Spin-s Spin-Glass Phases in the d=3 Ising Model

E. Can Artun\(^1\) and A. Nihat Berker\(^{1,2}\)

\(^1\)Faculty of Engineering and Natural Sciences, Kadir Has University, Cibali, Istanbul 34083, Turkey
\(^2\)Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

All higher-spin (\(s \geq 1/2\)) Ising spin glasses are studied by renormalization-group theory in spatial dimension \(d = 3\). The \(s\)-sequence of global phase diagrams, the chaos Lyapunov exponent, and the spin-glass runaway exponent are calculated. It is found that, in \(d = 3\), a finite-temperature spin-glass phase occurs for all spin values, including the continuum limit of \(s \to \infty\). The phase diagrams, with increasing spin \(s\), saturate to a limit value. The spin-glass phase, for all \(s\), exhibits chaotic behavior under rescalings, with the calculated Lyapunov exponent of \(\lambda = 1.93\) and runaway exponent of \(\gamma_R = 0.24\), showing simultaneous strong-chaos and strong-coupling behaviors. The ferromagnetic-spin-glass-antiferromagnetic phase transitions occurring around \(p_c = 0.37\) and 0.63 are unaffected by \(s\), confirming the percolative nature of this phase transition.

I. INTRODUCTION: SPIN-S ISING SPIN-GGLASS SYSTEMS

Frozen disorder of the interactions introduces many qualitatively and quantitatively new effects to statistical mechanical systems, such as the immediate (i.e., with infinitesimal disorder) conversion of first-order phase transitions into second-order phase transitions \([1, 2]\) or the creation of an entirely new phase such as the spin-glass phase \([3]\). The latter occurs under frozen (quenched) competing interactions causing local minimum-energy degeneracies dubbed frustration \([4]\). The signature of the spin-glass phase is the appearance of a chaotic sequence of interactions \([5, 17]\) under the successive scale changes of a renormalization-group transformation. This translates to a chaotic spin-spin correlation function, as function of distance, at a given scale. \([18]\) The spin-glass phase and its rescaling chaos appears with the introduction, by rewiring, of infinitesimal frustration to the Mattis phase \([19]\) obtained by random local spin redefinitions (gauge transformations) in the usual ferromagnetic or antiferromagnetic phase \([20]\). On the other hand, strong chaos, signalled by a large Lyapunov exponent, of the spin-glass phase in fully frustrated systems continues \([23]\) until the lower-critical dimension \(d_c \simeq 2.5\) of the spin-glass phase \([21, 27]\). Thus both gradual \([20]\) or abrupt \([25]\) onsets of chaos are seen.

Most spin-glass studies have been on the classical spin \(s = 1/2\) Ising model, where locally \(s_i = \pm 1\). \([20]\) Spin-glass studies have also been done on \(q\)-state clock models and their continuum limit the XY model \([30, 31]\), chiral (helical \([32]\)) Potts and clock models, in fact leading to a chiral spin-glass Potts \([33]\) and clock \([34, 35]\) phases, and quantum Heisenberg models \([36]\). The position-space renormalization-group method appears to be a method suited for such studies, where the rescaling behavior of the distribution of the quenched random interactions is followed and analyzed \([37]\). This is best effected (Fig. 2) by use of the Migdal-Kadanoff approximation \([38, 39]\) or, equivalently, the exact recursion of a hierarchical lattice \([40, 43]\). In the current work, we quantitatively and globally study, in spatial dimension \(d = 3\), the Ising spin glass for all spins \(s = 1/2, 1, 3/2, 2, 5/2, \ldots\) to the limiting value \(s \to \infty\), obtaining the global \(s\)-sequence phase diagram (Fig. 1) and chaotic behaviors.

![Fig. 1](image-url)

**FIG. 1.** Calculated phase diagrams of the spin-s Ising spin glasses in \(d = 3\). From top to bottom, \(s = 1/2, 1, 3/2, 2, 5/2, \ldots\) to \(s \to \infty\). There is an accumulation, from above, of the phase diagrams at the lowermost, but still at finite-temperature, phase diagram of the continuum limit \(s \to \infty\).

The spin-s Ising model is defined by the Hamiltonian

\[
-\beta \mathcal{H} = \sum_{\langle ij \rangle} J_{ij} (s_i/s)(s_j/s),
\]

where \(\beta = 1/kT\), at each site \(i\) of the lattice the spin \(s_i = \pm 1/2, \pm 1, \pm 3/2, \ldots, \pm s\), and \(\langle ij \rangle\) denotes summation over all nearest-neighbor site pairs. The division by \(s\) is done to conserve the energy scale across the different spin-s models and thereby make meaningful temperature comparisons between them. Note that for \(s = 1/2\), this formalism yields the much studied \(s_i/s = \pm 1\) case. The bond \(J_{ij}\) is ferromagnetic \(+J > 0\) or antiferromagnetic \(-J\) with respective probabilities \(1 - p\) and \(p\). Under renormalization-group transformation, this "double-delta" distribution of interactions is not conserved. A
more complicated distribution of interactions ensues and is kept track of, as explained below.

![Diagram](image)

FIG. 2. (a) Migdal-Kadanoff approximate renormalization-group transformation for the $d = 3$ cubic lattice with the length-rescaling factor of $b = 3$. In this intuitive approximation, bond moving is followed by decimation. (b) Exact renormalization-group transformation of the $d = 3$, $b = 3$ hierarchical lattice for which the Migdal-Kadanoff renormalization-group recursion relations are exact. The construction of a hierarchical lattice proceeds in the opposite direction of its renormalization-group solution. From \[34, 40].

II. METHOD: RENORMALIZATION-GROUP FLOWS OF THE QUENCHED PROBABILITY DISTRIBUTION OF THE INTERACTIONS

Under renormalization group, for $s > 1/2$, the Hamiltonian does not conserve its form in Eq.(1). Thus, for any $s$, the Hamiltonian is most generally expressed as

$$-\beta H = \sum_{(i,j)} E(s_i, s_j),$$

(2)

With no loss of generality, for each $<ij>$, the same constant is subtracted from all terms $E(s_i, s_j)$, so that the largest energy $E(s_i, s_j)_{\text{max}}$ of the spin-spin interaction is zero (and all other $E(s_i, s_j) < 0$). This formulation makes it possible to follow global renormalization-group trajectories, necessary for the calculation of phase boundaries, Lyapunov exponent, and runaway exponent, without running into numerical overflow problems. As the local renormalization-group transformation, the Migdal-Kadanoff approximate transformation \[38, 39\] and, equivalently, the exact transformation for the $d = 3$ hierarchical lattice \[40, 42\] is used (Fig. 2). Recent works using exactly soluble hierarchical models are in Refs. \[44-52\]. The length rescaling factor of $b = 3$ is used, to preserve under renormalization group the ferromagnetic-antiferromagnetic symmetry of the system. This local transformation consists in bond moving followed by decimation, with the above-mentioned subtraction after each local bond moving and decimation, giving the local renormalized energies $E'(s_i, s_j) \leq 0$. In our notation, all renormalized quantities are designated by a prime.

The quenched randomness is included by keeping, as a distribution, 10000 sets of the nearest-neighbor interaction energies $E(s_i, s_j)$. At the beginning of each renormalization-group trajectory, this distribution is formed from the double-delta distribution characterized by interactions $\pm J$ with probabilities $p, (1-p)$. 10000 local renormalization-group transformations determine each subsequent distribution as explained below.

The local renormalization-group transformation is simply expressed in terms of the transfer matrix $T(s_i, s_j) = e^{E(s_i, s_j)}$: Bond moving consists of multiplying elements at the same position of $b^{-1} = 9$ transfer matrices randomly chosen from the distribution,

$$\overline{T}(s_i, s_j) = \prod_{k=1}^{9} T_k(s_i, s_j),$$

(3)

so that a distribution of 10000 bond-moved transfer matrices is generated. Decimation consists of matrix multiplication of three randomly chosen bond-moved transfer matrices,

$$T' = \overline{T}_1 \cdot \overline{T}_2 \cdot \overline{T}_3,$$

(4)

so that a distribution of 10000 renormalized transfer matrices is generated. Phases are determined by following trajectories to their asymptotic limit: The asymptotic limit transfer matrices of trajectories starting in the ferromagnetic phase all have 1 in the corner diagonals and 0 at all other positions. The asymptotic limit transfer matrices of trajectories starting in the antiferromagnetic phase all have 1 in the corner anti-diagonals and 0 at all other positions. The asymptotic limit transfer matrices of trajectories starting in the disordered phase all have 1 at all other positions. Phase diagrams are obtained by numerically determining the boundaries, in the unrenormalized system, of these asymptotic flows.

III. RESULTS: GLOBAL S-SEQUENCE PHASE DIAGRAM AND SATURATION

The calculated phase diagrams of the spin-$s$ Ising spin glasses in $d = 3$ are shown in Fig. 1. From top to bottom, the phase diagrams are for spin-$s = 1/2, 1, 3/2, 2, 5/2, 3, \ldots$ to $s \to \infty$. There is an accumulation, from above, of the phase diagrams at the lowermost, but still at finite-temperature, phase diagram of the continuum limit $s \to \infty$.

The calculated ferromagnetic (at $p = 0$) and spin-glass (at $p = 0.5$) phase transition temperatures as a function of spin value $s$ are given in Fig. 3. With increasing $s$ both transition temperatures saturate around $s \simeq 4$. A similar behavior was found in $q$-state clock models saturating at the continuum XY model transition temperature. \[43\]
FIG. 3. The calculated ferromagnetic (at \( p = 0 \)) and spin-glass (at \( p = 0.5 \)) phase transition temperatures as a function of spin value \( s \). Note that with increasing \( s \) both transition temperatures saturate around \( s \approx 4 \). A similar behavior was found in \( q \)-state clock models.[43]

IV. RESULTS: CHAOS FOR ALL SPINS S, LYAPUNOV EXPONENT AND RUNAWAY EXPONENT

For all spin-\( s \), the renormalization-group trajectories starting within the spin-glass phase are asymptotically chaotic, as seen in Fig. 4, where the consecutively renormalized (combining with neighboring interactions) values at a given location \( \langle ij \rangle \) are followed. For the interaction \( K_{ij} \), we have used the difference between the largest value (which is 0 by construction) and the lowest value in \( E(s_i, s_j) \). \( \overline{K} \) is the average of this interaction over the entire distribution at the given renormalization-group step. The chaotic behavior is strong, as measured by the Lyapunov exponent [53, 54]

\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{dx_{k+1}}{dx_k} \right|, \tag{5}
\]

where \( x_k = K_{ij} / \overline{K} \) at step \( k \) of the renormalization-group trajectory. Eliminating the first 100 renormalization-group steps as crossover from initial conditions to asymptotic behavior and using the next 1500 steps, Eq.(5) yielded \( \lambda = 1.93 \) for all spins \( s \).

In addition to strong chaos, the renormalization-group trajectories show asymptotic strong coupling behavior,

\[
\overline{K^7} = b^{9\rho} \overline{K}, \tag{6}
\]

where \( y_R > 0 \) is the runaway exponent [23]. Again using 1500 renormalization-group steps after discarding 100 steps, we find \( y_R = 0.24 \) for all spins \( s \). Note that this is a “weak” strong coupling behavior, as the stronger runaway exponent of the ferromagnetic and antiferromagnetic phases is \( y_R = d - 1 = 2 \).

V. CONCLUSION

We have calculated the global spin-\( s \) sequence of phase diagrams for all spins \( s = 1/2, 1, 3/2, 2, 5/2, 3, ..., s \to \infty \) for the Ising spin-glass system in spatial dimension \( d = 3 \). The phase diagrams, all with a finite-temperature spin-glass phase, for increasing spin \( s \) saturate to the limit value of \( s \to \infty \). For all spins \( s \), the spin-glass phase has renormalization-group trajectories that are chaotic, with calculated Lyapunov exponent \( \lambda = 1.93 \) and runaway exponent \( y_R = 0.24 \), thus simultaneously showing strong chaotic and “weak” strong-coupling behaviors.

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