The spin-3/2 Ising model, with nearest-neighbor interactions only, is the prototypical system with two different ordering species, with concentrations regulated by a chemical potential. Its global phase diagram, obtained in $d = 3$ by renormalization-group theory with flows subtended by 40 different fixed points, presents a very rich structure containing eight different ordered and disordered phases, with more than fourteen different types of phase diagrams in temperature and chemical potential. It exhibits phases with orientational and/or positional order. It also exhibits quintuplet phase transition reentrances. Universality of critical exponents is conserved across different renormalization-group flow basins, via redundant fixed points. One of the phase diagrams contains a plastic crystal sequence, with positional and orientational ordering encountered consecutively as temperature is lowered. The global phase diagrams also contains double critical points, order-order critical lines, critical endpoints, usual and “inverted” bimodal critical points, tricritical points, and phase boundary quintuple reentrance. The 4-state Potts permutation-symmetric subspace is contained in this model.

PACS numbers: 64.60.Cn, 05.50.+q, 61.43.-j, 75.10.Hk

I. INTRODUCTION

The spin-3/2 Ising model, with nearest-neighbor interactions only, exhibits intricate but physically suggestive phase diagrams, as for example shown in Fig. 1(f) including three separate ferromagnetic phases and an only positionally ordered phase, new special points, a conservancy of the universality principle of critical exponents via the redundant fixed-point mechanism, and a temperature sequence of stepwise positional and orientational ordering as in plastic crystals. Other phase diagram cross-sections of the global phase diagram, with eight different ordered and disordered phases, include order-order double critical points, order-order critical lines, critical endpoints, usual and “inverted” bimodal critical points, tricritical points, and tetracritical points. The permutation-symmetric 4-state Potts subspace is lodged in this model.

The Hamiltonian of the spin-1/2 Ising model, $-\beta \mathcal{H} = \sum_{\langle ij \rangle} J_{ij} s_i s_j$, where at each site $i$ there is a spin $s_i = \pm 1$ and the sum is over all pairs of nearest-neighbor sites, generalizes for the spin-1 Ising model to

$$-\beta \mathcal{H} = \sum_{\langle ij \rangle} [J_{ij} s_i s_j + K s_i^2 s_j^2 - \Delta (s_i^2 + s_j^2)],$$

where $s_i = \pm 1, 0$. Eq.(1) constitutes the most general spin-1 Ising model with nearest-neighbor interactions only and no externally imposed symmetry breaking in the ordering degrees of freedom. The global understanding of the phase diagram of the spin-1 Ising model played an important role through applicability to many physical systems that incorporate non-ordering degrees of freedom ($s_i = 0$) as well as ordering degrees of freedom ($s_i = \pm 1$). The next qualitative step is the global study of a model that has two different types of local ordering degrees of freedom, namely the spin-3/2 Ising model:

$$-\beta \mathcal{H} = \sum_{\langle ij \rangle} [(J_{1} P_{1} P_{j} + J_{13} (P_{i} Q_{j} + Q_{i} P_{j}) + J_{3} Q_{i} Q_{j}) s_i s_j + (K_{1} P_{1} P_{j} + K_{13} (P_{i} Q_{j} + Q_{i} P_{j}) + K_{3} Q_{i} Q_{j}) s_i^2 s_j^2 - \Delta (s_i^2 + s_j^2)],$$

where $s_i = \pm 3/2, \pm 1/2$, is the most general spin-3/2 Ising model with only nearest-neighbor interactions and no externally imposed symmetry breaking in the ordering degrees of freedom. The projection operators in Eq.(2) are $P_i = 1 - Q_i = 1(0)$ for $s_i = \pm 1/2(\pm 3/2)$.

One of the models defined above, the spin-1/2 Ising model has a single critical point on the temperature $J^{-1}$ axis. The spin-1 Ising model, in the temperature and chemical potential $\Delta / J$ plane, has three different types of phase diagrams when the biquadratic interaction $K$ is non-negative. When negative biquadratic interactions are considered, nine different types of phase diagrams are obtained from mean-field theory. We find in our current work on the spin-3/2 Ising model, using renormalization-group theory, an extraordinarily rich solution, with numerous types of phase diagrams in temperature and chemical potential, exhibiting first- and second-order phase transitions between the variously ordered and disordered phases.

The Hamiltonian of Eq.(2) is expressed as $-\beta \mathcal{H} = \sum_{\langle ij \rangle} -\beta \mathcal{H}_{ij}$ and the transfer matrix is the exponentiated nearest-neighbor Hamiltonian $\exp(-\beta \mathcal{H}_{ij})$. The renormalization-group treatment of the system, for spatial dimension $d$ and length-rescaling factor $b$, is ef-
The renormalization-group flows to different basins, critical exponent universality is sustained via redundant fixed points.

Magnetically disordered. The point $I = 2$ is disordered with predominantly $|s| = 3/2$, and $|s| = 3/2$, respectively. The phase $F_{13}$ is disordered and magnetically disordered. The point $C$ is an ordinary critical point and the point $C_2$ is a double critical point. The points $L_1$ and $L_3$ are critical endpoints. The point $R_3$ is a tricritical point. The points $M_1$ and $M_3$ are tetracritical points. The points $I$, $I_{13}$, and $I_4$ each separate two segments of second-order phase transitions, between the same two phases, where in spite of renormalization-group flows to different basins, critical exponent universality is sustained via redundant fixed points.

![Graphs of Temperature versus chemical potential interactions](image)

FIG. 1: (Color online) Temperature versus chemical potential phase diagrams of the spin-$3/2$ Ising model for $d = 3$, for (a) $K/J = 2.75$, (b) $K/J = 1.20$, (c) $K/J = 0.35$, (d) $K/J = 0.06$, (e) $K/J = -0.50$, (f) $K/J = -2.50$. The first- and second-order phase transitions are drawn with dotted and full curves, respectively. The phases $F_1$ and $F_3$ are ferromagnetically ordered with predominantly $|s| = 1/2$ and $|s| = 3/2$, respectively. The phases $D_1$ and $D_3$ are disordered with predominantly $|s| = 1/2$ and $|s| = 3/2$, respectively. The phase $F_{13}$ is positionally and ferromagnetically ordered and the phase $A_3$ is positionally ordered and magnetically disordered. The point $C$ is an ordinary critical point and the point $C_2$ is a double critical point. The points $L_1$ and $L_3$ are critical endpoints. The point $R_3$ is a tricritical point. The points $M_1$ and $M_3$ are tetracritical points. The points $I$, $I_{13}$, and $I_4$ each separate two segments of second-order phase transitions, between the same two phases, where in spite of renormalization-group flows to different basins, critical exponent universality is sustained via redundant fixed points.

![Graphs of Zero-temperature phase transitions](image)

FIG. 2: (Color online) Zero-temperature ($J, J_{13} \to \infty$) phase diagrams. The first- and second-order phase transitions are drawn with dotted and full curves, respectively. The phases $F_1$ and $F_3$ are ferromagnetically ordered with predominantly $|s| = 1/2$ and $|s| = 3/2$, respectively. The phase $F_{13}$ is positionally and ferromagnetically ordered. The phases $A_3$ and $A_3$ are antiferromagnetically ordered with predominantly $|s| = 1/2$ and $|s| = 3/2$, respectively. The points $B_2$ are zero-temperature bicritical points which, being at zero temperature, accommodate boundary lines at finite angles.

II. GLOBAL PHASE DIAGRAM

We start by studying $J_1 = J_{13} = J_3 \equiv J$ and $K_1 = K_{13} = K_3 \equiv K$. Thus, $1/J$ is proportional to temperature and will be used as our temperature variable. In our system, one of the ordering species is $|s| = 3/2$, the other one is $|s| = 1/2$. The chemical potential $\Delta/J$ (dividing out inverse temperature) controls the relative amounts of each ordering species. The biquadratic inter-
action $K/J$ (again dividing out inverse temperature) controls the separation/mixing of the two ordering species. Fig. 1 shows the effects of the biquadratic interaction on the global phase diagram.

The temperature versus chemical potential phase diagram for large $K/J$, where separation is favored, is illustrated in Fig. 1(a) with $K/J = 2.75$. In this phase diagram, two ferromagnetically ordered phases $F_3$ and $F_1$ are seen at low temperatures, each rich in one of the ordering species, namely respectively rich in $|s_i| = 3/2$ and $|s_i| = 1/2$. Upon increasing temperature, each ferromagnetic phase undergoes a second-order phase transition to the disordered (paramagnetic) phase that is rich in the corresponding species, respectively $D_3$ and $D_1$. By changing the chemical potential $\Delta/J$, three different first-order fixed points are induced between phases rich in different species: A four-phase coexistence line between the ferromagnetic phases $F_3$ and $F_1$ at low temperatures, a three-phase coexistence line between the ferromagnetic phase $F_3$ and the disordered phase $D_1$ at intermediate temperatures, and a two-phase coexistence line between the disordered phases $D_3$ and $D_1$ at high temperatures. Each of the different first-order fixed points are given in Table I. The latter first-order transition terminates at high temperature at the isolated critical point $C$. At intermediate temperatures, both second-order transition lines terminate on the first-order boundary, at critical endpoints $L_3$ and $L_1$. The corresponding hybrid fixed points, which include both first-order ($y_1 = d$) and second-order ($0 < y_2 < d$) characteristics, are given in Table I.

As the biquadratic coupling strength $K/J$ is decreased from large positive values, lessening the tendency of the two ordering species to separate, the first-order phase transition line between their respective disordered phases shrinks, so that the isolated critical point $C$ and the upper critical endpoint $L_3$ approach each other and merge, to form the tricritical point $R_3$. The resulting phase diagram is illustrated in Fig. 1(b) with $K/J = 1.20$.

At lower values of the biquadratic coupling strength $K/J$, a narrow band of the ferromagnetic $F_1$ phase appears, decoupled from the main $F_3$ region, between the $F_3$ and $D_3$ regions, as seen in Fig. 1(c) for $K/J = 0.35$. The first-order phase boundary between this narrow $F_1$ region and the $F_3$ region extends, at lower temperature, to the upper critical endpoint $L_1$ and, at higher temperature, to an isolated critical point $C_2$ as seen in the inset in Fig. 1(c). This isolated critical point, totally imbedded in ferromagnetism, is thus a double critical point, as it mediates between the positively magnetized $F_3$ and $F_1$ and, separately, between the negatively magnetized $F_3$ and $F_1$. Due to this order-order critical point, it is possible to go continuously, without encountering a phase transition, between the ordered $F_3$ and $F_1$ phases. The second-order phase boundary extending to the upper $L_1$ is composed of to segments, on each side of the point $I$, separately subtended by the $F_3 - D_3$ and $F_1 - D_1$ critical fixed points. The universality principle of the critical exponents is sustained here by the redundant [70] fixed-point mechanism: Although these two fixed points are globally separated in the renormalization-group flow diagram, they have identical critical exponents (which is furthermore shared by the fixed point of $I$), as seen in Table I.

At lower values of $K/J$, the two critical endpoints $L_1$ merge and annihilate. A single second-order phase boundary, between the $F_3$ or $F_1$ ordered phase at low temperature and the $D_3$ or $D_1$ disordered phase at high temperature, extends across the entire phase diagram, as seen in Fig. 1(d) for $K/J = -0.06$. The universality principle is sustained along this second-order phase boundary by the redundancy of the fixed points, as explained above. A single first-order boundary forms between the $F_3$ and $F_1$ ordered phases, disconnected from the second-order boundary to the $D_3$ and $D_1$ disordered phases.
As the biquadratic coupling strength $K/J$ is further decreased, increasing the tendency of the two ordering species to mix, the first-order boundary between $F_3$ and $F_1$ shrinks to zero temperature and thus disappears. In fact, ordered mixing appears: A sublattice-wise (i.e., positionally) ordered, as well as magnetically (i.e., orientationally) ordered ferrimagnetic phase $F_{13}$ appears at $K/J \leq -1/4$. In this phase, one of two sublattices is predominantly $|s_i| = 3/2$ and the other sublattice is predominantly $|s_i| = 1/2$, and the system is magnetized. As illustrated in Fig. 1(e) for $K/J = -0.50$, the $F_{13}$ phase occurs at low temperatures and intermediate chemical potentials. The phase boundary between $F_{13}$ and $F_3$ or $F_1$ is second-order and remarkably extends to zero-temperature. Similar phase diagrams have been seen in $d = 2, 3$. [7, 11]

For even more negative values of $K/J$, a portion of the sublattice-ordered phase has erupted through the ferromagnetic ordering lines and in the process lost ferromagnetic ordering, as illustrated in Fig. 1(f) with $K/J = -2.50$. Thus, a new (antiquadrupolar) phase $A_q$ appears, that is sublattice-wise (positionally) ordered, but paramagnetic. Each ordering species predominantly occurs in one of the two sublattices, with no preferred spin orientation. This regime offers a phase diagram topology including four different ordered phases. Two of the ordered phases are orientationally ordered, one phase is positionally ordered, and one phase is both orientationally and positionally ordered. Note that, at intermediate chemical potentials, as temperature is lowered, the sequence of disordered, then only positionally ordered, finally positionally and orientationally ordered phases is encountered, as in plastic crystal systems. [7]

Second-order phase transition lines meet at the tetracritical point $N_1$. The inset shows that the phase diagram topology near the maximal temperatures of the phase $F_{13}$ is yet unaltered from Fig. 1(e). In this phase diagram region, as $(J_1 - J_3)/J_{13}$ is further increased, the phase $A_q$ replaces $F_{13}$, which is eliminated, as seen for $(J_1 - J_3)/J_{13} = 2.75$. The tetracritical point moves, as $N_1$ to the less negative crystal-field side of $F_{13}$. A seen in the phase diagrams for $(J_1 - J_3)/J_{13} = 3.70$, many phase-transition reentrances occur in the neighborhood of the tetracritical point $N_1$. As temperature is lowered, the phase transitions to $F_1$ are quintuply reentrant. The phase transitions to $F_{13}$ are singly reentrant. It should be noted that our current calculation, showing these complicated multiple reentrances, is exact for the hierarchical lattice with spatial dimension $d = 3$. As $(J_1 - J_3)/J_{13}$ is further increased, the multicritical point $N_1$ splits into usual and "inverted" bicritical points $B$ connected by a first-order transition line between the antiferromagnetic phase $A_3$ and the ferromagnetic phase $F_1$, as seen for $(J_1 - J_3)/J_{13} = 4.75$. As $(J_1 - J_3)/J_{13}$ is further increased, the higher-temperature bicritical point splits into a tricritical point $R_0$ and a critical endpoint $L_4$ and the lower-temperature unusual bicritical point disappears, along with the phase $F_{13}$, at zero temperature, as seen for $(J_1 - J_3)/J_{13} = 8.00$.

As seen in Fig. 1(e) or around the phase $A_q$, the segments on each side of the points $I_{13}$ and $I_A$ having different fixed points but same critical exponents (Table I). Similar phase diagrams have been seen in $d = 2, 3$. [7, 11]

The calculated finite-temperature global phase diagram given in Fig. 1 agrees with the zero-temperature phase diagram given in the left panel of Fig. 2, calculated by ground-state energy crossings. It is seen that the zero-temperature phase diagram includes a zero-temperature bicritical point $B_Z$ at $K/J = -1/4, \Delta/J = 3/16$.

III. DIFFERENTIATED SPECIES COUPLING

The spin-3/2 Ising model carries an even richer structure of phase diagrams, accessed by differentiating the interaction constants in Eq. (2). We give here two sequences of phase diagrams with $J_1 > J_3$, $J_{13} = (J_1 + J_3)/2$. 

A. Phase Diagrams Evolving from
Figure 3 shows phase diagrams with $K/J_{13} = 1.50$. As $(J_1 - J_3)/J_{13}$ is increased, the second-order transition temperature to the ferromagnetic phase $F_3$ falls below the second-order transition temperature to the ferromagnetic phase $F_1$, as seen for $(J_1 - J_3)/J_{13} = 1.75$. Eventually, the ferromagnetic phase $F_3$ disappears at zero temperature and an antiferromagnetic phase $A_3$, predominantly with $|s_1| = 3/2$, appears from zero temperature, as seen for $(J_1 - J_3)/J_{13} = 2.20$. As $(J_1 - J_3)/J_{13}$ is further increased, the second-order transition temperature to the antiferromagnetic phase $A_3$ moves above the second-order transition temperature to the ferromagnetic phase $F_1$, as seen for $(J_1 - J_3)/J_{13} = 3.00$.

The finite-temperature phase diagrams of Fig. 3 are consistent with the corresponding zero-temperature phase diagram, in the middle panel of Fig. 2. Conversely and not shown here, when $(J_1 - J_3)/J_{13}$ is made negative, an antiferromagnetic phase $A_1$, predominantly with $|s_1| = 1/2$, similarly appears, as seen in Fig. 2.

B. Phase Diagrams Evolving from the Figure 1(e) Topology

Figure 4 shows phase diagrams with $K/J_{13} = -0.50$. As $(J_1 - J_3)/J_{13}$ is increased, the second-order transition temperature to the ferromagnetic phase $F_3$ is depressed and the antiferromagnetic phase $A_3$ appears for more negative values of the crystal-field interaction $\Delta/J_{13}$, where the spin magnitude $|s_1| = 3/2$ is more favored, as seen for $(J_1 - J_3)/J_{13} = 2.05$. Four second-order phase transition lines meet at the tetracritical point $N_3$. The inset shows that the phase diagram topology is unaltered near the maximal temperatures of the phase $F_1$. In this phase diagram region, as $(J_1 - J_3)/J_{13}$ is further increased, $A_3$ replaces $F_3$, which is eliminated, as seen for $(J_1 - J_3)/J_{13} = 2.75$. The tetracritical point moves, as $N_1$, to the lower boundary of the antiferromagnetic phase $F_1$. As seen in the phase diagrams for $(J_1 - J_3)/J_{13} = 3.70$, many phase transition reentrances occur in the neighborhood of the tetracritical point $N_1$: As temperature is lowered, the phase transitions to $F_1$ are quintuply reentrant. The phase transitions to $F_1$ are singly reentrant. Previous to quadruply reentrant phase transitions have been found for liquid crystal systems [72–77] and surface systems [78]. More phase reentrances have been calculated in the high-$T_C$ superconductivity $tJ$ model [79]. It should be noted that our current calculation, showing these complicated reentrances, is exact for the hierarchical lattice with spatial dimension $d = 3$. As $(J_1 - J_3)/J_{13}$ is further increased, the tetracritical point $N_1$ splits into two bicritical points $B$ connected by a first-order transition line between the antiferromagnetic phase $A_3$ and the ferromagnetic phase $F_1$, as seen in Fig. 4 for $(J_1 - J_3)/J_{13} = 4.75$. At a (non-zero-temperature) bicritical point, normally, two high-temperature second-order boundaries and one low-temperature first-order boundary meet tangentially. [80] In our present phase diagram, the opposite temperature ordering occurs at the upper-temperature bicritical point. Thus, this is an "inverted bicritical point". As $(J_1 - J_3)/J_{13}$ is further increased, the higher-temperature bicritical point splits into a tricritical point $R_3$ and a critical endpoint $L_1$ and the lower-temperature inverted bicritical point disappears, along with the phase $F_1$ at zero temperature, as seen in Fig. 4 for $(J_1 - J_3)/J_{13} = 8.00$.

Thus, both multicritical points, bicritical and tetracritical, of the classic coupled-order-parameter problem [80] is contained within the spin-3/2 Ising model. The finite-temperature phase diagrams of Fig. 4 are consistent with the corresponding zero-temperature phase diagram, in the right panel of Fig. 2.

IV. 4-STATE POTTs SUBSPACE

Returning to the most general spin-3/2 Ising Hamiltonian in Eq.(2), for

$$4J_1 = 36J_3 = K_1 = 81K_3, \quad J_{13} = K_{13} = \Delta = 0,$$  (3)
the model reduces to the 4-state Potts model, with Hamiltonian
\[ -\beta H = \frac{J_1}{2} \sum_{(ij)} \delta(s_i, s_j), \] (4)
where the Kronecker delta function is \( \delta(s_i, s_j) = 1(0) \) for \( s_i = s_j \neq s_j \).

Figure 5 gives the calculated phase diagram in temperature \( 1/J_1 \) and chemical potential \( \Delta/J_1 \), for the 4-state Potts side conditions \( 4J_1 = 36J_3 = K_1 = 81K_3, J_{13} = K_{13} = 0 \). In this figure, \( \Delta/J_1 = 0 \) is the 4-state Potts subspace, where the system is permutation symmetric with respect to the 4 states \( s_i = \pm 3/2, \pm 1/2 \). Varying \( \Delta/J_1 \) from zero gives the "symmetric bicritical" phase diagram around the Potts multicritical point. This phase diagram is exact for the \( d = 3 \) hierarchical lattice, but approximate for the cubic lattice. For the cubic lattice, from 3-state Potts model analogy \[83\], we expect short first-order segments on each phase boundary leading to the 4-state Potts transition, which occurs as first-order with 5-phase coexistence. In renormalization-group theory, this first-order behavior is revealed by the accounting of local disorder as effective vacancies \[83\ [84\]. For the spin-3/2 Ising model, the resulting renormalization-group flows would be in the space of the spin-2 Ising model. In Fig. 5, for \( 3.5J_1 = 36J_3 = K_1 = 81K_3 \) and \( 4.5J_1 = 36J_3 = K_1 = 81K_3 \), shown in the left and right insets respectively, the symmetric Potts transition point \( P_4 \) is replaced, respectively, by tricritical points \( R_3 \) and critical endpoints \( L_1 \) and \( L_3 \) in asymmetric phase diagrams.

V. CONCLUSION

Renormalization-group theory reveals that the spin-3/2 Ising model in \( d = 3 \) has a rich phase diagram, with eight different orientationally and/or positionally ordered and disordered phases; first- and second-order phase transitions; double critical points, order-order critical lines, critical endpoints, usual and "inverted" bicritical points, tricritical points, tetracritical points, and quintuple phase boundary reentrance. Fourteen different phase diagram topologies, in the temperature and chemical potential variables, are presented here. The renormalization-group flows yielding this multicritical, multistructural global phase diagram are governed by 40 different fixed points (Table I).

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### II. Singly Unstable Fixed Points: Attractors of Second-Order Phase Boundaries and their Relevant Exponent $y_T$

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### III. Singly Unstable Fixed Points: Attractors of First-Order Phase Boundaries with Relevant Exponent $y_T = d$

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### IV. Singly Unstable Fixed Points: Attractors of Smooth Continuation (Null) Lines

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### V. Multiply Unstable Fixed Points: Attractors of Multicritical Points and their Leading 2 Relevant Exponents $y_{T_1}, y_{T_2}$

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### Table I: Fixed points underpinning the renormalization-group flows determining the global phase diagram of the $s = 3/2$ Ising model. In this Table, the matrix elements are $t = 0.9243, u = 0.9474, v = 0.9300, w = 0.4099$. The fixed point for the isolated critical point in Fig. 1(a) has thermal exponent $y_T = 0.9260$ along the first-order transition direction and magnetic 4-Potts critical point in Fig. 1(b) has thermal exponent $y_T = 0.9260$ along the second-order transition direction.