The tensor renormalization-group (TRG) method, developed by Levin and Nave, brings systematic improbability to the position-space renormalization-group method and yields essentially exact results for phase diagrams and entire thermodynamic functions. The method, previously used on systems with no quenched randomness, is extended in this study to systems with quenched randomness. Local magnetizations and correlation functions as a function of spin separation are calculated as tensor products subject to renormalization-group transformation. Phase diagrams are extracted from the long-distance behavior of the correlation functions. The approach is illustrated with the quenched bond-diluted model on the triangular lattice. An accurate phase diagram is obtained in temperature and bond-dilution probability, for the entire temperature range down to the percolation threshold at zero temperature.

PACS numbers: 75.10.Nr, 05.10.Cc, 64.60.ah, 64.60.De

I. INTRODUCTION

The tensor renormalization-group (TRG) method developed by Levin and Nave is a highly useful update of the traditional position-space renormalization-group approaches. While these founding approaches relied on uncontrolled approximations that were often system-specific, the TRG is general in scope—it works on any classical two-dimensional lattice Hamiltonian with local interactions—and its accuracy can be systematically improved to converge on the exact thermodynamic results. Along with these advantages, the method fits within the conceptual framework of traditional renormalization-group theory: it is a mapping between Hamiltonians on the original and coarse-grained lattices, and phase transition behavior can be extracted from flows of the Hamiltonians as the transformation is iterated.

The initial TRG study demonstrated the power of the approach in the context of the triangular-lattice Ising model. Since then it has proven a versatile tool for a variety of classical systems, including the frustrated Ising model on a Shastry-Sutherland lattice, relevant to magnetization plateaus in rare-earth tetraborides, and the zero-hopping limit of a model for ultra-cold bosonic polar molecules on a hexagonal optical lattice. Moreover, the ideas behind the TRG method have become the kernel for developments in two-dimensional quantum systems, most notably tensor-entanglement renormalization group for studying symmetry breaking and topological phase transitions, and accurate methods to calculate ground-state expectation values. Beyond the precision of the method, a key factor spurring the growth of tensor RG applications in both classical and quantum cases is computational efficiency: the CPU cost of carrying out TRG scales linearly with lattice size.

Given these promising characteristics, TRG is a natural candidate for tackling models with quenched randomness—a field where extracting accurate phase diagram information is a significant challenge. The current study presents the first example of TRG applied to such a system with frozen disorder, namely the percolative system of the bond-diluted triangular-lattice Ising ferromagnet, yielding, as seen in Fig. 1, a highly accurate global phase diagram, down to zero temperature, where it connects with the percolation transition.

Our paper is organized as follows: Sec. II develops the TRG method for a general quenched random system. Sec. III illustrates this tensor network mapping in particular for the bond-diluted model and shows how to extract physical observables such as spin-spin correlation functions. Sec. IV uses this method, together with finite-size scaling relations for the correlation functions, to derive our main result: The phase diagram in terms of temperature vs. bond dilution probability. Close agreement with the known critical temperature curve is achieved even at a relatively low order of the TRG approximation (i.e., a small cutoff parameter). Our work opens up future possibilities for the extensive use of TRG in quenched disordered systems, as argued in the concluding remarks of Sec. V.

II. TRG METHOD FOR QUENCHED RANDOM SYSTEMS

A. The Tensor Network

As in earlier studies, we focus here on classical Hamiltonians associated with hexagonal-lattice tensor networks, though the method that we develop for...
quenched random systems is readily generalized to other geometries like the square and kagomé lattices. We consider a general Hamiltonian that involves local interactions expressed in terms of bond degrees of freedom, such that each bond has $d$ possible states and the partition function of the system has the form

$$
Z = \sum_{i_1, \ldots, i_K = 1}^d T_{i_1i_2i_3}T_{i_3i_4i_5}\cdots T_{i_K-2i_K-1i_K},
$$

(1)

where, for each of the $N$ sites in the hexagonal lattice, a real-valued tensor $T_{i_n,i_{n+1}}$ is a Boltzmann weight depending on the configuration of the three bonds meeting at the site. The bond degrees of freedom correspond to each tensor index running from 1 to $d$. These bond indices are labeled $i_1$ through $i_K$, for the total of $K = 3N/2$ bonds in the lattice. Although the tensor can have as many as $d^3$ distinct non-zero elements, in practice some bond configurations may be disallowed for a given Hamiltonian, corresponding to zero-valued tensor elements.

To facilitate the description of the TRG procedure, the hexagonal lattice is constructed as illustrated in Fig. 2. At the $n$th step, we replace each vertex with a hexagon, with the initial structure denoted $n = 0$. We impose periodic boundary conditions, such that the top and bottom edges are equivalent, as well as the left and right edges, so that the lattice effectively lies on the surface of a torus. After the $n$th step, the system has $N = 8 \cdot 3^n$ sites and $K = 4 \cdot 3^{n+1}$ bonds. The TRG method involves a renormalization-group transformation that reverses this construction process, mapping the system at step $n$ to one at step $n - 1$.

The hexagonal lattice of any size can be decomposed into two sublattices $A$ and $B$, such that the nearest neighbors of one type belong to the other type. As an example, we label the sublattices in the $n = 0$ panel of Fig. 2. We distinguish the sublattice tensors with superscripts, $T^A_{i_m,i_{m+1}}$ or $T^B_{i_m,i_{m+1}}$. In the partition function sum of Eq. (1), each bond index $i_m$ appears twice, once within an $A$ tensor, and once within the neighboring $B$ tensor linked through that bond. Thus evaluating $Z$ consists of performing $K$ tensor contractions.

In addition to the bond variables, the general system we consider has quenched random degrees of freedom, though for notational simplicity we shall not explicitly show the dependence of $T$ on these. Physical observables $Q$ will be expressed as $\langle Q \rangle$, where $\langle \cdot \rangle$ denotes the configurational average over the bond degrees of freedom and $\langle \cdot \rangle$ denotes the configurational average over the quenched disorder.

### B. The TRG Transformation

The TRG transformation consists of two steps, known as rewiring and decimation. In the rewiring step, the bonds of every pair of neighboring tensors $T^A$ and $T^B$ are reconnected, rewriting them as a contraction of two new tensors $S^A$ and $S^B$. The reconnection pattern is illustrated in Fig. 3(a,b) and can be broken down into three basic cases (highlighted in different colors) involving different orientations of the initial $T^A$ and $T^B$ tensors. In our graphical convention, the vertex where three solid lines meet is a $T$ tensor and the vertex where three dashed lines meet is an $S$ tensor. Indices on a tensor, i.e., $T^A_{ijk}$, correspond to bonds labeled $i, j, k$ arranged...
counter-clockwise around the tensor, with the first index marking the vertical bond for the $T$ tensors and the horizontal bond for the $S$ tensors. Thus for example the three rewirings shown in Fig. 3(b) denote the mathematical identities

$$\text{Case 1}: \quad \sum_{k=1}^{d} T^A_{mk} T^B_{jki} = \sum_{\nu=1}^{d^2} S^A_{\nu j} S^B_{\nu mi},$$

$$\text{Case 2}: \quad \sum_{k=1}^{d} T^A_{klm} T^B_{kij} = \sum_{\nu=1}^{d^2} S^A_{\nu j\nu} S^B_{\nu mi},$$

$$\text{Case 3}: \quad \sum_{k=1}^{d} T^A_{lmk} T^B_{tjk} = \sum_{\nu=1}^{d^2} S^A_{\nu j\nu} S^B_{\nu mi},$$

Note that the $S$ tensors have two indices which run up to $d$ (labeled by Latin letters) and one index that runs up to $d^2$ (labeled by a Greek letter). The reason why $S^A$ and $S^B$ must have this structure comes from the following derivation, which also illustrates how one can explicitly calculate these tensors.

We shall describe the derivation for case 1, since the other two cases are analogous. The first line of Eq. (2) can be expressed as a $d^2 \times d^2$ matrix equation, $M = S^A(S^B)^T$, where $M_{\alpha\beta} \equiv \sum_{k} T^A_{mk} T^B_{jki}$. Here we use composite indices $\alpha$ and $\beta$ with $d^2$ states defined as $\alpha \equiv (j, l)$ and $\beta \equiv (m, i)$. As a real-valued matrix, $M$ has a singular value decomposition of the form $M = U \Sigma V^T$, where $U, V$ are orthogonal matrices and $\Sigma$ is a diagonal matrix containing the $d^2$ singular values of $M$. Once the singular value decomposition of $M$ is calculated, the elements of $S^A$ and $S^B$ are given by $S^A_{\alpha\nu} = \sqrt{\Sigma_{\alpha\alpha}} U_{\alpha\nu}$, $S^B_{\beta\nu} = \sqrt{\Sigma_{\beta\beta}} V_{\beta\nu}$, where $\Sigma_{\alpha\alpha}$ is the $\nu$th singular value, adopting the ordering convention from largest to smallest with increasing $\nu$.

After all $T^A$ and $T^B$ pairs are rewired, we have a so-called martini lattice of $S^A$ and $S^B$ tensors, shown in Fig. 3(c). The final step of the TRG transformation is decimation, which traces over the degrees of freedom in the triangles of the martini lattice, substituting for each triangle a renormalized tensor $T^{A'}$ or $T^{B'}$. Graphically, Fig. 3(d) shows the decimation of three $S^A$ tensors to form $T^{A'}$ and of three $S^B$ tensors to form $T^{B'}$. The corresponding expressions in terms of tensor components are

$$\sum_{j,l,h=1}^{d} S^A_{\nu jl} S^A_{\nu hj} S^B_{\nu \delta j \delta} = T^{A'}_{\nu \delta},$$

$$\sum_{m,h,k=1}^{d} S^B_{\nu \epsilon m} S^B_{\nu h} S^B_{\nu \delta m h} = T^{B'}_{\nu \delta}.$$
as $D$ is increased [1, 8]. With this cutoff, the maximum size of the tensors is bounded as the TRG procedure is iterated and we can extract numerically thermodynamic information from flows within a finite-dimensional space of real-valued tensor elements.

### III. TRG FOR QUENCHED RANDOMNESS: THE BOND-DILUTED ISING MODEL

#### A. The Bond-Diluted Ising Hamiltonian and Its Mapping onto a Tensor Network

The general Hamiltonian for a quenched random Ising system is

$$-\beta \mathcal{H} = \sum_{\langle ij \rangle} \left[ J_{ij} s_i s_j + H_{ij} (s_i + s_j) \right], \quad s_i = \pm 1, \quad (4)$$

where $\beta = 1/k_BT$, $J_{ij}$ and $H_{ij}$ are respectively the local spin-spin coupling and magnetic field for sites $i$ and $j$, and $\langle ij \rangle$ denotes a sum over nearest-neighbor pairs of sites. Although this Hamiltonian encompasses a variety of models, all the way to the random-field spin glass [21], we shall here focus on a the bond-diluted Ising case, where the interaction constants $J_{ij}$ are distributed with a quenched probability $\mathcal{P}(J_{ij})$ of the form

$$\mathcal{P}(J_{ij}) = p \delta(J_{ij}) + (1-p) \delta(J_{ij} - J). \quad (5)$$

Here $J > 0$, implying ferromagnetism, and $p$ is the fraction of missing bonds. While we restrict our attention to the zero magnetic field subspace, $H_{ij} = 0$, formally the local fields will be kept in the Hamiltonian in order to take derivatives to obtain thermodynamic functions.

Starting with the Hamiltonian of Eq. (4) on a triangular lattice, a duality transformation allows us to express the partition function as a hexagonal-lattice tensor network. Each triangle in the triangular lattice corresponds to a tensor, with up triangles associated with a $T^A$ and down triangles with a $T^B$, as shown in Fig. 4. For spin variables $s_i$, $s_j$, $s_k$ in a given triangle in the manner illustrated in the figure, we define corresponding edge variables $\sigma_m$ as the products of neighboring $s_m$, i.e., for the type A triangle, $\sigma_1 = s_3 s_1$, $\sigma_2 = s_1 s_2$, $\sigma_3 = s_2 s_3$ and for the type B triangle, $\sigma_1 = s_1 s_2$, $\sigma_2 = s_2 s_3$, $\sigma_3 = s_3 s_1$. Since $s_m = \pm 1$ and $\sigma_m = \pm 1$, we can now introduce a composite index $i_m \equiv (5 - \sigma_m - 2s_m)/2$ which runs from 1 to 4 and describes the four possible states of the $m$th triangle edge. Letting $J_m$ be the coupling $J_{ij}$ associated with this edge and $H_m = H_{ij}$ be the edge magnetic field, then the tensors for the two triangles types are:

$$T^A_{i_1 i_2 i_3} = e^{\frac{1}{2} \beta \left( \sum_{m=1}^{i_3} J_m s_m \mp H_m \right) \mathcal{P}(\sigma_1 \sigma_2 \sigma_3)} \cdot \mathcal{P}(\sigma_1 \sigma_3) \mathcal{P}(\sigma_2 s_1 s_2), \quad (6)$$

$$T^B_{i_1 i_2 i_3} = e^{\frac{1}{2} \beta \left( \sum_{m=1}^{i_3} J_m s_m \mp H_m \right) \mathcal{P}(\sigma_1 \sigma_2 \sigma_3)} \cdot \mathcal{P}(\sigma_1 s_1 s_2) \mathcal{P}(\sigma_2 s_3 s_1),$$

where $\mathcal{P}(x) \equiv (1 + x)/2$ is a projection operator. The $P$ factors in the tensors remove the bond states that do not correspond to a physically allowable spin configuration. As a result of the projection operators, only 8 out of the 64 elements in the tensor are nonzero. These are listed, for the first renormalization step, in the 3rd and 6th columns of Table I for $T^A$ and $T^B$ respectively.

#### B. Local Magnetization and Spin-Spin Correlation Function

In order to derive expressions for thermodynamic quantities in the tensor formalism, let us now restrict the notation $T^A$ and $T^B$ to tensors in the zero magnetic field subspace. We place a local magnetic field $H_k$ only at a single location $k$. Let us call the two tensors which share this bond $\bar{T}^A$ and $\bar{T}^B$. These are the only two tensors in the system whose components are modified by the local field. The corresponding partition function is

$$Z = \sum_{i_1, \ldots, i_K} \bar{T}^A_{i_1 i_2 i_3} \bar{T}^B_{i_4 i_5 i_3} \cdots \bar{T}^A_{i_k i_l i_m} \bar{T}^B_{i_k i_l i_m} \cdots \bar{T}^B_{i_{K-2} i_{K-1} i_k}.$$

(7)

Without loss of generality we take the contraction of the $\bar{T}^A$ and $\bar{T}^B$ tensors to be Case 2 in Eq. (2), since the derivation proceeds analogously for the other Cases.

The local magnetization is $m_k = \langle (s_i + s_j)/2 \rangle \equiv \langle S_k \rangle$ for the sites $i, j$ associated with the bond $k$. In terms of the local magnetic field $H_k$, the magnetization $m_k$ is given by the derivative

$$m_k = \frac{1}{2} \frac{\partial \ln Z}{\partial H_k} \Big|_{H_k=0},$$

$$= \frac{1}{2^2} \sum_{i_1, \ldots, i_K} \{ T^A_{i_1 i_2 i_3} T^B_{i_4 i_5 i_3} \cdots D^A_{i_k i_l i_m} D^B_{i_k i_l i_m} \cdots \}, \quad (8)$$

where the differentiated tensors are

$$D^A_{i_k i_l i_m} = \frac{\partial \bar{T}^A_{i_k i_l i_m}}{\partial H_k} \Big|_{H_k=0}, \quad D^B_{i_k i_l i_m} = \frac{\partial \bar{T}^B_{i_k i_l i_m}}{\partial H_k} \Big|_{H_k=0}. \quad (9)$$
The nonzero elements of $D^A$ and $D^B$ are shown, for the first renormalization step, in the 4th and 7th columns of Table II.

After taking the average over the disorder, the first and second terms in the brackets on the right-hand side of Eq. 8 are equivalent, so that

$$ [m_k] = \langle S_k \rangle = \left[Z^{-1} \sum_{i_1, \ldots, i_K} T^A_{i_1i_2i_3} T^B_{i_4i_5i_3} \cdots D^A_{i_ki_{i_m}i_k} T^B_{i_{i_n}i_{i_m}} \cdots \right]. \quad (10) $$

A similar derivation for the correlation function yields

$$ \langle S_k S_l \rangle = \left[Z^{-1} \sum_{i_1, \ldots, i_K} T^A_{i_1i_2i_3} T^B_{i_4i_5i_3} \cdots D^A_{i_ki_{i_m}i_k} T^B_{i_{i_n}i_{i_m}} \cdots \right]. \quad (11) $$

We shall be interested in long-range correlations, as an indicator of thermodynamic phase behavior. In this case, the four individual $s_i$ spin-spin correlations that make up the $\langle S_k S_l \rangle$ are approximately equal: $\langle S_k S_l \rangle \approx \langle s_i s_j \rangle$, where $s_i$ is either of the spins contributing to $S_k$ and $s_j$ is either of the spins contributing to $S_l$. Hence we shall use $\langle S_k S_l \rangle$ and $\langle s_i s_j \rangle$ interchangeably in the rest of the text.

C. Details of the Numerical Implementation

To calculate the long-range spin-spin correlation function $\langle s_i s_j \rangle$, we start with a finite hexagonal lattice after $n$ construction steps, with size varying between $n = 7\cdots10$ steps ($N = 17496 - 472392$ tensors). The bonds $k$ and $l$ are chosen to be at the maximum separation within the lattice, taking periodic boundary conditions into account. For a given realization of the disorder, the sum on the right-hand side of Eq. 11 is evaluated by doing $n$ TRG transformations, which yields the contraction in terms of four renormalized tensors in the $n = 0$ structure. These last four tensors are directly contracted. A similar process yields the value of the partition function $Z$ which is the denominator in Eq. 11. The configurational average is taken over 200–300 realizations, implemented by randomly assigning the $J_{ij}$ on the initial lattice according to the probability distribution in Eq. 5. The tensors on the original lattice, i.e., in Eqs. 6 and 9, have index range $d = 4$. For subsequent tensors, we use a cutoff parameter $D = 8 - 14$.

Some tensor elements tend to grow exponentially in magnitude as the TRG transformation is iterated, which poses potential numerical difficulties. To counteract this, we take advantage of the fact that we can always factor out a constant from each tensor without changing the physics. For each tensor during each TRG iteration, the factor extracted is equal to $\min(T_{\text{max}}, 2)$ where $T_{\text{max}}$ is the maximum absolute value of the tensor elements. Keeping an upper bound of 2 on this extracted factor would otherwise lead to other numerical artifacts.

IV. RESULTS

Representative results for the long-distance spin-spin correlation function $\langle s_i s_j \rangle$ as a function of temperature $T$ are shown in Figs. 3 and 4. The former shows curves for various tensor network sizes $N$ using cutoff $D = 8$, while the latter varies the cutoff $D$ at fixed size $N = 157464$. Away from the critical
TRG method for bond dilution probability \( p \) parameter \( D \) the system: at low \( 1/J \) temperature, where widely separated spins are uncorrelated, \( \left\langle s_i s_j \right\rangle \approx \left( s_i \right)^2 \), and we expect distinct limiting behaviors for the two different thermodynamic phases in the system: at low \( 1/J \) in the ferromagnetically ordered phase \( \left\langle s_i s_j \right\rangle \to 1 \), while at high \( 1/J \) in the paramagnetic phase \( \left\langle s_i s_j \right\rangle \to 0 \). The temperature region where one sees a smooth transition between these two regimes for finite systems, illustrated in Figs. 4 and 6, gives a rough indication of the phase transition temperature \( 1/J_c \) in the thermodynamic limit. With increasing \( N \) in Fig. 5 and increasing \( D \) in Fig. 6 the transition becomes sharper, as our truncations converge toward the exact result for an infinite system. The probability \( p \) at which these results are calculated is smaller than the threshold value \( p_c \approx 0.653 \) [22], above which the triangular lattice no longer percolates. For \( p > p_c \) we would not see a transition region: the paramagnetic phase exists at all temperatures, since islands of ordered spins of size \( \sim O(N) \) become exponentially improbable.

To obtain an accurate estimate of the exact transition temperature \( 1/J_c \), we can employ the following finite-size scaling relation, which describes the ratios of the correlation functions at three different system sizes \( N_1, N_2, \) and \( N_3 \) when \( J = J_c \) [23]:

\[
\frac{\ln \left( \frac{g(N_2)}{g(N_1)} \right)}{\ln \left( \frac{N_2}{N_1} \right)} = \frac{\ln \left( \frac{g(N_3)}{g(N_2)} \right)}{\ln \left( \frac{N_3}{N_2} \right)},
\]

where \( g(N) \) is the long-distance correlation function \( \left\langle s_i s_j \right\rangle \) for network size \( N \). For the \( j \)th system, at the temperature region where \( g(N_j) \) decays rapidly to zero (\( J \) just smaller than \( J_c \)), the decay is approximately exponential in \( J \):

\[
\ln(g(N_j)) \approx A_j J - B_i,
\]

for some constants \( A_i \) and \( B_i \). This exponential behavior for three different system sizes is shown in Fig. 7 for \( p = 0.25 \) and 0.55. To calculate \( A_i \) and \( B_i \), we do a weighted linear least squares fit to \( \ln(g(N_j)) \) vs. \( J \) data in a region of \( J \) where the relative uncertainty (from the configurational average) for the data points is less than 15%. Plugging Eq. 13 into Eq. 12 with \( J = J_c \), we can solve for \( J_c \) in terms of the \( A_i \), \( B_i \), and \( N_i \):

\[
J_c = \frac{(B_2 - B_1) \ln \left( \frac{N_2}{N_1} \right) + (B_2 - B_3) \ln \left( \frac{N_3}{N_2} \right)}{(A_1 - A_2) \ln \left( \frac{N_2}{N_1} \right) + (A_3 - A_2) \ln \left( \frac{N_3}{N_2} \right)}.
\]

Carrying out this calculation across the entire \( p \) range for \( N_1 = 17496, N_2 = 52488, \) and \( N_3 = 157464 \) at \( D = 12 \), we obtain the phase diagram shown in Fig. 1. For comparison we also plot the same phase diagram obtained from a rigorous approximation scheme for the bond-diluted Ising model free energy [17], which can be considered exact on the scale of the figure. The agreement is quite close, with an average relative deviation of 1%. Two values along the curve are known exactly: \( 1/J_c = 4/\ln 3 = 3.641 \) [24] at \( p = 0 \) and the curve goes to \( 1/J_c = 0 \) at the percolation threshold \( p = p_c = 1 - 2 \sin(\pi/18) = 0.653 \) [22]. Our results deviate from these exact values by 0.3% and 0.4% respectively.
FIG. 7: (Color online) Data points show the log of the long-distance spin-spin correlation, \( \ln \langle \sigma_i \sigma_j \rangle \), as a function of interaction strength \( J \) for three different system sizes \( N \) and two different bond dilution probabilities \( p \) (top panel: \( p = 0.25 \), bottom panel: \( p = 0.55 \)). The weighted least squares linear fits, shown as solid lines, yield the coefficients \( A_i \) and \( B_i \) in Eq. (13), which allow one to estimate \( J_c \) through finite size scaling, Eq. (14). The resulting values of \( J_c \) are 0.388 (\( p = 0.25 \)) and 0.871 (\( p = 0.55 \)).

V. CONCLUSIONS

We have shown how the TRG approach provides an efficient and precise method for calculating thermodynamic properties of a quenched random classical model—the triangular-lattice bond-diluted Ising Hamiltonian. By expressing the partition function and related quantities such as spin-spin correlation functions in terms of tensor networks, they can be readily evaluated through TRG for large lattice sizes. In combination with finite-size scaling ideas, the result is a precise estimate of the phase diagram. If desired, convergence to the exact critical properties can be achieved by increasing the cutoff parameter defining the index range of the tensors.

The bond-diluted Ising model is only a first step in the exploration of disordered systems using TRG: the methods presented here are easily extended to frustrated Hamiltonians exhibiting spin-glass behavior and the resulting complex multicritical phase structures. The numerical accuracy of the technique will be a valuable feature in probing analytical conjectures on the exact locations of spin-glass multicritical points [25–28].

Acknowledgments

This research was supported by the Alexander von Humboldt Foundation, the Scientific and Technological Research Council of Turkey (TÜBİTAK), and the Academy of Sciences of Turkey. Computational resources were provided by the Gilgamesh cluster of the Feza Gürsey Research Institute.

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