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DOCTORAL GENERAL EXAMINATION
PART 1
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SOLUTIONS

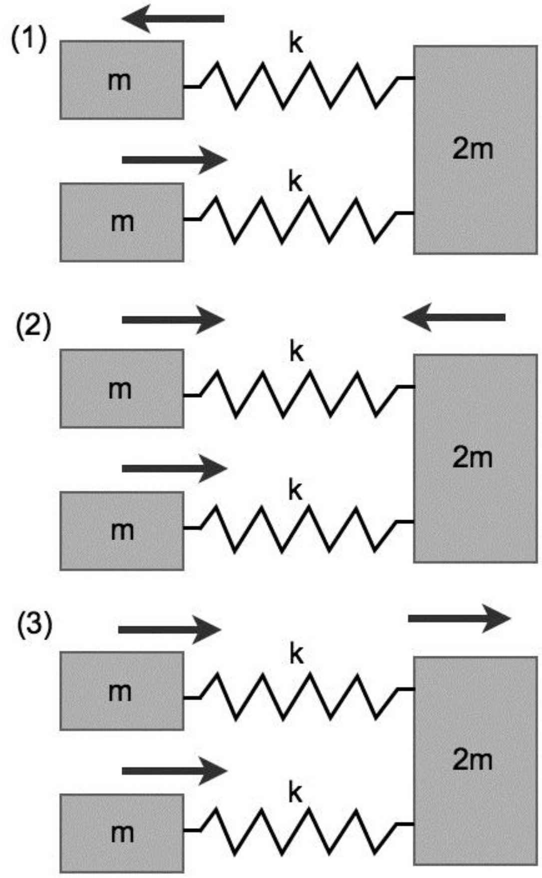
GROUP I – Solutions

I-1 Coupled Oscillators

In a normal mode, all the masses oscillate with the same frequency and the same phase. There are three normal modes for a system with three masses. Because of high symmetry of the problem, in this case, all of these can be found without doing any calculations. First normal mode corresponds to the case where small masses on the left moves by equal and opposite amount and the mass 2m on the right is stationary. The frequency of this normal mode is the same as a single mass m connected to a spring with spring constant k i.e. $\omega_1 = \sqrt{\frac{k}{m}}$.

The second normal mode corresponds to the motion in which two small masses on the left move together by the same amount, and the mass 2m moves by the equal but opposite amplitude. Since the two small mass moves together, we can consider them as one mass with value 2m. So the problem boils down to two masses with value 2m joined together with a spring of total spring constant 2k. The frequency of this motion is $\omega_2 = \sqrt{\frac{2k}{m}}$.

Finally, the last normal mode corresponds to the case where all the masses are displaced by the same amount, they will not return back to their equilibrium positions, so $\omega_3 = 0$.



The sketch on the right and the values of frequencies are enough to get the full credit.

I-2 CMB versus CνB

To find the ratio of densities, one needs to integrate over the full phase space (in momentum) according to their particle's statistics (Bose-Einstein or Fermi-Dirac). The generic integral to solve is

$$n(T) = \int_0^\infty f(T,E)\frac{d^3p_i}{(2\pi)^3}$$

where

$$f(T,E) = \frac{1}{e^{(E-\mu)/T} \pm 1}$$

where E is the energy of the particle, $d^3p = 4\pi p^2 dp$ is the momenta of the particle, and T is the temperature. Since the particle is relativistic, $E \cong p$. In this case, we let the chemical potential μ be zero, so the term in the exponent is proportional to p. Letting $z = p/T$, one finds...

$$n(T) = \frac{T^3}{2\pi^2}\int_0^\infty \frac{z^2}{e^z \pm 1}dz$$

Immediately one finds the T^3 dependence. Using the integration formulas from the front, one also finds.

$$n_\gamma(T) = \frac{T_\gamma^3 \zeta(3)}{\pi^2} \text{ Bose} - \text{Einstein}$$

$$n_\nu(T) = \frac{3T_\nu^3 \zeta(3)}{4\pi^2} \text{ Fermi} - \text{Dirac}$$

The ratio is thus given by

$$\frac{n_\nu}{n_\gamma} = \frac{3}{4}(\frac{T_\gamma}{T_\nu})^3$$

I wouldn't expect the students to necessarily remember that there are three neutrino species, but of course if students include the number of neutrinos ($N\nu$) in their answer, that would be correct as well.

I-3 Finding the potential

The Schroedinger equation is

$$\mathcal{H}\psi(r,t) = i\hbar \frac{\partial \psi(r,t)}{\partial t}$$

where

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

In this problem $\frac{d\psi}{dx}$ is discontinuous at x = 0 so the potential must have a delta function at the origin; it may also have a constant background. Obviously it is a bound state. That tells us that $V(x) = E_0 - V_0\delta(x)$ so we need to find the coefficient V_0 and the constant E_0. To do so, we can integrate the Schroedinger equation across the origin, letting $\varepsilon \to 0$.

$$\begin{aligned} -\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon}\frac{\partial^2\psi}{\partial x^2} + \int_{-\epsilon}^{+\epsilon} V(x)dx &= i\hbar\int_{-\epsilon}^{+\epsilon}\frac{\partial\psi}{\partial t} \\ -\frac{\hbar^2}{2m}\left(\left.\frac{\partial\psi}{\partial x}\right|_{\epsilon} - \left.\frac{\partial\psi}{\partial x}\right|_{-\epsilon}\right) - V_0\,\psi(0) + 2\,\epsilon\,E_0 &= 2\,\epsilon\,b\,\hbar\,\psi(0) \end{aligned}$$

giving $V_0 = \hbar a$.

Substituting into the Schroedinger equation, we find $E_0 = \hbar b + 1/2ma^2$ Thus,

$$\boxed{V(x) = \hbar\,b + \frac{1}{2}m\,a^2 - \hbar\,a\,\delta(x)}$$

I-4 Charged Sphere

Capacity of a charged sphere is $C = 4\,\pi\epsilon_0 R$. Stored energy is $W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\,\frac{1}{4\,\pi\epsilon_0}\,\,\frac{Q^2}{R}$.

Increasing the radius by dR yields

$$4\pi R^2\frac{F}{A}dR = \frac{dW}{dR}\,dR = -\,\frac{1}{2}\,\frac{1}{4\,\pi\epsilon_0}\,\frac{Q^2}{R^2}dR$$

Thus, $|\frac{F}{A}| = \frac{1}{4\,\pi\epsilon_0}\,\,\frac{Q^2}{8\pi R^4}$. The force points radially outwards everywhere.

GROUP II - Solutions

II-1 House of cards

Using symmetry, consider one card leaning against a vertical wall. Make sure that the total torque on the card vanishes, using the edge of the card touching the vertical surface as the axis. Torque due to gravity is $\tau_g = L/2\, mg \cos(\alpha)$. Torque from friction between card and horizontal surface is $\tau_F = L\, mg\, \mu \sin(\alpha)$. Requiring the total torque to vanish yields: $\frac{L}{2} mg \cos(\alpha) = L\, mg\, \mu \sin(\alpha)$ and thus $\alpha = \arctan \frac{1}{2\mu}$.

II-2 Harmonic oscillator in thermal equilibrium

The partition function can be written as:

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\hbar\omega} = e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} e^{-\beta n\hbar\omega} \\ &= e^{-\beta\hbar\omega/2} \frac{1}{1-e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}} = \frac{1}{2\sinh(\beta\hbar\omega/2)} \end{aligned}$$

The average energy can now be easily calculated:

$$\langle E \rangle = -\frac{\partial \log Z}{\partial \beta} = \frac{\hbar\omega}{2} \coth(\beta\hbar\omega/2)$$

For $\beta\hbar\omega \ll 1$, which is the high-temperature limit, $\coth(\beta\hbar\omega/2) \approx 2/(\beta\hbar\omega)$. The average energy takes the form $\langle E\rangle \approx kT$.

For $\beta\hbar\omega \gg 1$, which is the very-low-temperature limit, $\coth(\beta\hbar\omega/2) \approx 1$. The average energy takes the form $\langle E\rangle \approx \hbar\omega/2$, which is precisely the zero-point energy of the oscillator.

II-3 Sudden Expansion

Initially the particle has energy $E_1 = (\pi\hbar)^2/(2ma^2)$ and a wave function $\phi_1 = \sqrt{2/a}\, sin(\pi x/a)$. The second excited state of the expanded box has $E_3' = (\pi\hbar)^2/(2ma^2)$ and the wave function is $\Psi_3 = \sqrt{2/3a}\, sin(\pi x/a)$.

The probability that we want is:

$$|\langle \Psi_3 | \phi_1 \rangle|^2 = \frac{4}{3a^2} \left| \int_0^a \sin\frac{\pi x}{a} \sin\frac{\pi x}{a} dx \right|^2 = \frac{4}{3a^2}\frac{a^2}{4} = \frac{1}{3}$$

II-4 Interference and Diffraction from N slits

(a) The number of slits N on which the beam is incident.

There are four minima between the principal maximas. Therefore, N = 5 (Note that there will be N – 1 minima for an N slit interference pattern). (Full credit can be given for this statement)

One can also see this by thinking about adding N electric field vectors in the complex phase diagram, minima occurs when these vectors form a regular polygon, this happens when the angle between the vectors equal to $n\frac{2\pi}{N}$, where n is an integer that runs from 1 to N-1, these corresponds to the minima observed between the peaks. For n=0 and n=N, all the vectors are aligned and this corresponds to the principal maxima. Therefore, there will be N-1 minima between the peaks.

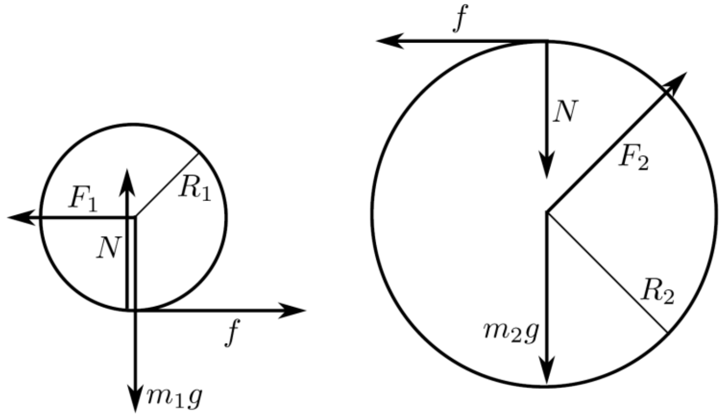
(b) The ratio d/D.

First, recall that interference peaks appear at $\sin\theta = \frac{n\lambda}{d}$, while diffraction minima appear at $\sin\theta = \frac{m\lambda}{D}$ where n and m are integers. From the interference pattern above, we can see 0th, 1st, 2nd, and 3rd peaks, but cannot see the 4th peak. This means that 4th peak is cancelled due to the diffraction minima. So we notice that the 1st diffraction minima and 4th interference peak are at the same position. Therefore, $\frac{4\lambda}{d} = \frac{\lambda}{D}$ and therefore $\frac{d}{D} = 4$.

GROUP III – Solutions

III-1 Two cylinders

The full force diagram for this problem is given below. Note that all the forces, except friction exert no torque. The force F_2 must be present in order to prevent m_2 from rolling to the left or falling down (F_2 counters the friction force f on m_2, as well as the normal force N from m_1 and its own gravity m_2g.) The force F_1 counteracts the friction force on m_1 and prevents m_1 from moving right. You may wonder why F_1 has no vertical component. This is because the statement of the problem sets $N = m_1g$, and no other vertical forces are present. Thus, the axle of m_1 can only provide horizontal forces, and can hold no weight in the vertical direction.



If we define θ_1 and θ_2 in the same directions as ω_1 and ω_2, respectively, the torque equations are:

$$fR_1 = -\frac{1}{2}m_1R_1^2\ddot{\theta}_1 \qquad fR_2 = \frac{1}{2}m_2R_2^2\ddot{\theta}_2 \qquad f = \mu m_1 g$$

The rolling without slipping condition asserts that the same linear distance (arc length) must be traversed by the two cylinders in the same amount of time: $s = v_1t = v_2t$, where $v_1 = R_1\omega_1$ and $v_2 = R_2\omega_2$. This simply yields:

$$R_1\omega_1 = R_2\omega_2$$

Since each cylinder is undergoing a constant angular acceleration/deceleration, we can write the angular frequencies as:

$$\omega_1(t) = \Omega - \frac{2\mu g}{R_1}t \qquad \omega_2(t) = \frac{2\mu g}{R_2}\frac{m_1}{m_2}t$$

Using the rolling without slipping condition, we find:

$$R_1\left(\Omega - \frac{2\mu g}{R_1}t_r\right) = R_2\left(\frac{2\mu g}{R_2}\frac{m_1}{m_2}t_r\right) \qquad t_r = \frac{m_2\Omega R_1}{2g\mu(m_1+m_2)}$$

III-2 Isothermal compression

$$\left(P+\frac{a}{V^2}\right)(V-b)=Nk_BT$$

$$\Rightarrow P=\frac{Nk_BT}{V-b}-\frac{a}{V^2}$$

For isothermal compression, T is constant:

$$đW=-PdV$$

$$\begin{aligned} W &= -\int_{V_1}^{V_2}\left(\frac{Nk_BT}{V-b}-\frac{a}{V^2}\right)dV \\ &= \left[-Nk_BT\ln(V-b)-\frac{a}{V}\right]_{V_1}^{V_2} \\ &= Nk_BT\ln\left(\frac{V_1-b}{V_2-b}\right)-a\left(\frac{1}{V_2}-\frac{1}{V_1}\right) \end{aligned}$$

III-3 1D Harmonic Oscillator

Start from $\Delta x\,\Delta p\ \geq \frac{\hbar}{2}$. In the ground state, <x> = <p> = 0 and $(\Delta x^2) < \langle x^2\rangle$ (same for p).

Thus $\langle x^2\rangle\langle p^2\rangle \geq \frac{\hbar^2}{4}$.

Now, $E=\langle H\rangle=\frac{1}{2m}\langle p^2\rangle+\frac{1}{2}m\omega^2\langle x^2\rangle \geq \frac{1}{2m}\langle p^2\rangle+m\omega^2\frac{\hbar^2}{4}/\langle p^2\rangle$

Finding the minimum vs $\langle p^2\rangle$ gives

0 = $\frac{1}{2m}-m\omega^2\frac{\hbar^2}{8}/\langle p^2\rangle^2$

Thus, $\langle p^2\rangle_{\min}=\frac{1}{2}m\omega\hbar,$ and $E\geq\frac{1}{4}\omega\hbar+\frac{1}{4}\omega\hbar=\frac{1}{2}\omega\hbar$

III-4 Magnetic Monopole:

Since

$$\nabla\cdot\mathbf{B}=4\pi\,\rho_m$$

the field must be given by

$$\mathbf{B}=\frac{\Gamma}{r^2}\,\hat{\mathbf{r}}$$

For a particle of mass *m*, charge *q* and velocity **v** the equation of motion must be:

$$m\frac{d\mathbf{v}}{dt}=q\frac{\mathbf{v}}{c}\times\mathbf{B}=\frac{q\Gamma}{cr^2}\mathbf{v}\times\hat{\mathbf{r}}.$$

Depending on the signs of q, Γ and the direction of **v**, the particle will experience a force $\pm(q\Gamma|\mathbf{v}|/cr^2)$ that is perpendicular to the plane of **r** and **v**. For $|\mathbf{v}|=0$, the particle is at rest. If we increase $|\mathbf{v}|$ from zero, to first order the particle will move in a circle in a plane perpendicular to the local **B**, with a radius given by $R=|\mathbf{v}|/\omega_{\text{cyclotron}}$, where $\omega_{\text{cyclotron}}=qB/mc=q\Gamma/r^2mc$. This approximation to the motion will be more or less correct as long as $R \ll r$, which is equivalent to $|\mathbf{v}| \ll q\Gamma/rmc$. This inequality defines a "low speed."

GROUP IV - Solutions

IV-1 Gyroscope

Let Ω be the precession frequency and T be the tension in the string. Then

$$T\cos\beta = Mg$$
$$T\sin\beta = M\Omega^2(l + L\sin\beta)$$
$$\Omega I_0\omega_s = Mgl.$$

This gives

$$lT\cos\beta = \Omega I_0\omega_s.$$

For small β a number of things become simpler:

$$T = Mg, \qquad \beta = \frac{\Omega^2 l}{g}, \qquad \Omega = \frac{Mgl}{I_0\omega_s}.$$

Then the result is

$$\beta = \frac{M^2 l^3 g}{I_0^2\omega_s^2}.$$

IV-2 Hot Atoms

The probability an atom will be in a state of energy $E = h\frac{c}{\lambda}$ is proportional to the degeneracy of the state, i.e. 2l + 1, and a Boltzmann factor. Thus,

$$\mathcal{R} = 3e^{-hc/\lambda k_B T}$$

which leads to

$$\boxed{T = \frac{hc}{\lambda k_B \ln(3/\mathcal{R})}}$$

IV-3 Harmonic Oscillator in three dimensions

We have $n=n_1+n_2+n_3$, with $n_i = 0,1,2, \ldots$.
For a given n choose a particular n_1. Then $n_2+n_3=n-n_1$.
There are $n-n_1+1$ possible pairs $\{n_2,n_3\}$: n_2 can take on the values 0 to $n-n_1$, and for each n_2 the value of n_3 is fixed.
The degree of degeneracy is therefore:

$$g_n = \sum_{n_1=0}^{n}(n-n_1+1) = \sum_{n_1=0}^{n}(n+1) - \sum_{n_1=0}^{n} n_1 = (n+1)(n+1) - \frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

IV-4 Relativistic capacitor

In the lab frame:

$$E_z = 4\pi\sigma \qquad E_x = E_y = 0$$
$$B_x = B_y = B_z = 0$$

In moving observer's frame:

$$\begin{aligned} E_x' &= E_x \\ E_y' &= \gamma (E_y - \beta B_z) \\ E_z' &= \gamma (E_z + \beta B_y) \\ B_x' &= B_x \\ B_y' &= \gamma (B_y + \beta E_z) \\ B_z' &= \gamma (B_z - \beta E_y) \end{aligned} \quad \Rightarrow \quad \begin{aligned} E_x' &= E_y' = 0 \\ E_z' &= \gamma E_z = 4\pi\gamma\sigma \\ B_x' &= B_z' = 0 \\ B_y' &= \gamma \beta E_z = 4\pi\gamma\beta\sigma \end{aligned}$$

Notice that $E_z' = \gamma E_z = 4\pi\gamma\sigma$ is basically because the moving observer sees measured charge density due to Lorentz contraction.

GROUP V – Solutions

V- 1 The Sopwith Camel

Suppose the engine rotates CW. Then the angular momentum vector **L** of the engine points straight ahead and a right turn will cause a change $\delta\mathbf{L}$ that points to the right. The airframe must then exert a torque τ on the engine that points to the right. By Newton's 3rd law, the engine exerts a torque on the airframe that points to the left; by the right hand rule, this will force the plane into a dive. As this is consistent with the behavior of the plane, the engine rotates CW.

Similar tendencies exist for any rotating engine or propeller, even a modern jet engine. It was a problem with the Camel because the rotating engine had a large moment of inertia and its mass was a significant fraction of the mass of the plane. The plane had another drawback in that it lacked a throttle; the only way to control the engine speed was to temporarily cut the ignition!

V-2 C_P/C_V for Gases

(a) One expects (the question does not ask to derive this, so it is enough to know it)

$$C_p = C_V + P\left(\frac{\partial V}{\partial T}\right)_P .$$

For an ideal gas of N molecules one expects $PV = NkT$, thus

$$C_p = C_V + Nk$$

A monatomic ideal gas (no internal degrees of freedom) has $\frac{1}{2}kT$ of energy for each of the 3 translational degrees of freedom, or $C_V = \frac{3}{2}Nk$ giving $\gamma = \frac{5}{3} = 1.67$.

For the diatomic molecules there are 2 rotational degrees of freedom and two vibrational degrees of freedom with each of the 4 additional degrees of freedom contributing $\frac{1}{2}k$ to C_V . One might expect that $\gamma = \frac{9}{7} = 1.29$. However, at 100 ^{0}C the vibrational degrees are not excited; That would give $\gamma = \frac{7}{5} = 1.4$. Thus the numbers in the table are consistent with expectations.

(b) As the temperature is raised, γ will not change for the monatomic molecules. However, it should fall to 1.29 for the diatomic molecules as the vibrational modes become excited. (The question does not ask this, but not until $T \gtrsim 1500$ K does this happen for H_2 and O_2.)

V-3 Uncertainty of S_x

$$\chi = A\begin{pmatrix} 2 \\ 1-i \end{pmatrix}$$

Normalize χ:

$$\chi^\dagger\chi = 1 = |A|^2(2 \quad 1+i)\begin{pmatrix} 2 \\ 1-i \end{pmatrix} = |A|^2(4+1+1) = 6|A|^2$$

$$\Rightarrow A = \frac{1}{\sqrt{6}}$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$< S_x >= \chi^\dagger S_x \chi = \frac{1}{6}\frac{\hbar}{2}(2 \quad 1+i)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1-i \end{pmatrix} = \frac{\hbar}{3}$$

$$< S_x^2 >= \chi^\dagger S_x^2 \chi = \frac{1}{6}\frac{\hbar^2}{4}(2 \quad 1+i)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1-i \end{pmatrix} = \frac{\hbar^2}{4}$$

$$\sigma_{S_x} = \sqrt{< S_x^2 > - < S_x >^2} = \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{9}} = \frac{\sqrt{5}}{6}\hbar$$

V-4 Scaling a circuit

Assume the capacitor, no matter how constructed, scales like a parallel capacitor (C ≡ Area/separation). Thus, $C' = \gamma C$. The inductance of the coil will be proportional to $N\Phi/i$, where Φ is the magnetic flux through one turn of the coil carrying a current i. For a long solenoid of length l and radius r, the magnetic field is uniform inside the solenoid (and zero outside) and equals $\mu_0 iN/l$. Thus, $\Phi = \mu_0 iN\pi r^2/l$ and the inductance is $L = \mu_0\pi N^2 r^2/l$, which also scales as γ.

As LC will scale like γ^2 , the resonant frequency will scale like $1/\gamma$.

Alternatively, we know that Maxwell's equations are scale invariant. So, if one takes Maxwell's equations describing a particular system, and scales all the dimensions (spatial and tempo- ral) by γ, the set of solutions of the resulting system are just the solutions for the original system, but with all the dimensions (spatial and temporal) scaled by γ. Of course, if the temporal parts are scaled by γ, that means that the frequencies of all solutions are scaled by $1/\gamma$.