\vec{r}_{jk}}{r_{jk}^2} + \sum_k \frac{\pi J}{2} q_i q_k \frac{a^2}{r_{jk}^2} + \sum_k \pi J q_i q_k \left(\frac{\pi J}{2} q_i q_k - 1\right) \frac{(\vec{a}\cdot\vec{r}_{jk})^2}{r_{jk}^4} + \right.$$

$$\left. + \sum_{(k \neq l)} \pi^2 J^2 \cancel{q_i} q_k \cancel{q_i} q_l \frac{(\vec{a}\cdot\vec{r}_{jk})(\vec{a}\cdot\vec{r}_{jl})}{r_{jk}^2\, r_{jl}^2} \right\} =$$

$$= a\,da\, 2\pi \left\{ 1 + \cancel{\sum_k \frac{\pi J}{2} q_i q_k \frac{a^2}{r_{jk}^2}} + \sum_k \pi J q_i q_k \left(\frac{\pi J}{2} q_i q_k \cancel{-1}\right) \frac{a^2}{r_{jk}^2} \frac{1}{2} + \right.$$

$$\left. + \sum_{(k \neq l)} \pi^2 J^2 q_k q_l \frac{a^2}{2} \frac{\vec{r}_{jk}\cdot\vec{r}_{jl}}{r_{jk}^2\, r_{jl}^2} \right\} =$$

$$= 2\pi a\,da \left\{ 1 + \frac{\pi^2 J^2 a^2}{4} \sum_k \frac{1}{r_{jk}^2} + \frac{\pi^2 J^2 a^2}{2} \sum_{(k \neq l)} q_k q_l \frac{\vec{r}_{jk}\cdot\vec{r}_{jl}}{r_{jk}^2\, r_{jl}^2} \right\}.$$

Note:
① k, l do not include i, j
② $q_i q_j \ln \frac{r_{ij}}{a} = \Theta(da)$ can be ignored compared to other interaction

$\cos\theta_{jk} \cos\theta_{jl} = \frac{1}{2}[\cos(\theta_{jk} - \theta_{jl}) + \cos(\theta_{jk} + \theta_{jl})]$
$= \frac{1}{2}[\cos\theta_{lk} + \cos(2\theta_j - \theta_k ...)]$

Now do the d^2r_j integration:

$$\int_{\bar D_{ji}} d^2r_j \int_{\delta_{ij}} d^2 r_i \exp\left\{\pi J\sum_k q_i q_k \ln\frac{r_{ik}}{a} + \pi J \sum_k q_j q_k \ln \frac{r_{jk}}{a}\right\} \simeq$$

$$\simeq 2\pi a\,da \int_{\bar D_{ji}} d^2r_j\left\{1+\frac{\pi^2J^2}{4}a^2\sum_k\frac{1}{r_{jk}^2}+\frac{\pi^2J^2}{2}a^2\sum_{(k\neq \ell)} q_kq_\ell\frac{\vec r_{jk}\cdot\vec r_{j\ell}}{r_{jk}^2\, r_{j\ell}^2}\right\}\simeq$$

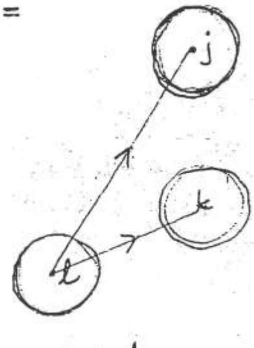
$$\simeq 2\pi a\,da \left\{A + \frac{\pi^2J^2}{4}a^2\sum_k 2\pi\ln\frac{L}{a} + \frac{\pi^2J^2}{2}a^2\sum_{(k\neq \ell)}q_kq_\ell\int_{\bar D_{ji}}d^2r_j \frac{\vec r_{jk}\cdot\vec r_{j\ell}}{r_{jk}^2\, r_{j\ell}^2}\right\}$$

really $\int_{\bar D_{ji}} d^2r_j \frac{\vec r_{jk}\cdot\vec r_{j\ell}}{r_{jk}^2\, r_{j\ell}^2} \Rightarrow \int_{\bar D_{ji}} d^2r_j \frac{\vec r_{jk}\cdot(-\vec r_j)}{r_{jk}^2\, r_j^2} = \int_{\bar D_{ji}} d^2r_j\frac{(\vec r_k-\vec r_j)\cdot(-\vec r_j)}{(r_j^2+r_k^2-2\vec r_j\cdot\vec r_k)\, r_j^2} =$

$\int d^2r_j \frac{1}{r_j^2}$ $= \int_{\bar D_{ji}} dr_j\, r_j\, d\theta \frac{(r_j^2 - r_k r_j\cos\theta)}{(r_j^2+r_k^2-2r_jr_k\cos\theta)\,r_j^2}\simeq$

Assuming large r_j dominates

$$\simeq \int_{r_k}^{L} dr_j\, 2\pi\,\frac{1}{r_j} = 2\pi\ln\frac{L}{r_k} \Rightarrow 2\pi\ln\frac{L}{r_{k\ell}}$$



Assumption born out.

Substituting.

$$\int_{\bar D_{ji}} d^2r_j \int_{\delta_{ij}} d^2 r_i \exp\left\{\pi J\sum_k q_i q_k \ln\frac{r_{ik}}{a} + \pi J \sum_k q_j q_k \ln \frac{r_{jk}}{a}\right\} \simeq$$

$$\simeq 2\pi a\,da\left\{A + \frac{\pi^2J^2a^2}{4}\sum_k 2\pi\ln\frac{L}{a} + \frac{\pi^2J^2}{2}a^2\sum_{(k\neq\ell)}q_kq_\ell\, 2\pi\ln\frac{L}{r_{k\ell}}\right\} =$$

$$= 2\pi a\,da\left\{A+\frac{\pi^3}{2}J^2a^2\left[\sum_k \ln\frac{L}{a} + \frac{2}{2}\sum_{k\neq \ell} q_kq_\ell\ln\frac{L}{r_{k\ell}}\right]\right\}=$$

$$= 2\pi a\,da\left\{A+\frac{\pi^3}{2}J^2a^2\left[-\sum_{k\neq\ell}q_kq_\ell\ln\frac{r_{k\ell}}{a} + \cancelto{0}{\sum_{k\neq \ell}q_kq_\ell\ln\frac{L}{a}+\sum_k\ln\frac{L}{a}}\right]\right\}=$$

$$= 2\pi a\,da\left\{A - \frac{\pi^3}{2}J^2a^2\sum_{k\neq\ell}q_kq_\ell\ln\frac{r_{k\ell}}{a}\right\}.$$

by overall neutrality

$$-\beta V_{2n} = \pi J \sum_{i\neq j} q_i q_j \ln \frac{r_{ij}}{a}$$

$$-\beta \mathcal{H}_{2n} = -\beta V_{2n} + 2n\Delta$$

$$Z = \sum_{n=0}^{\infty} \frac{e^{2n\Delta}}{(n!)^2} \frac{1}{(a^2)^{2n}} \int_{\{D\}_{2n}} \{d^2 r\}_{2n}\, e^{-\beta V_{2n}} \simeq$$

$$\simeq \sum_{n=0}^{\infty} \frac{e^{2n\Delta}}{(n!)^2} \frac{1}{(a^2)^{2n}} \Big\{ \int_{\{D'\}_{2n}} \{d^2 r\}_{2n}\, e^{-\beta V_{2n}} + n^2 \int_{\{D'\}_{2n-2}} \{d^2 r\}_{2n-2}\, e^{-\beta V_{2n-2}}\, 2\pi a\, da \Big[A - \frac{\pi^3}{2} J^2 a^2 \sum_{k\neq \ell} q_k q_\ell \ln \frac{r_{k\ell}}{a} \Big] \Big\}$$

$$\simeq \sum_{n=0}^{\infty} \frac{e^{2n\Delta}}{(n!)^2} \frac{1}{(a^2)^{2n}} \int_{\{D'\}_{2n}} \{d^2 r\}_{2n}\, e^{-\beta V_{2n}} + \sum_{n=1}^{\infty} \frac{e^{2n\Delta}}{(n-1)!^2} \frac{1}{(a^2)^{2n-2}} \int_{\{D'\}_{2n-2}} \{d^2 r\}_{2n-2}\, e^{-\beta V_{2n-2}}\, 2\pi \frac{da}{a^3} \Big[A - \frac{\pi^3}{2} J^2 a^2 \sum_{k\neq\ell} q_k q_\ell \ln \frac{r_{k\ell}}{a} \Big]$$

$$= \sum_{n=0}^{\infty} \frac{e^{2n\Delta}}{(n!)^2} \frac{1}{(a^2)^{2n}} \int_{\{D'\}_{2n}} \{d^2 r\}_{2n}\, e^{-\beta V_{2n}} + \sum_{n=0}^{\infty} \frac{e^{2(n+1)\Delta}}{(n!)^2} \frac{1}{(a^2)^{2n}} \int_{\{D'\}_{2n}} \{d^2 r\}_{2n}\, e^{-\beta V_{2n}}\, 2\pi \frac{da}{a^3} \Big[A - \frac{\pi^3}{2} J^2 a^2 \sum_{k\neq\ell} q_k q_\ell \ln \frac{r_{k\ell}}{a} \Big]$$

$$= \sum_{n=0}^{\infty} \frac{e^{2n\Delta}}{(n!)^2} \frac{1}{(a^2)^{2n}} \int_{\{D'\}_{2n}} \{d^2 r\}_{2n}\, e^{-\beta V_{2n}} \Big\{ 1 + e^{2\Delta} 2\pi \frac{da}{a^3} \Big[A - \frac{\pi^3}{2} J^2 a^2 \sum_{k\neq\ell} q_k q_\ell \ln \frac{r_{k\ell}}{a} \Big] \Big\} \simeq$$

$$\simeq \sum_{n=0}^{\infty} \frac{e^{2n\Delta}}{(n!)^2} \frac{1}{(a^2)^{2n}} \int_{\{D'\}_{2n}} \{d^2 r\}_{2n}\, e^{-\beta V_{2n} + e^{2\Delta} 2\pi \frac{da}{a^3} \left[A - \frac{\pi^3}{2} J^2 a^2 \sum_{k\neq\ell} q_k q_\ell \ln \frac{r_{k\ell}}{a} \right]} \qquad \#6$$

$\#7$

$$\left(\frac{1}{a^2}\right)^{2n} = \left(\frac{1}{a'^2}\right)^{2n} \left(1 + \frac{da}{a}\right)^{4n} \simeq \left(\frac{1}{a'^2}\right)^{2n} \left(1 + 4n \frac{da}{a}\right) \simeq \left(\frac{1}{a'^2}\right)^{2n} e^{4n\, da/a}$$

$$-\beta V_{2n}(J,a) = \pi J \sum_{i\neq j} q_i q_j \ln \frac{r_{ij}}{a} = \pi J \sum_{i\neq j} q_i q_j \ln \frac{r_{ij}}{a'} + \pi J \sum_{i\neq j} q_i q_j \ln \frac{a'}{a} =$$

$$= -\beta V_{2n}(J,a') + \pi J \sum_i q_i \sum_j q_j \ln \frac{a'}{a} - \pi J \sum_i q_i^2 \ln \frac{a'}{a}$$

$$\simeq -\beta V_{2n}(J,a') - 2n\, \pi J \frac{da}{a}$$

$$Z \simeq \sum_{n=0}^{\infty} \frac{e^{2n(\Delta + 2\frac{da}{a} - \pi J \frac{da}{a})}}{(n!)^2} \frac{1}{(a'^2)^{2n}} \int_{\{D'\}_{2n}} \{d^2 r\}_{2n}\, e^{\pi J \sum_{i\neq j} q_i q_j \ln \frac{r_{ij}}{a'} - e^{2\Delta} \pi^4 \frac{da}{a} J^2 \sum_{k\neq \ell} q_k q_\ell \ln \frac{r_{k\ell}}{a'} + e^{2\Delta} 2\pi \frac{da}{a}}$$

Δ — Big question: will vortex chem potential increase (more vortices) or decrease?

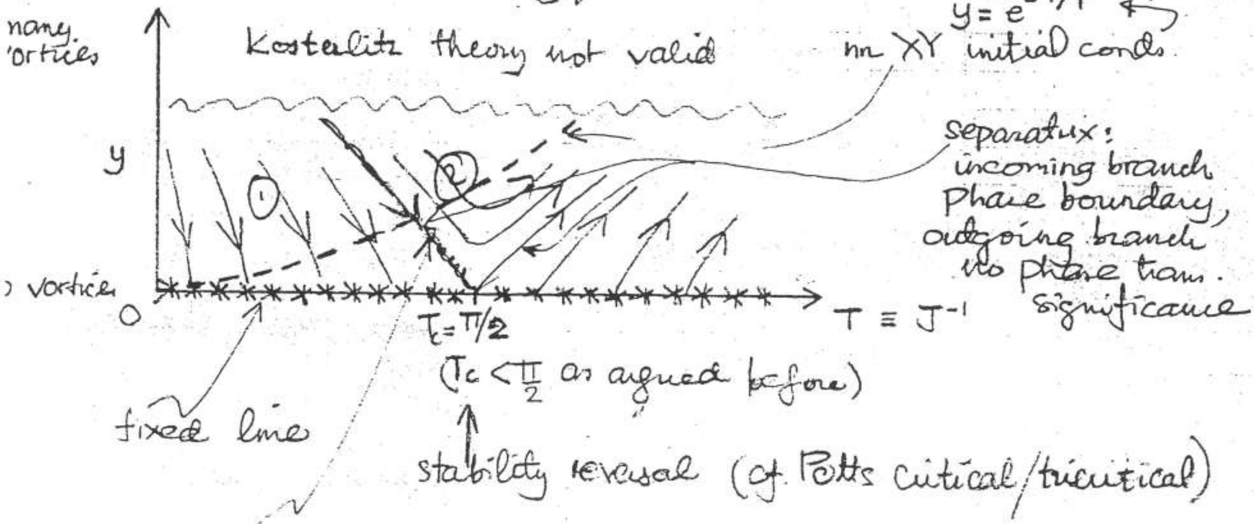
Approximations self-consistent as long vortices do not proliferate

Can now read-off recursion relations:

$$\frac{d\Delta}{d\ln a} = 2 - \pi J \qquad \text{Vortex Fugacity: } y \equiv e^{\Delta}: \qquad \boxed{\frac{dy}{d\ln a} = y\left(2 - \frac{\pi}{T}\right)}$$ flow

$$\frac{dJ}{d\ln a} = -e^{2\Delta}\pi^3 J^2 \qquad \text{Temperature: } T \equiv J^{-1}: \qquad \boxed{\frac{dT}{d\ln a} = y^2\pi^3}$$ eqs.

$$\frac{dG}{d\ln a} = e^{2\Delta} 2\pi \qquad \text{Additive free energy / site}: \qquad \frac{dG}{d\ln a} = y^2 2\pi$$



Phase boundary line.

$y = e^{\Delta} \simeq e^{-4J} = e^{-4/T}$ Initial conditions of the nearest-neighbor XY model (dashed in figure).

) Low temperature phase: Bound-vortex phase:
for length scale coarse enough, no vorticity detected.
Fixed line: algebraic order. A continuum of criticality. $\oplus\ominus$ no vorticity
instead of sink.

) High temperature phase: Free-vortex phase:

$\oplus \ominus \ominus \oplus$ Total charge $\sim \sqrt{N_{\text{vortex}}}$

Differential recursion relations ⇐ infinitesimal rescaling
Continuum flows. Still in position-space, (as opposed to momentum-s

Eigenvalue analysis:

$$J' - J^* = b^y (J - J^*)$$
$$d(J - J^*) = (b^y - 1)(J - J^*)$$
$$d(J - J^*) = (e^{y \ln b} - 1)(J - J^*)$$
$$\simeq y \ln b \, (J - J^*)$$
$$\simeq y \, d \ln a \, (J - J^*)$$

$b = 1 + \frac{da}{a} = 1 + d \ln a$

$\ln b \simeq d \ln a$

$$\frac{d(J - J^*)}{d(\ln a)} = y\,(J - J^*)$$

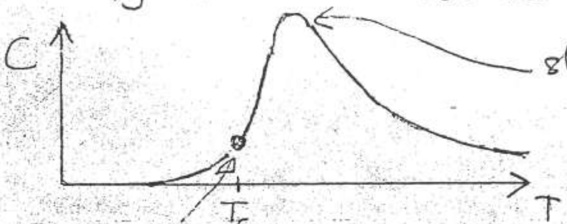
Read off eigenvalue exponents from coefficient of linear t

Here: $y = 0$ Cf. page 132a.

$\nu = 1/y = \infty$. $\xi \sim (T - T_c)^{-\nu}$ diverges faster than any power law.

Can calculate $\xi \sim e^{+A/t^{1/2}}$

$\alpha = 2 - d/y = -\infty$. Essential singularity: invisible



short-range order

$T(C_{max}) > T_c$

vortex unbinding
a very delicate thermodynamic effect.

A.N.B. and D.R. Nelson, Phys. Rev. B 19, 2488 (1979)

#7

#8

Migdal-Kadanoff Approximation:

Very good, originally thought to be exact!

Let $u(\theta) = e^{V(\theta) - V(0)}$ Fourier transform: $f(p) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ip\theta} u(\theta)$

Eigenvalue exponent analysis: cf. page 132.

Change variables: $t \equiv 2 - \pi/T$ $\qquad dt = +\frac{\pi}{T^2}dT = (t-2)^2\frac{1}{\pi}\,dT$

$u \equiv 2\pi y$

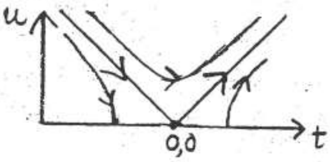
Flow: $\frac{dt}{d\ln a} = (t-2)^2 \frac{1}{\pi} \frac{u^2}{4\pi^2} \pi^3 = \frac{(t-2)^2}{4} u^2$

$\frac{du}{d\ln a} = ut$

Examine the terminus of the phase transition region, on fixed line (i.e. the separatrix close to the fixed line)
↳ bounces back at $t = u = 0$

To leading order in t, u:

$\frac{dt}{d\ln a} = u^2 \quad , \quad \frac{du}{d\ln a} = ut$



$\frac{dt^2}{d\ln a} - \frac{du^2}{d\ln a} = 2(tu^2 - uut) = 0$

$t^2 - u^2 =$ constant along flows, hyperbolic structure.

Also, clearly see that linear terms missing, $y = 0$.

$$u'(\theta) = A^{-1} \int_0^{2\pi} \frac{d\varphi}{2\pi}\, u^2(\varphi)\, u^2(\theta-\varphi) , \qquad A = \int_0^{2\pi} \frac{d\varphi}{2\pi}\, u^4(\varphi) , = \sum_{p=-\infty}^{\infty} \left[\sum_{q=-\infty}^{\infty} f(q) f(p-q) \right]$$

or, equivalently:

$$f'(p) = A^{-1} \left[\sum_{q=-\infty}^{\infty} f(q) f(p-q) \right]^2 .$$

Apparent asymptotic fixed-line behavior for $J \gtrsim 1.0$ ($> 2/\pi$, as required by flow diagram)

After a few iterations, stationary at

$$f^*(p) = e^{-p^2/2\tilde{J}} \Big/ \sum_{q=-\infty}^{\infty} e^{-q^2/2\tilde{J}}$$ Villain potential.

Example: For $\tilde{J}^{-1} = \frac{1}{2} \tilde{J}_c^{-1} \approx \frac{1}{2}\left(\frac{\pi}{2}\right)$,

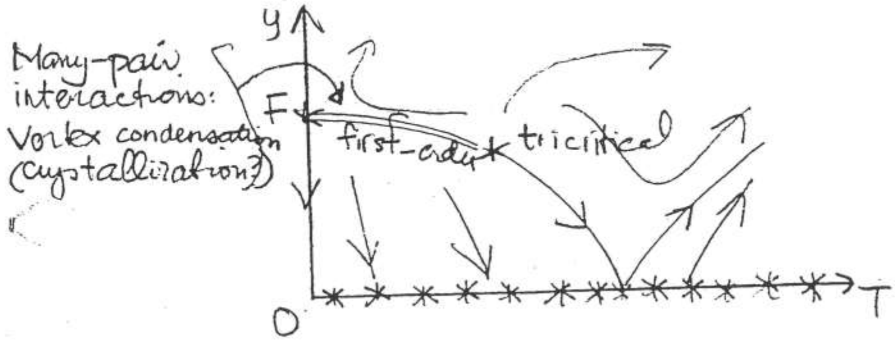
takes over 12,000 rescalings to finally drift to infinite-temperature sink ($V(\theta) \sim 10^{-38}$)

By contrast, for $\tilde{J}^{-1} = \frac{3}{2} J_c^{-1}$, it takes 11 iterations for the same.

Refs. J.V. José, L.P. Kadanoff, S. Kirkpatrick, and D.R. Nelson Phys. Rev. B 16, 1217 (1977)

A.N.B. and D.R. Nelson, Phys. Rev. B 19, 2488 (1979)

Open topic of research: Possible scenario for first-order transition:



VIII. Tentative outline for remaining lectures

✓5. Kadanoff variational PSRG.
Optimal choice of projection operator.
Highly accurate exponents.

✓4. Migdal-Kadanoff PSRG
Extreme simplicity
Good for Global phase diagram

✓3. Superfluidity – a review.

2. Blume-Emery-Griffiths model for ^3He-^4He.
Model-building for physical situations.
Global results. Tricritical phenomena

— 1. Adsorbed systems
Comparison with experiments

0. Potts models
Exact critical and tricritical exponents → experiments

(IX. The XY Model in d=2 and Algebraic Order
X. Dynamics. The Glauber model.)

Kadanoff's Variational PSRG

Kadanoff, PRL 34, 1005 (1975).
Kadanoff, Houghton, and Yalabik, J Stat. Phys. 14, 171 (1976).

Preliminary: Feynman, Stat. Mech. p. 66.

$$F = \ln \operatorname{Tr} e^{\mathcal{H}} = \ln \operatorname{Tr} e^{\mathcal{H}_0 + V} =$$

$$= \ln \operatorname{Tr} e^{\mathcal{H}_0}\left(1 + V + \frac{1}{2}V^2 + \ldots\right) =$$

$$= \ln \operatorname{Tr} e^{\mathcal{H}_0} + \ln \frac{\operatorname{Tr} e^{\mathcal{H}_0}\left(1 + V + \frac{1}{2}V^2 + \ldots\right)}{\operatorname{Tr} e^{\mathcal{H}_0}} =$$

$$= F_0 + \ln\left\{1 + \langle V\rangle_0 + \frac{1}{2}\langle V^2\rangle_0 + \ldots\right\} =$$

$$= F_0 + \langle V\rangle_0 + \frac{1}{2}\left[\langle V^2\rangle_0 - \langle V\rangle_0^2\right] + \ldots \equiv F_0 + F_1 + F_2 + \ldots$$

Prove $(\mathrm{Tr}\, e^{\mathcal{H}_0})(\mathrm{Tr}\, e^{\mathcal{H}_0} V^2) \geq (\mathrm{Tr}\, e^{\mathcal{H}_0} V)(\mathrm{Tr}\, e^{\mathcal{H}_0} V)$

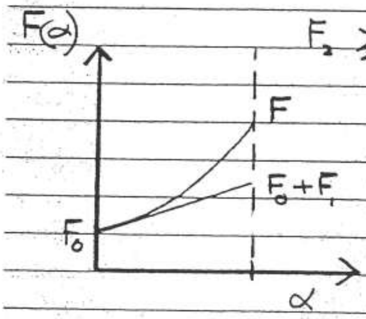
Consider any two states (a, b):

$$e^{\mathcal{H}_0(a) + \mathcal{H}_0(b)} \left[V^2(a) + V^2(b)\right] \geq e^{\mathcal{H}_0(a) + \mathcal{H}_0(b)} \left[2 V(a) V(b)\right]$$

$$\langle V^2 \rangle_0 \geq \langle V \rangle_0^2$$

or

$$\langle (V - \langle V \rangle)^2 \rangle \geq 0$$



$F_2 \geq 0$

$$H = H_0 + \alpha V$$

$$F(\alpha) = F_0 + \alpha F_1 + \alpha^2 F_2 + \ldots$$

At an arbitrary α:

$$F(\alpha + \gamma) = F_0(\alpha) + \gamma F_1(\alpha) + \gamma^2 F_2(\alpha)$$

$$F_2(\alpha) \geq 0$$

$$F''(\alpha) = \frac{d^2}{d\gamma^2} F(\alpha + \gamma)\Big|_{\gamma=0} = 2 F_2(\alpha) \geq 0$$

$\therefore$ $$\boxed{F \geq F_0 + F_1 = F_0 + \langle V \rangle_{H_0}}$$

$\mathcal{H} = \mathcal{H}_0 + V$ or $\mathcal{H}_0 = \mathcal{H} - V$

Interchange roles:

$$\begin{array}{ccc} \mathcal{H} & & \mathcal{H}_0 \\ \mathcal{H}_0 & \Longleftrightarrow & \mathcal{H} \\ V & & -V \end{array}$$

$$F_0 \geq F + \langle -V \rangle_H$$

$$\boxed{F_0 + \langle V \rangle_H \geq F \geq F_0 + \langle V \rangle_{H_0}}$$

~ NvL cluster

NvL cumulant

Kadanoff:

$$e^{\mathcal{H}'\{s'\}} = \sum_{\{s\}} P(\{s'\},\{s\})\, e^{\mathcal{H}\{s\}} = \sum_{\{s\}} e^{T\{s',s\} + \mathcal{H}\{s\}}$$

Identifications

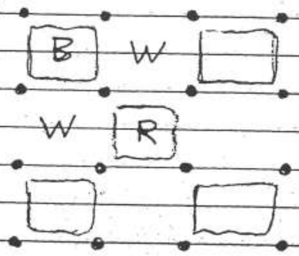
above	here
F	$\ln \operatorname{Tr} \exp \mathcal{H}'\{s'\}$
$\mathcal{H}$	$T\{s',s\} + \mathcal{H}\{s\}$
V	$V\{s\}$
$\mathcal{H}_0 = \mathcal{H} - V$	$\mathcal{H}_0\{s',s\} = T\{s',s\} + \mathcal{H}\{s\} - V\{s\}$
F_0	$\ln \operatorname{Tr} \exp \mathcal{H}'_B\{s'\}$

use: $F \leq F_0 + \langle V \rangle_H$

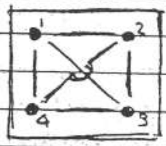
choose V: $\langle V \rangle_H = 0$

$F \leq F_0(p)$ Optimize with respect to p.

#18



inside each square



$$J(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1) = J s^2$$
$$K(s_1 s_3 + s_2 s_4) = K(s)^2$$
$$M s_1 s_2 s_3 s_4 = M s^4$$

$$\mathcal{H}\{s\} = \sum_{B,R,W} \{ J s^2 + K(s)^2 + M s^4 \}$$

#17

$$T\{s',s\} = \sum_{B} t_B(s'; s_1, s_2, s_3, s_4)$$

caret: Feynman quantities

Identifications

$$\hat{F} = \ln \mathrm{Tr}' \, \mathrm{Tr} \; e^{T\{s',s\} + \mathcal{H}\{s\}} = \ln \mathrm{Tr} \, e^{\mathcal{H}\{s\}}$$

$$\hat{\mathcal{H}} = T\{s',s\} + \mathcal{H}\{s\}$$

$$\hat{\mathrm{Tr}} = \mathrm{Tr}' \, \mathrm{Tr}$$

$\hat{V} = V\{s\}$, the bond-moving potential

$$\hat{\mathcal{H}}_0 = \hat{\mathcal{H}} - \hat{V} = T\{s',s\} + \mathcal{H}\{s\} - V\{s\}$$

$$\hat{F}_0 = \ln \hat{\mathrm{Tr}} \, e^{\hat{\mathcal{H}}_0} = \ln \mathrm{Tr}' \, \mathrm{Tr} \; e^{T\{s',s\} + \mathcal{H}\{s\} - V\{s\}} = \ln \mathrm{Tr}' \, e^{\mathcal{H}'_A\{\}}$$

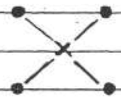
$$\hat{F} \le \hat{F}_0 + \langle \hat{V} \rangle_{\hat{\mathcal{H}}}$$

$$\langle \hat{V} \rangle_{\hat{\mathcal{H}}} \sim \mathrm{Tr}' \, \mathrm{Tr} \; V\{s\} \, e^{T\{s',s\} + \mathcal{H}\{s\}}$$

$$= \mathrm{Tr} \, V\{s\} \, e^{\mathcal{H}\{s\}} = 0 \,. \checkmark$$

$$t_B(s'; s_1, s_2, s_3, s_4) = s' p (s_1+s_2+s_3+s_4) + q_0 + q_2 [s^2 + (s')^2] + q_4 s^4$$

Inside blue squares



Z-conservation: A sufficient condition is

$$1 = \sum_{s'} e^{t_B(s', s)} = e^{q_0 + q_2 [s] + q_4 s^4} \, 2\cosh(p\Sigma) \equiv Q^{-1}(s) \, 2\cosh(p\Sigma)$$

$$\left.\begin{aligned} Q(++++) &= 2\cosh 4p = e^{-q_0 - 6q_2 - q_4} \\ Q(+++-) &= 2\cosh 2p = e^{-q_0 + 0 + q_4} \\ Q(++--) &= 2 \qquad\quad\; = e^{-q_0 + 2q_2 - q_4} \end{aligned}\right\} \text{determine } q \text{ in terms of } p.$$

Only parameter is p.

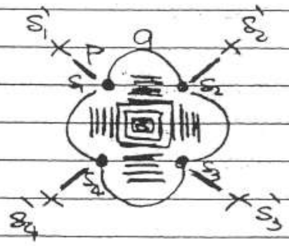
Note possibility of $s' p s_1 s_2 s_3$, not used.

$$V\{s\} = \sum_{W,B} \{Js^2 + K(s^2) + Ms^4\} + \sum_B \{q_0 + q_2 [s^2] + q_4 s^4\} +$$
$$-3\sum_R \{Js^2 + K(s^2) + Ms^4\} - \sum_R \{q_0 + q_2 [s^2] + q_4 s^4\}$$

Note: $\langle (s_1 s_2)_B \rangle_H = \langle (s_1 s_2)_R \rangle_H$ etc.

Thus $\langle V \rangle_H = 0$.

$\mathcal{H}_0\{s', s\} = T\{s', s\} + \mathcal{H}\{s\} - V\{s\}$ is composed of the following units:



$$\mathcal{H}_0\{s', s\} = \sum_R \Big[4\{Js^2 + K(s)^2 + Ms^4\} +$$
$$+ \{q_0 + q_2 [s^2] + q_4 s^4\} +$$
$$+ p (s_1' s_1 + s_2' s_2 + s_3' s_3 + s_4' s_4) \Big] =$$
$$= \sum_R \mathcal{H}_{0R}(s'_{1-4}, s_{1-4})$$

$$Tr' e^{\mathcal{H}_B\{s'\}} \overset{\ln Tr'}{=} \sum_{\{s\}} e^{\mathcal{H}_0\{s', s\}} + \cancel{\langle V \rangle_H}^{\,0}$$

The sum is doable because it factorizes, into each unit as shown above.

$$\mathcal{H}'_B\{s'\} = \sum_{\substack{\text{all new}\\ \text{squares}\\ N}} \mathcal{H}'_{BN}(s'_{1-4})$$

$$e^{\mathcal{H}'_{BN}(s'_{1-4})} = \sum_{s_{1-4}} e^{\mathcal{H}_{OR}(s'_{1-4}, s_{1-4})} = 16 \text{ terms} \equiv R(s'_{1-4}) =$$
$$= e^{J' s'^2 + K'(s'^2) + M' s'^4 + \tilde{G}}$$

$$R_0 = R(++++) = e^{4J' + 2K' + M' + \tilde{G}}$$
$$R_1 = R(+++-) = e^{-M' + \tilde{G}}$$
$$R_2 = R(++--) = e^{-2K' + M' + \tilde{G}}$$
$$R_3 = R(+-+-) = e^{-4J' + 2K' + M' + \tilde{G}}$$

$$\tilde{G} = \frac{1}{8} \ln R_0 R_1^4 R_2^2 R_3 = \tilde{G}_p(J, K, M)$$
$$J' = \frac{1}{8} \ln R_0/R_3 = J'_p(J, K, M)$$
$$K' = \frac{1}{8} \ln R_0 R_3/R_2^2 = K'_p(J, K, M)$$
$$M' = \frac{1}{8} \ln R_0 R_2^2 R_3/R_1^4 = M'_p(J, K, M)$$

Dimensionless free energy per original site: At fp

$$f^* = \frac{1}{N^*} \ln Z = \sum_{n=1}^{\infty} \frac{1}{b^{dn}} \tilde{G}^* = \tilde{G}^* \frac{1/4}{1 - 1/4} = \frac{G_p^*}{3}$$

$$O = \frac{\partial G_p^*}{\partial p} :$$

	$d=2$ (4 types of interactions)		$d=3$ (22 types)	
$y_T^{-1} = \nu =$	0.999	(1)	0.944	(0.938)
$\frac{y_H}{d - y_H} = \delta =$	15.04	(15)	4.818	(5.0 ± 0.2)

$$\delta^{-1} = \frac{d}{y_H} - 1 \Rightarrow y_H = 1.87531 \quad (1.875)$$

Global results: maximize along trajectory

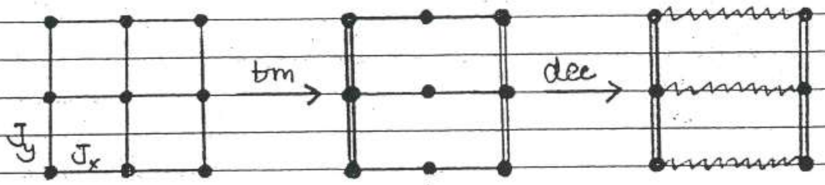
$$f = \sum_{n=1}^{\infty} \frac{\tilde{G}_{p_n}\left(J^{(n-1)}_{p_{m<n}}, K^{(n-1)}_{p_m}, M^{(n-1)}_{p_m}\right)}{b^{dn}}$$

Has not been developed.

Migdal-Kadanoff RG (MKRG)

A.A Migdal, Zh. Eksp. Teor. Fiz. 69, 1457 (1975)
[Sov. Phys. - JETP 42, 743 (1976)]

L.P. Kadanoff, Ann. Phys. (N.Y.) 100, 359 (1976);
Rev. Mod. Phys. 49, 267 (1977).



J_x, J_y $\qquad$ J_x, $\hat{J}_y = 2J_y$ $\qquad$ $\tanh \hat{J}_x = (\tanh J_x)^2$, $\hat{J}_y = 2J_y$

Repeat along y-direction. $\qquad J'_x = 2\hat{J}_x$, $\quad \tanh J'_y = (\tanh \hat{J}_y)^2$

Combining, the following recursion relations relations are obtained for d=2, b=2:

$$\tanh\left(\frac{J'_x}{2}\right) = (\tanh J_x)^2$$

$$\tanh(J'_y) = (\tanh 2J_y)^2$$

d=2, arbitrary b:

$$\tanh\left(\frac{J'_x}{b}\right) = (\tanh J_x)^b$$

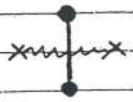
$$\tanh J'_y = (\tanh bJ_y)^b$$

Arbitrary d, b:

$$\tanh\left(\frac{J'_i}{b^{d-i}}\right) = (\tanh b^{i-1} J_i)^b$$

Note: Anisotropy is introduced, even alternating does not help.

MKRG and duality ($d=2$)

Recall duality:  $e^{-2\tilde{J}_x} = \tanh J_y$

or, equivalently

$e^{-2J_y} = \tanh \tilde{J}_x$.

$$J_y \xrightarrow{\text{dual}} \tanh \tilde{J}_x = e^{-2J_y}$$

$$\xrightarrow{\text{MKRG}} \tanh \frac{\tilde{\tilde{J}}_x}{b} = (\tanh \tilde{J}_x)^b = e^{-2bJ_y}$$

Equivalent relation: $\tanh bJ_y = e^{-2\tilde{\tilde{J}}_x/b}$

$$\xrightarrow{\text{dual}} \tanh \tilde{\tilde{\tilde{J}}}_y = e^{-2\tilde{\tilde{J}}_x} = (\tanh bJ_y)^b$$

Compare with direct MKRG: $\tanh \tilde{J}_y = (\tanh bJ_y)^b$

[Considered (bm/dec) MKRG; in PS, (dec/bm)]

Dual transformation and MKRG transformation commute.

$\tilde{J} = D(J)$ $\qquad\qquad$ $J' = R(J)$

$D^{-1} = D$: $\quad J = D(\tilde{J})$

① We just showed $\quad D\{R[D(J)]\} = R(J)$

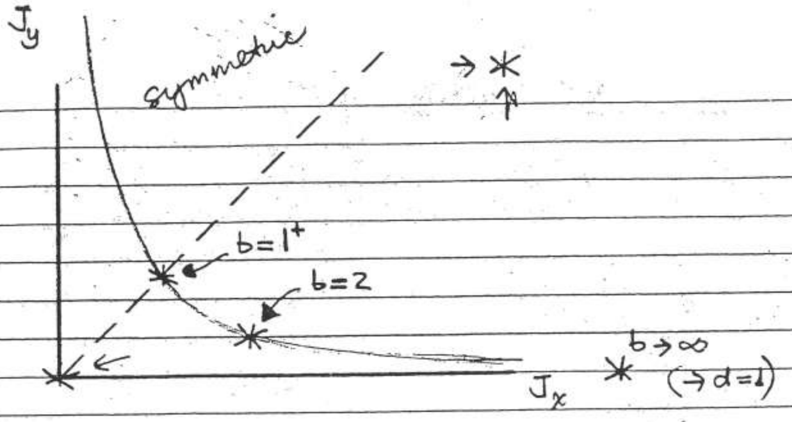
$$R[D(J)] = D[R(J)]$$

② Consider self-dual point $\quad D(J_0) = J_0$

$$D[R(J_0)] = R[D(J_0)] = R(J_0)$$

$J_0' = R(J_0)$ is also self-dual.

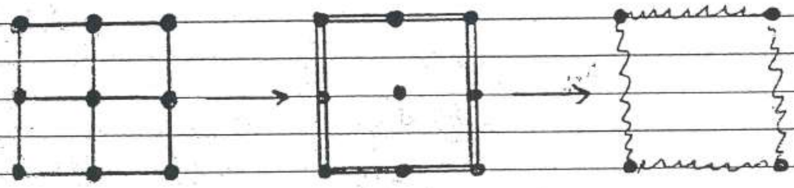
Self-dual space is closed under RG.



(Scanning b) exact phase boundary obtained.

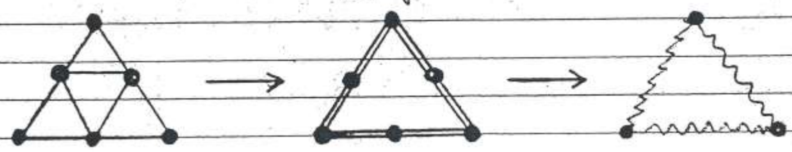
Exponents: $y_T \sim -20\%$ $\quad$ $y_H \sim -2\%$

Isotropic MKRG $\qquad$ #18

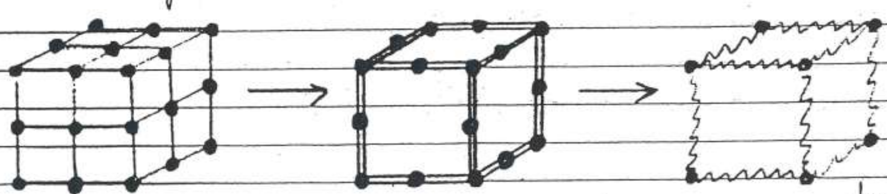


#17

Again justified by connectivity.
Exact T_c when $b \to 1^+$
Otherwise, reasonably good. T_c 10% for $b=2$.



Also hexagonal and other lattices.



d-dimension
hypercube:
Bond-moving
$\hat{J} = b^{d-1} J$
Decimation:
$\tanh J' = (\tanh \hat{J}$

Also, other types of spin objects – important applicability.

1) <u>As a function of d:</u>

First look at strong coupling behavior: $J \to \infty$.

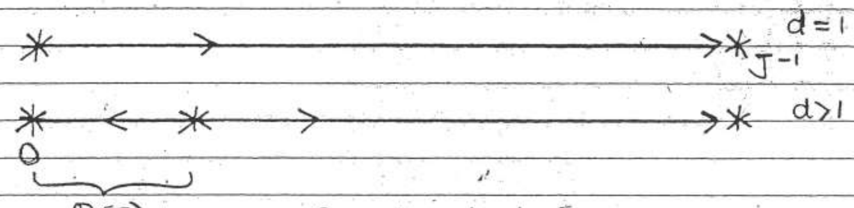
$$1-2e^{-2J'} \simeq 1-\cancel{2}b\, e^{-2b^{d-1}J}$$

$$e^{-J'} \simeq \sqrt{b}\, e^{-b^{d-1}J}$$

$$J' \simeq b^{d-1}J \; (-\tfrac{1}{2}\ln b) \quad ,, \qquad b>1.$$

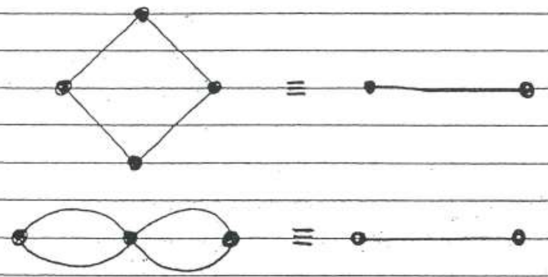
$J^* = \infty$ stable for $d>1$.

$J^* = \infty$ marginally unstable at $d=1$.



Can do $d=1+\epsilon$ expansion.

2) <u>Hierarchical model</u> for which MKRG is exact



"Physically realizable approximation"
Basic thermodynamics is satisfied, e.g. positive specific heat.

Also, infinitely many families of exactly soluble models, with finite temperature T_c's, non-classical.

Interesting results for competing interactions.

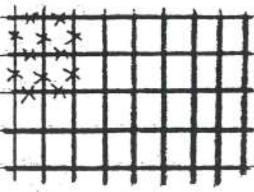
Lattice Gauge Theories

K.G. Wilson, Phys. Rev. D10, 2445 (1974)
L.P. Kadanoff, Rev. Mod. Phys. article.

Global symmetry: Ising $s_i' = \sigma s_i$ $\qquad \sigma = \pm 1$, i-independent

also global rotation in n-vector models.

Local symmetry: "Plaquette" model:



$$-\beta\mathcal{H} = J \sum_{\text{plaquettes}} s_{12} s_{23} s_{34} s_{41}$$

Local gauge transformation:

$$s_{ij}' = \sigma_i s_{ij} \sigma_j, \qquad \sigma_i = \pm 1 \text{ independently of each other}$$

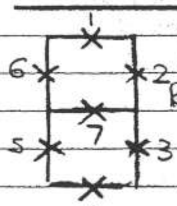
In words, flip all spins of one incoming vertex.

cf. classical electrodynamics:

$$\vec{A}'(r) = \vec{A}(r) + \vec{\nabla}\psi(r)$$

$$s_{ij}' \qquad s_{ij} \qquad \sigma_i$$

Exact rescaling in two dimensions:



$$R = \sum_{s_7} e^{J(s_1 s_2 s_7 s_6 + s_3 s_4 s_5 s_7)} = e^{J(s_1 s_2 s_6 + s_3 s_4 s_5)} + e^{-J(s_1 s_2 s_6 + s_3 s_4 s_5)}$$

$$\uparrow s_3^2$$

$$s_2' = s_2 s_3, \qquad s_4' = s_5 s_6$$

$$R = e^{J(s_1 s_2' s_6 \cancel{s_3} + \cancel{s_3} s_4 s_5)} + e^{-J(s_1 s_2' s_6 \cancel{s_3} + \cancel{s_3} s_4 s_5)} =$$

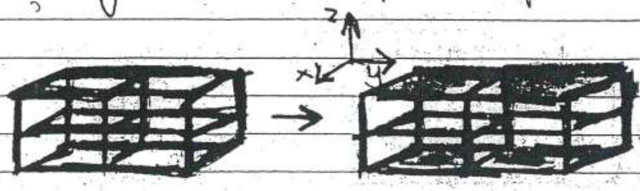
$$= e^{J(s_1 s_2' s_4' \cancel{s_5} + s_4 \cancel{s_5})} + e^{-J(s_1 s_2' s_4' \cancel{s_5} + s_4 \cancel{s_5})} =$$

$$= e^{J(s_1 s_2' s_4' + s_4)} + e^{-J(s_1 s_2' s_4' + s_4)}$$

$$R = e^{J}\, e^{J s_1 s_2' s_4' s_4} + e^{-J s_1 s_2' s_4' s_4} e^{-J} = \underset{s_1' = s_1 \quad s_3' = s_4}{R(s_1 s_2' s_4' s_4)} = e^{J' s_1 s_2' s_4' s_4 + \tilde{G}}$$

$$\tanh J' = (\tanh J)^2$$

<u>Three dimensions: Migdal–Kadanoff transformation:</u>



$$\tanh J'_{xy} = (\tanh 2 J_{xy})^2$$

$$\tanh \frac{J'_{yz}}{2} = (\tanh J_{yz})^2$$

$$\tanh \frac{J'_{zx}}{2} = (\tanh J_{zx})^2$$

d-dimensional gauge model $\xleftrightarrow{MK}$ (d-1)-dimensional spin models

10. Dynamics: Stochastic Models; Detailed Balance; Dynamic Universality Classes

"Time-Dependent Statistics of the Ising Model"

R.J. Glauber, J. Math Phys. 4, 294 (1963).

- The individual spins of the Ising model are assumed to interac with an external agency (e.g. a heat reservoir) that causes them to change their states randomly with tim "stochastic model"
- The transition probabilities for any one spin depend on the values of the neighboring spins.

Single-spin system

No magnetic field, therefore no preference between $s = \pm 1$.

Rate, per unit time, at which the spin flips $\equiv \alpha/2$.

The probability $p(s,t)$ that the spin takes on the value s at t obeys the equation:

$$\frac{d}{dt} p(s,t) = -\frac{\alpha}{2} p(s,t) + \frac{\alpha}{2} p(-s,t).$$

Normalization: $p(1,t) + p(-1,t) = 1$.

$$m(t) \equiv \langle s(t) \rangle = p(1,t) - p(-1,t).$$

Substituting above: $\frac{d}{dt} m(t) = -\alpha m(t)$.

$\therefore$ the mean spin decays exponentially with a relaxation time of $1/\alpha$ from whatever value it is known to have initially:

$$m(t) = m(0)\, e^{-\alpha t}$$

$$p(s,t) = \frac{1}{2}\left[1 + s m(t)\right]$$

<u>Many-spin system</u> $d=1$ with periodic boundary cond.

$w_i(s_i) \equiv$ prob. per unit time that the (i)th spin flips from s_i to $-s_i$, while the others remain momentarily fixed.

$$\boxed{\frac{d}{dt} p(s_1, \ldots, s_N, t) = -\left[\sum_i w_i(s_i)\right] p(s_1, \ldots, s_N, t) + \sum_i w_i(-s_i)\, p(s_1, \ldots, -s_i, \ldots, s_N, t)}$$

"<u>master equation</u>": solution would contain the most complete (t-dependent) description of the system.

<u>Correspondence with the Ising model</u>

$w_i(s_i)$ depends on neighboring spins as well as s_i.

Tendency for each spin to align itself parallel to its nearest neighbors:

$$w_i(s_i) = \frac{1}{2}\alpha\left\{1 - \frac{\gamma}{2} s_i (s_{i-1} + s_{i+1})\right\}, \qquad |\gamma| \leq 1 .$$

$\alpha \equiv$ (inverse) time scale on which all transitions take place. No analog in the equilibrium theory of the Ising model.

However, connection of γ: $\quad -\beta\mathcal{H} = J \sum_i s_i s_{i+1}$,

At equilibrium: $$\frac{p(-s_i)}{p(s_i)} = \frac{e^{-Js_i(s_{i-1}+s_{i+1})}}{e^{Js_i(s_{i-1}+s_{i+1})}} \overset{\text{detailed}}{\underset{\text{balance}}{=}} \frac{w(s_i)}{w(-s_i)}$$

$$e^{\pm Js_i(s_{i-1}+s_{i+1})} = \cosh\left[Js_i(s_{i-1}+s_{i+1})\right]\left\{1 \pm s_i \tanh\left[J(s_{i-1}+s_{i+1})\right]\right\} =$$

$$= \cosh\left[Js_i(s_{i-1}+s_{i+1})\right]\left\{1 \pm s_i \frac{s_{i-1}+s_{i+1}}{2} \tanh[2J]\right\}$$

Identify: $\quad \gamma = \tanh[2J]$.

Reduction of the Probality Function

$$m_i(t) = \langle s_i(t)\rangle = \sum_{\{s\}} s_i \, p(s_1,\dots,s_N,t)$$

Master equation: Compactified:

$$\frac{d}{dt} p(s_1,\dots,s_N,t) = -\sum_i s_i \sum_{s_i'} s_i' \, w(s_i') \, p(s_1,\dots,s_i',\dots,s_N,t) \,.$$

$\sum_{\{s\}} s_j$:
$$\frac{d}{dt} m_j(t) = -\sum_{\{s\}} \sum_{s_j'} s_j' \, w(s_j') \, p(s_1,\dots,s_j',\dots,s_N,t) =$$
$$= -2 \sum_{\{s\}} s_j \, w(s_j) \, p(s_1,\dots,s_j,\dots,s_N,t) =$$
$$= -2 \langle s_j(t) \, w(s_j(t))\rangle \,.$$

Substituting for the transition rate:

$$\boxed{\frac{d}{dt} m_j(t) = -\alpha\left\{m_j(t) - \frac{\gamma}{2}\left[m_{j-1}(t) + m_{j+1}(t)\right]\right\}}$$

Most general solution for $N \to \infty$: $m_j(t) = e^{-\alpha t} \sum_{i=-\infty}^{\infty} m_i(0) \, I_{j-i}(\gamma\alpha t)$

Bessel functions of imaginary argument

For finite N:
$$m_j(t) = e^{-\alpha t} \sum_{i=1}^{N} \sum_{k=-\infty}^{\infty} m_i(0) \, I_{j-i+kN}(\gamma\alpha t)$$ "echoes"

In all cases, total magnetization:

$$m(t) = \frac{1}{N} \sum_j m_j(t) = e^{-\alpha(1-\gamma)t} \, m(0) \,.$$ No permanent magnetization.

Effect of spin interaction: reduce decay coefficient from α to $\alpha(1-\gamma)$.
(Increase, when interaction antiferromagnetic.)

Renormalization-Group Solution

$$\frac{d}{dt} m_3(t) = -\alpha \left\{ m_3(t) - \frac{\gamma}{2}\left[m_2(t) + m_4(t)\right]\right\}$$

α sets the inverse time scale.

Fourier transform: $(\alpha - i\omega) m_3(\omega) = \frac{\alpha\gamma}{2}\left[m_2(\omega) + m_4(\omega)\right]$

$$(\alpha - i\omega)^2 m_3(\omega) = \left(\frac{\alpha\gamma}{2}\right)^2 \left[m_1(\omega) + 2m_3(\omega) + m_5(\omega)\right]$$

$$\left[(\alpha - i\omega)^2 - \frac{\alpha^2\gamma^2}{2}\right] m_3(\omega) = \left(\frac{\alpha\gamma}{2}\right)^2 \left[m_1(\omega) + m_5(\omega)\right]$$

Interested in long time, therefore small frequencies.
Keep $O(\omega)^1$: close to equilibrium

$$\left[\alpha^2\left(1 - \frac{\gamma^2}{2}\right) - 2i\omega\cancel{\alpha}\right] m_3(\omega) = \frac{\alpha^2\gamma^2}{4}\left[m_1(\omega) + m_5(\omega)\right]$$

$$\left[\frac{\alpha}{2}\left(1 - \frac{\gamma^2}{2}\right) - i\omega\right] m_3(\omega) = \frac{\alpha\gamma^2}{8}\left[m_1(\omega) + m_5(\omega)\right]$$

Recursion relations: $\alpha' = \frac{\alpha}{2}\left(1 - \frac{\gamma^2}{2}\right)$.

$$\gamma' = \frac{\cancel{\alpha}\gamma^2}{\cancel{4}_{2}} \frac{\cancel{2}/\cancel{\alpha}}{(1 - \gamma^2/2)} = \frac{\gamma^2/2}{1 - \gamma^2/2}$$

Fixed points: $\gamma^* = 0$ stable.

Also, $2 - \gamma^{*2} = \gamma^*$

$\gamma^{*2} + \gamma^* - 2 = 0$

$\gamma^* = -2$ unphysical

$\gamma^* = 1$

$\gamma' \simeq \frac{1-2\epsilon}{1+2\epsilon} \simeq 1 - 4\epsilon$ unstable

$\rightarrow 1 - 2e^{-4J'} \simeq 1 - 8e^{-4J}$

$e^{-J'} = \sqrt{2}\, e^{-J} = b^{y_T} e^{-J}$

$y_T = 1/2$, as before

$$\underset{1}{*} \longleftarrow \overset{\gamma}{\longrightarrow} \longrightarrow \underset{0}{*}$$

$\tanh 2J$

At $\gamma^* = 1$, $\alpha' = \frac{\alpha}{4}$.

As J is mapped onto J', to maintain the dynamics, α has to be mapped onto $\alpha' = \alpha/4$ (i.e., slowed down).

low consider "experimentally" scanning temperature J^{-1} for a given system: (J', α) will be 4 times faster than (J, α).

me scales, i.e. characteristic times: $\tau' = \tau/4$

$\tau'/\tau = b^{-2} = (\xi'/\xi)^2 \rightarrow$ dynamic exponent $z = 2$ (for $d=1, n=1$)

Critical slowing - down

Temperature divergence of characteristic times:

$$\tau \sim \xi^{z} \sim t^{-\nu z}$$, where t is temperature deviation from T_c.

$d=1$, $n=1$ result: $\tau \sim \left(e^{-J}\right)^{-2\cdot 2} = e^{4J}$.

Previously, dynamic scaling hypothesis:

B.I. Halperin and P.C. Hohenberg, PRL 19, 700 (1967).
P.C. Hohenberg and B.I. Halperin, RMP 49, 435 (1977)

Time-dependent correlation function:

$$\Gamma(\vec{r},t) = \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^d}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \Gamma(\vec{k},\omega).$$

For $k^{-1}, \xi \gg a$,

$$\Gamma(\vec{k},\omega) = \text{func}\left(k\xi, \frac{\omega}{\omega(k,\xi)}\right)$$ where the characteristic frequency $\omega(k,\xi) = k^z f(k\xi)$

homogeneous functions

Dynamic Universality Classes

BLE I. Some dynamical models treated by renormalization-group methods.

Model	Designation	System	Dimension order of parameter	Non-conserved fields	Conserved fields	Non-vanishing Poisson bracket
laxational	A	Kinetic Ising anisotropic magnets	n	ϕ	None	None
	B	Kinetic Ising uniaxial ferromagnet	n	None	ϕ	None
	C	Anisotropic magnets structural transition	n	ϕ	m	None
ıid	H	Gas–liquid binary fluid	1	None	ϕ $\mathbf{j}$	$\{\phi, \mathbf{j}\}$
nmetric planar agnet	E	Easy-plane magnet, $h_z=0$	2	ϕ	m	$\{\phi, m\}$
ymmetric planar agnet	F	Easy-plane magnet, $h_z \neq 0$ superfluid helium	2	ϕ	m	$\{\phi, m\}$
tropic tiferromagnet	G	Heisenberg antiferromagnet	3	ϕ	m	$\{\phi, \mathrm{m}\}$
tropic rromagnet	J	Heisenberg ferromagnet	3	None	ϕ	$\{\phi, \phi\}$

tical behavior, *suffice* to determine the universality ss for critical dynamics. The extent to which this *versality hypothesis* has been tested by theory and periment will be discussed below.

f one accepts the universality hypothesis then it sufes to study the simplest dynamical *model* consistent h the known static behavior and with a given set of nservation laws, Poisson-bracket relations, symme-, etc. If we can solve that model for its critical beior, then we know the critical behavior of all mems of the class. In order to *test the universality asnption,* one may add all possible perturbations to the del, consistent with the conservation laws, etc., and to convince oneself that these perturbations are relevant" at the critical point. In practice, it has y been possible to study small perturbations from the aple systems, for both static and dynamic behavior.[3] The field-theoretic models generally studied with ipled-mode or renormalization group methods are *nimicroscopic,* in that they describe the behavior of variables remaining after one has first integrated r all variations with wavelengths short compared to utoff Λ^{-1}. The scale of length Λ^{-1} is intermediate been the microscopic scale $a \approx 1$ Å, and the macroscopic le defined by the correlation length ξ near T_c. The nsition from a microscopic Hamiltonian to the semicroscopic models has not been carried out explicitly either the statics or the dynamics in most cases. eed, the difficulties involved in such a *microscopic ivation* are comparable to the problem of calculating transport coefficients and short-wavelength spectm of a dense liquid away from the critical point, starting from interparticle potentials. Fortunately, however, this information does not seem to be needed tc obtain the critical behavior, at least insofar as one is concerned with "universal" properties. The steps leading from a microscopic Hamiltonian to the correspondin semimicroscopic representation will be discussed further in Sec. IV.D, for a model of a displacive transition in an interacting phonon system.

Some semimicroscopic models belonging to different universality classes are listed in Table I, along with physical systems to which they correspond. These models will be described, and their dynamic critical behavior will be examined, in Secs. IV–VII below. Some key results of the renormalization group treatment of the models are summarized in Table II.

II. THE SYMMETRIC BINARY FLUID—A SIMPLE EXAMPLE

A. Hydrodynamics

Many of the physical ideas mentioned in the Introduction, can be illustrated with a simple example—the critical point for phase separation (consolute point) in a hypothetical binary fluid whose properties are symmetr with regard to interchange of the two constituents A and B. We shall consider the properties of this system for $T \geq T_c$, at the critical concentration which is $c = \frac{1}{2}$ by symmetry.

The binary fluid has six conserved densities,[4] viz., th energy density e, the mass densities ρ_A and ρ_B for the two constituents A and B, respectively, and the three components of the momentum density $\mathbf{j}$. Small, long-

Superfluidity

Superfluidity has been observed in ^4He.

(1.) At 2.17 °K (λ-point), the specific heat shows a log singular

This is thought to be Bose condensation.

(2.) The low-temperature phase, He II, has remarkable

hydrodynamic properties.

It can flow through fine channels with no pressure drop.

(zero viscosity)

But rotating cylinder transfers momentum (non-zero viscosity).

① Wave function $\psi_k = \text{const} \sum_{j=1}^{N} e^{ikr_j} \psi_0$

Superfluidity $\leftrightarrow$ occupancy of $|\psi_0\rangle$.

2) Consider single particle wave function $\psi(r)$

At each site i, an amplitude and a phase.

$$\psi(r) = A(r)\, e^{i\varphi(r)}$$

A^2 measures the presence or absence of ^{4}He. Because now we have included ^{3}He.

<u>Blume-Emery-Griffiths model</u> ($s=1$ Ising model)

Phys. Rev. A $\underline{4}$, 1071 (1971)

$$-\beta\mathcal{H} = J\sum_{\langle ij\rangle} s_i s_j + K\sum_{\langle ij\rangle} s_i^2 s_j^2 - \Delta \sum_i s_i^2, \qquad s_i = \pm 1, 0$$

Most general nearest-neighbor, even Hamiltonian.

Odd interaction: $H\sum_i s_i + L\sum_{\langle ij\rangle}(s_i s_j^2 + s_i^2 s_j)$

$$n_4 = \langle s_i^2\rangle \equiv \langle t_i\rangle \qquad t \equiv s_i^2$$

$$n_3 = 1 - \langle s_i^2\rangle = 1 - \langle t_i\rangle$$

$\psi = \langle s_i\rangle$ superfluid order parameter

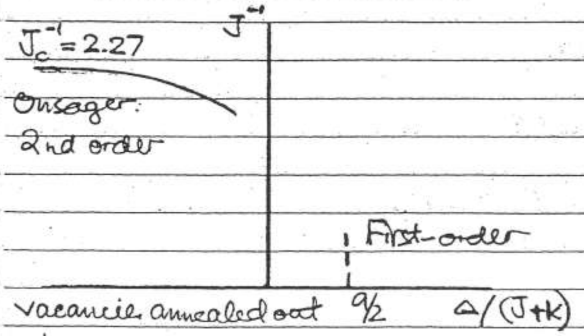
$K = K_{33} + K_{44} - 2K_{34}$ "classical interaction"

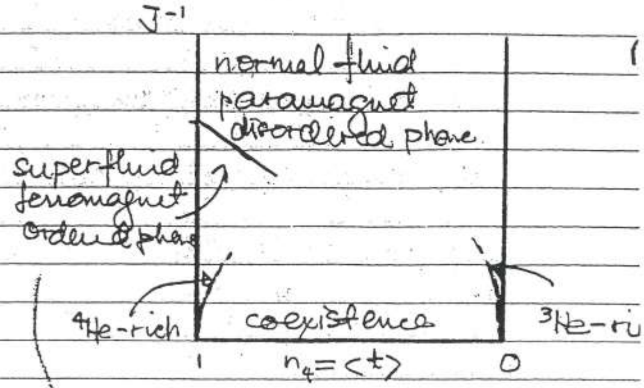
$\Delta = (\mu_3 - \mu_4)/kT + q(K_{33} - K_{44})$ $\sim$ chemical potential difference

$J > 0$ is a potential which promotes superfluidity.

#20

Exact information I





$t = 1,\ s = \pm 1$

$K t_i t_j \to K$ additive constant

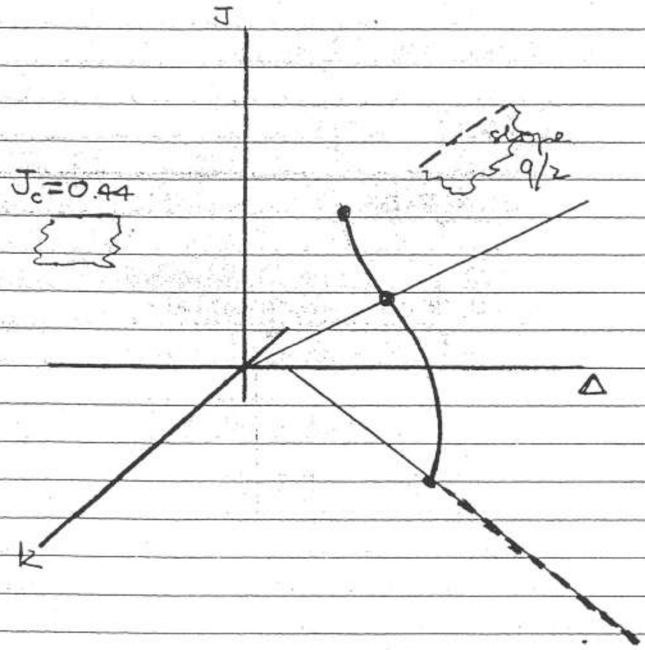
$$\frac{9}{2}(J+K) \sim \Delta$$

$$-\beta\mathcal{H}(\uparrow\uparrow\uparrow) = -\beta\mathcal{H}(\downarrow\downarrow\downarrow) = \left[\frac{9}{2}(J+K) - \Delta\right] N \begin{matrix} \nearrow +\infty \\ \searrow -\infty \end{matrix}$$

#19

$$-\beta\mathcal{H}(000) = 0$$

Infinite ground state energy crossing.



Exact information 2

$J=0$ Griffiths symmetry (1967)

$$-\beta\mathcal{H} = K\sum_{\langle ij\rangle} s_i^2 s_j^2 - \Delta\sum_i s_i^2 \qquad s_i = 0, \pm 1$$

$$= K\sum_{\langle ij\rangle} t_i t_j - \Delta\sum_i t_i$$

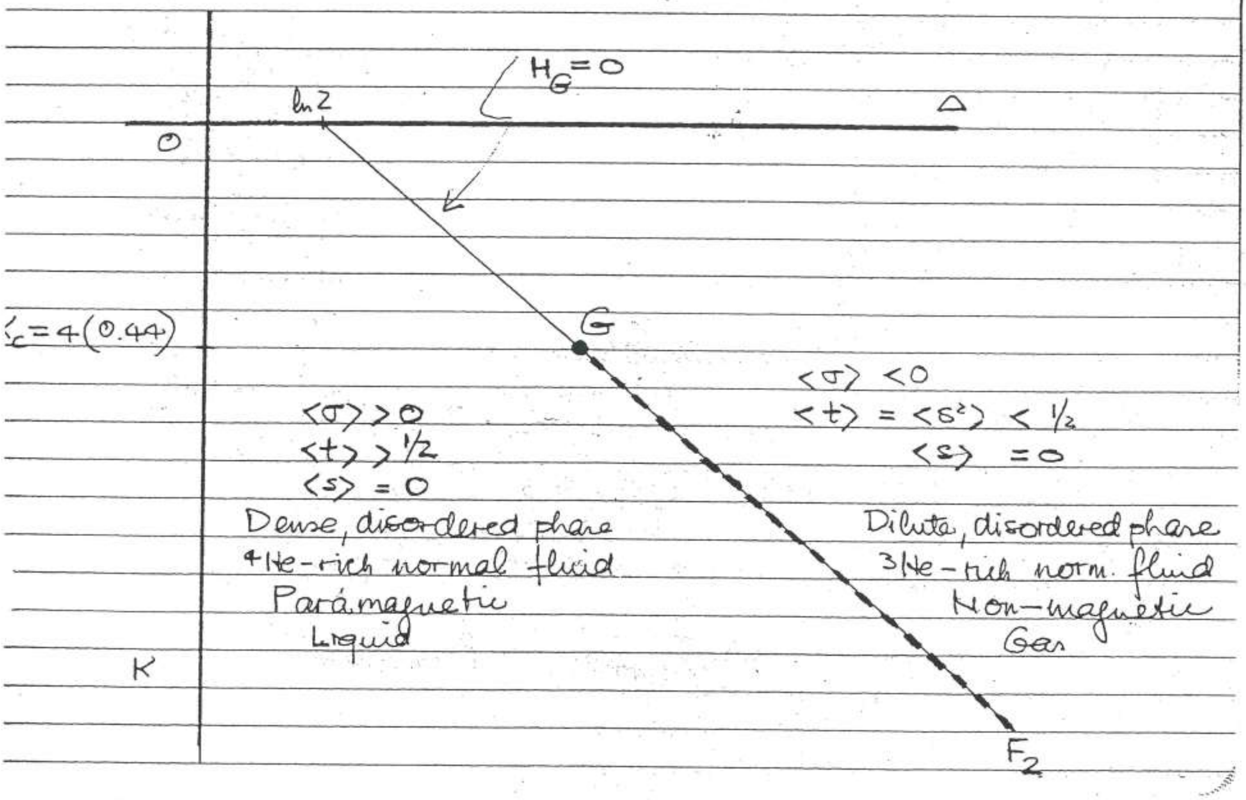
$$Z(J=0) = \sum_{\{s\}} e^{-\beta\mathcal{H}\{s\}} = \sum_{\{t\}} e^{-\beta\mathcal{H}\{t\}}\, 2^{\sum_i t_i} = \sum_{\{t\}} e^{-\beta\mathcal{H}_G}$$

$$-\beta\mathcal{H}_G = K\sum_{\langle ij\rangle} t_i t_j + (\ln 2 - \Delta)\sum_i t_i$$

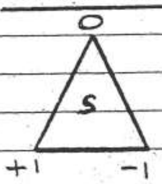
Let $\sigma_i = 2t_i - 1 = \pm 1$. spin-1/2 Ising variable

$$-\beta\mathcal{H}_G = J_G\sum_{\langle ij\rangle}\sigma_i\sigma_j + H_G\sum_i\sigma_i$$

$$J_G = K/4, \qquad H_G = qK/4 + (\ln 2 - \Delta)/2.$$



Exact information 3



$$K = 3J \qquad \Delta = 2qJ$$

$$-\beta\mathcal{H} = \sum_{\langle ij\rangle} E(s_i, s_j)$$

$$E(s_i,s_j) = Js_is_j + Ks_i^2s_j^2 - \frac{\Delta}{q}(s_i^2+s_j^2) =$$

$$= J\left[s_is_j - s_i^2s_j^2 - 2(s_i^2-s_j^2)^2\right]$$

$E(+,+) = 0$ $\qquad E(s_i = s_j) = 0$

$E(-,-) = 0$

$E(0,0) = 0$

$E(+,-) = -2J$ $\qquad E(s_i \neq s_j) = -2J$

$E(+,0) = -2J$

$$-\beta\mathcal{H} = \sum_{\langle ij\rangle} 2J(\delta_{s_is_j} - 1) \Rightarrow 2J\sum_{\langle ij\rangle}\delta_{s_is_j} \qquad s_i = -1, 0, +1$$

$$= 2J\sum_{\langle ij\rangle}\delta_{u_iu_j} \qquad u_i = a, b, c$$

3-state Potts model
Different universality cl

From duality, transition at $2J = 1.005 = \ln(1+\sqrt{3})$

Mean-field theory

1. Review of procedure

a) $F = -kT\ln Z \leq -kT\{\mathrm{Tr}\,\rho\mathcal{H} - \mathrm{Tr}\,\rho\ln\rho\} \equiv \psi(\rho)$ $\qquad$ Gibbs inequa
where ρ is any density matrix
(non-neg, Hermitian, $\mathrm{Tr}\,\rho = 1$).

Equality when $\rho = \dfrac{e^{-\beta\mathcal{H}}}{\mathrm{Tr}\,e^{-\beta\mathcal{H}}} = \rho_{canonical}$

b) Mean-field approximation

Use most general single-site density matrix $\rho = \prod_i \rho_i$.

Minimize $\Psi(\rho_i)$.

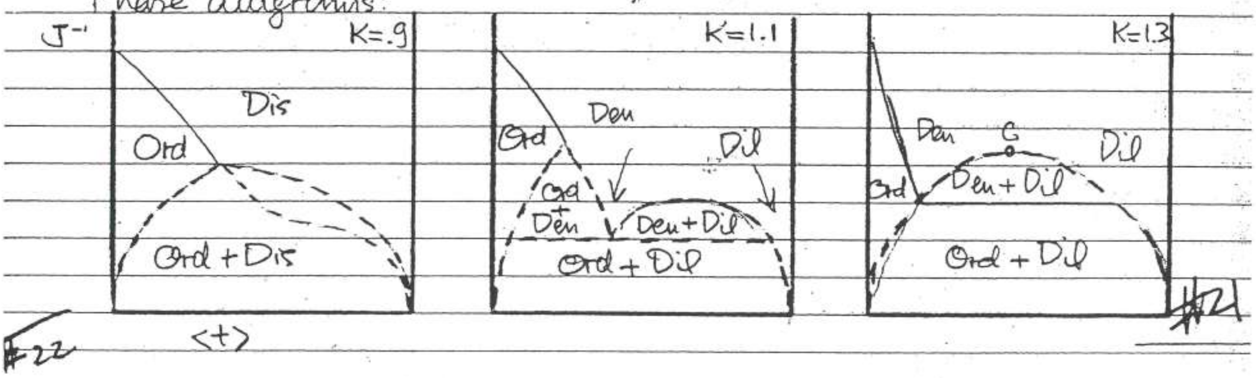
$$\rho_i = R\, e^{h s_i + g s_i^2}$$
$$R = \text{normalization} = (1 + e^{h+g} + e^{-h+g})^{-1}.$$

Minimize $\Psi(h,g)$

Turn out that
$$h = qJ\langle s\rangle$$
$$g = qK\langle s^2\rangle - \Delta,$$
phenomenological mean fields.

Two variable minimization $\rightarrow$ simplifies to one variable when $K=0$.

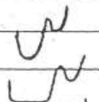
Phase diagrams:

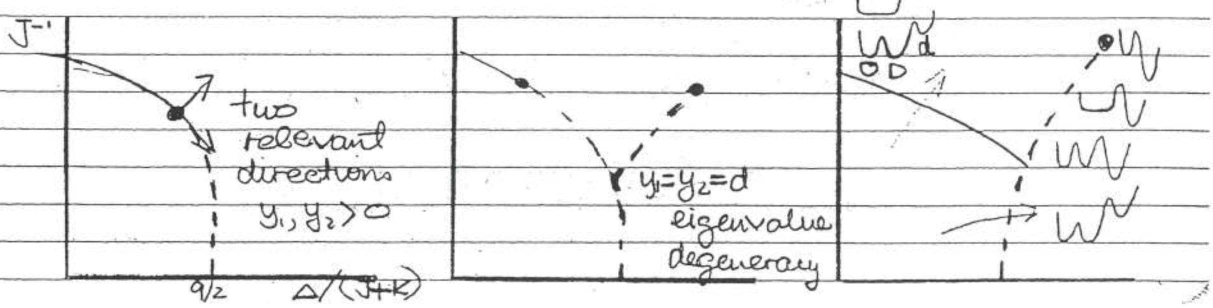


Tricritical point
$\langle ss\rangle - \langle s\rangle^2 \rightarrow \xi = \infty$
$\langle s^2 s^2\rangle - \langle s^2\rangle \rightarrow \xi = \infty$

Triple point
(Quadruple point)

Critical end-point





Migdal-Kadanoff recursion

$$-\beta\mathcal{H} = \sum_{\langle ij\rangle} \bar{\mathcal{H}}_{ij}$$

$$\bar{\mathcal{H}}_{ij} = J s_i s_j + K s_i^2 s_j^2 - \frac{\Delta}{q}(s_i^2 + s_j^2)$$

$$\tilde{J} = b^{d-1} J = 2J$$
$$\tilde{K} = 2K$$
$$\tilde{\Delta} = 2\Delta$$

$$x \equiv e^{J} \qquad y \equiv e^{K} \qquad z \equiv e^{-\Delta/q}$$

Decimation:

$$e^{\bar{\mathcal{H}}'_{13}} = \sum_{s_2} e^{\mathcal{H}_{12}+\mathcal{H}_{23}} = R(s_1 s_3) = e^{J' s_1 s_3 + K' s_1^2 s_3^2 - \frac{\Delta'}{q}(s_1^2+s_3^2) + \tilde{G}}$$

$$\phantom{R(++) = z^4\{}\ s_2 = 0 \qquad +1 \qquad\qquad -1$$

$$R(++) = z^4\{1 + z^4 x^4 y^4 + z^4 x^{-4} y^4\} = e^{J' + K' - 2\Delta'/q + \tilde{G}}$$

$$R(+-) = z^4\{1 + z^4 y^4 + z^4 y^4\} = e^{-J' + K' - 2\Delta'/q + \tilde{G}}$$

$$R(+0) = z^2\{1 + z^4 x^2 y^2 + z^4 y^2\} = e^{-\Delta'/q + \tilde{G}}$$

$$R(00) = \{1 + z^2 + z^2\} = e^{\tilde{G}}$$

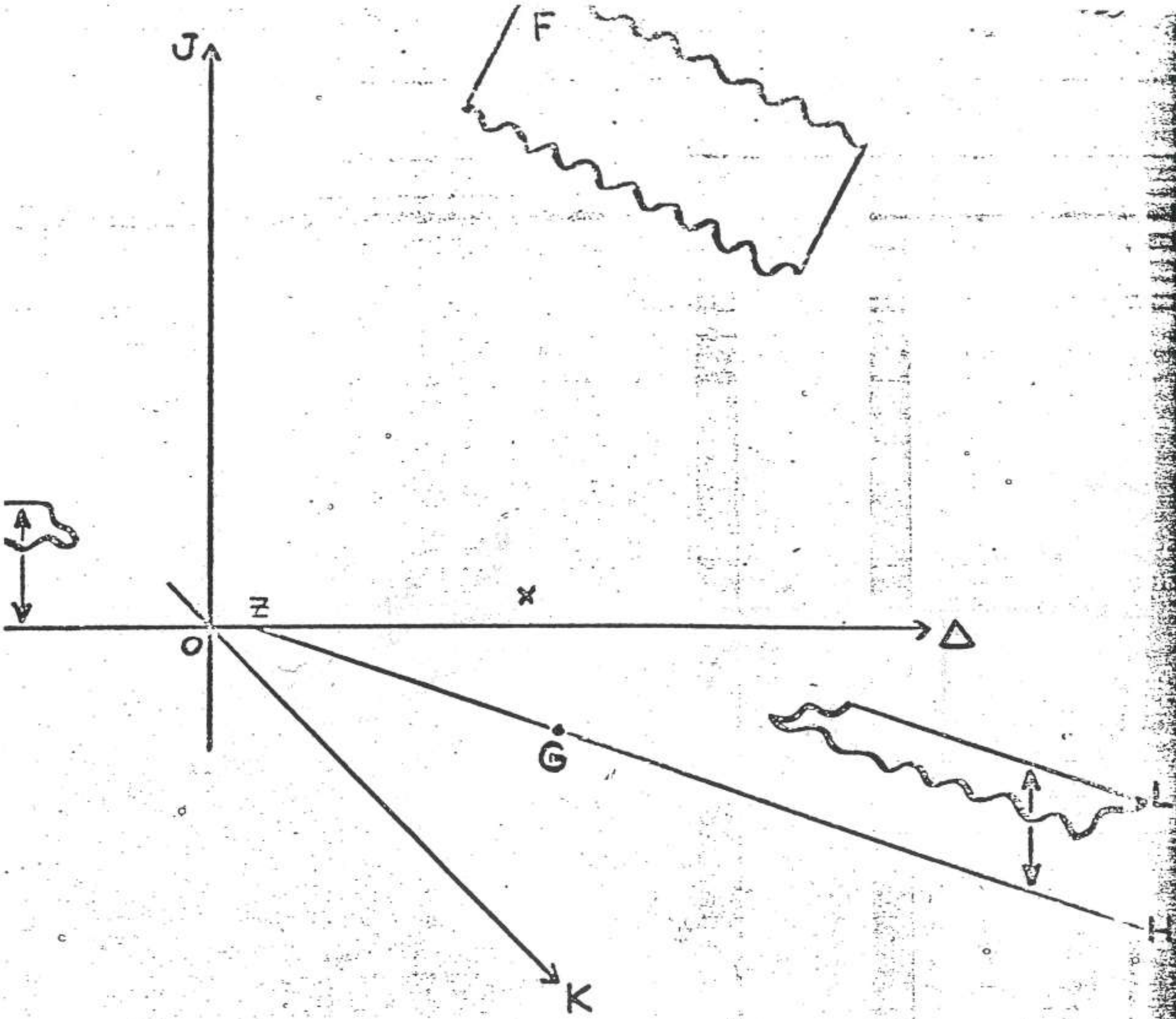
$$\tilde{G} = \ln R(00)$$

$$-\Delta'/q = \ln R(+0)/R(00)$$

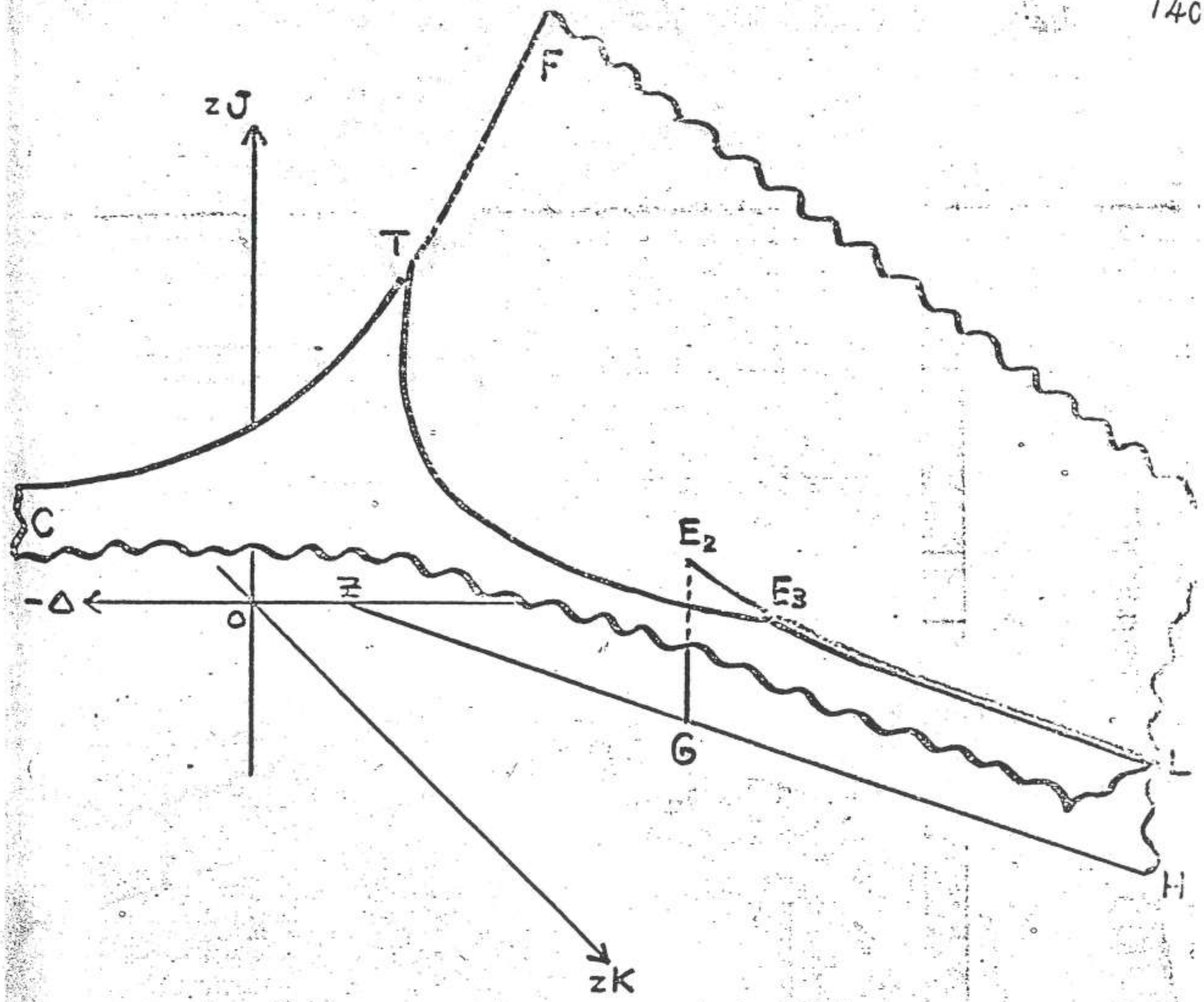
$$J' = \frac{1}{2}\ln R(++)/R(+-)$$

$$K' = \frac{1}{2}\ln R^2(00)\, R(++)\, R(+-)/R(+0)^2$$

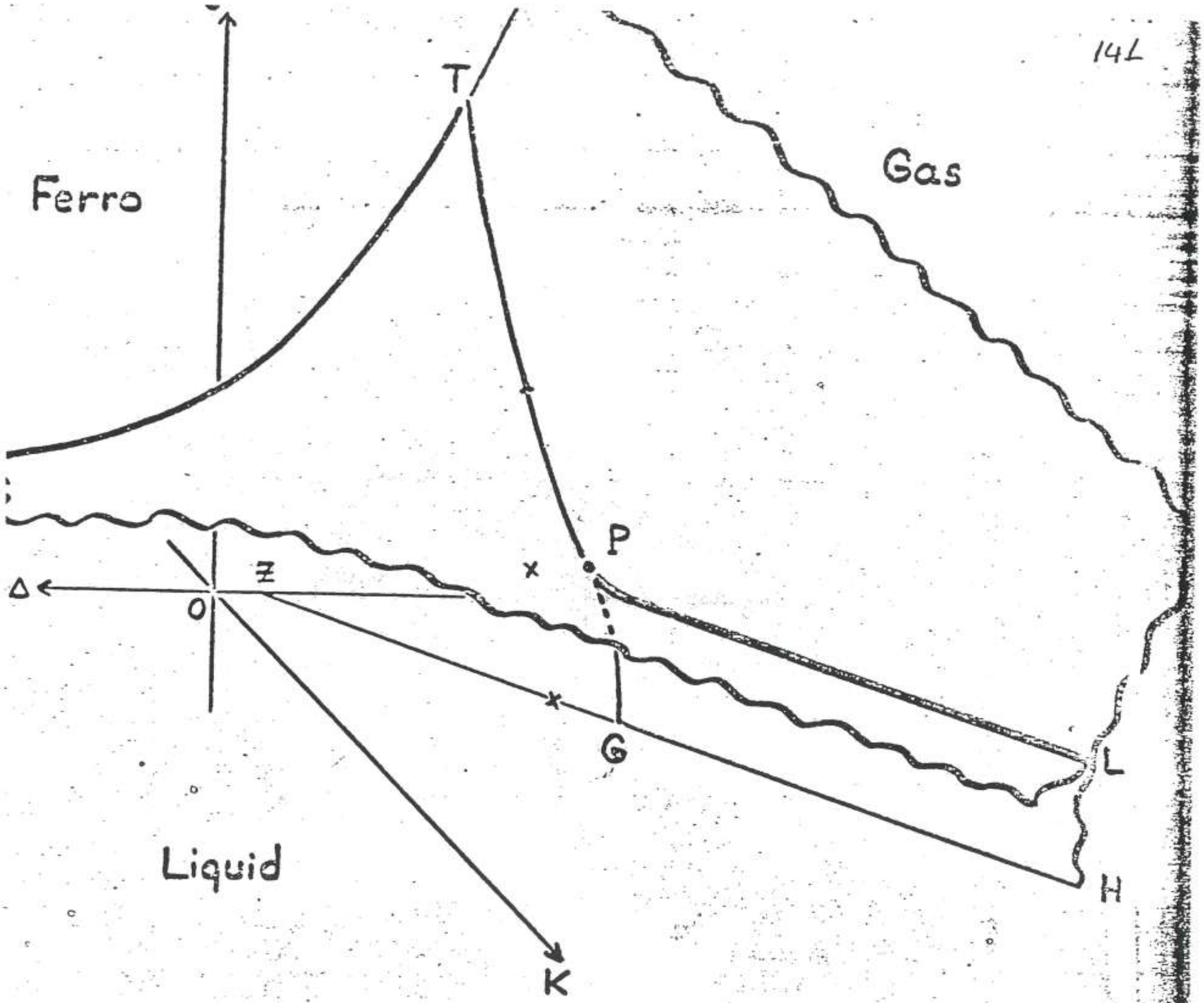
Cluster calculation: A.N.B. and M.Wortis, Phys. Rev. B 14, 4946 (1976)



Exact information

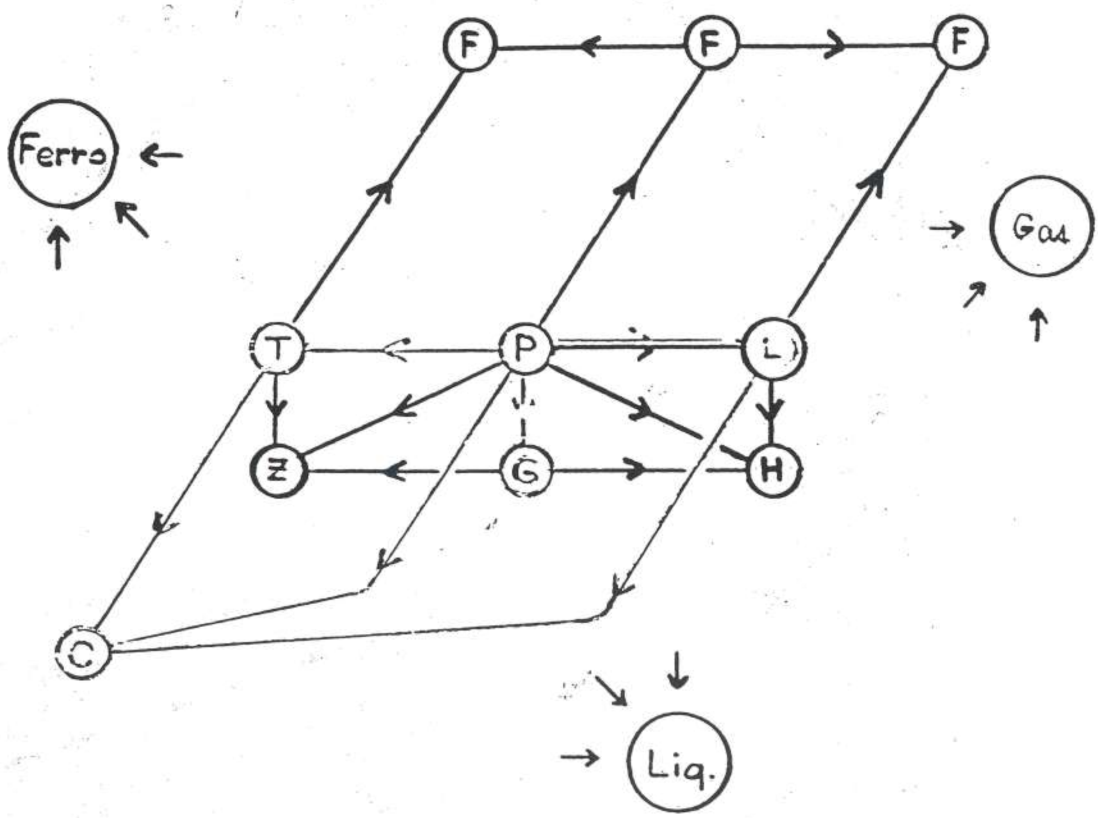


Mean-field theory



Position-space RG

First-order
Critical
Tricritical



Fixed point connectivity

Information Processing and
the Phase Transition Models of Statistical Mechanics

I. Neural Networks
Hopfield
Amit, Modeling Brain Function (Cambridge UP, 1989)
Heidelberg Colloquium on Glassy Dynamics (Springer-Verlag, 1987)
Domany, Kinzel, Sompolinsky

II. Simulated Annealing
S. Kirkpatrick, C.D. Gelatt, and M.F. Vecchi, Science 220, 671 (1983)
articles in Heidelberg Colloquium

III. Information Transmission (Coding-decoding)
N. Sourlas, Nature 339, 693 (29 June 1989)
(H.G.E. Hentschel, p. 662 of same issue)

Phase transition models of statistical mechanics:
M. Mézard, G. Parisi, and M.A. Virasoro, Spin-Glass Theory and Beyond
(World Scientific, Singapore, 1987)

B. Müller and J. Reinhardt, Neural Networks (Springer-Verlag, 1990)
E. Domany, J.L. van Hemmen, and K. Schulten,
Models of Neural Networks (Springer-Verlag, 1991)
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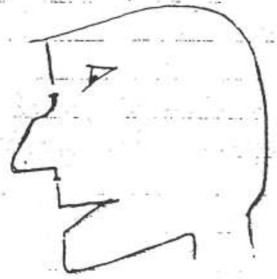
Neural Computation (Addison-Wesley, 1991).

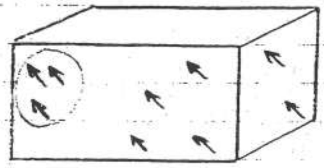
P. Rujan, Phys. Rev. Lett. 70, 2968 (1993).
N. Sourlas, Europhys. Lett. 25, 159 (1994).
N. Sourlas, in Statistical Mechanics of Neural Networks,
ed. by L. Garrido (Springer-Verlag, 1990).

I. Neural Networks

A. The Basic Idea

. Retrieval





Nose → $\boxed{\text{brain}}$ → face

Few spin states specified → $\boxed{\text{magnet}}$ → all spin states

The brain can store many memories

A "spin-glass" magnet can store many global spin states

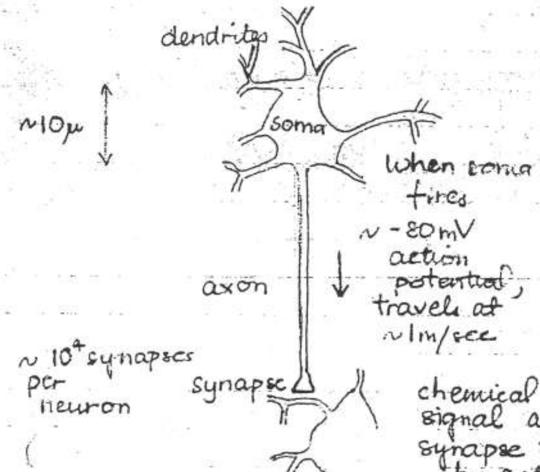
Other phenomena:

. Sequential recall

. Forgetting (or fuzzy recall)

B. The Biological System: Coupled Neurons

Brain: 10^{12} neurons (nerve cells)



chemical signal across synapse triggered by action potential.

Collective phenomena in a network of many, relatively simple constituen

⇓

statistical mechanics

A cell fires due to cumulative effect of incoming signals, which are excitatory or inhibitory

Retractory period ~ 6 μsec

recovery period during which cell cannot fire.

C. Developing the Neural Network Models

1. Faithful representation

Cumulative signal at neuron i $\equiv$ post-synaptic potential $\equiv$

$N_i \rightarrow$ sum over all neurons j presynaptic to neuron i.

$$h_i = \sum_{j=1}^{N_i} J_{ij}\,\sigma_j \,, \quad \text{where}$$

$$\sigma_j = \begin{cases} 1 & \text{for neuron } j \text{ firing} \\ 0 & \text{not firing.} \end{cases}$$

J_{ij} is the synaptic efficacy (coupling constant)

$J_{ij} > 0$ excitatory

$J_{ij} < 0$ inhibitory

$$\sigma_i(t+1) = \Theta(h_i(t) - u_i) \,, \quad \text{where}$$

$\Theta(x \gtrless 0) = \begin{cases} 1 \\ 0 \end{cases}$ and u_i is the threshold of neuron i.

2. Simplifications

(a) $N_i = N$ full connectivity

(b) $J_{ij} = J_{ji}$ symmetric connectivity

(asymmetric connectivity, $J_{ij} \neq J_{ji}$ will be investigated later.)

$s_i \equiv 2\sigma_i - 1 = \pm 1$, Ising spins

$\sigma_i \equiv \frac{1}{2}(s_i + 1)$

$$s_i(t+1) = \text{sign}(\bar{h}_i(t) - \bar{u}_i) \,, \quad \text{where}$$

$\text{sign}(x \gtrless 0) = \pm 1$

$$\bar{h}_i = \frac{1}{2}\sum_{j=1}^{N} J_{ij}\, s_j \,, \quad \bar{u}_i = u_i - \frac{1}{2}\sum_i J_{ij}$$

absorb

(c) $\bar{u}_i = 0$.

3. Energy Consider the following spin Hamiltonian

$$\mathcal{H} = -\sum_{ij} J_{ij}\, s_i s_j = -\sum_i h_i s_i$$

at zero temperature.

Each spin i feels the field $h_i = \sum_j J_{ij} s_j$ due to the other spins.

At zero temperature, the system evolves to minimum energy by each spin aligning with its field, $s_i(t+1) = \text{sign}(h_i(t))$

4. Temperature (a) Neurons operate in a noisy environment:

- At the synapse, the number of vesicles discharging chemical quanta upon arrival of the action potential varies (according to a Poisson probability distribution.)
- The size of the chemical quanta varies according to a Gaussian probability distribution.
- The mean of these phenomena determines J_{ij}, but there is also a variance.

Probabilities:

$$P(h_i^{actual}) = \sqrt{\frac{2}{\pi T^2}}\, e^{-(h_i^{act}-h_i)^2/(T^2/2)} \quad \text{, width } T.$$

$$P(h_i^{act}>0) = \int_0^\infty dh_i^{act}\, P(h_i^{act}) = \frac{1}{2}\left[1+\mathrm{erf}\left(\frac{h_i}{T/\sqrt{2}}\right)\right]$$

$$P(h_i^{act}<0) = 1 - P(h_i^{act}>0) = \frac{1}{2}\left[1-\mathrm{erf}\left(\frac{h_i}{T\sqrt{2}}\right)\right]$$

$$P(s_i(t+1)) = \frac{1}{2}\left[1+s_i\,\mathrm{erf}\left(\frac{h_i}{T\sqrt{2}}\right)\right] = \frac{1}{2}\left[1+\mathrm{erf}\left(\frac{h_i s_i}{T\sqrt{2}}\right)\right] \underset{\text{to } 1\%}{\simeq} \frac{1}{2}\left[1+\tanh\left(\frac{h_i s_i}{T}\right)\right]$$

$$= \frac{e^{h_i s_i/kT}}{e^{h_i/kT}+e^{-h_i/kT}}$$

desor!

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x dt\, e^{-t^2}$$

$$\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{h_i s_i}{T\sqrt{2}}\right)\right] \overset{\text{small } |x|}{\simeq} \frac{1}{2}\left[1+\frac{2}{\sqrt{\pi}}\frac{h_i s_i}{T\sqrt{2}}\right] = \frac{1}{2}\left[1+\sqrt{\frac{2}{\pi}}\frac{h_i s_i}{T}\right]$$

$$P(s_i=-1\to+1) \sim e^{h_i/kT}, \qquad P(s_i=+1\to-1) \sim e^{-h_i/kT}$$

Detailed balance theorem: This system relaxes to the equilibrium distribution function:

$$P(\{s_i\}) = Z^{-1} e^{-\sum_i h_i s_i/kT} = Z^{-1} e^{-\sum_{i,j} J_{ij} s_i s_j/kT} \qquad \text{Gibbs distribution}$$

The normalization, $Z = \sum_{\{s_i\}} e^{-\sum_{i,j} J_{ij} s_i s_j/kT}$ is called the partition function.

Boltzmann-H theorem: The system monotonically decreases its free energy

$F = -kT \ln Z$, until the Gibbs distribution is reached (where F is a minimum)

5. Dynamics (a) Asynchronous: Each s_i is tested sequentially for a flip.

More applicable to the neural system.

Detailed balance theorem valid (process is Markov chain) as above.

(b) Synchronous. All $\{s_i\}$ tested, then updated together.
Application of detailed balance is modified:

$P(\{s_i\}) = Z^{-1} e^{\sum_i \ln(2\cosh(\sum_j J_{ij} s_j/kT))}$, corresponding to note: $J_{ii} = 0$.

$\mathcal{H} = -kT \sum_i \ln(2\cosh(\sum_j J_{ij} s_j/kT))$.

$F = -kT \ln Z$, from the new normalization, is minimized.
($T \to 0$: reduce to asynchronous case.)

D. Retrieval of Memories: Basic (Hopfield-Little) Model

1. Storage of Memories

(a) Memory μ: $\{s_i\} = \{\xi_i^\mu\}$.

Storage of memories = choosing $\{J_{ij}\}$.

(Thus, the brain learns by modifying its synapses.)

(b). A single memory: Take $J_{ij} = \frac{1}{N}\xi_i\xi_j$ (Mattis model) of spin-glasses

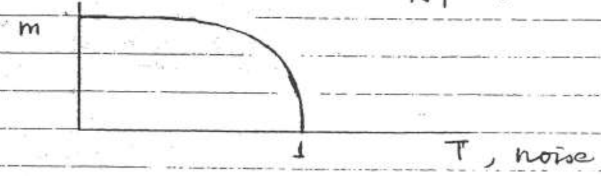
$\mathcal{H} = -\sum_{i,j} J_{ij} s_i s_j = -N$, absolute minimum when $\{s_i\} = \{\xi_i\}$.

One other minimum (degenerate): $\{s_i\} = \{-\xi_i\}$.
Reverse memory also stored.

The system evolves by a Monte Carlo algorithm:
For s_i, a random number $R[0,1]$ is compared to $P(s_i = \text{flipped})$
$R \lessgtr P(\text{flipped}) \rightarrow$ flip accepted / rejected. (heath bath) Glauber (1963)

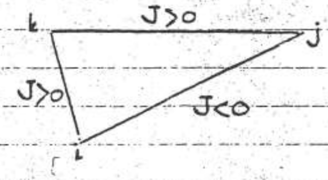
Recall of the memory is measured by: $m = \frac{1}{N}\sum_i s_i \xi_i$

When equilibrium is reached:



(c) Many memories: $J_{ij} = \frac{1}{N} \sum_{\mu=1}^{P} \xi_i^\mu \xi_j^\mu$

"Built-in noise":

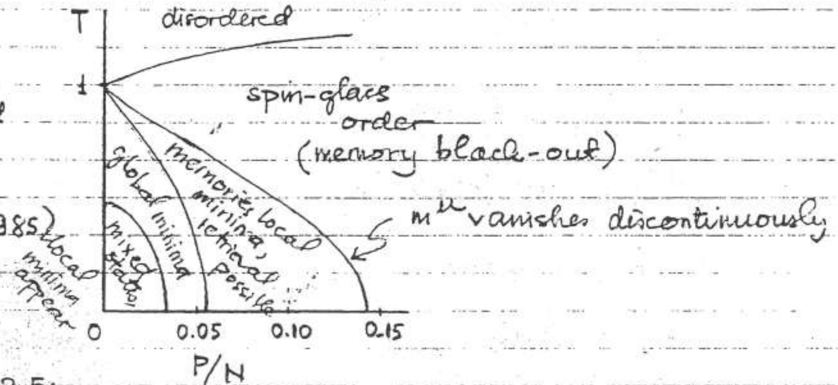


cannot be simultaneously satisfied.

This "frustration" characterizes spin-glass models.
Minima, other than memories, of the energy (T=0) and free energy (T>0), and limit retrieval.

Phase diagram:

Numerical and analytic
Amit, Gutfreund, and Sompolinsky (1985)
PRL 55, 1530



: e.g. van Hemmen, pp. 559, 562 in Heidelberg colloquium (1987) Figs. 2, 5;
"clipped" model: $J_{ij} = \text{sign} \sum_\mu \xi_i^\mu \xi_j^\mu$ Robustness
20×20 Ising spins; 30% noise level
25 patterns; Hamming distance 126 $\xrightarrow{3\text{MCSS}}$ 0

Theory also stable to the more realistic asymmetric connectivity $J_{ij} \neq J_{ji}$,
Hertz, Grinstein, and Solla. / Heidelberg (1987)
although no free energy construction, but dynamics well defined. In fact, spinglass states suppressed: Memory retrieval may be improved.

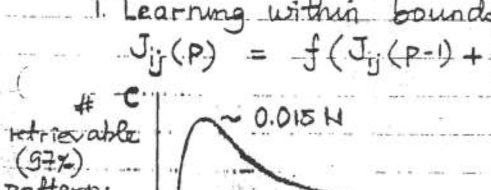
Further development of models:

E. Learning and Forgetting

1. "Learning within bounds": Parisi; Toulouse / Heidelberg (1987) (1986)

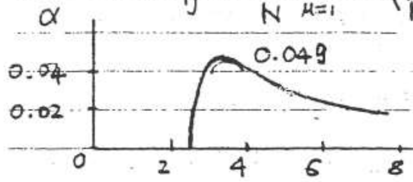
$J_{ij}(P) = f\left(J_{ij}(P-1) + \frac{\epsilon}{N} \xi_i^P \xi_j^P\right)$, $\quad f(x) = \begin{cases} A & \text{for } x>A \\ x & A>x>-A \\ -A & -A>x \end{cases}$

favors latest storages.



2. "Marginalist Learning": Nadal, Toulouse, Changeux and Dehaene (1986)

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \Lambda\left(\frac{\mu}{N}\right) \xi_i^\mu \xi_j^\mu, \qquad \Lambda(x) = \epsilon \exp\left(-\frac{\epsilon^2}{2}x\right)$$



$\mu = 1$ latest memory.

Analytic

3. 7 ± 2 Rule: Humans presented with lists of items (e.g. numbers), recall such lists soon after presentation if they do not exceed 7 ± 2 items.

$7 \pm 2 \sim 0.02 \cdot N$ optimally from above, gives
$N \sim 350 \pm 100$ neurons.

On the low side. May indicate less than optimal operation.
(10^4 synapses per neuron) But, perhaps, should use long-term, practised memories...

F. Correlated Memories

1. Breakdown Kinzel Figs. 8, 9, 10; pp. 210-1 (1985)
Z. Phys. B 60, 205 (1985) Non-zero overlaps with several patterns no good.

2. Learning by Eliminating Frustrated Bonds Kinzel (1985)
(a) Start with random J_{ij}.
(b) If $J_{ij}\,\xi_i^{(\mu)}\xi_j^{(\mu)} < 0$ for any μ, set $J_{ij} = 0$.
Inspired by neurobiology: synapses get modified by dying.
Kinzel, Fig. 11, p. 211. (1985)

3. Projection Matrix Personnaz, Guyon, and Dreyfus (1985)

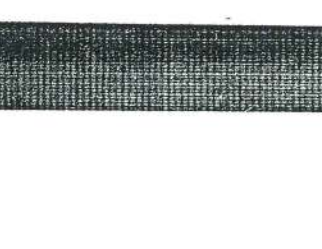
$\sum_j J_{ij}\,\xi_j^\mu = \xi_i^\mu$, for all μ, would guarantee the stability of all stored patterns.

This is accomplished by $J_{ij} = \frac{1}{N}\sum_{\bar\mu,\bar\nu=1}^{p} \xi_i^{\bar\mu}\xi_j^{\bar\nu} A^{-1}_{\bar\mu\bar\nu}$, $\quad A_{\bar\mu\bar\nu} = \frac{1}{N}\sum_{l}^{N} \xi_l^{\bar\mu}\xi_l^{\bar\nu}$

$$\sum_j J_{ij}\,\xi_j^\mu = \sum_j \frac{1}{N}\sum_{\bar\mu,\bar\nu} \xi_i^{\bar\mu}\xi_j^{\bar\nu} A^{-1}_{\bar\mu\bar\nu}\,\xi_j^\mu = \sum_{\bar\mu,\bar\nu} \xi_i^{\bar\mu} A^{-1}_{\bar\mu\bar\nu} A_{\bar\nu\mu} = \sum_{\bar\mu} \xi_i^{\bar\mu}\,\delta_{\bar\mu\mu} = \xi_i^\mu$$

Set $J_{ii} = 0$ by hand (Kanter and Sompolinsky 1986), to obtain free energy model. non-local

phase diagram:



inefficiency storage in field model due to finite size, $(N)^{-1/2}$ and J_{ii}

Large N, uncorrelated patterns → reduces to Hopfield model,
Larger storage capacity, which is why it works well.

$$\sum_i J_{ii} = \sum_i \frac{1}{N}\sum_{\bar\mu,\bar\nu} \xi_i^{\bar\mu}\xi_i^{\bar\nu} A^{-1}_{\bar\mu\bar\nu} = \sum_{\bar\mu}\delta_{\bar\mu\bar\mu} = N\alpha$$

$$S_1 = \mathrm{sign}\left(\sum_{j\neq 1} J_{1j}\,\xi_j^\mu + J_{11}S_1\right) \simeq \mathrm{sign}\left((1-\alpha)\,\xi_1^\mu + \alpha S_1\right)$$

G. Sequential Memories

$$h_i(t) = \sum_j J_{ij} s_j(t) + \sum_j J_{ij}^{\times} s_j(t-\tau) =$$

$$= \sum_j \left(\frac{1}{N} \sum_{\mu=1}^{p} \xi_i^{\mu} \xi_j^{\mu} \right) s_j(t) + \sum_j \left(\frac{\lambda}{N} \sum_{\mu=1}^{p} \xi_i^{\mu+1} \xi_j^{\mu} \right) s_j(t-\tau)$$

projection matrix method: $\left(\frac{\lambda}{N} \sum_{\bar{\mu},\bar{\nu}}^{p} \xi_i^{\bar{\mu}+1} \xi_j^{\bar{\nu}} A^{-1}_{\bar{\mu}\bar{\nu}} \right)$

- Amit, Fig. 5.5, p. 234 (1989)

Collective Phenomena in Neural Networks

J.L. van Hemmen and R. Kühn, in Domany et al. (1991).

A. Hopfield Model

$$J_{ij} = \frac{1}{N} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu}$$

$$\mathcal{H}_N = -\frac{1}{2} \sum_{i,j} J_{ij} s_i s_j = -\frac{1}{2} N \sum_{\mu} \left(\frac{1}{N} \sum_{i=1}^{N} \xi_i^{\mu} s_i \right)^2 = -\frac{1}{2} N \sum_{\mu} m_{\mu}^2 ,$$

where $m_{\mu} = \frac{1}{N} \sum_{i=1}^{N} \xi_i^{\mu} s_i$ are the "overlaps".

$s_i = \xi_i^{\mu}$ for all $i \Leftrightarrow m_{\mu} = 1$.

In all other cases, $m_{\mu} < 1$.

If ξ_i^{μ} and s_i are uncorrelated, $m_{\mu} = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$ by the central limit theorem.

As discussed previously, the neural network obeys statistical mechanics: $P(s_i(t+1)) = \dfrac{e^{h_i(t) s_i(t+1)/kT}}{e^{h_i(t)/kT} + e^{-h_i(t)/kT}}$

and proceeds to an equilibrium characterized by the Gibbs distribution.

B. Single Overlap

$$\mathcal{H}_N = -\frac{1}{2} N m^2 .$$

Free energy per spin: $f = -\frac{kT}{N} \ln \sum_{\{s_i\}} e^{-\mathcal{H}_N/kT} = -\frac{kT}{N} \ln Z_N$.

Calculation: (1) By completing the square:

$$e^{\frac{1}{2}\lambda a^2} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 + x\sqrt{\lambda}a} , \quad \text{so that}$$

$$Z_N = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \sum_{\{s_i\}} e^{x\sqrt{\beta N}\, m} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \prod_i \left(\sum_{s_i} e^{x\sqrt{\beta/N}\, \xi_i s_i} \right) =$$

$$= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \left[2\cosh x\sqrt{\beta/N} \right]^N = \int_{-\infty}^{\infty} \sqrt{\frac{N}{2\pi}}\, dy\, e^{N\left[-\frac{y^2}{2} + \ln 2\cosh y\sqrt{\beta}\right]} .$$

$$-\beta f = \frac{1}{N} \ln Z_N = \frac{1}{N} \ln \int_{-\infty}^{\infty} \sqrt{\frac{N}{2\pi}}\, dy\, e^{N\left[-\frac{y^2}{2} + \ln 2\cosh y\sqrt{\beta}\right]}$$

$N \to \infty$, saddle-point integration:

$$-\beta f \simeq \frac{1}{\cancel{N}} \operatorname*{Supremum}_{y} \cancel{N}\left[-\frac{y^2}{2} + \ln 2\cosh y\sqrt{\beta}\right].$$

(2) By using a <u>density function</u>:

$$Z_N = \sum_{\{S_i\}} e^{\beta N m^2/2} = \int_{-\infty}^{\infty} dm\, \mathcal{D}_N(m)\, e^{\beta N m^2/2}$$

$$\sum_{k=0}^{N} \underbrace{\binom{N}{k}}_{\Delta k} e^{\beta N m^2(k)/2},$$ where k is the # of spins along ξ_i,
$m(k) = \frac{1}{N}(k-(N-k)) = (2k-N)/N.$
$N(m+1)/2 = k.$
$\frac{N}{2} dm = \Delta k$

$$\mathcal{D}_N(m) = \frac{N}{2}\binom{N}{N(m+1)/2} = \frac{N}{2} \exp\left\{\ln N! - \ln N(1+m)/2! - \ln N(1-m)/2!\right\} =$$

Stirling formula
$$\simeq \frac{N}{2} \exp\Big\{-\cancel{N} + N \cancel{\ln N} + \tfrac{1}{2}\ln N + \tfrac{1}{2}\ln 2\pi +$$
$$-\tfrac{1}{2}\ln 2\pi \quad + \cancel{N\left(\tfrac{1+m}{2}\right)} - N\left(\tfrac{1+m}{2}\right)\ln \cancel{N}\left(\tfrac{1+m}{2}\right) - \tfrac{1}{2}\ln N\left(\tfrac{1+m}{2}\right)$$
$$-\tfrac{1}{2}\ln 2\pi \quad + \cancel{N\left(\tfrac{1-m}{2}\right)} - N\left(\tfrac{1-m}{2}\right)\ln \cancel{N}\left(\tfrac{1-m}{2}\right) - \tfrac{1}{2}\ln N\left(\tfrac{1-m}{2}\right)\Big\}$$

$$\simeq \frac{1}{2}\sqrt{\frac{N}{2\pi}} \exp\left\{-\frac{N}{2}\left[(1+m)\ln(1+m) + (1-m)\ln(1-m) - 2\ln 2\right]\right\} \equiv \frac{1}{2}\sqrt{\frac{N}{2\pi}}\, e^{-Nc^*(m)} =$$

$= \mathcal{D}_N(m)$ for $|m| \leq 1$.
$\mathcal{D}_N(m) = 0$ for $|m| > 1$.

$$-\beta f = \frac{1}{N} \ln \int_{-\infty}^{\infty} dm \frac{1}{2}\sqrt{\frac{N}{2\pi}}\, e^{N[\beta m^2/2 - c^*(m)]} =$$ again, by saddle-point integration:

$$\simeq \frac{1}{\cancel{N}} \sup_{m} \cancel{N}\left[\beta m^2/2 - c^*(m)\right].$$

Supremum is realized for $\beta m = \frac{d}{dm} c^*(m) = \frac{1}{2} \ln \frac{1+m}{1-m} = \tanh^{-1} m$

<u>Self-consistent equation</u>: $m = \tanh(\beta m)$

C. <u>Many Overlaps</u>

Let $W_N^{(\mu)} = \sum_{i=1}^{N} \xi_i^{\mu} S_i$, $\vec{W} = (W_N^{(1)}, \dots, W_N^{(2)})$.

Theorem: $\mathrm{Prob}\{m_\mu \leq \frac{W_N^{(\mu)}}{N} \leq m_\mu + dm_\mu ; 1 \leq \mu \leq p\} \sim e^{-Nc^*(\vec{m})} d\vec{m}$,

where $d\vec{m} = \prod_\mu dm_\mu$ and $c^*(\vec{m}) = \sup_{\vec{t}} [\vec{m}\cdot\vec{t} - c(\vec{t})]$, the Legendre transform of the cumulant generating function

$$c(\vec{t}) = \lim_{N\to\infty} \frac{1}{N} \ln \langle e^{\vec{t}\cdot\vec{W}_N} \rangle_{\{s_i\}}$$

R.S. Ellis, Entropy, Large Deviations, and Statistical Mechanics (Springer, 1985).

E.g. Ising spins with $p = 1$ (previous calculation)

$$c(t) = \lim_{N\to\infty} \frac{1}{N} \ln \frac{1}{2^N} \sum_{\{s_i\}} e^{t\sum_i \xi_i s_i} = \lim_{N\to\infty} \frac{1}{\cancel{N}} \ln [\cosh t]^{\cancel{N}}$$

$$c^*(m) = \sup_t [mt - \ln \cosh t].$$

Supremum realized at $m = \tanh t$, $t = \tanh^{-1} m$

$$c^*(m) = m \tanh^{-1} m - \ln \cosh \tanh^{-1} m =$$
$$= \frac{m}{2} \ln \frac{1+m}{1-m} - \ln \cosh \frac{1}{2} \ln \frac{1+m}{1-m} =$$
$$= \frac{m}{2} \ln \frac{1+m}{1-m} - \ln \frac{1}{2}\left\{\sqrt{\frac{1+m}{1-m}} + \sqrt{\frac{1-m}{1+m}}\right\} =$$
$$= \frac{m}{2} \ln \frac{1+m}{1-m} - \ln \frac{1}{\cancel{2}} \frac{1+\cancel{m}+\cancel{1}-\cancel{m}}{\sqrt{1-m^2}} =$$
$$= \frac{1}{2}\{m \ln(1+m) - m \ln(1-m) + \ln(1+m) + \ln(1-m)\}$$

, as in density function on previous page.

Neural networks: p patterns characterized by $\{\xi_i^\mu\}$

In general, $-\beta \mathcal{H}_N = N F(\vec{m})$.

$$c(\vec{t}) = \lim_{N\to\infty} \frac{1}{N} \ln \frac{1}{2^N} \sum_{\{s_i\}} e^{\sum_\mu t_\mu \sum_i \xi_i^\mu s_i} = \lim_{N\to\infty} \frac{1}{N} \ln \prod_{i=1}^N \cosh \sum_\mu t_\mu \xi_i^\mu =$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \ln \cosh \sum_\mu t_\mu \xi_i^\mu = \langle \ln \cosh \sum_\mu t_\mu \xi^\mu \rangle_\xi.$$

$$-\beta f = \lim_{N\to\infty} N^{-1} \ln \int d\vec{m} \exp\{N[F(\vec{m}) - c^*(\vec{m})]\} = \sup_{\vec{m}} [F(\vec{m}) - c^*(\vec{m})].$$

Supremum realized at $\vec{\nabla} F(\vec{m}) = \vec{\nabla} c^*(\vec{m})$.

$$c^*(\vec{m}) = \sup_{\vec{t}} \left[\vec{m}\cdot\vec{t} - c(\vec{t})\right] \quad \text{realized at} \quad \vec{m} = \vec{\nabla} c(\vec{t}).$$

Inverse Legendre transform:

$$c(\vec{t}) = \sup_{\vec{m}} \left[\vec{m}\cdot\vec{t} - c^*(\vec{m})\right] \quad \text{realized at} \quad \vec{t} = \vec{\nabla} c^*(\vec{m})$$

Self-consistent eq.: $\vec{m} = \vec{\nabla} c\left(\vec{\nabla} c^*(\vec{m})\right) = \vec{\nabla} c\left(\vec{\nabla} F(\vec{m})\right)$.

When it is satisfied, $\vec{t} = \vec{\nabla} c^*(\vec{m}) = \vec{\nabla} F(\vec{m}) \longrightarrow$ $c^*(\vec{m}) = \sup_{\vec{t}} \left[\vec{m}\cdot\vec{t} - c(\vec{t})\right] = \vec{m}\cdot\vec{\nabla} F(\vec{m}) - c(\vec{\nabla} F(\vec{m}))$;

$$-\beta f = \max_{\vec{m}} \left[F(\vec{m}) - \vec{m}\cdot\vec{\nabla} F(\vec{m}) + c(\vec{\nabla} F(\vec{m}))\right].$$

D. The Hopfield Model with Several Patterns

From above, (p. 11) $-\beta \mathcal{H}_N = \frac{\beta N}{2} \sum_\mu m_\mu^2 \equiv N F(\vec{m})$, $\quad F(\vec{m}) = \frac{\beta}{2} \vec{m}^2$.

Self-consistent equation: $\vec{m} = \vec{\nabla} c\left(\vec{\nabla} F(\vec{m})\right) = \vec{\nabla} c(\beta \vec{m})$

$$\boxed{\vec{m} = \left\langle \left(\tanh \sum_\mu \beta m_\mu \xi^\mu\right) \vec{\xi} \right\rangle_{\xi}}.$$

$$-\beta f = \max_{\vec{m}} \left[\frac{\beta}{2}\vec{m}^2 - \vec{m}\cdot\beta\vec{m} + c(\beta\vec{m})\right] = \max_{\vec{m}} \left[-\frac{\beta}{2}\vec{m}^2 + c(\beta\vec{m})\right] =$$

$$= \max_{\vec{m}} \left[-\frac{\beta}{2}\vec{m}^2 + \langle \ln\cosh \beta\vec{m}\cdot\vec{\xi}\rangle_{\xi}\right]$$

Many solutions:

i) Retrieval states: Solutions that have a non-zero overlap with a single pattern only.

$m_\mu = m_\gamma \delta_{\mu\gamma}$

Self-consistent equation: $m_\gamma = \langle (\tanh \beta m_\gamma \xi^\gamma) \xi^\gamma \rangle = \tanh \beta m_\gamma$.

Non-zero solution for $\beta > 1$.

$\beta \to \infty$: $m \to 1$ exponentially.

Only for uncorrelated patterns $\left[m_\nu \simeq \beta \sum_\mu \langle \xi_\nu \xi_\mu \rangle m_\mu\right]$ $\rightarrow$ $m_{\bar\gamma} = \langle (\tanh \beta m_\gamma \xi^\gamma) \xi^{\bar\gamma} \rangle = \tanh \beta m_\gamma \langle \xi^\gamma \xi^{\bar\gamma} \rangle =$

ii) n-symmetric states: mixture states:

$$\vec{m} = m(\underbrace{1, \ldots, 1}_{n\ 1\text{'s}}, 0, \ldots, 0) \quad \text{and permutations.}$$

$$m = \left\langle \left(\tanh \beta m \sum_{\mu=1}^{n} \xi^\mu\right) \xi_1 \right\rangle = \left\langle \left(\tanh \beta m \sum_{\mu=1}^{n} \xi^\mu\right)\left(\frac{1}{n}\sum_{\mu=1}^{n} \xi^\mu\right)\right\rangle.$$

$$m = \frac{1}{n}\left\langle \left(\sum_{\mu=1}^{n} \xi^\mu\right) \tanh \beta m \sum_{\mu=1}^{n} \xi^\mu \right\rangle.$$

For $\beta \to \infty$ $\quad m = \frac{1}{n}\left\langle \left|\sum_{\mu=1}^{n} \xi^\mu\right| \right\rangle \sim n^{-1/2}$ for large n.

iii) Asymmetric (mixture) states:

$\vec{m} = (m_1, \dots, m_1, m_2, \dots, m_2, m_3, \dots \; 0, \dots, 0)$

could also be called $(n_1, n_2, \dots)$-symmetric states.

In general, m_a occurs n_a times.

$$m_a = \frac{1}{n_a} \left\langle \left(\sum_{\nu_a=1}^{n_a} \xi^{\nu_a} \right) \tanh \sum_{\nu} \beta m_\nu \xi^{\nu} \right\rangle .$$

tal numbers

Total number of retrieval states: p

Total number of symmetric states: $\displaystyle\sum_{n=2}^{p} \binom{p}{n} = (1+1)^p - 1 - p = 2^p - 1 - p.$

Total number of (n_1, n_2)-symmetric states: $n_{12} = n_1 + n_2$

or n_{12} odd, $\displaystyle\frac{1}{2}\binom{p}{n_{12}} \sum_{n_1=1}^{n_{12}-1} \binom{n_{12}}{n_1} = \frac{1}{2}\binom{p}{n_{12}}(2^{n_{12}} - 2) = \binom{p}{n_{12}}(2^{n_{12}-1} - 1).$

(This assumes that n_1, n_2 determine m_1, m_2, etc., as will be seen below).

Conclude that there are very many mixture state solutions of the self-consistent equation.

Fortunately, often, they are not maxima, i.e. not stable under the dynamics:

$$-\beta f = \max_{\vec{m}} \left[-\frac{\beta}{2} \vec{m}^2 + c(\beta \vec{m}) \right] \equiv R(\vec{m}).$$

$$R(\vec{m}) = -\frac{\beta}{2} \vec{m}^2 + \left\langle \ln \cosh \beta \sum_{\nu} m_\nu \xi^\nu \right\rangle$$

$$\frac{\partial R}{\partial m_\nu} = -\beta m_\nu + \left\langle \left(\tanh \beta \sum_{\nu'} m_{\nu'} \xi^{\nu'} \right) \beta \xi^\nu \right\rangle$$

$$\boxed{\beta^{-1} \frac{\partial^2 R}{\partial m_\mu \partial m_\nu} = -\delta_{\mu\nu} + \beta \left\langle \xi^\mu \xi^\nu \left(\cosh \beta \sum_{\nu'} m_{\nu'} \xi^{\nu'} \right)^{-2} \right\rangle \quad \text{must have all eigenvalues negative.}}$$

(17.1)

Bifurcation: Small Overlap Analysis $\quad \sum_i \xi_i^\mu = 0$ memories.

lf-consistent equation:

$$\vec{m} = \langle \vec{\xi} \tanh \beta \vec{m} \cdot \vec{\xi} \rangle \simeq \langle \vec{\xi} \, \beta \vec{m} \cdot \vec{\xi} \rangle - \frac{1}{3} \langle \vec{\xi} \, \beta^3 (\vec{m} \cdot \vec{\xi})^3 \rangle$$

$$\vec{m} = \beta \vec{m} - \frac{\beta^3}{3} \langle \vec{\xi} (\vec{m} \cdot \vec{\xi})^3 \rangle .$$

Leading term: Bifurcation occurs at $\beta_c = 1$.

$$\vec{m} \simeq \beta \vec{m} - \frac{1}{3} \langle \vec{\xi} (\vec{m} \cdot \vec{\xi})^3 \rangle, \text{ to leading orders.}$$

$$(\beta - 1) \cancel{m_\nu} = \frac{1}{3} \sum_{\nu_i} m_{\nu_1} m_{\nu_2} m_{\nu_3} \langle \xi_\nu \xi_{\nu_1} \xi_{\nu_2} \xi_{\nu_3} \rangle = \frac{1}{3} \left(3 \sum_{\nu_2} \cancel{m_\nu} m_{\nu_2}^2 - 2 m_\nu^{\cancel{3}\,2} \right)$$

$\beta - 1 = \vec{m}^2 - \frac{2}{3} m_\nu^2$ $\quad\therefore$ No asymmetric state bifurcation.

$\beta - 1 = (n - 2/3)\, m^2$

$m^2 = (\beta - 1)/(n - 2/3)$ $\quad\therefore$ Retrieval state $(n=1)$ for $\beta > 1$.

Also, n-symmetric state for $\beta > 1$.

Note: disordered state, $\vec{m} = \vec{0}$ is always a solution. These are bifurcations from it.

Stability:

$$\beta^{-1} \frac{\partial^2 R}{\partial m_\mu \partial m_\nu} = -\delta_{\mu\nu} + \beta \langle \xi^\mu \xi^\nu [1 - (\vec{m}\cdot\vec{\xi})^2] \rangle =$$

$$= (\beta - 1)\delta_{\mu\nu} - \sum_{\nu_i} m_{\nu_1} m_{\nu_2} \langle \xi^\mu \xi^\nu \xi^{\nu_1} \xi^{\nu_2} \rangle =$$

$$= \delta_{\mu\nu}(\beta - 1 - nm^2) - (1 - \delta_{\mu\nu})\, m_\mu m_\nu\, 2 =$$ substituting for $m \neq 0$ solutions:

$$= \delta_{\mu\nu}\left(-\tfrac{2}{3} m^2\right) - (1 - \delta_{\mu\nu})\, 2 m_\mu m_\nu =$$

$\therefore$ Retrieval state is a maximum (exists at $\beta > 1$).

Disordered state is a maximum for $\beta < 1$, not for $\beta > 1$.

n-symmetric states: Eigenvalue problem:

$$= -\frac{2m^2}{3} \begin{array}{c} \overbrace{\qquad\qquad}^{n \times n} \\ \left[\begin{array}{cccc|ccc} 1 & 3 & 3 & 3 & & & \\ 3 & 1 & 3 & 3 & & 0 & \\ 3 & 3 & 1 & 3 & & & \\ 3 & 3 & 3 & 1 & & & \\ \hline & & & & 0 & 0 & 0 \\ & 0 & & & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \end{array}\right] \\ p \times p \end{array}$$

Try vector $\vec{\begin{bmatrix} x \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$

$x + 3(n-1) = \lambda x$

$3x + 1 + 3(n-2) = \lambda$

$(\lambda - 3n + 5) + 9n - 9 =$
$= \lambda(\lambda - 3n + 5)$

$\lambda^2 + (4 - 3n)\lambda + (4 - 6n) = 0$

$\lambda = \frac{1}{2}\{(3n-4) \pm [(3n-4)^2 + 4(6n-4)$

There is a positive eigenvalue for all $n > 1$.

$\therefore$ n-symmetric state not maxima.

<u>Low-Temperature Analysis of n-Symmetric States</u> : $\beta \to \infty$

$$\beta^{-1} \frac{\partial^2 R}{\partial m_\mu \partial m_\nu} = -\delta_{\mu\nu} + \beta \langle \xi^\mu \xi^\nu (\cosh \beta m \sum_{\nu'}^{n} \xi^{\nu'})^{-2} \rangle$$

for n odd $= -\delta_{\mu\nu}$ <u>Maxima</u>

fraction$_n$ of sites with $\sum_{\nu'} \xi^{\nu'} = 0$

for n even $= -\delta_{\mu\nu} + \beta \langle \xi^\mu \xi^\nu [\ \delta(0, \sum_{\nu'=1}^{n} \xi^{\nu'})] \rangle = [\beta\ \ f - 1]\,\delta_{\mu\nu}$

still not maxima

$\beta \to \infty$ Asymmetric States ? $\vec{m} = \langle \vec{\xi} \tanh \beta \sum_{\mu} m_{\mu} \xi^{\mu} \rangle$

$n_1 = 1, \quad n_2 = 2:$

$m_1 = \left[\tanh\beta(m_1 + 2m_2) + 2\tanh\beta m_1 + \tanh\beta(m_1 - 2m_2) \right]/4$ $\nearrow 1$ $\to 3/4 \times$ $\searrow 1/2$

$m_2 = \left[\tanh\beta(m_1 + 2m_2) + \tanh\beta(-m_1 + 2m_2) \right]/4$ $\nearrow 0$ $\to 1/4 \times$ $\searrow 1/2$

no.

Solutions:
$(1,0,0)$
$(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})$
$(0;\tfrac{1}{2},\tfrac{1}{2})$

$n_1 = 2, \quad n_2 = 3:$

$\sqrt{3/8}$

$n_1 = \left[\tanh\beta(2m_1 + 3m_2) + 3\tanh\beta(2m_1 + m_2) + 3\tanh\beta(2m_1 - m_2) + \tanh\beta(2m_1 - 3m_2) \right]/16$

$n_2 = \left[\tanh\beta(2m_1 + 3m_2) + \tanh\beta(2m_1 + m_2) - \tanh\beta(2m_1 - m_2) - \tanh\beta(2m_1 - 3m_2) + 2\tanh\beta\, 3m_2 + 2\tanh\beta m_2 \right]/16$

$\tfrac{1}{2}$, $7/16$, $9/16$, 0, $1/4$, $5/16$, $7/16$, $1/2$

solutions
$;\tfrac{1}{2}; \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4})$
$;\tfrac{1}{2}; 0, 0, 0)$
$0; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), \; (3/8, 3/8; 3/8, 3/8, 3/8)$

$m_1 = 1/2, \quad m_2 = 1/4$ a solution. Maximum $\sqrt{3/8}$

E. Hopfield Model with Infinitely Many Patterns $(p = \alpha N, \; N \to \infty)$

$J_{ij} = N^{-1} \sum_{\lambda} K_{\lambda} \xi_i^{\lambda} \xi_j^{\lambda}$, each pattern λ given weight K_{λ}.

Consider the possibility of having

"condensed" patterns μ : $\lim_{N\to\infty} \frac{1}{N} \sum_i \xi_i^{\mu} s_i = m^{\mu} \neq 0$

"non-condensed" patterns ν : $m^{\nu} = 0$

$-\beta H_N = \frac{\beta}{2N} \sum_{\mu} K_{\mu} \left(\sum_i \xi_i^{\mu} s_i \right)^2 + \frac{\beta}{2N} \sum_{\nu} K_{\nu} \left(\sum_i \xi_i^{\nu} s_i \right)^2 .$

$\{\xi_i^{\lambda}\}$ are random (uncorrelated) patterns

$$-\beta f = \lim_{N\to\infty} N^{-1} \ln Z \Rightarrow \lim_{N\to\infty} N^{-1} \frac{1}{2^{pN}} \sum_{\{\xi_i^{\lambda}\}} \ln Z_N \equiv \lim_{N\to\infty} N^{-1} \langle \ln Z_N \rangle .$$

$$\langle \ln Z_N \rangle \overset{?}{=} \frac{d}{dn} \ln \langle Z_N^n \rangle \Big]_{n=0} = \frac{\frac{d}{dn}\langle e^{n \ln Z_N} \rangle}{\langle Z_N^n \rangle} \Big]_{n=0} = \frac{\langle Z_N^n \ln Z_N \rangle}{\langle Z_N^n \rangle} \Big]_{n=0} \checkmark$$

$$= \lim_{n\to 0} \frac{\langle Z_N^n \rangle - 1}{n}$$

$$\equiv -N\beta \frac{d}{dn} \Phi_N(n) \Big]_{n=0} \quad ; \quad -\beta f = \lim_{N\to\infty} -\beta \frac{d}{dn} \Phi_N(n) \Big]_{n=0}$$

$-\beta \Phi_N(n) \equiv N^{-1} \ln \langle Z_N^n \rangle$ = free energy of n replicas of system.

Replica trick

Statistical Mechanics of the Spin-Glass Problem

$Z = \sum_{\{s_i\}} e^{-\sum_{ij} J_{ij} s_i s_j}$ hard to evaluate, because inhomogeneous

Average the free energy (the generating function) over inhomogeneity.

$f = kT \sum_{\{\xi_i^\mu\}} \ln Z$

$\ln Z = \lim_{n \to 0} \frac{Z^n - 1}{n}$

Calculate at finite n, then take $n \to 0$ (non-rigorous)

$\sum_{\{\xi_i^\mu\}} Z^n$ can be readily summed.

$\Rightarrow$ n coupled "replicas of system, $\{s_i^n\}$.
Now homogeneous problem.
Mean-field theory.

Temporal versus Thermodynamic Stability

Temporal evolution:

Recall: $P(s_i(t+1)) = \dfrac{e^{s_i(t+1)h_i(t)/kT}}{e^{h_i(t)/kT} + e^{-h_i(t)/kT}}$ Glauber dynamics

$$h_i = \sum_j J_{ij} s_j = \sum_j \frac{1}{N} \sum_\mu \xi_i^\mu \xi_j^\mu s_j = \sum_\mu \xi_i^\mu m_\mu .$$

Sequential (asynchronous) dynamics

$$m_\mu(t+\Delta t) = \frac{1}{N}\sum_i \xi_i^\mu s_i(t) + \frac{1}{N}\sum_i \frac{1}{N} \xi_i^\mu [\tanh \beta h_i(t) - s_i(t)]$$

$$m_\mu(t) + \Delta t\, \dot m_\mu(t) = m_\mu(t) + \frac{1}{N^2}\sum_i \xi_i^\mu \tanh \beta \vec\xi_i \cdot \vec m(t) - \frac{1}{N} m_\mu(t)$$

$$\Delta t\, \dot m_\mu(t) = \frac{1}{N} \langle \xi^\mu \tanh \beta \vec\xi \cdot \vec m(t) \rangle - \frac{1}{N} m_\mu(t) .$$

ΓN trials per unit time $\rightarrow$ $\Delta t = 1/\Gamma N$.

$$\dot m_\mu(t) = \Gamma\left[\langle \xi^\mu \tanh \beta \vec\xi \cdot \vec m(t) \rangle - m_\mu(t)\right] .$$

Thermodynamic function

$$-\beta f = \max_{\vec m} \left[-\frac{\beta}{2} \vec m^2 + c(\beta \vec m) \right]$$

$$R(\vec m) \equiv -\frac{\beta}{2} \vec m^2 + c(\beta \vec m) = -\frac{\beta}{2} \vec m^2 + \langle \ln \cosh \beta \vec\xi \cdot \vec m \rangle$$

$$\dot R(\vec m) = \left[-\beta \vec m + \beta \langle \vec\xi \tanh \beta \vec\xi \cdot \vec m \rangle \right] \cdot \dot{\vec m} = \frac{\beta}{\Gamma} \dot{\vec m}^2 \geqslant 0 .$$

$\therefore$ System relaxes to a local or global maximum of $R(\vec m)$.
"$R(\vec m)$ is a Lyapunov function for the Glauber dynamics."

Dynamic stability $\equiv$ thermodynamic (meta)stability.

3. The replicas $\sigma = 1, \ldots, n$ have the same $\{\xi_i^\lambda\}$

The replicas are otherwise decoupled. m_μ maximally (over σ) condensed set.

$$\langle Z^n \rangle = \sum_{\{s_{i\sigma}\}} \langle e^{\frac{\beta}{2N}\sum_{\mu\sigma} K_\mu (\sum_i \xi_i^\mu s_{i\sigma})^2} e^{\frac{\beta}{2N}\sum_{\nu\sigma} K_\nu (\sum_i \xi_i^\nu s_{i\sigma})^2} \rangle =$$ since uncorrelated patterns.

$$= \sum_{\{s_{i\sigma}\}} \langle e^{\frac{\beta}{2N}\sum_{\mu,\sigma} K_\mu (\sum_i \xi_i^\mu s_{i\sigma})^2} \rangle \langle e^{\frac{\beta}{2N}\sum_{\nu\sigma} K_\nu (\sum_i \xi_i^\nu s_{i\sigma})^2} \rangle .$$

Consider non-condensed factor: $\prod_\nu \langle e^{\frac{\beta}{2N}\sum_\sigma K_\nu (\sum_i \xi_i^\nu s_{i\sigma})^2} \rangle =$ completing the square

$$= \prod_\nu \langle \int \frac{\prod_\sigma dx_{\nu\sigma}}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_\sigma x_{\nu\sigma}^2 + \sum_\sigma x_{\nu\sigma}\sqrt{\beta K_\nu/N}\sum_i \xi_i^\nu s_{i\sigma}} \rangle =$$

$$= \prod_\nu \int \frac{d\vec{x}_\nu}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_\sigma x_{\nu\sigma}^2} \prod_i \cosh\left(\sqrt{\beta K_\nu/N} \sum_{\sigma=1}^{n} x_{\nu\sigma} s_{i\sigma} \right) \simeq$$

$$\simeq \prod_\nu \int \frac{d\vec{x}_\nu}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_\sigma x_{\nu\sigma}^2} \prod_i \left[1 + \frac{\beta K_\nu/N}{2} \left(\sum_{\sigma=1}^{n} x_{\nu\sigma} s_{i\sigma} \right)^2 \right] \simeq$$

$$\simeq \prod_\nu \int \frac{d\vec{x}_\nu}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_\sigma x_{\nu\sigma}^2} \prod_i e^{\frac{\beta K_\nu/N}{2}\left(\sum_{\sigma=1}^{n} x_{\nu\sigma} s_{i\sigma}\right)^2} =$$

$$\equiv \prod_\nu \int \frac{d\vec{x}_\nu}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_{\sigma\sigma'} x_{\nu\sigma} Q^\nu_{\sigma\sigma'} x_{\nu\sigma'}} =$$

$\prod_\nu \exp -\frac{1}{2} \ln \det Q^\nu_{\sigma\sigma'}$

$$Q^\nu_{\sigma\sigma'} \equiv \delta_{\sigma\sigma'} - \frac{\beta K_\nu}{N} \sum_{i=1}^{N} s_{i\sigma} s_{i\sigma'} .$$

Gaussian integral

$$= \prod_\nu \frac{1}{|\det Q^\nu_{\sigma\sigma'}|^{1/2}} =$$ as long as $Q^\nu_{\sigma\sigma'}$ is real, symmetric, and positive definite.

$$= \exp -\frac{1}{2}\sum_\nu \ln \det Q^\nu_{\sigma\sigma'} \Big[= \exp -\frac{1}{2}\sum_\nu \ln \prod_\alpha Q^\nu_\alpha$$ product over eigenvalues

$$= \exp -\frac{1}{2}\sum_\nu \sum_\alpha \ln Q^\nu_\alpha = \exp -\frac{1}{2}\sum_\nu \mathrm{Tr} \ln Q^\nu_{\sigma\sigma'} . \Big]$$

$$\langle Z^n \rangle = \sum_{\{s_{i\sigma}\}} \langle e^{\frac{\beta}{2N}\sum_{\mu\sigma} K_\mu (\sum_i \xi_i^\mu s_{i\sigma})^2 - \frac{1}{2}\sum_\nu \ln \det Q^\nu_{\sigma\sigma'}} \rangle .$$

Define two sets of order parameters:

$$m_{\mu\sigma} = \frac{1}{N}\sum_{i=1}^{N} \xi_i^\mu s_{i\sigma} , \qquad q_{\sigma\sigma'} = \frac{1}{N}\sum_{i=1}^{N} s_{i\sigma} s_{i\sigma'}$$ defined as an order parameter for $\sigma \neq \sigma'$.

$q_{\sigma\sigma'} = q_{\sigma'\sigma}$, one and the same order parameter.

$$\Theta\left(\frac{p}{N}=\alpha\right)$$

$-\beta\Phi_N(n) = N^{-1}\ln\langle Z_N^n\rangle$. As in previous treatment,

$$-\beta\mathcal{H}_N = NF(\overleftrightarrow{m},\overleftrightarrow{q}), \quad F(\overleftrightarrow{m},\overleftrightarrow{q}) = \frac{\beta}{2}\sum_{\mu\sigma}K_\mu m_{\mu\sigma}^2 - \frac{1}{2N}\sum_{\nu}\ln\det\Phi^\nu_{\sigma\sigma'},$$

$$\Phi^\nu_{\sigma\sigma'} = \delta_{\sigma\sigma'}(1-\beta K_\nu) - (1-\delta_{\sigma\sigma'})\beta K_\nu q_{\sigma\sigma'}$$

Density function, $\mathcal{D}_N(\overleftrightarrow{m},\overleftrightarrow{q}) \sim e^{-Nc^*(\overleftrightarrow{m},\overleftrightarrow{q})}$,

$$c^*(\overleftrightarrow{m},\overleftrightarrow{q}) = \sup_{\overleftrightarrow{t},\overleftrightarrow{u}}\left[\overleftrightarrow{m}:\overleftrightarrow{t} + \overleftrightarrow{q}:\overleftrightarrow{u} - c(\overleftrightarrow{t},\overleftrightarrow{u})\right],$$

$$c(\overleftrightarrow{t},\overleftrightarrow{u}) = \frac{1}{N}\ln\frac{1}{2^{Nn}}\sum_{\{s_{i\sigma}\}} e^{\sum_{\mu\sigma}t_{\mu\sigma}\sum_i\xi_i^\mu s_{i\sigma} + \sum_{\sigma,\sigma'}u_{\sigma\sigma'}\sum_i s_{i\sigma}s_{i\sigma'}} =$$

$$= \frac{1}{N}\sum_{i=1}^N \ln\frac{1}{2^n}\sum_{\{s_\sigma\}} e^{\sum_{\mu\sigma}t_{\mu\sigma}\xi_i^\mu s_\sigma + \sum_{\sigma\sigma'}u_{\sigma\sigma'}s_\sigma s_{\sigma'}} =$$

$$= \left\langle \ln\frac{1}{2^n}\sum_{\{s_\sigma\}} e^{\sum_{\mu\sigma}t_{\mu\sigma}\xi^\mu s_\sigma + \sum_{\sigma\sigma'}u_{\sigma\sigma'}s_\sigma s_{\sigma'}}\right\rangle, \text{ so that}$$

$$-\beta\Phi_N(n) = N^{-1}\ln\langle Z_N^n\rangle = N^{-1}\ln Z_N^n =$$

$$= \sup_{\overleftrightarrow{m},\overleftrightarrow{q}}\left[F(\overleftrightarrow{m},\overleftrightarrow{q}) - c^*(\overleftrightarrow{m},\overleftrightarrow{q})\right], \text{ realized at } \vec{\nabla}F(\overleftrightarrow{m},\overleftrightarrow{q}) = \vec{\nabla}c^*(\overleftrightarrow{m},\overleftrightarrow{q}).$$

Consider term $\frac{\partial}{\partial q_{\sigma\sigma'}}\; -\frac{1}{2N}\sum_\nu \ln\det\Phi^\nu_{\sigma\sigma'} =$

$$= -\frac{1}{2N}\sum_\nu\frac{(-1)^{\sigma+\sigma'}\operatorname{cov}(\Phi^\nu)_{\sigma\sigma'}}{\det\Phi^\nu_{\sigma\sigma'}}\frac{\partial\Phi^\nu_{\sigma\sigma'}}{\partial q_{\sigma\sigma'}}\,2 = \cancel{-}\frac{1}{\cancel{2}N}\sum_\nu(\Phi^\nu)^{-1}_{\sigma\sigma'}(\cancel{-}\beta K_\nu)\cancel{2} = \frac{\beta}{N}\sum_\nu K_\nu(\Phi^\nu)^{-1}_{\sigma\sigma'}$$

contribution from $\Phi^\nu_{\sigma'\sigma}$

$$\vec{\nabla}F(\overleftrightarrow{m},\overleftrightarrow{q}) = \begin{pmatrix}\beta K_\mu m_{\mu\sigma}\\ \frac{\beta}{N}\sum_\nu K_\nu(\Phi^\nu)^{-1}_{\sigma\sigma'}\end{pmatrix} \leftarrow \text{for } \sigma\neq\sigma'.$$

$$\begin{pmatrix}\overleftrightarrow{m}\\ \overleftrightarrow{q}\end{pmatrix} = \vec{\nabla}c\left(\vec{\nabla}F(\overleftrightarrow{m},\overleftrightarrow{q})\right) \qquad \underline{\text{Self-consistent equation}}$$

<u>Replica Symmetry</u> Assume that all replicas are equivalent: The symmetry between them is not broken.

$m_{\mu\sigma} = m_\mu$, $q_{\sigma\sigma'} = q$ satisfy the self-consistent equation.

$$\Phi^\nu_{\sigma\sigma'} = (1-\beta K_\nu + \beta K_\nu q)\,\delta_{\sigma\sigma'} - \beta K_\nu q\,|1\rangle\langle 1|, \quad |1\rangle = \underbrace{\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}}_{n \text{ elements}}$$

Try $(\Phi^\nu)^{-1}_{\sigma\sigma'} = A\delta_{\sigma\sigma'} + B\,|1\rangle\langle 1|$

can also use $\sum_{\sigma\sigma'}|\sigma\rangle\langle\sigma'|$ and $\delta_{\sigma\sigma'} = \sum_\sigma|\sigma\rangle\langle\sigma|$.

$$\sum_{\bar\sigma}(Q^\nu)^{-1}_{\sigma\bar\sigma}Q^\nu_{\bar\sigma\sigma'} = A(1-\beta K_\nu+\beta K_\nu q)\,\delta_{\sigma\sigma'} - A\beta K_\nu q\,|1\rangle\langle 1| +$$
$$+ B(1-\beta K_\nu+\beta K_\nu q)\,|1\rangle\langle 1| - B\beta K_\nu q\,|1\rangle n\langle 1| = \delta_{\sigma\sigma'}$$

$$A = (1-\beta K_\nu+\beta K_\nu q)^{-1}, \quad B = A\beta K_\nu q/(1-\beta K_\nu+\beta K_\nu q-\beta K_\nu q n)$$
$$\overset{n\to 0}{=} \beta K_\nu q/(1-\beta K_\nu+\beta K_\nu q)^2.$$

$$\frac{\beta}{N}\sum_\nu K_\nu (Q^\nu)^{-1}_{\sigma\sigma'} \overset{\sigma\neq\sigma'}{=} \frac{\beta^2 q}{N}\sum_\nu K_\nu^2/(1-\beta K_\nu+\beta K_\nu q)^2 \equiv \beta^2 q r.$$

For $K_\nu=1$ (unweighted patterns), $r=\alpha/(1-\beta+\beta q)^2$.

5. Self-consistent equations: — having made sum unrestricted.

$$c(\overleftrightarrow{t},\overleftrightarrow{u}) = \left\langle \ln \frac{1}{2^n}\sum_{\{s_\sigma\}} e^{\sum_{\mu,\sigma} t_{\mu\sigma}\xi^\mu s_\sigma + \frac{1}{2}\sum_{\sigma,\sigma'} u_{\sigma,\sigma'} s_\sigma s_{\sigma'}} \right\rangle .$$

$$\frac{\partial}{\partial t_{\mu\sigma}} c(\overleftrightarrow{t},\overleftrightarrow{u}) = \left\langle \frac{\sum_{\{s_\sigma\}} \xi^\mu s_\sigma\, e^{\sum_{\mu'\sigma'} t_{\mu'\sigma'}\xi^{\mu'} s_{\sigma'} + \frac{1}{2}\sum_{\sigma'\sigma''} u_{\sigma'\sigma''} s_{\sigma'} s_{\sigma''}}}{\sum_{\{s_\sigma\}} e^{\sum_{\mu'\sigma'} t_{\mu'\sigma'}\xi^{\mu'} s_{\sigma'} + \frac{1}{2}\sum_{\sigma'\sigma''} u_{\sigma'\sigma''} s_{\sigma'} s_{\sigma''}}} \right\rangle =$$

Evaluated at $t_{\mu\sigma} = \beta K_\mu m_\mu$, $u_{\mu\sigma} = \beta^2 q r$; — $\sigma'=\sigma''$ terms cancel in numerator and denominator

$$= \left\langle \frac{\sum_{\{s_\sigma\}} \xi^\mu s_\sigma\, e^{\beta\sum_{\mu'\sigma'} K_{\mu'} m_{\mu'}\xi^{\mu'} s_{\sigma'} + \frac{\beta^2 q r}{2}(\sum_{\sigma'} s_{\sigma'})^2}}{\sum_{\{s_\sigma\}} e^{\beta\sum_{\mu'\sigma'} K_{\mu'} m_{\mu'}\xi^{\mu'} s_{\sigma'} + \frac{\beta^2 q r}{2}(\sum_{\sigma'} s_{\sigma'})^2}} \right\rangle =$$

Completing the square,

$$= \left\langle \frac{\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{\{s_\sigma\}} \xi^\mu s_\sigma\, e^{\beta\sum_{\mu'\sigma'} K_{\mu'} m_{\mu'}\xi^{\mu'} s_{\sigma'} + x\beta\sqrt{qr}\sum_{\sigma'} s_{\sigma'}}}{\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{\{s_\sigma\}} e^{\beta\sum_{\mu'\sigma'} K_{\mu'} m_{\mu'}\xi^{\mu'} s_{\sigma'} + x\beta\sqrt{qr}\sum_{\sigma'} s_{\sigma'}}} \right\rangle =$$

$$= \left\langle \frac{\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \xi^\mu \sinh(\beta K\vec{m}\cdot\vec{\xi} + \beta\sqrt{qr}\,x)\cosh^{n-1}(\beta K\vec{m}\cdot\vec{\xi}+\beta\sqrt{qr}\,x)}{\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cosh^n(\beta K\vec{m}\cdot\vec{\xi}+\beta\sqrt{qr}\,x)} \right\rangle =$$

Setting $n=0$,

$$= \boxed{\left\langle \int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \xi^\mu \tanh(\beta K\vec{m}\cdot\vec{\xi}+\beta\sqrt{qr}\,x) \right\rangle = m_\mu .}$$

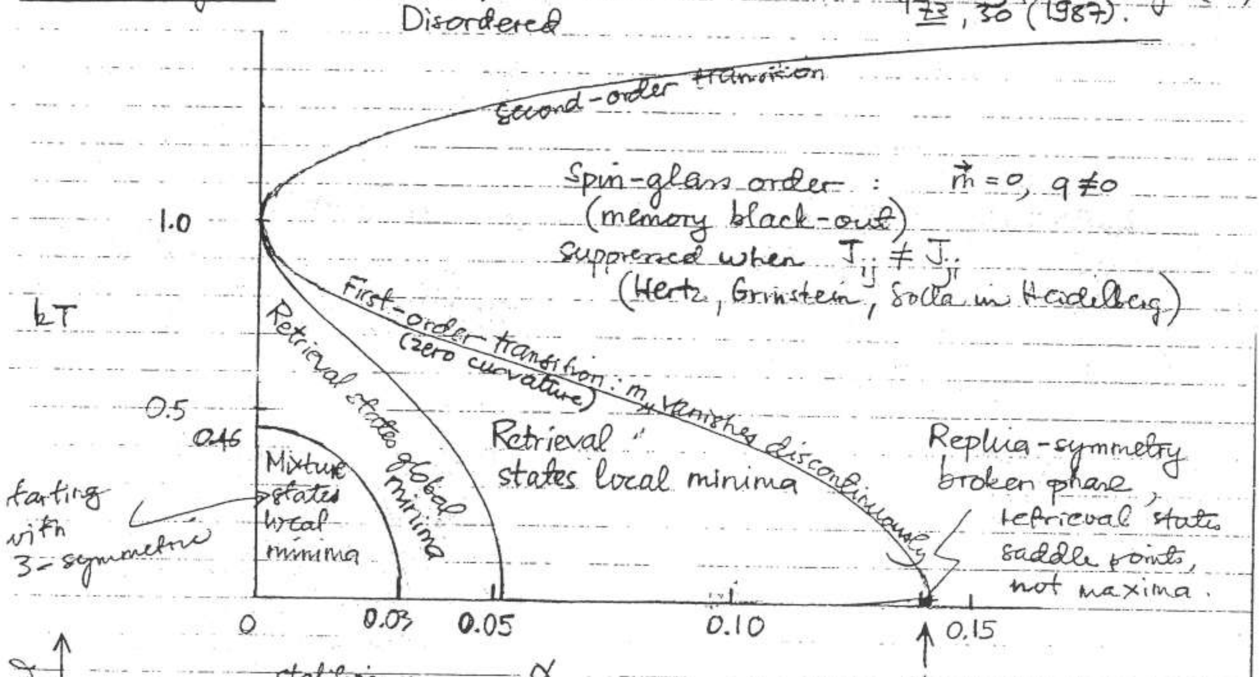
Similarly,
$$\frac{\partial}{\partial u_{\sigma\sigma'}} c(\overleftrightarrow{t},\overleftrightarrow{u}) = \left\langle \frac{\sum_{\{s_\sigma\}} s_\sigma s_{\sigma'}\, e^{\sum_{\mu\sigma''} t_{\mu\sigma''}\xi^\mu s_{\sigma''} + \frac{1}{2}\sum_{\sigma''\sigma'''} u_{\sigma''\sigma'''} s_{\sigma''} s_{\sigma'''}}}{\sum_{\{s_\sigma\}} e^{\sum_{\mu\sigma''} t_{\mu\sigma''}\xi^\mu s_{\sigma''} + \frac{1}{2}\sum_{\sigma''\sigma'''} u_{\sigma''\sigma'''} s_{\sigma''} s_{\sigma'''}}} \right\rangle$$

leads to
$$\boxed{q = \left\langle \int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tanh^2(\beta K\vec{m}\cdot\vec{\xi}+\beta\sqrt{qr}\,x) \right\rangle}$$

. For $\alpha = 0 \rightarrow r = 0$, the self-consistent equation for non-infinite number of patterns is recovered.

. For $\alpha > 0$, the non-condensed patterns (ν) act as Gaussian noise on the condensed patterns (μ), in the presence of spin-glass order ($q \neq 0$).

Phase Diagram D.J. Amit, H. Gutfreund, and H. Sompolinsky, Ann. Phys. (N.Y.) 173, 30 (1987).



$\alpha_c = 0.138$

Thermal noise

when $\{S_i\} = \{\xi_i^\mu\}$ — stabilizes μ — noise destabilizes

$$h_i = \sum_j J_{ij}\xi_j^\mu = \xi_i^\mu + \frac{1}{N}\sum_{j,\,\nu\neq\mu} \xi_i^\nu \xi_j^\nu \xi_j^\mu \sim \xi_i^\mu \pm \frac{\sqrt{Np}}{N} = \xi_i^\mu \pm \sqrt{\alpha}.$$

Statistical noise generated by multitude of patterns.

Numerical simulation: $\alpha_c = 0.143 \pm 0.001$
G.A. Kohring, J. Stat. Phys. 59, 1077 (1990)

Correlation: $\xi_i^\nu \xi_j^\mu \rightarrow \xi_i^\nu \xi_j^\nu = 1$
more noise.

Onset of retrieval states: first-order, determined numerically.
Can do low temperature expansion, and follow incrementally numerically.
Onset of spin-glass phase: second-order

$$q = \int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tanh^2 \beta\sqrt{qr}\,x \simeq \beta^2 q r \int \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} x^2 = \beta^2 q r \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{4} 2\sqrt{2} = \beta^2 q r$$

$$1 = \beta^2 r = \beta^2 \alpha / (1-\beta+\beta q)^2 \simeq \alpha / (\beta^{-1} - 1)^2$$

$kT = 1 + \sqrt{\alpha}$.

II. Optimization by Simulated Annealing — Kirkpatrick, Gelatt, and Vec (1983).

1. Traveling salesman problem

$$E = \sum_{i=1}^{N} (\vec{r}_i - \vec{r}_{i+1})^2 .$$

- Incremental search gets caught in local minima.

- Monte Carlo search: A given proposed step (e.g., interchanging two cities on itenerary) accepted if $\Delta E < 0$.

for $\Delta E > 0$, Random number $[0,1]$ is compared to $e^{-\Delta E/kT}$ for acceptance ($R < \exp$)

Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller (1953)

Evolves to Gibbs distribution.

- T gradually reduced during annealing procedure.

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2. Also in computer chip design, circuit: partitioning: circuits onto chips, placement: chips onto cards. wiring, including level assignment

E can have several terms, with different weights, such as wire length, bends, overlaps, element overlap etc.

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3. Guyon et al., Figs. 2,3; p. 378 in Heidelberg Colloquium (1987)
Kirkpatrick et al, Figs. 4, 17+18, 22, 23, 28 (1983)

Formal equivalences possible — including minimization of the spin-glass free energy. These are the so-called np complete problems,

systematic solution would require moves more numerous than any power of N.
Many local minima.
However, many minima near-degenerate with global minimum.

Simulated annealing: not problem specific, unlike heuristics.
Now its own heuristics: genetic evolution: mutation and mating

III. Information Transmission (Coding-decoding) Sourlas (1989)

Input : N bits of information $\{\xi_i\}$ $\xi_i = \pm 1$

Coded message (input to the channel) : $\{J^{\circ}_{i_1,\dots,i_p} = \xi_{i_1} \cdots \xi_{i_p}\}$

output of the channel : $\{J_{i_1,\dots,i_p} = J^{\circ}_{i_1,\dots,i_p} + \underbrace{\delta_{i_1,\dots,i_p}}_{\text{noise}}\}$

Decoding : Finding the ground state of the Hamiltonian

$$\mathcal{H} = -\sum_p \sum_{i_1,\dots,i_p=1}^{N} C^{(p)}_{i_1,\dots,i_p} J_{i_1,\dots,i_p} \sigma_{i_1} \cdots \sigma_{i_p} \,,$$

a spin-glass.

A particular code in the family is specified by the connectivity matrices C.

The error probability is $1-m$.

From spin-glass statistical mechanics, it is claimed that, some of these codes are the first known codes to saturate Shannon's cost-performance bound:

$$\frac{\text{number of information bits}}{\text{total number of transmitted bits}} = R \leqslant \frac{1}{2} \log\left(1 + \frac{E_s}{E_n}\right)$$

E_s ← energy/bit, source τv^2

E_n ← energy/bit, noise

Variance

IV. A New Development:

Neural-Network Learning Strategy for the Traveling Salesman Problem
(better results than Simulated Annealing) Chen (1989)

Cities can be viewed as "neurons" connected by synapses.

Initially, a synaptic strength is assigned to each pair of cities:
$w_{ij} = e^{-d_{ij}/T}$. (probabilistic greedy)

Trial and adaption: i_1 picked at random,
i_2 is picked with probability $P_{i_2 i_1} \sim w_{i_2 i_1}$
i_3 " $P_{i_3 i_2} \sim w_{i_3 i_2}$, etc.

After (m)th tour is completed, synapses in tour are modified:
$w_{ij}(m) = w_{ij}(m-1)\, e^{\alpha[d(m-1)-d(m)]/T}$.

Thus, If tour length is shorter, synapses are strengthened, and vice-vera

50-city tour : 3 seconds of IBM 3090 to achieve $d = 5.78$.
(simulated annealing : 12 seconds $d = 5.80$)
Chen, Figs. 1 and 2 .