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DEPARTMENT OF PHYSICS

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DOCTORAL GENERAL EXAMINATION
PART 1
February 1, 2002

FIVE HOURS

Note: this is a correct version of the General I exam given Spring 2002.
The corrections were made on June 13, 2002.

1. This examination is divided into five sections, each consisting of four problems. Answer all the problems. Each problem is worth 5 points, thus the maximum score for the exam is 100.
2. Use a separate fold of paper for each problem. Write your name and the problem number (IV-3 for example) on each fold. A diagram or sketch as part of the answer is often useful, particularly when a problem asks for a quantitative response.
3. Read the problem carefully and do not do more work than is necessary. For example "give" and "sketch" do not mean "derive".
4. Calculators may be used but are not necessary.
5. No books, notes or reference materials may be used.

Group I

1. **Dipole antenna**
   A dipole antenna of length l is driven by a current source $I(t) = I_o \cos\omega t$. The dipole is oriented along $\hat{z}$ and generates an EM wave. Assuming $l << \lambda$, find the vector potential $\vec{A}(\vec{r}, t)$ for the propagating wave.

2. **Parking orbit**
   A spacecraft of mass m is approaching Planet X (of mass M) at radial velocity $\dot{r} = v$ and impact parameter b. At this time, the spacecraft is a distance a from Planet X. Give the condition on b for a bound orbit in terms of M, m, a and/or v and physical constants.

3. **Interacting bosons**
   Two identical bosons are placed in an infinite 1-dimensional square well of length L. They interact weakly with one another via the potential

   $$V(x_1, x_2) = -LV_o\delta(x_1 - x_2).$$

   Use first order perturbation theory to calculate the effect of the particle-particle interaction on the ground state energy.

4. **Solar corona**
   One model for the solar corona assumes the heat transport satisfies the radially symmetric steady state diffusion equation

   $$\vec{\nabla} \cdot k(T)\vec{\nabla}T = 0$$

   where the diffusion $k(T)$ constant is proportional to $T^{5/2}$. If $T \propto r^n$ is a solution, find n.

Group II

1. **Ring in a magnetic field**
   A ring of radius a and mass M carries an immobile charge Q which is uniformly distributed around the ring. Before $t = 0$, the ring is stationary in a magnetic field $\vec{B} = B_o\hat{z}$, where the z axis is perpendicular to the plane of the ring. Starting at $t = 0$, the magnetic field is given by $\vec{B}(t) = B_o e^{-\alpha t}\hat{z}$. Find the angular momentum of the ring as $t \to \infty$. Explain the dependence of your result on α.

2. **van der Waals gas**
   For a gas obeying van der Waals equation

$$P = \frac{nRT}{V-b} - \frac{a}{V^2}$$

   find $(\partial E/\partial V)_T$ where E is the internal energy of the gas..

3. **Barely bound particle**
   Consider a particle of mass M in the following potential

$$V(x) = V_o\Theta(x) - W\delta(x)$$

   where $V_o > 0$, $W > 0$ and $\Theta(x)$ is the unit step (Heavyside) function. Note that both the step function and the delta function are at $x = 0$. Find the minimal value for W for which the ground state energy is $E = 0$ (i.e. particle is barely bound).

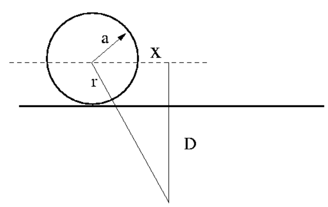
4. **Bessel function**
   The generating function of Bessel's equation is

$$g(x,t) = e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

   Use the generating function to find a series expression for $J_n(x)$.

Group III

1. **Oscillating disk**
A uniform disk of moment of inertia I and mass M rolls without slipping on a flat surface. An attractive force of magnitude $F = -kr^{-\alpha}\hat{r}$ acts between the center of the disk and a fixed point a distance D below the locus of the center of the disk. If $k > 0$ and $|x/D| << 1$, for what values of α to stable oscillations occur? Give the frequency of small oscillations.



2. **Vibrations of a monatomic crystal**
Calculate the entropy of the lattice vibrations of a monatomic crystal as described by the Einstein theory, $\epsilon_r = \hbar\omega(r + 1/2)$, $r = 0, 1, 2, 3, 4, ..$ at low temperatures, $\hbar\omega >> kT$.

3. **Radially confined quantum system** A free particle of mass M is confined to move on a circle. At time $t = 0$ its wave function is given by [1]
$$\psi(\phi, 0) = A\left[e^{2i\phi} + 2\cos\phi\right]$$
where ϕ gives the position of the particle on the circle. At time t_o the angular momentum of the particle is measured. What are the possible results of this measurement and what are the corresponding probabilities for each possible outcome?

4. **Novel spaceship**
A novel propulsion system for a space ship (of mass m) consists of a large hollow cavity whose walls are maintained at a high temperature T with a small hole of area A through which electromagnetic radiation escapes. Find the acceleration of the rocket in terms of A, T and fundamental constants.

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[1] The wave function given on the Spring 2002 General I exam was incorrect.

Group IV

1. **Adiabatic oscillator**
A simple harmonic oscillator has a mass m and spring constant k. At $t = 0$, the oscillator is moving with velocity v at $x = 0$. At this time, the spring constant is changed according to $k(t) = k_o - \alpha t$ where $\alpha << \omega = \sqrt{k/m}$ at all times relevant to this problem. Find the average energy per unit time expended by the mechanism changing the spring constant at $t = 0$ in terms of m, v, k_o, α and fundamental constants.

2. **Black hole entropy**
The temperature of a black hole of mass M is given by

$$kT = \frac{\hbar c^3}{8\pi MG}.$$

Find the entropy given the energy $E = Mc^2$. Assume $S \to 0$ as $M \to 0$.

3. **Cartesian resonator cavity**
A rectangular cavity resonator is fabricated in the form of a hollow metallic box with inside dimensions a along the $\hat{x}$ and $\hat{y}$ axis and $a/2$ in the $\hat{z}$. Find the magnetic field of the lowest frequency mode assuming the largest field anywhere in the cavity is E_o.

4. **Diffraction limited lens**
A plane wave of wavelength λ is incident on a circular lens of focal length f and diameter D and $f/D > 1$. Ideally, the image at the focal plane would be a point, but diffraction defocusses the image to a disk. Estimate the size of the disk.

Group V

1. **Off center cylinder**
   A cylinder of radius a and mass m contains a point mass, also of mass m located a distance $a/2$ from the symmetry axis. The cylinder is placed on an incline, which is initially horizontal, but is very slowly raised. Assuming the cylinder cannot slide on the incline, at what inclination angle α does the cylinder begin to roll down the incline?

2. **Non-ideal gas**
   A theoretical model for a certain real (non-ideal) gas gives the following expressions for the internal energy and the pressure

$$\begin{aligned} E(T,V) &= aV^{-2/3} + bV^{2/3}T^2 \\ P(T,V) &= \frac{2}{3}aV^{-5/3} + \frac{2}{3}bV^{-1/3}T^2. \end{aligned}$$

   Find the entropy S in terms of T and V.

3. **Quantum mechanical translation**
   A quantum mechanical translation operator $T(a)$ translates a system in space

$$T(a)\psi(x) = \phi(x+a).$$

   Exhibit a derivation for an expression for $T(a)$.

4. **Refrigerator magnets**
   With the exam, you have two refrigerator magnets. Using only your observations of the two magnets, answer the following questions:

   (a) Sketch the magnetization inside the magnetic material.

   (b) Sketch the magnetic field outside the magnetic material. Why is the magnet made like this?

   (c) Estimate the magnetization M of the magnetic material based on an estimate of how much force you need to apply to get the magnets apart.