Color-Flavor Locking and Chiral Symmetry Breaking in High Density QCD

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Abstract

We propose a symmetry breaking scheme for QCD with three massless quarks at high baryon density wherein the color and flavor $SU(3)_{\rm color} \times SU(3)_L \times SU(3)_R$ symmetries are broken down to the diagonal subgroup $SU(3)_{\rm color+L+R}$ by the formation of a condensate of quark Cooper pairs. We discuss general properties that follow from this hypothesis, including the existence of gaps for quark and gluon excitations, the existence of Nambu-Goldstone bosons which are excitations of the diquark condensate, and the existence of a modified electromagnetic gauge interaction which is unbroken and which assigns integral charge to the elementary excitations. We present mean-field results for a Hamiltonian in which the interaction between quarks is modelled by that induced by single-gluon exchange. We find gaps of order hundreds of MeV for plausible values of the coupling. We discuss the effects of nonzero temperature, nonzero quark masses and instanton-induced interactions on our results.

1 Introduction

The behavior of matter at high quark density is interesting in itself and is relevant to phenomena in the early universe, in neutron stars, and in heavy-ion collisions. Unfortunately, the presence of a chemical potential makes lattice calculations impractical, so our understanding of high density quark matter is still rudimentary. Even qualitative questions, such as the symmetry of the ground state, are unsettled.

In an earlier paper [1] we considered an idealization of QCD, supposing there to exist just two species of massless quarks. We argued that at high density there was a tendency toward spontaneous breaking of color (color superconductivity). Specifically, we analyzed a model Hamiltonian with the correct symmetry structure, abstracted from the instanton vertex, and found an instability toward the formation of Cooper pairs of quarks leading to a sizable condensate of the form

$$\langle q_i^\alpha C \gamma^5 q_j^\beta \rangle \;\propto\; \epsilon_{ij}\epsilon^{\alpha\beta 3} \;, \tag{1.1}$$

where the Latin indices signify flavors and the Greek indices signify colors. The choice of "3" for the preferred direction in color space is, of course, conventional. In the presence of the condensate (1.1) the color symmetry is broken down to $SU(2)$, while chiral $SU(2)_L \times SU(2)_R$ is left unbroken. Other authors have reached similar conclusions [2, 3], and color breaking condensates have also been studied in a single-flavor model [4]. There is an obvious asymmetry in the *ansatz* (1.1) between the two "active" colors that participate in the condensation, and the third, passive one.[1] Some such asymmetry is inevitable in a world with two light quarks due to the mismatch between the number of colors and the number of flavors.

If the strange quark were heavy relative to fundamental QCD scales, the idealization involved in assuming two massless flavors would be entirely innocuous. In reality, this is not the case as we now explain. We are interested in densities above the transition at which the ordinary chiral condensate $\langle \bar{q}q \rangle$ vanishes (or, in the case of $\langle \bar{s}s \rangle$, is greatly reduced.) This means that the strange quark mass is approximately equal to its Lagrangian mass m_s, which is of order 100 MeV. Color superconductivity involves quarks near the Fermi surface. As we are interested in chemical potentials which are large compared to m_s, strange quarks cannot be neglected. Furthermore, one can expect condensates at the Fermi surface to be essentially unaffected by the presence of quark masses that are small compared to the chemical potential, and this has been demonstrated explicitly for the condensate (1.1) in the two flavor model [5]. As we are interested in chemical

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[1]We did find a very much weaker tendency toward condensation of the quarks of the third color in an isoscalar axial vector channel [1]. The third color quarks, therefore, need not be completely passive but this phenomenon makes the asymmetry between their behavior and that of the first two colors even more pronounced.

potentials which are large compared to m_s, taking $m_s = 0$ is a reasonable starting point for the physics of interest to us here.

Thus on both formal and physical grounds it is of considerable interest to consider an alternative idealization of QCD, supposing there to exist three species of massless quarks. We shall argue that upon making this idealization, there is a symmetry breaking pattern, generalizing (1.1), which has several interesting and attractive features including the participation of quarks of all flavors and colors and the breaking of both color and flavor symmetries, and present model calculations which indicate that this pattern is likely to be energetically favorable.

2   Proposed Ordering

The symmetry of QCD with three massless quarks is $\mathrm{SU}(3)_{\mathrm{color}} \times \mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{U}(1)_{\mathrm{B}}$. The SU(3) of color is a local gauge symmetry, while the chiral flavor SU(3) symmetries are global, and the final factor is baryon number. Our proposal is that at high density this symmetry breaks to the diagonal SU(3) subgroup of the first three factors – a purely global symmetry.

A condensate invariant under the diagonal SU(3) is

$$\langle q_{Lia}^{\alpha} \, q_{Ljb}^{\beta} \, \epsilon^{ab} \rangle = -\langle q_{Ri}^{\alpha \dot{a}} \, q_{Rj}^{\beta \dot{b}} \, \epsilon_{\dot{a}\dot{b}} \rangle = \kappa_1 \delta_i^{\alpha} \delta_j^{\beta} + \kappa_2 \delta_j^{\alpha} \delta_i^{\beta} \; , \tag{2.1}$$

where we have written the condensate using two component Weyl spinors. The explicit Dirac indices make the L's and R's superfluous here, but we will often drop the indices. The mixed Kronecker δ matrices are invariant under matched vectorial color/flavor rotations, leaving only the diagonal SU(3) unbroken. That is, the LL condensate "locks" $\mathrm{SU}(3)_{\mathrm{L}}$ rotations to color rotations, while the RR condensate locks $\mathrm{SU}(3)_{\mathrm{R}}$ rotations to color rotations. As a result, the only remaining symmetry is the global symmetry $\mathrm{SU}(3)_{\mathrm{color}+\mathrm{L}+\mathrm{R}}$ under which one makes simultaneous rotations in color and in vectorial flavor. In particular, the gauged color symmetries and the global axial flavor symmetries are spontaneously broken.

The condensates in (2.1) can be written using Dirac spinors as $\langle q_i^{\alpha} C \gamma^5 q_j^{\beta} \rangle$, and therefore constitute a Lorentz scalar. Note that it might be possible for the LL and RR condensates in (2.1) to differ other than by a sign. This would violate parity. As discussed further below, we have seen no sign of this phenomenon in the model Hamiltonians which we consider. Note also that when $\kappa_1 = -\kappa_2$, the condensate can be rewritten as $\langle q_i^{\alpha} C \gamma^5 q_j^{\beta} \rangle \propto \epsilon_{ijI} \epsilon^{\alpha\beta I}$, which is a natural generalization of the two-flavor ordering (1.1). This would mean that the Cooper pair wave functions would be antisymmetric under exchange of either color or flavor. We find, however, that solutions of the coupled gap equations for κ_1 and κ_2 do not have solutions with $\kappa_1 = -\kappa_2$. The Cooper pairs in the condensate (2.1) are symmetric under

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simultaneous exchange of color and flavor, but are not antisymmetric under either color or flavor exchange.

The ordering (2.1) is not the only possibility. One could first of all try a state which is antisymmetric under simultaneous exchange of color and flavor, but which has spin one. Based on the results of Ref. [1], this is very unlikely to compete with (2.1), because not all momenta at the Fermi surface can participate equally. A much more plausible possibility is that, when one allows m_s to differ from the up and down quark masses, one will have a less symmetric condensate in which the strange quarks are distinguished from the up and down quarks. Although we have argued above that the strange quark mass in nature is light enough that $m_s = 0$ is a reasonable starting point for estimating the magnitude of the superconductor gap, it is certainly the case that once m_s is set to its physical value, the symmetry of the condensate will be reduced.[2]

To forestall a possible difficulty, let us remark that the formal violation of flavor and baryon number by our condensate does not imply the possibility of large genuine violation of flavor or baryon number in the context of a heavy ion collision or a neutron star, any more than ordinary superconductivity (with Cooper pairs) involves violation of lepton number. The point is that the condensates occur only in a finite volume, and the conservation equation can be integrated over a surface completely surrounding and avoiding this volume, allowing one to track the conserved quantities. There is, however, the possibility of easy transport of quantum numbers into or out of the affected volume, and thus of large dynamical fluctuations in the normally conserved quantities in response to small perturbations. These are the typical manifestations of superfluidity.

In the following sections we shall present quantitative calculations, based on an interaction Hamiltonian modelling single-gluon exchange, which indicate that symmetry breaking of this form, with substantial condensates, becomes energetically favorable at any density which is high enough that the ordinary $\bar{q}q$ condensate vanishes. Before describing these calculations in detail, however, we would like to make a number of general qualitative remarks.

Motivation: As already discussed, our proposal here for three flavors is a natural generalization of what we and others have already found to be favorable for two flavors. Since it retains a high degree of symmetry – the residual $SU(3)$, as well as

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[2] As an exercise, we have added a four-fermion interaction involving up and down quarks only, modelled on the 't Hooft vertex in the two flavor theory. This mocks up (some of) the effects of the true six-fermion 't Hooft vertex at nonzero m_s, as long as the coupling constant is taken to be proportional to m_s. We find five independent gap parameters instead of two. That is, the condensates are less symmetric, as expected. As the coupling of the instanton-like four-fermion interaction is increased, the condensate changes smoothly from (2.1) to (1.1). A full treatment using the six-fermion interaction remains to be done, but we expect this crossover to occur for strange quark masses of order μ.

space-time rotation symmetry – different flavor sectors and different parts of the Fermi surfaces all make coherent contributions, thus taking maximal advantage of the attractive channel. The proposed ordering bears a strong resemblance to the B phase of superfluid ^{3}He, which is the ordered state of liquid ^{3}He favored at low temperatures. In that phase, atoms form Cooper pairs such that vectors associated with orbital and nuclear spin degrees of freedom are correlated, breaking an SO(3)×SO(3) symmetry to the diagonal subgroup.

Broken chiral symmetry: Chiral symmetry is spontaneously broken by a new mechanism: locking of the flavor rotations to color.

Gap: The condensation produces a gap for quarks of all three colors and all three flavors, for all points on the Fermi surface. Thus there are no residual low energy single-particle excitations. The color superconductivity is "complete". This has immediate consequences in neutron stars. The rate of neutrino emission from matter in this phase is exponentially suppressed for temperatures less than of order the gap. In the two flavor theory, this rate was only reduced by a factor 2/3, unless the third color quarks were also able to condense. Note also that because quarks of all colors are able to participate in the condensate, one expects larger gaps in the three-flavor theory than in the two-flavor theory.

Higgs phenomenon and Nambu-Goldstone bosons: The symmetry has been reduced by 17 generators. Of these, eight were local generators. The corresponding quanta – seven gluons and one linear combination of the eighth gluon and the photon of electromagnetism – all acquire mass according to the Higgs phenomenon. Eight other generators correspond to the broken chiral symmetry. They generate massless collective excitations, the Nambu-Goldstone bosons. The broken symmetry generators are given by the axial charges, just as in the standard discussion of chiral symmetry breaking in QCD at zero density. Goldstone's theorem entails the existence of massless bosons with the quantum numbers of the broken symmetry generators, again as at zero density. They must be states of negative parity and zero spin. Finally, there is an additional scalar Nambu-Goldstone particle associated with spontaneous breakdown of baryon number symmetry. Of course the Nambu-Goldstone bosons associated with chiral symmetry breaking form an octet under the unbroken SU(3), while the Nambu-Goldstone boson associated with baryon number violation is a singlet.

The octet Nambu-Goldstone bosons are created by acting with an axial charge operator on a diquark condensate which spontaneously breaks U(1)$_\mathrm{B}$. This means that they do not have well-defined baryon number. This is evident once one realizes that a propagating $\bar{q}\gamma_5 q$ oscillation can become a propagating $\bar{q}C\bar{q}$ (or qCq) oscillation by an interaction with the $\langle \bar{q}C\gamma_5\bar{q}\rangle$ (or $\langle qC\gamma_5 q\rangle$) condensate in which a Nambu-Goldstone boson associated with the breaking of U(1)$_\mathrm{B}$ is excited. In a lattice simulation, the octet Nambu Goldstone modes could be created by

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inserting qCq, $\bar{q}C\bar{q}$ or $\bar{q}\gamma_5 q$ operators. In lattice simulations at finite density using the quenched approximation [6], the appearance of states which are massless in the chiral limit but which carry baryon number has now been understood [7] and demonstrates that the theory being simulated is not in fact the $N_F \to 0$ limit of QCD. However, the observation of baryonic pions seems to persist even in lattice simulations with four flavors of dynamical quarks [8]. A possible explanation of these lattice results is that a phase analogous to the one we propose is present.

One might be concerned with the use of diquark operators, which are not gauge invariant, as interpolating fields for the Nambu-Goldstone bosons. We do not think this is a serious difficulty; but in any case, we could alternatively use operators of the form $\bar{q}\gamma_5 q$ and $(qqq)^2$ for the chiral symmetry octet and baryon number singlet, respectively.

Residual $\mathbb{Z}_2 \times \mathbb{Z}_2$, axial baryon number, and parity:

Our model Hamiltonian is symmetric under $U(1)_V \times U(1)_A$. Our condensate breaks this down to a $\mathbb{Z}_2^L \times \mathbb{Z}_2^R$ that changes the sign of the left/right- handed quark fields. In real three-flavor QCD, $U(1)_A$ is anomalous, and is broken to $\mathbb{Z}_6$ by instantons. The condensate would then break $U(1)_V \times \mathbb{Z}_6$ down to the common $\mathbb{Z}_2$ which flips the sign of all quark fields.

Because our Hamiltonian does not explicitly violate $U(1)_A$, we cannot use it to infer the relative phase of the LL and RR condensates. In our previous work on the two-flavor case we used a model Hamiltonian abstracted from the instanton, that did violate the anomalous symmetries. There we found that the parity-conserving choice of relative phases was favored. This interaction will still be present, of course, and we expect that the parity-conserving phase will still be favored dynamically in real QCD. Hence, we choose the minus sign in (2.1) which makes the condensate a Lorentz scalar. Upon making this choice, the Nambu-Goldstone bosons associated with axial flavor symmetry breaking will be pseudoscalar, and the Nambu-Goldstone boson associated with baryon number violation will be scalar. In the 3-flavor theory, the 't Hooft vertex has six fermion legs. This means that it is irrelevant, in the sense that its coefficient is reduced as modes are integrated out and only those closer and closer to the Fermi surface are kept. Thus in practice its effects might become quite small at high density. In that case there will be an additional light pseudoscalar, associated with axial baryon number, which is a singlet under the unbroken $SU(3)$ and whose mass does not vanish in the chiral limit.

Effect of quark masses: Quark mass terms are present in the real world. They lift the masses of the pseudoscalar octet of Nambu-Goldstone bosons. One can consider this effect along the lines set out by Gell-Mann, Oakes and Renner for conventional chiral symmetry breaking. Explicit breaking of chiral symmetry occurs through inclusion of symmetry breaking operators in the effective

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Lagrangian. The addition of an $m_q \bar{q}q$ operator to the Lagrangian yields $m_{NG}^2 \propto m_q$ if $\langle \bar{q}q \rangle$ is nonzero. If $\mathbb{Z}_2^L$ were a valid symmetry, this contribution would vanish. Then one must go to higher order and consider the operator proportional to $m_q^2(\bar{q}q)^2$. The expectation value of this operator is nonzero, as it can be written as a product of $\langle qC\gamma_5 q \rangle$ and $\langle \bar{q}C\gamma_5\bar{q} \rangle$ condensates, and so we obtain a contribution to $m_{NG}^2 \propto m_q^2$, which is likely small (but see below).

The scalar Nambu-Goldstone boson associated with baryon number violation of course remains strictly massless, since baryon number symmetry is not violated by quark mass terms.

Effects of instanton-induced interactions: We have already seen that these interactions, although small, are needed to fix the relative sign of the condensates with differing helicities and to give a (small) mass to the η'-like boson. In addition, by breaking $\mathbb{Z}_2^L$, they induce a further qualitative modification. The 't Hooft vertex can connect an incident $\bar{q}_L$ and q_R to the product of the $\langle \bar{q}_R\bar{q}_R \rangle$ and $\langle q_L q_L \rangle$ condensates. This means that in the presence of both the condensate (2.1) and the instanton-induced interaction, one can have an ordinary $\langle \bar{q}_R q_L \rangle$ condensate. This breaks no new symmetries: chiral symmetry is already broken. As noted above, the 't Hooft vertex is irrelevant at the Fermi surface. The gap equation for the ordinary chiral condensate will not have a log divergence, and so the induced $\langle \bar{q}_R q_L \rangle$ condensate can be either nonzero or zero. If it is nonzero, one must analyze this condensate coupled with the coexisting diquark condensates, for example using the methods of Ref. [5]. We leave this analysis to future work, but note here that this effect introduces a contribution to $m_{NG}^2 \propto m_q$.

Modified electromagnetism: Though the standard electromagnetic symmetry is violated by our condensate, as are all the color gauge symmetries, there is a combination of electromagnetic and color symmetry that is preserved. This is possible because the electromagnetic interaction is traceless and vectorial in flavor $SU(3)$. Consider the gauged $U(1)$ which is the sum of electromagnetism, under which the charges of the quarks are (2/3,-1/3,-1/3) depending on their flavor, and the color hypercharge gauge symmetry under which the charges of the quarks are (-2/3,1/3,1/3) depending on their color. It is a simple matter to check that the condensates (2.1) are invariant under this rotation. There is therefore a massless Abelian gauge boson, corresponding to a modified photon. In this superconductor, therefore, although seven gluons and one linear combination of gluon and photon get a mass and the corresponding non-Abelian fields display the Meissner effect, there is a massless modified photon and a corresponding magnetic field which can penetrate the matter.

Under this modified electromagnetism, the quark charges are compounded from the (2/3, -1/3, -1/3) of their flavor and (-2/3, 1/3, 1/3) of their color and thus are all integral, as are the charges of the Nambu-Goldstone bosons. Specifically,

four of the Nambu-Goldstone bosons have charges ± 1, and the rest are neutral. These are just the charges carried by the ordinary octet mesons under ordinary electromagnetism. Four of the massive vector bosons arising from the color gluons via the Higgs mechanism have charges ± 1, and the rest are neutral. The true Nambu-Goldstone boson associated with spontaneous baryon number violation is neutral.

It is amusing that with quark masses turned on, all hadronic excitations (single fermion excitations, massive gauge bosons, and pseudo-Nambu-Goldstone bosons) with charges under the modified electromagnetism acquire a gap. Therefore, at zero temperature a propagating modified photon with energy less than the lightest charged mode cannot scatter, and the ultradense material is transparent. Presumably some relatively small density of electrons will be needed to ensure overall charge neutrality (since the strange quark is a bit heavier than the others), and this metallic fraction will dominate the low-energy electromagnetic response.

Thermal properties: For temperatures less than the mass of the pseudo-Nambu-Goldstone bosons associated with chiral symmetry breaking, and less than the gap, the specific heat is dominated by the exact Nambu-Goldstone boson of broken baryon number. Charged excitations are exponentially suppressed. This is quite unlike the physics expected at high densities in the absence of a gap, where one expects massless gluons and quasi-particle excitations with arbitrarily low energies. We find neither. At temperatures above the pseudo-Nambu-Goldstone mass, but below the gap, the electromagnetic response will be dominated by the pseudo-Nambu-Goldstone bosons. As four of them are charged under the modified U(1), they can emit and scatter the modified photons. At these temperatures, the matter is no longer transparent.

Although in this paper we only present calculations at $T = 0$, our proposal clearly raises interesting issues for the phase diagram, which can be explored as has been done in the two-flavor theory[5]. Here we will only make a few simple remarks. In the theory with three massless quarks, there will be a phase transition at some μ-dependent temperature, above which the diquark condensate vanishes and chiral symmetry and U(1)$_B$ are restored. Even with nonzero quark masses, this transition involves the restoration of the global U(1)$_B$ symmetry, and cannot be an analytic crossover. The existence of a broken U(1) in the high density phase also ensures that it cannot be connected to the zero density phase without singularity, despite the fact that both phases exhibit broken chiral symmetry. Presumably there is in fact a first-order transition at low temperatures as a function of increasing chemical potential, with a bag-model interpretation, similar to the one we found for two flavors[1]. However now the phase interior to the bag has *less* symmetry than the phase outside! To join them, one will need to quantize the collective coordinates of the broken symmetry generator for baryon number.

We turn now to our model calculation of the two gaps κ_1 and κ_2 appearing in (2.1).

3 Model Hamiltonian

In our previous study of two-flavor diquark condensation, we used the instanton vertex for our NJL model. In the three flavor case, instantons are not the dominant source of attractive interactions in the diquark channel, since the instanton vertex now has 6 legs, and cannot be saturated by diquark condensates.

However single-gluon exchange does provide an attraction between quarks, and so we use an NJL model containing a four-fermion interaction with the color, flavor, and spinor structure of single-gluon exchange:

$$
\begin{aligned}
H &= \int d^3x\, \bar{\psi}(x)(\slashed{\nabla} - \mu\gamma_0)\psi(x) + H_I, \\
H_I &= K \sum_{\mu,A} \int d^3x\, \mathfrak{F}\, \bar{\psi}(x)\gamma_\mu T^A \psi(x)\, \bar{\psi}(x)\gamma^\mu T^A \psi(x) ,
\end{aligned}
\tag{3.1}
$$

where T^A are the color $SU(3)$ generators. In real QCD the interactions become weak at high momentum, so we have included a schematic form factor $\mathfrak{F}$. When we expand H_I in momentum modes we will make this factor explicit, via a form factor $F(p)$ on each leg of the interaction vertex. We will explore both power-law and smoothed-step profiles for F:

$$
F(p) = \left(\frac{\Lambda^2}{p^2 + \Lambda^2}\right)^\nu, \qquad \text{or} \quad F(p) = \left(1 + \exp\left[\frac{p-\Lambda}{w}\right]\right)^{-1} ; \tag{3.2}
$$

As noted above, a plausible scenario is that at high densities a diquark condensate will form, breaking the color and flavor symmetries down to their diagonal $SU(3)$ subgroup. In Section 4 we solve the mean-field gap equations for such a condensate, in order to estimate the size of the gap parameters as a function of the interaction strength K. We derive the gap equations via the Bogoliubov-Valatin approach, which is equivalent to the variational method used in Ref. [1] but is perhaps simpler. In order to get some idea of the correct size of K, we perform a calculation (Appendix A) of the interaction H_I to the zero density chiral gap. We present our results in Section 5.

4 Color-Flavor Gap equations

Single gluon exchange cannot convert left-handed massless particles to right-handed, so we can rewrite the Hamiltonian in terms of Weyl spinors, following

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only the left-handed particles from now on. The computation for the right-handed particles would be identical. (There are also terms in H_I which couple left- and right-handed quarks, but these do not contribute to the gap equations for the LL and RR condensates of (2.1).) Explicitly displaying color $(\alpha, \beta \ldots)$, flavor $(i, j \ldots)$, and spinor $(a, \dot{a} \ldots)$ indices, and rewriting the color and flavor generators,

$$H_I = \frac{8}{3}K \int d^3x\, \mathfrak{F}\, (3\delta^\alpha_\delta \delta^\gamma_\beta - \delta^\alpha_\beta \delta^\gamma_\delta)(\psi^{\dagger i}_{\alpha \dot{a}} \psi^{\dagger j}_{\gamma \dot{c}} \epsilon^{\dot{a}\dot{c}} \psi^\beta_{ib} \psi^\delta_{jd} \epsilon^{bd}), \tag{4.1}$$

where all fields are left-handed. We give our spinor conventions in Appendix B. Now make the mean-field ansatz (2.1) for $|\psi\rangle$, the true ground state at a given chemical potential:

$$\begin{aligned}
\langle\psi|\psi^{\dagger i}_{\alpha \dot{a}} \psi^{\dagger j}_{\gamma \dot{c}} \epsilon^{\dot{a}\dot{c}}|\psi\rangle &= \frac{3}{16K} P^{ij}_{\alpha\gamma}, \\
P^{ij}_{\alpha\gamma} &= \tfrac{1}{3}(\Delta_8 + \tfrac{1}{8}\Delta_1)\delta^i_\alpha \delta^j_\gamma + \tfrac{1}{8}\Delta_1 \delta^i_\gamma \delta^j_\alpha ,
\end{aligned} \tag{4.2}$$

where the numerical factors have been chosen so that Δ_1 and Δ_8, which parameterize P and which are linear combinations of κ_1 and κ_2, will turn out to be the two gaps (up to a form factor — see the end of this section). This condensate is invariant under the diagonal $SU(3)$ that simultaneously rotates color and flavor. It therefore "locks" color and left-flavor rotations together as described in Section 2. The interaction Hamiltonian becomes

$$\begin{aligned}
H_I &= \tfrac{1}{2} \int d^3x\, \mathfrak{F}\, Q^{ij}_{\beta\delta}\, \psi^\beta_{ib} \psi^\delta_{jd} \epsilon^{bd} + \text{c.c.}, \\
Q^{ij}_{\beta\delta} &= \Delta_8 \delta^i_\delta \delta^j_\beta + \tfrac{1}{3}(\Delta_1 - \Delta_8)\delta^i_\beta \delta^j_\delta
\end{aligned} \tag{4.3}$$

Replacing indices i, β with a single color-flavor index ρ, we can simultaneously diagonalize the 9×9 matrices Q and P, and find that they have two eigenvalues,

$$\begin{aligned}
P_1 &= \Delta_8 + \tfrac{1}{4}\Delta_1, & P_2 \cdots P_9 &= \pm \tfrac{1}{8}\Delta_1 \\
Q_1 &= \Delta_1, & Q_2 \cdots Q_9 &= \pm \Delta_8
\end{aligned} \tag{4.4}$$

That is, eight of the nine quarks in the theory have a gap parameter given by Δ_8, while the remaining linear combination of the quarks has a gap parameter Δ_1. The Hamiltonian can be rewritten in this color-flavor basis in terms of particle/hole creation/annihilation operators $\bar{a}_\rho, \bar{b}_\rho$. We also expand in momentum modes using (B.4) and now explicitly include the form factors $F(p)$ described in Section 3.

$$\begin{aligned}
H &= \sum_{\rho, k>\mu} (k-\mu)\bar{a}^\dagger_\rho(\mathbf{k})\bar{a}_\rho(\mathbf{k}) + \sum_{\rho, k<\mu} (\mu - k)\bar{a}^\dagger_\rho(\mathbf{k})\bar{a}_\rho(\mathbf{k}) + \sum_{\rho, \mathbf{k}} (k+\mu)\bar{b}^\dagger_\rho(\mathbf{k})\bar{b}_\rho(\mathbf{k}) \\
&\quad + \tfrac{1}{2} \sum_{\rho, \mathbf{p}} F(p)^2 Q_\rho \mathrm{e}^{-i\phi(\mathbf{p})} \left(\bar{a}_\rho(\mathbf{p})\bar{a}_\rho(-\mathbf{p}) + \bar{b}^\dagger_\rho(\mathbf{p})\bar{b}^\dagger_\rho(-\mathbf{p}) \right) + \text{c.c.},
\end{aligned} \tag{4.5}$$

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where the perturbative ground state, annihilated by $\bar{a}_\rho$ and $\bar{b}_\rho$ is the Fermi sea, with states up to $p_F = \mu$ occupied. Finally, we change basis to creation/annihilation operators y and z for quasiparticles and quasiholes,

$$
\begin{aligned}
y_\rho(\mathbf{k}) &= \cos(\theta_\rho^y(\mathbf{k}))\bar{a}_\rho(\mathbf{k}) + \sin(\theta_\rho^y(\mathbf{k}))\exp(i\xi_\rho^y(\mathbf{k}))\bar{a}_\rho^\dagger(-\mathbf{k}) \\
z_\rho(\mathbf{k}) &= \cos(\theta_\rho^z(\mathbf{k}))\bar{b}_\rho(\mathbf{k}) + \sin(\theta_\rho^z(\mathbf{k}))\exp(i\xi_\rho^z(\mathbf{k}))\bar{b}_\rho^\dagger(-\mathbf{k})
\end{aligned}
\tag{4.6}
$$

where

$$
\begin{aligned}
\cos(2\theta_\rho^y(\mathbf{k})) &= \frac{|k - \mu|}{\sqrt{(k-\mu)^2 + F(k)^4 Q_\rho^2}}, & \xi_\rho^y(\mathbf{k}) &= \phi(\mathbf{k}) + \pi \\
\cos(2\theta_\rho^z(\mathbf{k})) &= \frac{k + \mu}{\sqrt{(k+\mu)^2 + F(k)^4 Q_\rho^2}}, & \xi_\rho^z(\mathbf{k}) &= -\phi(\mathbf{k}).
\end{aligned}
\tag{4.7}
$$

These values are chosen so that H has the form of a free Hamiltonian for quasiparticles and quasiholes:

$$
H = \sum_{\mathbf{k},\rho}\Bigg\{ \sqrt{(k-\mu)^2 + F(k)^4 Q_\rho^2}\; y_\rho^\dagger(\mathbf{k})y_\rho(\mathbf{k}) \\
+ \sqrt{(k+\mu)^2 + F(k)^4 Q_\rho^2}\; z_\rho^\dagger(\mathbf{k})z_\rho(\mathbf{k}) \Bigg\}
\tag{4.8}
$$

The true ground state $|\psi\rangle$ contains no quasiparticles:

$$
y_\rho(\mathbf{k})|\psi\rangle = z_\rho(\mathbf{k})|\psi\rangle = 0.
\tag{4.9}
$$

The gap equations follow from requiring that the mean field ansatz (4.2) hold in the quasiparticle basis. In other words, we use (4.6) and (4.7) to rewrite (4.2) in terms of quasiparticle creation/annihilation operators, and then evaluate the expectation value using (4.9). We get two gap equations,

$$
\begin{aligned}
\Delta_8 + \tfrac{1}{4}\Delta_1 &= \frac{16}{3}KG(\Delta_1) \\
\tfrac{1}{8}\Delta_1 &= \frac{16}{3}KG(\Delta_8)
\end{aligned}
\tag{4.10}
$$

where

$$
G(\Delta) = -\frac{1}{2}\sum_{\mathbf{k}}\left\{ \frac{F(k)^4\Delta}{\sqrt{(k-\mu)^2 + F(k)^4\Delta^2}} + \frac{F(k)^4\Delta}{\sqrt{(k+\mu)^2 + F(k)^4\Delta^2}} \right\}
\tag{4.11}
$$

From (4.8) we see that the physical gap, namely the minimum energy of the quasiparticles, is $F(\mu)^2\Delta$. Creating a quasiparticle-quasihole pair requires at least twice this energy. The Nambu-Goldstone excitations discussed at length in Section 2 are long space-dependent oscillations of the phase and the flavor quantum numbers of the condensate. They do not involve the excitation of quasiparticle-quasihole pairs.

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5 Results

We write our "one-gluon" coupling K in terms of a dimensionless strong coupling constant α:

$$K = \frac{4\pi\alpha}{\Lambda^2} \qquad (5.1)$$

If we think of this vertex as coming from integrating out the gluons, then we expect α to be roughly of order 1. As we will see below, the gap is insensitive to details of the form factor (3.2). Specifically, if we vary the scale Λ and the shape of the form factor, at the same time tuning the coupling to keep some physical quantity fixed, then the color gap varies very little. However, the gap is sensitive to the coupling α, and therefore to the choice of physical criterion used to fix α.

We use the zero-density chiral symmetry breaking to fix the coupling. If our single-gluon-exchange four-fermion vertex (3.1) is the only interaction in the theory, then the chiral gap, which can play the role of a constituent quark mass, is determined by the gap equation

$$\frac{32K}{3} \sum_{\mathbf{k}} \frac{F(k)^4}{\sqrt{k^2 + F(k)^4 \Delta_\chi^2}} = 1 \qquad (5.2)$$

derived in Appendix A. Since we are only keeping the part of the Hamiltonian that models single-gluon exchange, we likely overestimate its strength if we require that at zero density it give a phenomenologically reasonable value for Δ_χ, say 0.4 GeV. In reality, other terms in the Hamiltonian are at least partially responsible for the zero density chiral gap. For example, although instanton effects are small at high density, they certainly contribute to Δ_χ at zero density. We have therefore explored a range of couplings α.

In Figure 1 we show how the color-flavor gaps vary with chemical potential μ at fixed coupling: they are suppressed at low μ by the smallness of the Fermi surface, and at large μ by the fall-off of the form factor which implements the weakening of the interaction that occurs in an asymptotically free theory. In comparing our results to those in the two-flavor theory, note that in Ref. [1] we plotted the gap parameter Δ, rather than the smaller, but physical, gap $F^2(\mu)\Delta$. In the two-flavor theory with its single gap parameter, the critical temperature T_c^Δ above which the gap vanishes is quite close to its BCS value of $0.57F^2(\mu)\Delta(T = 0)$ [5]. Here, it remains to be seen how T_c^Δ, the transition at which chiral and color symmetries are restored, is related to the two zero temperature gaps. As in the two-flavor theory, we expect that at temperatures less than T_c^Δ, the chiral condensate is replaced by the superconducting condensate above a first order phase transition which occurs at some μ_0 which should be around 0.3 GeV for a reasonable phenomenology [1]. However we have not included sufficient interactions to treat the low density phase.
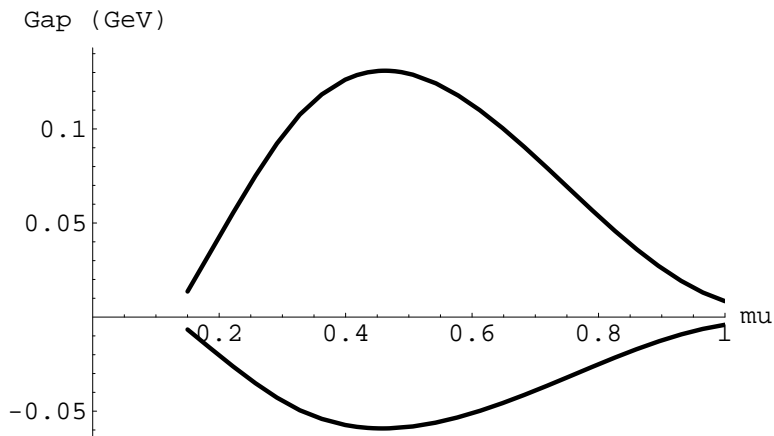
Figure 1: Physical color-flavor gaps as a function of chemical potential, for a power-law form factor with scale $\Lambda = 0.8$ GeV and sharpness $\nu = 1$. The upper curve is $\Delta_1 F^2(\mu)$ while the lower curve is $\Delta_8 F^2(\mu)$. The coupling was chosen to be $\alpha = 0.290$, which is half of the coupling at which our Hamiltonian would produce a zero density chiral gap of 0.4 GeV.

We therefore defer an estimate of the μ_0 above which the curves in Figure 1 are relevant. Suffice to say that the chemical potentials which are of physical interest are also those at which the gaps are not much less than their maximum values.

Requiring α to be half that at which our Hamiltonian would produce a zero density chiral gap of 0.4 GeV, as done in Figure 1, is an ad hoc criterion. In Figure 2 we therefore explore the sensitivity of color-flavor condensation to the coupling, by plotting the maximum of the gap. The strongest possible coupling is the one at which single-gluon-exchange alone gives the observed chiral gap of about 0.4 GeV. Were α this large, the color-flavor gap in the high density phase would be enormous — larger even than the zero density chiral gap. We see that even if we make the single-gluon-exchange interaction much weaker than that, the color-flavor condensate still forms. With α reduced by a factor of two as in Figure 1, the maximum of the gap is still large, around 130 MeV. The gaps we find are significantly larger than those found in the two-flavor theory with the 't Hooft vertex being the only interaction [1, 3, 5]. Single-gluon exchange is of course also attractive in the two-flavor diquark channel (1.1), which suggests that adding it to the two flavor theory could enlarge the gap there too.

In Table 1 we vary the scale Λ and the profile of the form factor, in each case fixing the coupling α to be half that at which the zero density chiral gap is 0.4 GeV. We see that once we use a physical criterion to fix α, the maximum gap
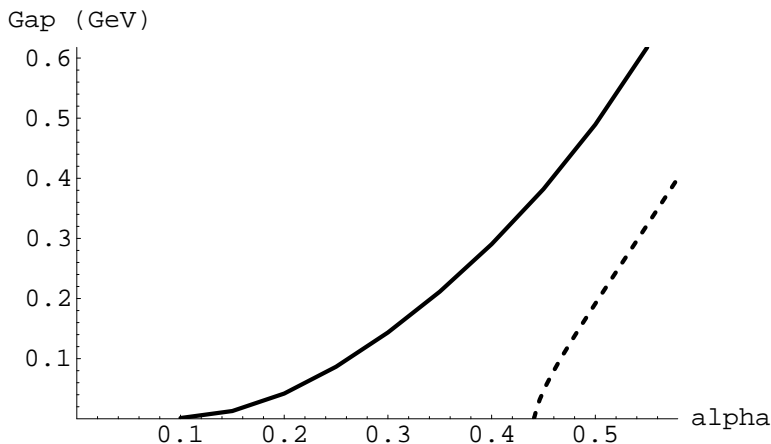
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Figure 2: Color-flavor gap $\Delta_1 F(\mu)^2$ at the μ at which it is largest (solid line) and zero-density chiral gap (dashed line) as a function of coupling α, for a power-law form factor with scale $\Lambda = 0.8$ and sharpness $\nu = 1$.

as a function of chemical potential depends only weakly on details of the form factor. Clearly, the most sensitive dependence is that shown in Figure 2, namely the dependence on the physical criterion by which α is fixed. Although the effects of varying the shape of the form factor are small in comparison, it is worth noticing that gaps twice those shown in Figures 1 and 2 can easily arise for form factors which are closer to step functions.

6   Concluding Remarks

In Section 2 we have explained the consequences of the symmetry breaking pattern which we propose for high density QCD. All quarks have a gap. All gluons get a mass. There is an octet of pseudoscalar approximate Nambu-Goldstone bosons reflecting the fact that chiral symmetry is broken because color and flavor rotations are locked, and a singlet truly massless scalar associated with the breaking of baryon number. There is a modified but still massless photon; with respect to this photon, one finds only integrally charged excitations.

Much remains to be done before a truly quantitative estimate of the magnitude of the gaps is possible, but they can easily be a few hundred MeV. It will be very interesting to add the six-fermion 't Hooft interaction, to add a nonzero strange quark mass, to treat simultaneous condensation in $\langle q C \gamma_5 q \rangle$ and $\langle \bar{q} q \rangle$ channels and to go beyond mean field theory. In order to study the long-wavelength physics of this phase, we need to combine a non-Abelian generalization of the Ginzburg-

Λ	ν	α	$\mu_{\max}$	Max gap
0.8	0.75	0.182	0.581	0.128
0.8	1.	0.290	0.458	0.131
0.8	1.25	0.406	0.395	0.132
0.5	0.75	0.204	0.365	0.111
0.5	1.	0.334	0.283	0.118
0.5	1.25	0.477	0.233	0.122
Λ	width w	α	$\mu_{\max}$	Max gap
0.8	0.05	0.126	0.589	0.284
0.8	0.1	0.162	0.523	0.248
0.8	0.2	0.245	0.477	0.192
0.8	0.4	0.390	0.543	0.154
0.5	0.05	0.182	0.332	0.240
0.5	0.1	0.255	0.291	0.193
0.5	0.2	0.397	0.313	0.143
0.5	0.4	0.500	0.430	0.125

Table 1: Color-flavor gap $(F(\mu)^2 \Delta_1)$, maximized with respect to μ, for a variety of form factors (see (3.2)). Top half: power-law; bottom half: smoothed step. In each case the coupling is fixed to be half of that at which the chiral gap would be 0.4 GeV at zero density. All numbers (except ν) are in GeV.

Landau approach familiar from superconductivity with a suitably modified chiral perturbation theory. This could yield insights into neutron star physics beyond the elementary observations that direct neutrino emission is suppressed, the hadronic material is transparent, and the thermal transport is dominated by a neutral superfluid component. A quantitative treatment relevant to heavy ion collisions would also require the introduction of different chemical potentials for each of the three flavors. The system will then tend to evolve toward equal chemical potentials, as in so doing it can maximize the color-flavor gap and lower its energy.

The present qualitative considerations are already striking. The high density environment may be quite different than previously expected — no massless gluons and no light quarks; physics dominated by an octet of light pseudo-Nambu-Goldstone bosons and a massless singlet. In other words, QCD at high densities may in many ways be much more similar to QCD at low densities than to a weakly-coupled quark-gluon plasma. If this is the case, it may only be as heavy ion collisions reach higher energies and create hotter and less dense conditions that they will be able to access an approximately chirally symmetric phase with light

quark and gluon degrees of freedom.

Appendix A   Chiral Gap equations at zero density

As in Section 4, use the Weyl-spinor form of the hamiltonian (4.1). Now make the mean-field ansatz that, for the true ground state $|\psi\rangle$,

$$\int d^3x \,\langle \psi | \psi_{L\alpha\dot{a}}^{\dagger\,i}(x) \psi_{Rj}^{\dagger\,\delta\dot{a}}(x) | \psi \rangle = \frac{3}{32K} \Delta_\chi \delta_j^i \delta_\alpha^\delta \;, \tag{A.1}$$

where Δ_χ will turn out to be the ordinary chiral gap which can play the role of a constituent quark mass. The interaction Hamiltonian becomes

$$H_I = \Delta_\chi \int d^3x \,\mathfrak{F}\, \psi_{R\beta}^{\dagger\,ib} \psi_{Lib}^{\beta} + \text{c.c.} \tag{A.2}$$

Using (B.4), the Hamiltonian can be rewritten

$$\begin{aligned}
H &= \sum_{\mathbf{k}} k \left(a_{Li\beta}^{\dagger}(\mathbf{k}) a_{Li\beta}(\mathbf{k}) + b_{Li\beta}^{\dagger}(\mathbf{k}) b_{Li\beta}(\mathbf{k}) \right) + L \to R \\
&\quad - \Delta_\chi \sum_{\mathbf{k}} F(k)^2 \,\exp(-i\phi(\mathbf{k})) \left(a_{R\beta}^{\dagger\,i}(\mathbf{k}) b_{Li}^{\dagger\,\beta}(-\mathbf{k}) + b_{R\beta}^{i}(\mathbf{k}) \alpha_i^{\beta}(-\mathbf{k}) \right) + \text{c.c.},
\end{aligned} \tag{A.3}$$

where the perturbative vacuum is annihilated by a_L, b_L, a_R, b_R. Finally, we change basis to creation/annihilation operators y and z for quasiparticles and quasiholes,

$$\begin{aligned}
y_L(\mathbf{k}) &= \cos(\theta_L(\mathbf{k})) a_L(\mathbf{k}) - \sin(\theta_L(\mathbf{k})) \exp(i\xi_L(k)) b_R^{\dagger}(-\mathbf{k}) &&\text{and } L \to R \\
z_L(\mathbf{k}) &= \sin(\theta_L(\mathbf{k})) \exp(i\xi_L(k)) a_L^{\dagger}(\mathbf{k}) + \cos(\theta_L(\mathbf{k})) b_R(-\mathbf{k}) &&\text{and } L \to R
\end{aligned} \tag{A.4}$$

where

$$\cos(2\theta_L(\mathbf{k})) = \cos(2\theta_R(\mathbf{k})) = \frac{k}{\sqrt{k^2 + F(k)^4 \Delta_\chi^2}} \tag{A.5}$$

$$\xi_L(\mathbf{k}) = \phi(\mathbf{k}), \qquad \xi_R(\mathbf{k}) = -\phi(\mathbf{k}) + \pi$$

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These values are chosen so that H has the form of a free Hamiltonian for quasiparticles and quasiholes of mass Δ_χ

$$H = \sum_{\mathbf{k},i,\beta} \sqrt{k^2 + F(k)^4 \Delta_\chi^2} \left(y_{Li\beta}^\dagger(\mathbf{k}) y_{Li\beta}(\mathbf{k}) + z_{Li\beta}^\dagger(\mathbf{k}) z_{Li\beta}(\mathbf{k}) \right) + L \to R \qquad \text{(A.6)}$$

The true vacuum $|\psi\rangle$ contains no quasiparticles:

$$y_L(\mathbf{k})|\psi\rangle = z_L(\mathbf{k})|\psi\rangle = 0, \qquad \text{and} \quad L \to R \qquad \text{(A.7)}$$

The gap equation

$$\frac{32K}{3} \sum_{\mathbf{k}} \frac{F(k)^4}{\sqrt{k^2 + F(k)^4 \Delta_\chi^2}} = 1 \qquad \text{(A.8)}$$

follows.

Appendix B   Spinor conventions

We use the spinor conventions of Ref. [9]. Using gamma matrices in the chiral representation,

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = -\gamma_i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \qquad \text{(B.1)}$$

quark fields are

$$\psi = \begin{pmatrix} \psi_{La} \\ \psi_R^{\dot{a}} \end{pmatrix}, \qquad \psi^\dagger = \begin{pmatrix} \psi_{L\dot{a}}^\dagger & \psi_R^{\dagger a} \end{pmatrix}. \qquad \text{(B.2)}$$

For the 2-component Weyl spinors ψ_L and ψ_R, indices are raised and lowered by

$$\varepsilon^{\dot{a}\dot{c}} = \varepsilon^{ac} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \text{(B.3)}$$

Expand the left-handed fermion field in modes in unit volume, with color index α and flavor index i:

$$\psi_{Li\alpha}(x) = \sum_{\mathbf{k}} \begin{pmatrix} -\sin(\theta/2)\,\mathrm{e}^{-i\phi} \\ \cos(\theta/2) \end{pmatrix} \left[a_{i\alpha}(\mathbf{k})e^{-ik\cdot x} + b_{i\alpha}^\dagger(\mathbf{k})e^{ik\cdot x} \right] \qquad \text{(B.4)}$$

where $k_0 = |\mathbf{k}|$, and θ, ϕ are the polar and azimuthal angles of $\mathbf{k}$, so under $\mathbf{k} \to -\mathbf{k}$, $\theta \to \pi - \theta$ and $\phi \to \pi + \phi$. Here and throughout, $\sum_{\mathbf{k}} \equiv \int d^3k/(2\pi)^3$.

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