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DOCTORAL GENERAL EXAMINATION

Solutions

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PART 1

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Group I

1. **One dimensional bound state**
   The ground state wavefunction is

$$\psi_o = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a} - \frac{\pi}{2}\right).$$

The first order correction to the energy is

$$\Delta E_o^1 = \int_{-a/2}^{a/2} \psi_o^* W \delta(x) \psi_o dx = \frac{2W}{a}.$$

The correction to the first excited state is zero to all orders; the wavefunction for the first excited state has a node at the origin. The first order perturbed energies are then

$$\begin{aligned} E_o^1 &= \frac{\hbar^2\pi^2}{2ma^2} + \frac{2W}{a} \\ E_1^1 &= \frac{4\hbar^2\pi^2}{ma^2} \end{aligned}$$

2. **Charged particle in a rectangular cavity**
   The parallel component of the electric field must vanish at the walls of the conductor, so the electric field has the form

$$\vec{E} = \vec{E}_o \cos\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi z}{c}\right) \hat{x}$$

where l, m, n are integers. The electric field must also satisfy the wave equation

$$\begin{aligned} \nabla^2 \vec{E} - \frac{1}{v_c^2}\frac{\partial^2 \vec{E}}{\partial t^2} &= 0 \\ \rightarrow \omega = v_c \pi \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)^{1/2} \end{aligned}$$

where v_c is the speed of light in vacuum. The smallest ω occurs when $m = n = 1$, $l = 0$ and $\vec{E}_o = E_o\hat{x}$, so the lowest frequency is

$$\omega = v_c \pi \left(\frac{1}{b^2} + \frac{1}{c^2}\right)^{1/2}.$$

Note we cannot choose $n = 1, m = l = 0$ because there is no choice for $\vec{E}_o$ which satisfies the boundary conditions.

The vector potential satisfies the wave equation

$$\nabla^2 \vec{A} - \frac{1}{v_c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J}$$

Labelling the eigenfunctions of the homogeneous wave equation as $\vec{A}_i = \vec{A}_{lmn}$, a solution for the inhomogeneous equation is

$$\vec{A}(\vec{x}', t) = \sum_i c_i(t)\vec{A}_i(\vec{x}') = \frac{4\pi}{c}\int \frac{\vec{J}(\vec{x})}{|\vec{x} - \vec{x}'|} d^3x.$$

The eigenvalues are orthogonal ($\int \vec{A}_i \cdot \vec{A}_j d^3x = \delta_{ij}$), so at $t = 0$

$$\int \vec{A}_o(\vec{x}') \cdot \sum_i \vec{A}_i(\vec{x}') d^3x' = c_o = \frac{4\pi}{c}\int \frac{\vec{A}_o(\vec{x}') \cdot \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x d^3x'$$

$$\rightarrow \vec{J} || \vec{A}_o \rightarrow \vec{J} || \hat{x}$$

to maximize c_o.

Other solutions:

- One may argue the electron and proton are moving apart along x and thus creating a changing dipole. This is correct although the problem explicitely says to ignore the coulomb fields. This gets half credit.
- One may use

  $$\vec{A}(\vec{x}) = \frac{1}{c}\int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

  which is incorrect; the $1/|\vec{x} - \vec{x}'|$ is the Green's function for free space. Half credit.
- One can make the argument based on symmetry: nothing distinguishes the y and z directions aside from the differing boundary conditions. The lowest mode is zero at these boundary, so the only direction that makes sense is the x direction. Full credit *if* the boundary conditions and the vector potential being zero there are explcitely mentioned.

3. **Temperature scale**

   Bose-Einestein condensate - nK - μK
   Cosmic microwave background - 2.7 K
   Liquid helium-4 at boiling - 4K
   Liquid nitrogen at boiling - 76K
   Hi T_c superconductor - $>$ 76K
   Room temperature - 300K
   CMB at recombination - 10^4K
   Core of the sun - 10^6K
   Quark-gluon plasma - 10^{12}K

4. **Relativistic Hamiltonian** The momentum conjugate to x is

$$p = \frac{\partial L}{\partial \dot{x}} = mc^2 \left(\frac{\dot{x}}{c^2\sqrt{1 - \dot{x}^2/c^2}} \right) = m\gamma\dot{x}$$

   which is the relativistic momentum. The Hamiltonian is

$$H = p\dot{x} - L(x,p) = \frac{p^2}{m\gamma} + \left(mc^2 \sqrt{1 - \frac{p^2c^2}{m^2\gamma^2c^4}} \right) = \sqrt{p^2c^2 + m^2c^4} = E.$$

   Non-relativistic means $mc^2 >> pc$, so

$$H = mc^2\sqrt{1 + \frac{p^2c^2}{m^2c^4}} \sim mc^2 + \frac{p^2}{2m}$$

   which gives the non-relativistic kinetic energy $T = p^2/2m$.

Group II

1. **Cosmological red-shift**
   From the expression for the infinitesimal interval ds, the expression for a line element is

$$d\vec{x} = \frac{a(t)dr}{\sqrt{1-kr^2}}\hat{r} + a(t)r^2 d\theta\hat{\theta} + a(t)r\sin\theta d\phi\hat{\phi}$$
$$\rightarrow l(t) = \int_0^{r_o} a(t)dr = r_o a(t) \rightarrow v = r_o\dot{a}.$$

2. **Carnot cycle between two bodies**
   The changes are reversible, so

$$dS_H = -dS_C$$

   then from the First Law and since the heat capacities are equal,

$$\frac{dQ_H}{T_H} = -\frac{dQ_C}{T_C} \rightarrow \frac{c_o dT_H}{T_H} = -\frac{c_o dT_C}{T_C}.$$

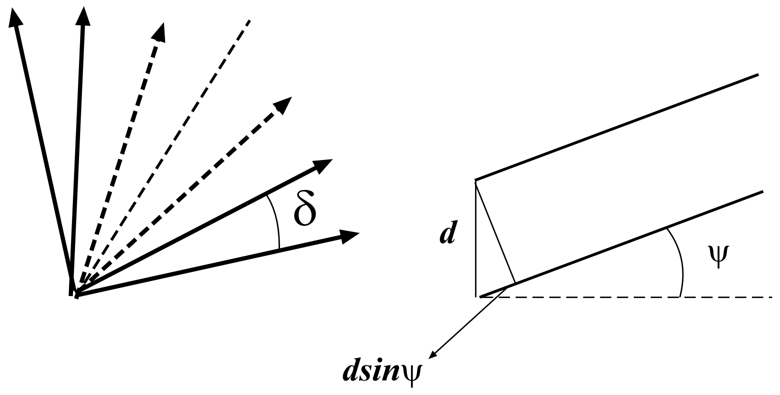
   Then,

$$\int_{T_H}^{T_F} dS_H = -\int_{T_C}^{T_F} dS_C \rightarrow \int_{T_H}^{T_F}\frac{dT}{T} = -\int_{T_C}^{T_F}\frac{dT}{T}$$
$$\rightarrow \ln\frac{T_F}{T_H} = -\ln\frac{T_F}{T_C} \rightarrow T_F = \sqrt{T_H T_C}.$$

3. **Force between comb and paper**
   The comb is negatively charged by running through your hair. An electric field is created by the charges on the comb which polarizes the charges in the small pieces of paper creating a polarization density. The force on a dipole is $\vec{F} = \vec{\nabla}\vec{p}\cdot\vec{E}$ which means either $\vec{p}$ or $\vec{E}$ must vary with position. In this case, since $\vec{p} \propto \vec{E}$, the force results from the non-uniformity of $\vec{E}$. The paper remains electrically neutral. For full credit, must mention induced dipole and non-uniform electrical field. Also msut mention range, which is a few cm (based on observation).

4. **Unequally spaced antenna array** If we choose as the reference distance the midpoint between the two nearest antennae, the two nearest antennae lie along the z axis at $\pm 2d\sin\psi$ and the furthest are $\pm 3d\sin\psi$. The relative electric field vectors far from the array are as shown at the left of Fig. 4, which phase difference $\delta = \lambda/d\sin\psi$. The total field is then given by the projection along the reference line, which gives

$$E_T = 2E_o\cos 2\delta + 2E_o\cos 3\delta = 4E_o\cos\delta\cos 5\delta.$$



The intensity for one antenna is $< I_o >= aE_o^2$ and the total current is $< I >= aE_T^2 = 16 < I_o > \cos^2 \delta \cos^2 5\delta$.

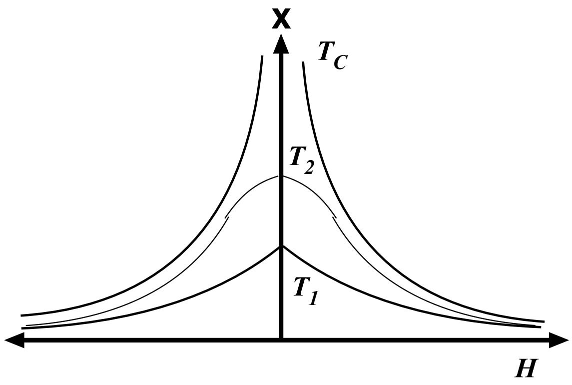


Figure 1:

Group III

1. **Ferromagnet** $T_1 < T_c < T_2$. The magnetic susceptibility is

$$\chi = \left(\frac{\partial M}{\partial H}\right)_T .$$

At the critical temperature T_c, χ becomes infinite at zero field, see Fig. 1

The units are

Quantity	SI	CGS
M	amp/m	Oersted
H	amp-turn/m	erg/gaus/cm^3
χ	turns	cm^{-3}

2. **Two state system** The time dependent Schrodinger equation is $i\hbar\partial\psi/\partial t = H\phi$ and we use

$$\psi = \alpha|+> + \beta|-> = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}.$$

Then

$$\begin{aligned} \alpha(t) &= \cos \pi t/T \\ \beta(t) &= \sin \pi t/T \\ \to i\hbar \begin{pmatrix} -\pi/T \sin \pi t/T \\ \pi/T \cos \pi t/T \end{pmatrix} &= \begin{pmatrix} 0 & -\frac{\pi}{T} \\ \frac{\pi}{T} & 0 \end{pmatrix} i\hbar\psi \\ \to H &= \frac{i\hbar\pi}{T} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

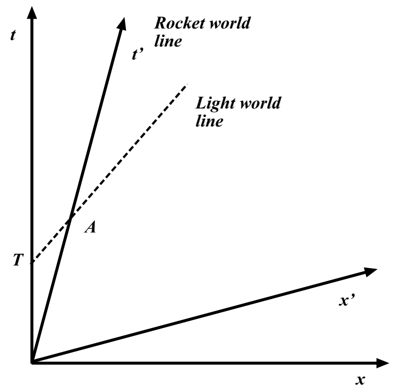


Figure 2:

The Hamiltonian

$$H = \frac{\hbar\pi}{T}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is also a valid solution, given the statement of the problem.

3. **Relativistic Rocket** In the stationary frame, the world line for the rocket is $x = \beta tc$ and the world line for the light emitted from the stationary frame is $x = c(t - T)$. These meet at point A in Fig. 2, which is given by $t_A = T/(1-\beta)$, $x_A = \beta cT/(1-\beta)$. The coordinates in the stationary frame are given in the rocket frame by

$$ct' = \gamma ct - \beta\gamma x \to t' = \sqrt{\frac{1+\beta}{1-\beta}} = T'.$$

If we treat the time interval in the rocket frame from $t' = 0$ to $t' = T'$ as one oscillation period, then $\nu' = 1/T'$ and the received frequency is then

$$\nu' = \frac{1}{T'} = \sqrt{\frac{1-\beta}{1+\beta}}\nu$$

where $\nu = 1/T$.

4. **Young's Modulus** The total force on the end is $F = PA$, where P is the applied pressure and A is the cross sectional area of the beam. The average atomic spacing is $r \sim 1/n^{1/3}$.

The potential energy is at a minimum, so energy change per atom is

$$\begin{aligned} U_a(r - \delta r) &= U_a(r) - \frac{\partial U_a}{\partial r}\delta r + \frac{1}{2}\frac{\partial^2 U_a}{\partial r^2}\delta r^2 \\ \rightarrow \delta U_a &= \frac{1}{2}\frac{\partial^2 U_a}{\partial r^2}\delta r^2 = \frac{Z_{eff}^2 e^2}{r^3}\delta r^2 = U_a \left(\frac{\delta l}{l}\right)^2 . \end{aligned}$$

The total energy change due to the compression is $\delta U = F\delta l = Pa\delta l$ and

$$\begin{aligned} Pa\delta l &= nal\delta U_a = nalU_a \left(\frac{\delta l}{l}\right)^2 \\ P &= nU_a \frac{\delta l}{l} \rightarrow E = nU_a = (Z_{eff}e)^2 n^{4/3} = 3 \times 10^{13} \mathrm{dyne/cm}^2 \end{aligned}$$

Group IV

1. **Particle in a conical well**
Take a as the radius around which the particle moves, then $a = h \tan\alpha$ and $l = amv$ is the angular momentum. The forces parallel to the wall must balance:

$$\begin{aligned} F_{||} &= \sin\alpha F_{centripital} - \cos\alpha F_{grav} \\ &= \frac{mv^2}{a}\sin\alpha - mg\cos\alpha \\ &= \frac{l^2}{mh^3\tan^3\alpha}\sin\alpha - mg\cos\alpha = 0 \\ \rightarrow\ h &= \left(\frac{l^2}{\tan^2\alpha m^2 g}\right)^{1/3}. \end{aligned}$$

$\dot{\alpha}/\alpha << 1/T$ meaning the wall moves slowly in comparison with the change in α. This is a classical adiabatic change, so $l = \oint pds$ (where $ds = ad\phi$ is a line element around the orbit) an adiabatic invariant and is constant. l is constant in any event, as the applied force can *only* be perpendicular to the motion of the particle. Either answer is acceptable.

2. **Yield of the first atomic bomb** The only combination of E, ρ and t which has units of length is

$$R(t) = \mu\left(\frac{Et^2}{\rho}\right)^{1/5}.$$

At $t = 1\text{ms}$, $R = 40\text{m}$,

$$E\mu^5 = \frac{R^5\rho}{t^2} = 10^{14}\text{J} = 25\text{kt}.$$

3. **Charged conductors over a grounded conducting plane** Use the method of images: remove the conducting plane and replace with line charges of $+\lambda$ a distance h below the line charges and a charge of $-Q$ below the point charge. Obviously, the parallel components of the electrostatic force cancels. The upward force from both the line charges is

$$F_l = 2\frac{2\lambda Q}{h\sqrt{5}}\cos\theta$$

and $\cos\theta = 2/\sqrt{5}$. The force from the point image charge is

$$F_Q = -\frac{Q^2}{4h^2}$$

Then the total force must sum to zero:

$$F_T = -\frac{Q^2}{4h^2} + \frac{8\lambda Q}{5h} = 0 \quad (1)$$
$$\rightarrow \quad Q = \frac{32\lambda h}{5}.$$

for the stability analysis, the force from the line charges (and their images) must be included. The, expand in δ/h, keeping only the linear term. An important check is for the leading termto cancel. Q is stable if the linear term is negative. If Q moves up by δ, then

$$\begin{aligned}
F_{-\lambda} &= -\frac{4\lambda Q\delta}{h^2 + delta^2} \sim -\frac{4\lambda Q\delta}{h^2} \\
F_{+\lambda} &= \frac{4\lambda Q(\delta + 2h}{h^2 + (\delta + 2h)^2} \sim \frac{8\lambda Q}{5h}(1 - \frac{3\delta}{10h}) \\
F_{-Q} &= -\frac{Q^2}{(2h + 2\delta)^2} \sim -\frac{Q^2}{4h^2}(1 - \frac{2\delta}{h})
\end{aligned}$$

The leading term is

$$\frac{8\lambda Q}{5h} - \frac{Q^2}{4h^2} = 0$$

when we put in $Q = 32\lambda h/5$, which is what we expect. The linear term works out to be

$$\frac{-1024\delta\lambda^2}{125h}$$

so the Q is stable in the vertical direction.

4. **Magnetic field from a 2D current distribution**
The Biot-Savart law says

$$d\vec{B} = \frac{Id\vec{l} \times \hat{r}}{r^2 c}.$$

For our current distribution, $Id\vec{l} = Idl_x\hat{x} + Idl_y\hat{y}$, then

$$\begin{aligned}
dB_x &= \frac{Idl_y r_z}{r^2 c} \\
dB_y &= -\frac{Idl_x r_z}{r^2 c} \\
dB_z &= \frac{Idl_x r_y - Idl_y r_x}{r^2 c}
\end{aligned}$$

so

$$\vec{B}(x, y, -z) = -a\hat{x} - b\hat{y} + c\hat{z}.$$

Group V

1. **Dipole antenna**

   a) No current can flow off the end of the dipole, so $I(|z| = d) = 0$. Since the driving voltage is sinusoidal, the current distribution must be $I(z,t) = I_o \sin(kd - k|z|)\cos(\omega t + \phi)$ where $k = 2\pi/\lambda$ and ϕ is a phase. Since the vector potential $\vec{A}$ (and hence the fields) is proportional to I, the radiated power is largest when $kd = \pi/2 \to d = \lambda/4$.

   b) $\vec{E} = -\vec{\nabla}\phi - (1/c)\partial\vec{A}/\partial t$. The upper and lower conductors have opposite charges, so the potential gradient is along $\hat{z}$. $\vec{A} \propto \vec{J}$, so $\vec{A}$ also lies along $\hat{z}$, so the polarization is along $\hat{z}$.

   c) We find the vector potential from the current density

   $$\vec{A}(\vec{x}) = \frac{1}{c}\int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'.$$

   in the far field, $r >> d$, so $\vec{A} \propto I/r$ and $\vec{B} = \vec{\nabla} \times \vec{A}$, so $|\vec{B}| \propto \sin\theta$ and the angular distribution of the radiated power goes like $\sin^2\theta$.

2. **Driven harmonic oscillator** There are two ways to do this problem. One may use the Green's function and carry out the integral

   $$x(t) = \int G(t,t')F(t')dt' = \frac{F_o}{m\omega_o^2}\left[1 - e^{-\beta t}\cos\omega_1 t - \frac{\beta e^{-\beta t}}{\omega_1}\sin\omega_1 t\right].$$

   Alternatively, you can find a solution for the homogeneous and inhomogeneous equations and match boundary conditions to get the same result. At $t \to \infty$, $x \to F_o/m\omega_o^2$.

   Note: during the exam, a correction was given: $\omega_1 = \sqrt{\omega^2 - \beta^2}$. Full credit should be given for either expression for ω_1 so long as the rest is correct.

3. **Hydrogen atom in a magnetic field** The $\vec{L} \cdot \vec{S}$ term is the spin-orbit interaction, the $\vec{S} \cdot \vec{B}$ term is the interaction of the dipole moment of the electron and the applied magnetic field and the $\vec{L} \cdot \vec{B}$ term is the interaction between the dipole moment of the orbit and the applied field.

   In the $|\vec{B}| \to \infty$ limit, the magnetic field exerts a torque on the electron and the orbit, so J is not a good quantum number, l, m_l, s, m_s are. In the $|\vec{B}| \to 0$ limit, s, j, l are all good quantum numbers.

4. **Fermions in a magnetic field** For a fermion of spin s, there are $2s+1$ states. Taking $\vec{B}(t)$ as the quantization axis at time t, the energy levels will be $E = -s\mu B_o, -(s-1)\mu B_o ... s\mu B_o$. $eB_o/m = \hbar\omega_p$ is the energy of the precession of the magnetic moment around the $\hat{z}$ axis. If $\hbar\omega_p >> \hbar\omega$, i.e. the energy of the precession is much greater than the energy of the magnetic moment and the applied field, the quantum state of the system in the $\hat{z}$ basis will remain the same; the spin vector will precess around the $\hat{z}$ axis at frequency ω_p.

   Note: buring the exam, the fermion was specified to be spin 1/2 with magnetic moment μ, which gives $|\mu B_o/\hbar| >> \omega$ full credit should be given either way.