

Classical Mechanics Problem 1: Central Potential Solution

a) Integrals of motion for a central potential $V(r)$:

$$\begin{aligned} \text{Angular Momentum} \quad L &= r v_t = r^2 \dot{\phi} \\ \text{Energy per unit mass} \quad E &= \frac{1}{2} (\dot{r}^2 + v_t^2) + V(r) = \frac{1}{2} \dot{r}^2 + V_{\text{eff}}(r) \end{aligned}$$

where v_t is the tangential velocity and V_{eff} is defined as

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2r^2}$$

If the orbit is circular, the distance of the test body from the origin is invariant: $\dot{r} = 0$, which implies that the body is always at the equilibrium-distance:

$$\frac{dV_{\text{eff}}}{dr} = 0 \quad \Rightarrow \quad \frac{dV}{dr} = \frac{L^2}{r^3} = \frac{v_t^2}{r} = r \dot{\phi}^2$$

then

$$\dot{\phi} = \omega_\phi = \frac{L}{r^2} = \left(\frac{1}{r} \frac{dV}{dr} \right)^{1/2}$$

so for the period we get

$$T_\phi = \frac{2\pi}{\omega_\phi} = 2\pi \left(\frac{1}{r} \frac{dV}{dr} \right)^{-1/2}$$

b) Write the orbit as in the statement of the problem:

$$r(t) = r_0 + \epsilon(t) \quad \text{with} \quad \frac{dV_{\text{eff}}}{dr}(r_0) = 0 \quad \text{and} \quad \epsilon^2 \ll r_0^2.$$

The energy per unit mass is now $E = \frac{1}{2} \dot{\epsilon}^2 + V_{\text{eff}}(r_0 + \epsilon)$, and since ϵ is small we may Taylor-expand the potential as

$$V_{\text{eff}}(r_0 + \epsilon) = V_{\text{eff}}(r_0) + \underbrace{\frac{dV_{\text{eff}}}{dr}(r_0)}_{=0} \epsilon + \frac{1}{2} \frac{d^2 V_{\text{eff}}}{dr^2}(r_0) \epsilon^2 + \mathcal{O}(\epsilon^3)$$

so then

$$E - V_{\text{eff}}(r_0) = \frac{1}{2} \dot{\epsilon}^2 + \frac{1}{2} \frac{d^2 V_{\text{eff}}}{dr^2}(r_0) \epsilon^2 + \mathcal{O}(\epsilon^3) = \text{const.}$$

In the above equation we readily recognize the equation of the simple harmonic oscillator with

$$\omega_r = \left(\frac{d^2 V_{\text{eff}}}{dr^2} \right)_{r=r_0}^{1/2}$$

and its general solution is

$$\epsilon(t) = \frac{\sqrt{E - V_{\text{eff}}(r_0)}}{\omega_r} \cos[\omega_r(t - t_0)]$$

where t_0 is an arbitrary constant.

Now return to writing ω_r in terms of $V(r)$ instead of $V_{\text{eff}}(r)$.

$$\omega_r^2 = \frac{d^2 V_{\text{eff}}}{dr^2} = \frac{d^2 V}{dr^2} + \frac{3L^2}{r^4} = \frac{d^2 V}{dr^2} + 3\omega_\phi^2 = \frac{d^2 V}{dr^2} + \frac{3}{r} \frac{dV}{dr}$$

$$\omega_r = \left(\frac{d^2 V}{dr^2} + \frac{3}{r} \frac{dV}{dr} \right)_{r=r_0}^{1/2} = \left[\frac{1}{r^3} \frac{d}{dr} \left(r^3 \frac{dV}{dr} \right) \right]_{r=r_0}^{1/2}$$

And the radial period is

$$T_r = \frac{2\pi}{\omega_r}$$

- c) Stability is determined by the sign of ω_r^2 . For stability: $\omega_r^2 > 0$, so

$$\frac{1}{r^3} \frac{d}{dr} \left(r^3 \frac{dV}{dr} \right) > 0$$

for the Yukawa-potential

$$V(r) = -\frac{GM}{r} e^{-kr}$$

so the condition is

$$\frac{1}{r^3} \frac{d}{dr} \left(r^3 \frac{dV}{dr} \right) = \frac{GM}{r^3} e^{-kr} [1 + kr - (kr)^2] > 0 \quad \Rightarrow \quad [1 + kr - (kr)^2] > 0$$

$$[1 + kr - (kr)^2] = \left(\frac{\sqrt{5}-1}{2} + kr \right) \left(\frac{\sqrt{5}+1}{2} - kr \right) > 0$$

which is satisfied only if

$$kr < \left(\frac{\sqrt{5}+1}{2} \right)$$

Therefore circular orbits are unstable for

$$kr > \left(\frac{\sqrt{5}+1}{2} \right)$$

■

d) The outermost stable circular orbit is at

$$r_0 = \left(\frac{\sqrt{5} + 1}{2k} \right)$$

its energy per unit mass is

$$E = V(r_0) + \frac{1}{2}(r_0\omega_\phi)^2 = V(r_0) + \frac{1}{2} \left(r \frac{dV}{dr} \right)_{r=r_0}$$

$$\frac{1}{2} \frac{GM}{r_0} e^{-kr_0} (kr_0 - 1) = \frac{GM}{r_0} e^{-kr_0} \left(\frac{\sqrt{5} - 1}{4} \right) > 0$$

If r_0 is decreased only slightly, $E > 0$ still and the orbit is absolutely stable ■

The effective potential for the Yukawa-potential has the form shown in Figure 1.

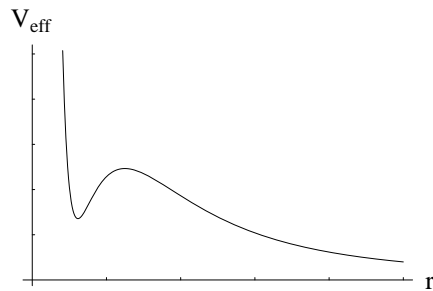
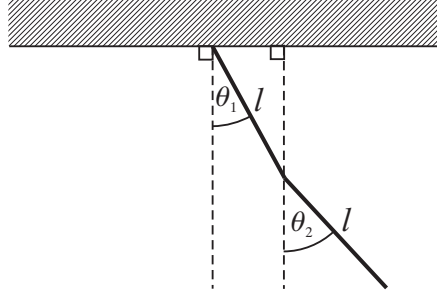


Figure 1: Effective potential against distance from the origin

Classical Mechanics Problem 2: Planar Double Pendulum
Solution



a) $L = T - V$

The moment of inertia for a uniform rod of length l and mass m is

$$I = \frac{1}{3}ml^2 \quad \text{about one of the ends}$$

and

$$I_c = \frac{1}{12}ml^2 \quad \text{about the rod's center}$$

The kinetic energy term we can decompose into three parts:

$$T = T_1 + T_{2,rot} + T_{2,trans}$$

where T_1 is the kinetic energy of the first rod, $T_{2,trans}$ is the translational energy of the center of mass of the second rod and $T_{2,rot}$ is its rotational energy about its center of mass. Then

$$T_1 = \frac{1}{6}ml^2\dot{\theta}_1^2$$

$$T_{2,rot} = \frac{1}{24}ml^2\dot{\theta}_2^2$$

and

$$T_{2,trans} = \frac{1}{2}m(\dot{x}_c^2 + \dot{y}_c^2)$$

where x_c and y_c are the coordinates of the second rod's center of mass, so

$$x_c = l \sin \theta_1 + \frac{l}{2} \sin \theta_2$$

$$y_c = -l \cos \theta_1 - \frac{l}{2} \cos \theta_2$$

from which

$$\dot{x}_c^2 + \dot{y}_c^2 = l^2 \left[\dot{\theta}_1^2 + \frac{1}{4}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2 (\sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2) \right]$$

The potential energies are simply $V_i = mgy_{c,i}$, where $y_{c,i}$ are the vertical coordinates of the rods' centers of mass. Since both rods are uniform, $y_{c,i}$ are simply the coordinates of the centers. Thus,

$$V_1 = -mg\frac{l}{2}\cos\theta_1; \quad V_2 = -mg\left(l\cos\theta_1 + \frac{l}{2}\cos\theta_2\right)$$

The full Lagrangian is then

$$L = T_1 + T_{2,rot} + T_{2,trans} - V_1 - V_2$$

$$= ml^2 \left[\frac{2}{3}\dot{\theta}_1^2 + \frac{1}{6}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) \right] + mgl \left[\frac{3}{2}\cos\theta_1 + \frac{1}{2}\cos\theta_2 \right]$$

- b) Expand the Lagrangian from part a) for small angles. The only function we have to deal with is

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \mathcal{O}(\theta^4)$$

Since we are going to look for normal modes with $\theta_j = \hat{\theta}_j \exp(i\omega t)$, where the $\hat{\theta}_j \ll 1$, we immediately see that in the term $\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$, the θ -dependence in the cosine can be dropped, because even the first θ -dependent term gives a fourth order correction. Then the approximate Lagrangian is

$$L = ml^2 \left[\frac{2}{3}\dot{\theta}_1^2 + \frac{1}{6}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2 \right] - mgl \left[\frac{3}{4}\theta_1^2 + \frac{1}{4}\theta_2^2 \right] + \text{const.}$$

The Euler-Lagrange equations are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_j} = \frac{\partial L}{\partial \theta_j}$$

so in the specific case:

$$\frac{4}{3}\ddot{\theta}_1 + \frac{1}{2}\ddot{\theta}_2 + \frac{g}{l} \left(\frac{3}{2}\theta_1 \right) = 0$$

$$\frac{1}{2}\ddot{\theta}_1 + \frac{1}{3}\ddot{\theta}_2 + \frac{g}{l} \left(\frac{1}{2}\theta_2 \right) = 0$$

if we now look for normal modes, as mentioned, the above set of equations takes the form

$$\begin{bmatrix} \left(\frac{4}{3}\omega^2 - \frac{3g}{2l} \right) & \frac{1}{2}\omega^2 \\ \frac{1}{2}\omega^2 & \left(\frac{1}{3}\omega^2 - \frac{1g}{2l} \right) \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = 0$$

Non-trivial solutions exist if the determinant of the matrix on the left is zero. Denoting $\omega^2 = \lambda g/l$, we can write this condition as

$$\left(\frac{4}{3}\lambda - \frac{3}{2} \right) \left(\frac{1}{3}\lambda - \frac{1}{2} \right) - \frac{\lambda^2}{4} = 0,$$

that is

$$\frac{7}{36}\lambda^2 - \frac{7}{6}\lambda + \frac{3}{4} = 0$$

whose solutions are

$$\lambda_{\pm} = 3 \pm \frac{6}{\sqrt{7}},$$

so finally

$$\omega_{\pm} = \left[\left(3 \pm \frac{6}{\sqrt{7}} \right) \frac{g}{l} \right]^{1/2}$$

c) To sketch the eigenmodes, find eigenvectors of the matrix in part **b**).

- $\omega^2 = \lambda_- g/l$ (low-frequency mode)

$$\hat{\theta}_2 = \left(\frac{3}{\lambda} - \frac{8}{3} \right) \hat{\theta}_1 = \frac{1}{3} (2\sqrt{7} - 1) \hat{\theta}_1$$

$(2\sqrt{7} - 1)/3 > 0$ and real, therefore the two pendula are in phase;

- $\omega^2 = \lambda_+ g/l$ (high-frequency mode)

$$\hat{\theta}_2 = \left(\frac{3}{\lambda} - \frac{8}{3} \right) \hat{\theta}_1 = \frac{1}{3} (-2\sqrt{7} - 1) \hat{\theta}_1$$

$(-2\sqrt{7} - 1)/3 < 0$ and real, therefore the two pendula are perfectly out of phase.

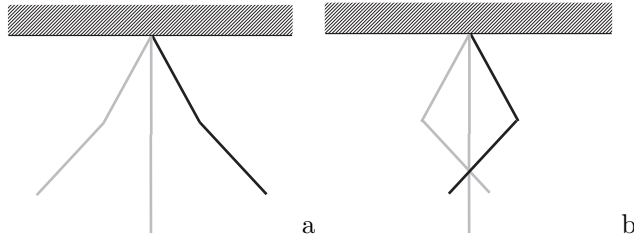


Figure 2: The low- (a) and high-frequency (b) normal modes of the planar double pendulum.

Electromagnetism Problem 1 Solution

- a) Normal modes are products of harmonic standing waves in the x , y and z directions. For their frequencies, we have

$$\omega = c\sqrt{k_x^2 + k_y^2 + k_z^2} = c\left[\left(\frac{\pi n_x}{a}\right)^2 + \left(\frac{\pi n_y}{b}\right)^2 + \left(\frac{\pi n_z}{b}\right)^2\right]^{1/2}; \quad n_x, n_y, n_z \in \mathbb{Z}^+$$

Since $a > b$, the lowest frequency has $n_x = 1$ and either $n_y = 1, n_z = 0$ or $n_y = 0, n_z = 1$ (note that $n_y = 0, n_z = 0$ does not satisfy the boundary condition $E_{\parallel, \text{at wall}} = 0$). Since we are told to pick the mode with $\vec{E} \parallel \hat{y}$, the boundary conditions require

$$\boxed{\vec{E}(r, t) = E_0 \hat{y} \sin \frac{\pi x}{a} \sin \frac{\pi z}{b} \cos \omega t}$$

The magnetic induction we can get from Faraday's Law:

$$\begin{aligned} \frac{\partial \vec{B}}{\partial t} &= -c(\nabla \times \vec{E}) = \hat{x}c \frac{\partial E_y}{\partial z} - \hat{z}c \frac{\partial E_y}{\partial x} \\ &= \left(\hat{x}cE_0 \frac{\pi}{b} \sin \frac{\pi x}{a} \cos \frac{\pi z}{b} - \hat{z}cE_0 \frac{\pi}{a} \cos \frac{\pi x}{a} \sin \frac{\pi z}{b}\right) \cos \omega t \end{aligned}$$

$$\boxed{\vec{B}(r, t) = -\frac{\pi c E_0}{\omega} \left(\frac{\hat{x}}{b} \sin \frac{\pi x}{a} \cos \frac{\pi z}{b} - \frac{\hat{z}}{a} \cos \frac{\pi x}{a} \sin \frac{\pi z}{b}\right) \sin \omega t}$$

where the frequency ω is (by the argument above)

$$\boxed{\omega = \pi c \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{1/2}}$$

- b) At a boundary of media, the discontinuity in the *normal* component of the electric field is 4π times the surface charge density σ , so

$$\begin{aligned} E_y(x, 0, z) &= 4\pi\sigma \\ \sigma(x, 0, z) &= \frac{E_0}{4\pi} \sin \frac{\pi x}{a} \sin \frac{\pi z}{b} \cos \omega t \\ \sigma(x, b, z) &= -\sigma(x, 0, z) \end{aligned}$$

and

$$\sigma(0, y, z) = \sigma(a, y, z) = \sigma(x, y, 0) = \sigma(x, y, b) \equiv 0$$

Similarly, at the boundary of media the discontinuity of the *tangential* component of the magnetic field is given by the surface current $\vec{\kappa}$

$$\hat{n} \times \vec{B} = \frac{4\pi}{c} \vec{\kappa}$$

where \hat{n} is a unit vector normal to the surface, so

$$\begin{aligned}\vec{\kappa}(x, 0, z) &= \frac{c^2 E_0}{4\omega} \left(\frac{\hat{z}}{b} \sin \frac{\pi x}{a} \cos \frac{\pi z}{b} + \frac{\hat{x}}{a} \cos \frac{\pi x}{a} \sin \frac{\pi z}{b} \right) \sin \omega t \\ \vec{\kappa}(x, b, z) &= -\vec{\kappa}(x, 0, z) \\ \vec{\kappa}(0, y, z) &= -\frac{c^2 E_0}{4\omega} \frac{\hat{y}}{a} \sin \frac{\pi z}{b} \sin \omega t \\ \vec{\kappa}(a, y, z) &= -\vec{\kappa}(0, y, z) \\ \vec{\kappa}(x, y, 0) &= -\frac{c^2 E_0}{4\omega} \frac{\hat{y}}{b} \sin \frac{\pi x}{a} \sin \omega t \\ \vec{\kappa}(x, y, b) &= -\vec{\kappa}(x, y, 0)\end{aligned}$$

- c) Since there is no charge on the $b \times b$ sides, the force there is purely magnetic and is given by

$$\begin{aligned}\vec{F}(t) &= \frac{1}{2c} \int_{b \times b} (\vec{\kappa} \times \vec{B}) d^2x \\ \vec{F}(x=0, t) &= -\frac{E_0^2 c^2 \pi}{8\omega^2 a^2} \hat{x} \int_0^b dy \int_0^b dz \sin^2 \frac{\pi z}{b} \sin^2 \omega t \\ &= \boxed{-\hat{x} \left(\frac{c}{4\omega} \frac{b}{a} E_0 \sin \omega t \right)^2} \\ \vec{F}(x=a, t) &= -\vec{F}(x=0, t)\end{aligned}$$

The forces point outwards from the box on both sides (as is indicated by the sign in the equation above).

- d) Start with the sides where $y = \text{const}$. The magnetic component of the force can be written as above

$$\begin{aligned}\vec{F}_{\text{mag}}(y=0, t) &= -\frac{E_0^2 c^2 \pi}{8\omega^2} \hat{y} \int_0^a dx \int_0^b dz \left(\frac{1}{a^2} \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi z}{b} + \frac{1}{b^2} \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi z}{b} \right) \sin^2 \omega t \\ &= -\hat{y} \frac{1}{2} \left(\frac{c}{2\omega} E_0 \sin \omega t \right)^2 \frac{1}{4\pi} \left(\frac{b}{a} + \frac{a}{b} \right)\end{aligned}$$

To simplify this result further, use ω from part a)

$$\omega^{-2} = (\pi c)^{-2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^{-1} = ab(\pi c)^{-2} \left(\frac{b}{a} + \frac{a}{b} \right)^{-1}$$

Then

$$\vec{F}_{\text{mag}}(y=0, t) = -\hat{y} \left(\frac{E_0}{4} \sin \omega t \right)^2 \frac{ab}{2\pi^3}$$

The electric component of the force can be written as

$$\begin{aligned}\vec{F}_{\text{el}}(y=0, t) &= \int \frac{1}{2} \sigma \vec{E} d^2x = \hat{y} \frac{E_0^2}{8\pi} \cos^2 \omega t \int_0^a dx \int_0^b dz \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi z}{b} \\ &= \hat{y} \frac{1}{2} \left(\frac{E_0}{4} \cos \omega t \right)^2 \frac{ab}{\pi^3}\end{aligned}$$

$$\begin{aligned}\vec{F}_{\text{tot}}(y=0, t) &= \vec{F}_{\text{el}}(y=0, t) + \vec{F}_{\text{mag}}(y=0, t) \\ &= \hat{y} \left(\frac{E_0}{4} \right)^2 \frac{ab}{2\pi^3} (\cos^2 \omega t - \sin^2 \omega t) \\ &= \boxed{\hat{y} \left(\frac{E_0}{4} \right)^2 \frac{ab}{2\pi^3} \cos 2\omega t}\end{aligned}$$

and

$$\boxed{\vec{F}_{\text{tot}}(y=b, t) = -\vec{F}_{\text{tot}}(y=0, t)}$$

There net force on the top and bottom sides oscillates between the inward and outward direction with half the period of the lowest frequency mode. In a time average, therefore, this force cancels.

Next, calculate the force on the sides where $z = \text{const}$. Again, there is no charge, therefore no electric component; the force is purely magnetic

$$\begin{aligned}\vec{F}(z=0, t) &= -\frac{E_0^2 c^2 \pi}{8\omega^2 b^2} \hat{z} \int_0^b dy \int_0^a dx \sin^2 \frac{\pi x}{a} \sin^2 \omega t \\ &= \boxed{-\hat{z} \frac{a}{b} \left(\frac{c}{4\omega} E_0 \sin \omega t \right)^2}\end{aligned}$$

$$\vec{F}(z=b, t) = -\vec{F}(z=0, t)$$

The magnetic force is pushing the $a \times b$ walls outwards, too (sign!).

- e) From the Maxwell stress tensor, the force per unit surface area is

$$\vec{f} = \frac{1}{4\pi} \vec{E}(\vec{E} \cdot \hat{n}) - \frac{E^2}{8\pi} \hat{n} + \frac{1}{4\pi} \vec{B}(\vec{B} \cdot \hat{n}) - \frac{B^2}{8\pi} \hat{n}$$

On the $x = \text{const}$. walls $\hat{n} = \pm \hat{x}$, $\vec{E} = 0$ and $\vec{B} \cdot \hat{x} = 0$, so

$$\vec{f}(x = \{0, a\}, t) = \mp \frac{B^2}{8\pi} \hat{x} = \mp \frac{E_0^2 c^2 \pi}{8\omega^2 a^2} \hat{x} \sin^2 \frac{\pi z}{b} \sin^2 \omega t$$

which is exactly the integrand from part c).

On the $y = \text{const.}$ walls $\hat{n} = \pm\hat{y}$, $\vec{E}(\vec{E} \cdot \hat{y}) = E^2\hat{y}$ and $\vec{B} \cdot \hat{y} = 0$, so we get

$$\begin{aligned}\vec{f}(y = \{0, b\}, t) &= \mp \frac{1}{8\pi} (E^2 - B^2)\hat{y} \\ &= \pm \left[\frac{E_0^2}{8\pi} \cos^2\omega t \sin^2\frac{\pi x}{a} \sin^2\frac{\pi z}{b} \right. \\ &\quad \left. - \frac{E_0^2 c^2 \pi}{8\omega^2} \left(\frac{1}{a^2} \cos^2\frac{\pi x}{a} \sin^2\frac{\pi z}{b} + \frac{1}{b^2} \sin^2\frac{\pi x}{a} \cos^2\frac{\pi z}{b} \right) \sin^2\omega t \right] \hat{y}\end{aligned}$$

the sum of the first two integrands from part d).

On the $z = \text{const.}$ walls $\hat{n} = \pm\hat{z}$, $(\vec{E} \cdot \hat{z}) = 0$ and $\vec{B} \cdot \hat{z} = 0$, so we get

$$\vec{f}(z = \{0, b\}, t) = \mp \frac{B^2}{8\pi} \hat{z} = \mp \frac{E_0^2 c^2 \pi}{8\omega^2 b^2} \hat{z} \sin^2\frac{\pi x}{a} \sin^2\omega t$$

the last integrand from part d).

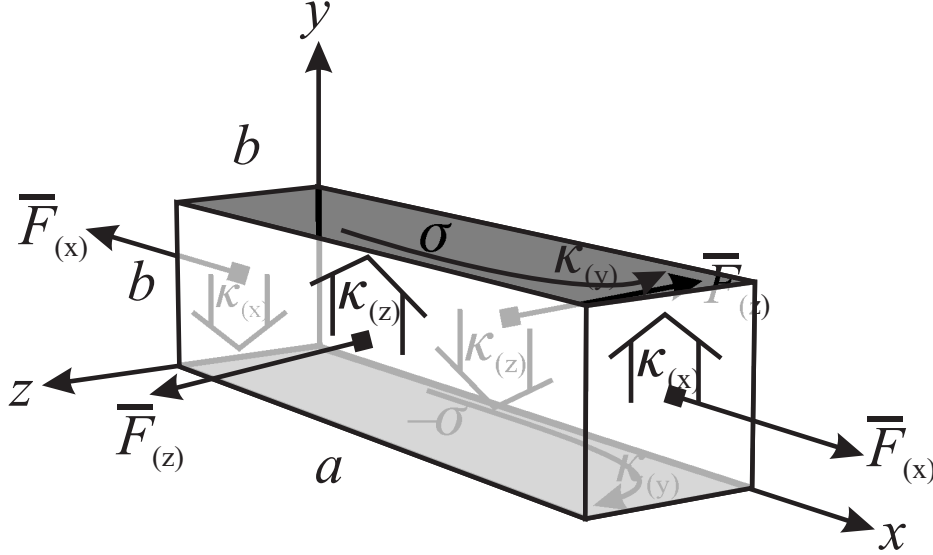


Figure 3: Average total forces, surface charges and surface currents on the cavity.

Electromagnetism Problem 2: Waves in a Dilute Gas Solution

(see Feynman Lectures on Physics, vol. II, chapter 32)

- a) The EM wave is travelling in the $\hat{\mathbf{x}}$ direction; it has a transverse electric field, so assume $\mathbf{E} \times \hat{\mathbf{y}} = 0$. Then the electron in the atom behaves classically as a damped, driven harmonic oscillator

$$m_e (\ddot{y} + \gamma \dot{y} + \omega_0^2 y) = -qE_0 e^{-i\omega t}$$

with the solution

$$y(t) = \frac{1}{\omega^2 - \omega_0^2 + i\gamma\omega} \frac{qE(t)}{m_e}.$$

For the dipole moment per unit volume:

$$P = n_a (-q)y = \frac{1}{\omega^2 - \omega_0^2 - i\gamma\omega} \frac{n_a q^2 E}{m_e}$$

Therefore the volume polarizability is, according to the definition given,

$$\boxed{\alpha(\omega) = \frac{P}{\epsilon_0 E} = \frac{1}{\omega^2 - \omega_0^2 - i\gamma\omega} \frac{n_a q^2}{\epsilon_0 m_e}}$$

(A quantum mechanical derivation would give this same expression multiplied by the oscillator strength f for the transition.)

- b) With no free charges or currents, Maxwell's equations read

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 0; & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0; & \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} \end{aligned}$$

and $\mathbf{B} = \mu_0 \mathbf{H}$, $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0(1 + \alpha)\mathbf{E}$ for a single frequency ω . This gives us the following wave-equation

$$\frac{\partial^2 \mathbf{D}}{\partial t^2} - \frac{1}{\mu_0 \epsilon_0 (1 + \alpha)} \nabla^2 \mathbf{D} = 0.$$

Now let \mathbf{D} be that of a plane wave: $\mathbf{D} \propto e^{i(kx - \omega t)}$. Then

$$\begin{aligned} k^2 &= \mu_0 \epsilon_0 (1 + \alpha) \omega^2 = (1 + \alpha) \frac{\omega^2}{c^2} \\ \Rightarrow &\boxed{n(\omega) = \sqrt{1 + \alpha(\omega)}} \end{aligned}$$

One can also get this result by using the microscopic \mathbf{E} , \mathbf{B} and \mathbf{P} fields:

$$\nabla \cdot \mathbf{E} = -\frac{1}{\epsilon_0} \nabla \cdot \mathbf{P}; \quad c^2 \nabla \times \mathbf{B} = \frac{\partial}{\partial t} \left(\frac{\mathbf{P}}{\epsilon_0} + \mathbf{E} \right)$$

$$\Rightarrow \frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2 \nabla^2 \mathbf{E} = -\frac{1}{\epsilon_0} \frac{\partial \mathbf{P}}{\partial t}$$

also

$$\frac{\partial^2 \mathbf{P}}{\partial t^2} + \gamma \frac{\partial \mathbf{P}}{\partial t} + \omega_0^2 \mathbf{P} = -\frac{n_a q^2}{m_e} \mathbf{E}.$$

Together these give us $k^2 = (1 + \alpha)\omega^2/c^2$ for a plane wave, as before. (Note that we are neglecting dipole-dipole interactions in the dilute gas.)

c) We start by noting that according to Fourier-analysis

$$\begin{aligned} E(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{i(kx - \omega(k)t)} \hat{E}(k) \\ \hat{E}(k) &= \int_{-\infty}^{\infty} dx e^{-ikx} E(x, 0) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx e^{-i(k-k_c)x - x^2/(2\sigma^2)} \\ &= e^{-i(k-k_c)^2\sigma^2/2} \end{aligned}$$

Now Taylor-expand $\omega(k)$ about $k = k_c$:

$$\begin{aligned} \omega(k) &= \omega(k_c) + \left(\frac{d\omega}{dk}\right)_{k_c} (k - k_c) + \mathcal{O}\{(k - k_c)^2\} \\ &\equiv k_c v_{ph} + v_g (k - k_c) + \mathcal{O}\{(k - k_c)^2\} \end{aligned}$$

where, by definition, v_{ph} and v_g are the phase- and group-velocities, respectively. Now let $K = k - k_c$. Then

$$E(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dK e^{ik_c(x - v_{ph}t) + iK(x - v_g t) - K^2\sigma^2/2}$$

$$\boxed{E(x, t) = e^{ik_c(x - v_{ph}t)} \frac{e^{(x - v_g t)^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}} = e^{ik_c(x - v_{ph}t)} N(x - v_g t, \sigma)}$$

d) From part c)

$$v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1} = \frac{c}{n} \left(1 + \frac{d \log n}{d \log \omega}\right)^{-1}.$$

For the dilute gas, $n = \sqrt{1 + \alpha} \approx 1 + \alpha/2$, which we will write as $n = n_r + in_i$ (for α is complex)

$$\begin{aligned} n_r &\approx 1 + \frac{n_a q^2}{2\epsilon_0 m_e} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \\ n_i &\approx \frac{n_a q^2}{2\epsilon_0 m_e} \frac{\gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \end{aligned}$$

Here the real part n_r of the index of refraction determines the dispersion, and the imaginary part n_i determines the absorption/gain coefficient. At $\omega = \omega_0$:

$$n_r = 1 \quad \text{and} \quad \frac{d \log n_r}{d \log \omega} = -\frac{n_a q^2}{2\epsilon_0 m_e \gamma^2}$$

$$v_g = \left(1 - \frac{n_a q^2}{2\epsilon_0 m_e \gamma^2}\right)^{-1} c \approx \left(1 + \frac{n_a q^2}{2\epsilon_0 m_e \gamma^2}\right) c$$

Note that $v_g > c$ at $\omega = \omega_0$. This is called anomalous dispersion. It does *not* violate causality because *signals* (information) cannot travel faster than the minimum of (v_{ph}, v_g) , and now $v_{ph} = c$ (since $n_r = 1$). Also, the waves are damped by the electronic resonance maximally at $\omega = \omega_0$.

Quantum Mechanics Problem 1 Solution

- The ground state will have no nodes, so we can pick the even part of the general solution of the free Schrödinger equation inside the well. Outside the well, square-integrability demands the solutions to vanish at infinity. The wave-function for the ground state is then

$$\begin{array}{ll} |x| < w & |x| > w \\ \psi(x) = \cos kx & \psi(x) = Ae^{-\alpha|x|} \end{array}$$

Both the wave-function and its derivative has to be continuous at the boundaries of the well:

$$\begin{array}{ll} \psi : & \cos kw = Ae^{-\alpha w} \\ \frac{d\psi}{dx} : & -k \sin kw = -\alpha Ae^{-\alpha w} \end{array} \Rightarrow k \tan kw = \alpha$$

Directly from Schrödinger's equation:

$$\begin{array}{ll} |x| < w & |x| > w \\ E = \frac{\hbar^2 k^2}{2m} & E = -\frac{\hbar^2 \alpha^2}{2m} + V_0 \end{array} \Rightarrow \frac{\hbar^2 k^2}{2m} = -\frac{\hbar^2 \alpha^2}{2m} + V_0$$

From which we get the transcendental equation:

$$\boxed{k \tan kw = \left[\frac{2mV_0}{\hbar^2} - k^2 \right]^{1/2}}$$

Let k^* denote the positive root of the equation above, and introduce the following notation:

$$k_c = \frac{\sqrt{2mV_0}}{\hbar} \quad \text{and} \quad k_{max} = \frac{\pi}{2w}.$$

Clearly, the LHS of the equation diverges at k_{max} ; and the RHS describes a circle with radius k_c , as shown in Fig. 4.

For the energy we have

$$\boxed{E = \frac{\hbar^2 k^{*2}}{2m}}$$

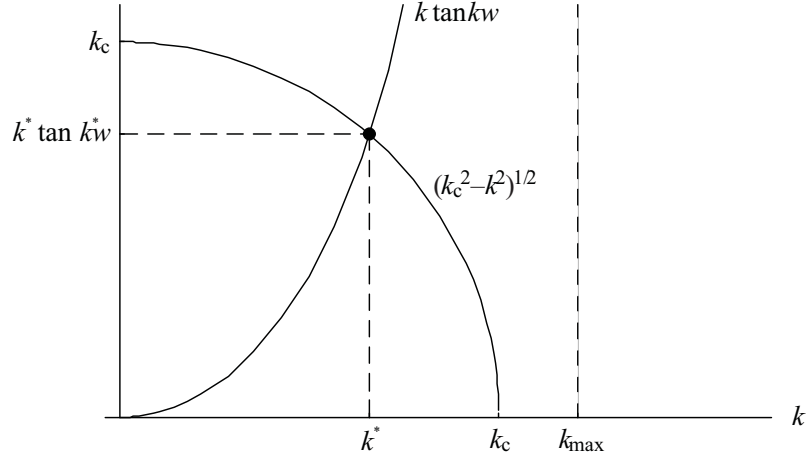


Figure 4: Graphical representation of the solution of the transcendental equation

2. Write the result of part 1 in the non-dimensional form:

$$kw \tan kw = \left[\frac{2mw^2V_0}{\hbar^2} - (kw)^2 \right]^{1/2}$$

According to the condition given in the statement of the problem, the radius of the circle on the RHS (that in Fig. 4) goes to infinity, therefore

$$k \rightarrow k_{\max}$$

and

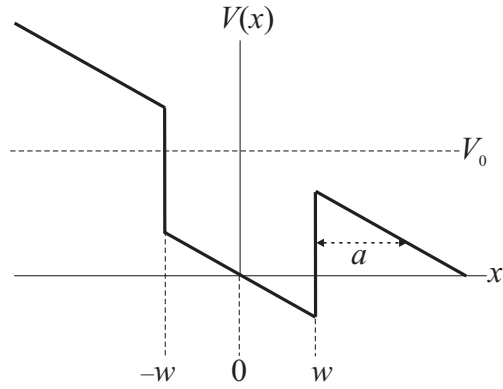
$$E \rightarrow \frac{\hbar^2 k_{\max}^2}{2m} = \frac{\hbar^2 \pi^2}{8mw^2}$$

3. The potential barrier on the low-potential side of the well, denoted a in the figure, will be finite (for any E), so the particle will eventually escape by the tunnel-effect.
4. $\Delta E = 0$, because the perturbation is odd (and therefore its integral with the square of the ground-state wave-function vanishes).
- 5.

$$F = \frac{1}{\hbar} \int_w^a dx \sqrt{2m(V(x) - B)}$$

$$V(x) = V_0 - e\mathcal{E}x$$

$$B = V_0 - e\mathcal{E}a \quad \Rightarrow \quad a = \frac{V_0 - B}{e\mathcal{E}}$$



$$\begin{aligned}
 F &= \frac{\sqrt{2m}}{\hbar} \int_w^a dx \sqrt{(V_0 - B) - e\mathcal{E}x} \\
 &= \frac{\sqrt{2m}}{\hbar} \frac{2}{3} \left(-\frac{1}{e\mathcal{E}} \right) (V_0 - B - e\mathcal{E}x)^{3/2} \Big|_w^a \\
 &= \boxed{\frac{\sqrt{2m}}{\hbar} \frac{2}{3} \frac{1}{e\mathcal{E}} (V_0 - B - e\mathcal{E}w)^{3/2}}
 \end{aligned}$$

6. Write the energy of the particle as

$$B = \frac{1}{2}mv^2.$$

Then

$$v^2 = \frac{2B}{m}.$$

The time it takes for the particle to bounce back and forth once is

$$T = \frac{4w}{v},$$

so it hits the right wall with frequency

$$\nu = \frac{v}{4w}$$

$$\Rightarrow \frac{\text{Probability to escape}}{\text{unit time}} = \frac{v}{4w} e^{-2F}$$

$$\Rightarrow \text{Lifetime} \sim \frac{4w}{v} e^{2F}$$

Quantum Mechanics Problem 2 Solution

1. Drop the t -label for simplicity. Then we have

$$H = B \begin{bmatrix} \cos \theta & \sin \theta e^{-i\omega t} \\ \sin \theta e^{i\omega t} & -\cos \theta \end{bmatrix},$$

and for the eigenvectors solve

$$\begin{bmatrix} \cos \theta & \sin \theta e^{-i\omega t} \\ \sin \theta e^{i\omega t} & -\cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \pm \begin{bmatrix} x \\ y \end{bmatrix}$$

with the normalization condition $|x|^2 + |y|^2 = 1$. From the vector-equation

$$y = e^{i\omega t} \frac{\sin \theta}{\cos \theta \pm 1} x$$

and with the normalization, we end up with

$$|+\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\omega t} \end{bmatrix} \quad |-\rangle = \begin{bmatrix} \sin(\theta/2) \\ -\cos(\theta/2)e^{i\omega t} \end{bmatrix}$$

2. Decompose the state-vector as

$$|\psi\rangle = c_+|+\rangle + c_-|-\rangle$$

and write Schrödinger's equation in terms of these vectors:

$$i\hbar \frac{d}{dt} |\psi\rangle = H|\psi\rangle$$

$$i\hbar \left[\dot{c}_+|+\rangle + c_+ \frac{d}{dt}|+\rangle + \dot{c}_-|-\rangle + c_- \frac{d}{dt}|-\rangle \right] = B [c_+|+\rangle - c_-|-\rangle]$$

or in the $(|+\rangle, |-\rangle)$ basis

$$i\hbar \frac{d}{dt} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} B - i\hbar \langle + | \frac{d}{dt} | + \rangle & -i\hbar \langle + | \frac{d}{dt} | - \rangle \\ -i\hbar \langle - | \frac{d}{dt} | + \rangle & -B - i\hbar \langle - | \frac{d}{dt} | - \rangle \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}$$

which, with the given concrete form of the vectors, is

$$i\hbar \frac{d}{dt} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} B + \hbar\omega \sin^2(\theta/2) & -\hbar\omega \cos(\theta/2) \sin(\theta/2) \\ -\hbar\omega \sin(\theta/2) \cos(\theta/2) & -B + \hbar\omega \cos^2(\theta/2) \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}.$$

Now use the identities

$$\sin^2(\theta/2) = \frac{1}{2}(1 - \cos \theta)$$

$$\cos^2(\theta/2) = \frac{1}{2}(1 + \cos \theta)$$

$$\sin(\theta/2) \cos(\theta/2) = \frac{1}{2} \sin \theta$$

to get

$$i\hbar \frac{d}{dt} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2B - \hbar\omega \cos \theta & -\hbar\omega \sin \theta \\ -\hbar\omega \sin \theta & -2B + \hbar\omega \cos \theta \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}.$$

Note that in the last equation we dropped the part of the Hamiltonian that was proportional to the identity, since that gives only a time dependent phase that is identical for the coefficients c_- , c_+ . This we can rewrite in the form:

$$i\hbar \frac{d}{dt} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} D_z & D_x \\ D_x & -D_z \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}.$$

with the solution

$$\begin{aligned} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} &= \exp \left\{ \frac{-i}{\hbar} \begin{bmatrix} D_z & D_x \\ D_x & -D_z \end{bmatrix} t \right\} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \cos \left(\frac{|\vec{D}|t}{\hbar} \right) - i \hat{D} \cdot \vec{\sigma} \sin \left(\frac{|\vec{D}|t}{\hbar} \right). \end{aligned}$$

And so

$$c_+ = \cos \left(\frac{|\vec{D}|t}{\hbar} \right) - i \frac{D_z}{|\vec{D}|} \sin \left(\frac{|\vec{D}|t}{\hbar} \right)$$

$$\boxed{|c_+|^2 = \cos^2 \left(\frac{|\vec{D}|t}{\hbar} \right) - i \frac{D_z^2}{D^2} \sin^2 \left(\frac{|\vec{D}|t}{\hbar} \right)}$$

3. For $B \gg \hbar\omega$, $D_z \rightarrow D$, so

$$|c_+|^2 \rightarrow 1$$

(Adiabatic theorem)

Statistical Mechanics and Thermodynamics Problem 1
Thermodynamics of a Non-Interacting Bose Gas
Solution

a)

$$n_p = \frac{1}{e^{\beta(E_p - \mu)} - 1} \qquad E_p = \frac{p^2}{2m}$$

At and below T_{BEC} $\mu = 0$. At *exactly* T_{BEC} , there are no atoms in the condensate and

$$N = \frac{V}{2\pi\hbar^3} \int \frac{d^3p}{e^{\beta p^2/(2m)} - 1} = (2\pi\hbar)^{-3} V \left(\frac{2m}{\beta}\right)^{3/2} \underbrace{4\pi \int_0^\infty \frac{x^2 dx}{e^{x^2} - 1}}_{=I_1}$$

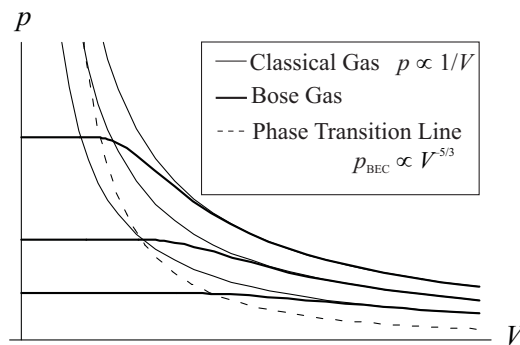
$$n = \frac{1}{2\pi^2} \left(\frac{2mkT}{\hbar^2}\right)^{3/2} I_1$$

$$kT_{\text{BEC}} = \left(\frac{2\pi^2 n}{I_1}\right)^{2/3} \frac{\hbar^2}{2m}$$

(2 points)

b) The above integral with $\mu = 0$ also applies below T_{BEC} , but it then gives the number of *non-condensed* atoms. So on an isotherm below V_{critical}

- $N_{\text{non-condensed}}$ is constant
 - T is constant
- $\Rightarrow p$ is constant
(think of the kinetic origin of pressure)



(2 points)

c)

$$\begin{aligned}
 U &= \frac{V}{2\pi\hbar^3} \int \frac{p^2}{2m} \frac{d^3p}{e^{\beta p^2/(2m)} - 1} = (2\pi\hbar)^{-3} V \left(\frac{2m}{\beta}\right)^{5/2} \frac{4\pi}{2m} \underbrace{\int_0^\infty \frac{x^4 dx}{e^{x^2} - 1}}_{=I_2} \\
 &= N_c \frac{I_2}{I_1} \left(\frac{2m}{\beta}\right) \frac{1}{2m} = N_c \frac{I_2}{I_1} kT \propto T^{5/2} \\
 c_v &= \frac{5}{2} \frac{I_2}{I_1} N_c k = \boxed{\frac{V}{2\pi^2} \left(\frac{2mkT}{\hbar^2}\right)^{3/2} k \left(\frac{5}{2} I_2\right) \propto T^{3/2}}
 \end{aligned}$$

(2 points)

d) From the reversibility of the Carnot-cycle:

$$\begin{aligned}
 dS_1 &= -dS_2 \quad \text{for 1 cycle} \\
 \Delta S_1 &= -\Delta S_2 \quad \text{for the entire process} \\
 dU &= TdS - \underbrace{pdV}_{=0} \Rightarrow T \frac{\partial S}{\partial T} \Big|_V = \frac{\partial U}{\partial T} \Big|_V = c_v = \underbrace{\frac{aT^{3/2}}{\text{from c)}} \\
 \Rightarrow \frac{dS}{dT} &= \frac{c_v}{T}
 \end{aligned}$$

Therefore the entropy transfer in the entire process is

$$\begin{aligned}
 \Delta S_i &= \int_{T_i}^{T_0} \frac{c_v}{T} dT = a \int_{T_i}^{T_0} T^{1/2} dT = \frac{2}{3} a (T_0^{3/2} - T_i^{3/2}) \\
 \Delta S_1 + \Delta S_2 &= 0 \Rightarrow \boxed{T_0^{3/2} = \frac{1}{2} (T_1^{3/2} + T_2^{3/2})}
 \end{aligned}$$

Heat transferred to F_1 :

$$Q_1 = \int_{T_1}^{T_0} T dS = \frac{2}{5} a (T_0^{5/2} - T_1^{5/2}).$$

Heat transferred from F_2 :

$$Q_2 = \int_{T_0}^{T_2} T dS = \frac{2}{5} a (T_2^{5/2} - T_0^{5/2}).$$

Therefore the total work done by the Carnot-machine is

$$\boxed{W = Q_2 - Q_1 = \frac{2}{5} a (T_1^{5/2} + T_2^{5/2} - 2T_0^{5/2})}$$

(4 points)

Statistical Mechanics and Thermodynamics Problem 2
Phase Transition in a Superconductor
Solution

a)

$$c_H \equiv \left. \frac{\partial Q}{\partial T} \right|_H = T \left. \frac{\partial S}{\partial T} \right|_H$$

$$dS = \left. \frac{\partial S}{\partial T} \right|_M dT + \left. \frac{\partial S}{\partial M} \right|_T dM$$

$$\left. \frac{\partial S}{\partial T} \right|_H = \left. \frac{\partial S}{\partial T} \right|_M + \left. \frac{\partial S}{\partial M} \right|_T \underbrace{\left. \frac{\partial M}{\partial T} \right|_H}_{=0}$$

where the last term is zero because M is independent of T . Then

$$c_H = T \left. \frac{\partial S}{\partial T} \right|_M = \left. \frac{\partial Q}{\partial T} \right|_M \equiv c_M$$

(2 points)

b) The transition takes place at constant T and H . The thermodynamic function whose variables are T and H is the Gibbs-potential:

$$dG = -SdT - MdH$$

$G_{\text{super}} = G_{\text{normal}}$ at every point on $H_C(T)$, so $dG_S = dG_N$ which we then write as

$$-S_S dT - M_S dH = -S_N dT - \underbrace{M_N}_{=0} dH$$

$$\left. \frac{dH}{dT} \right|_{\text{trans. line}} = \frac{dH_C}{dT} = \frac{S_N - S_S}{M_S} = \boxed{-\frac{4\pi}{V H_C(T)} (S_N - S_S)}$$

(3 points)

c) By the third law $S \rightarrow 0$ as $T \rightarrow 0$. But the figure shows $H_C(T=0)$ is finite. Therefore

$$\frac{dH_C}{dT} \rightarrow 0 \quad \text{as} \quad T \rightarrow 0.$$

The transition is second order where $S_N - S_S = 0$, that is, the latent heat equals zero.

$$S_N - S_S = -\frac{V}{4\pi} H_C(T) \frac{dH_C}{dT}$$

- At $T = 0$ the transition is second order because both entropies go to zero.
- At $T = T_C(H = 0)$ the transition is second order since $H_C(T) = 0$ and dH_C/dT is finite.
- At all other temperatures the transition is first order since both $H_C(T)$ and dH_C/dT are finite.

(2 points)

d) Use H and T as variables

$$dS(H, T) = \left. \frac{\partial S}{\partial H} \right|_T dH + \left. \frac{\partial S}{\partial T} \right|_H dT$$

$$\left. \frac{\partial S}{\partial H} \right|_T = \frac{c_H}{T} \qquad \left. \frac{\partial S}{\partial T} \right|_H \stackrel{\substack{\uparrow \\ \text{Maxwell} \\ \text{relation}}}{=} - \left. \frac{\partial M}{\partial T} \right|_H = 0$$

$$S = \int \frac{c_H}{T} dT = \frac{a}{3} T^3 V \qquad T < T_C$$

$$= \frac{b}{3} T^3 V + \gamma T V \qquad T > T_C$$

$$S_N - S_S = \left(\frac{b-a}{3} \right) T^3 V + \gamma T V \qquad T = T_C(H = 0)$$

$$\gamma = \left(\frac{b-a}{3} \right) T_C^2$$

$$\boxed{T_C(H = 0) = \left(\frac{3\gamma}{b-a} \right)^{1/2}}$$

(3 points)