MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Education Office
Room 4-352

Phone: (617) 253-4851
Fax: (617) 258-8319

DOCTORAL GENERAL EXAMINATION

PART II

August 30, 2002

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. Read both problems in each section carefully before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. Use a separate paper (i.e. folded sheet) for each problem. Write your name and the problem number (I.2 for example) on each folded sheet.
3. Calculators may be used.
4. No books or reference materials may be used.

Problem 1

CLASSICAL MECHANICS: CENTRAL POTENTIAL

A Newtonian test particle orbits in a central potential $V(r)$, i.e. the acceleration of the particle is $-\vec{\nabla}V$.

a) Determine the period of a circular orbit of radius r, in terms of $V(r)$ and its derivatives.

b) Suppose that the orbit is slightly noncircular, with $r(t) = r_0 + \epsilon(t)$ where r_0 is constant. Find the **general solution** for $\epsilon(t)$ in the limit $\epsilon^2 \ll r_0^2$; your answer should depend on the energy per unit mass E and on $V(r_0)$ and its derivatives. Determine the period of radial oscillation in terms of $V(r_0)$ and its derivatives.

c) Show that there are no stable circular orbits for the Yukawa potential

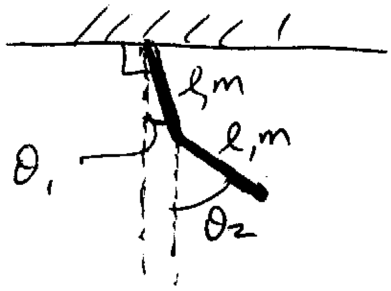
$$V(r) = -\frac{GM}{r}e^{-kr} , \qquad (1)$$

if $r > (2k)^{-1}(1 + \sqrt{5})$.

d) Show that it is possible to have bound orbits with positive energy for the Yukawa potential (with $V = 0$ at infinity).

Problem 2

CLASSICAL MECHANICS: PLANAR DOUBLE PENDULUM



A double pendulum consists of two identical rods hinged at the ends and mounted from a ceiling as shown in the diagram. The rods have mass m and length l and they have constant linear mass density (mass per unit length). There is a downward gravitational acceleration g. The pendula are constrained to move in a vertical plane. Throughout this problem you may assume small angles $\theta_1^2 \ll 1$ and $\theta_2^2 \ll 1$.

a) Find the Lagrangian of the system in terms of the two angles θ_1 and θ_1 and their time derivatives.

b) Find the frequencies of small oscillations.

c) Sketch the normal modes.

General Exam Part II (2002), Electro-Magnetism

Problem 1

Consider *the lowest* resonance mode of electromagntic field in a cavity made of an ideal conductor. The cavity shape is a box $a \times b \times b$, $b < a$.

a) Find the resonance frequency and the spatial distribution of the fields $\vec{E}(r,t)$, $\vec{B}(r,t)$. Choose the mode with $\vec{E} \parallel \hat{y}$.

b) Find the surface current and charge density on the cavity boundary. Make a shematic drawing.

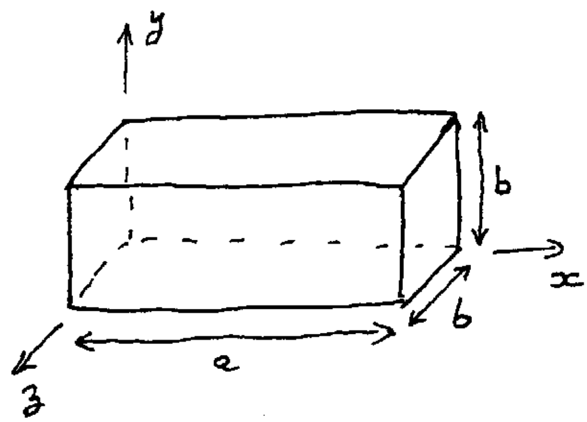
c) Find the electromagnetic force on the square sides $b \times b$. Indicate the direction of the force; show it on a drawing.

d) Find the force on the $a \times b$ sides.

e) By using the Maxwell stress tensor

$$T_{ij} = \frac{1}{8\pi}\left(2E_iE_j - \delta_{ij}E^2\right) + \frac{1}{8\pi}\left(2B_iB_j - \delta_{ij}B^2\right) \tag{1}$$

verify the results of the parts c) and d).



Problem 2.

ELECTROMAGNETISM: WAVES IN A DILUTE GAS

Electromagnetic waves travelling through a dilute gas induce a time-varying electric polarization. This problem considers waves travelling in the x-direction through a uniform gas. The gas is assumed to be sufficiently tenuous so that the molecules are non-interacting.

a) Suppose that the gas has number density n_a of atoms each having a single electronic resonance with angular frequency ω_0 and damping rate γ. That is, treat the atom as an electron on a spring. Derive an expression for the volume polarizability $\alpha(\omega) = P/(\epsilon_0 E)$ where $E \propto e^{-i\omega t}$ is the electric field and P is the polarization. Your expression for α should depend on n_a, e, ϵ_0, m_e, γ, ω_0, and ω.

b) Derive the dispersion relation $k^2 = \omega^2 n^2/c^2$ for plane waves and give an expression for the index of refraction $n(\omega)$.

c) At $t = 0$, the electric field is given by a Gaussian wavepacket

$$E(x, t = 0) = e^{ik_c x} N(x, \sigma) \ , \quad N(x, \sigma) \equiv (2\pi\sigma^2)^{-1/2} e^{-x^2/2\sigma^2} \ , \qquad (2)$$

whose Fourier transform is $\exp[-(k - k_c)^2\sigma^2/2]$. Find the electric field at a later time, $E(x, t)$. (Hint: use a first-order Taylor expansion of the dispersion relation. For this part of the problem, it is not necessary to know the particular form of the dispersion relation.)

d) Evaluate the group velocity for the dilute gas at $\omega = \omega_0$, assuming $|\alpha| \ll 1$. Is causality violated? Explain your results.

MIT General Exam

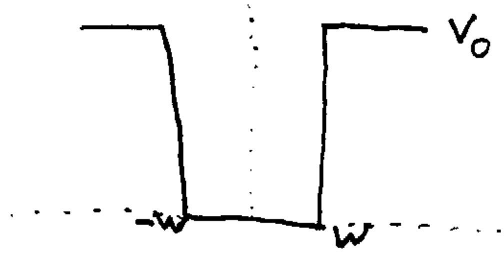
Fall 2002

Quantum Mechanics

Problem 1

A particle of mass m is trapped in a one-dimensional well of depth V_0 and width $2w$:

$$V(x) = \begin{cases} V_0 & x < -w \\ 0 & -w \le x \le w \\ V_0 & x > w\,. \end{cases}$$



The Hamiltonian is

$$H = \frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\,.$$

(1) Calculate the energy of the ground state. You should express your answer in terms of the root of a transcendental equation.

(2) What is the ground-state energy as

$$\frac{\hbar^2}{2mw^2V_0} \to 0\ ?$$

(3) Add an electric field so the total potential is now $V(x) - e\mathcal{E}x$. Sketch the new potential and explain why the particle will eventually escape from the well.

(4) What is the first-order shift in the ground-state energy due to the electric field?

(5) Let B denote the energy of a particle bound in the well in the presence of the electric field. Use the semiclassical approximation to calculate the barrier penetration factor in terms of B.

(6) Estimate the lifetime of the state in (5).

Problem 2

Consider a spin-1/2 particle interacting with a time-dependent magnetic field,

$$H(t) = \vec{B} \cdot \vec{\sigma} \ ,$$

where $\vec{B}(t) = B\big(\sin\theta\cos\omega t, \sin\theta\sin\omega t, \cos\theta\big)$.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(1) Find the instantaneous eigenstates $|+, t\rangle$ and $|-, t\rangle$ of $H(t)$ with eigenvalues $+B$ and $-B$, respectively.

(2) Let $|\psi, t\rangle$ obey the Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi, t\rangle = H(t) |\psi, t\rangle$$

with $|\psi, 0\rangle = |+, 0\rangle$. Calculate

$$P_+(t) \equiv \Big|\langle +, t | \psi, t\rangle\Big|^2$$

exactly. Hint: Expand $|\psi, t\rangle$ in terms of $|+, t\rangle$ and $|-, t\rangle$ and find the equation for the expansion coefficients. You may find the following equation helpful:

$$e^{i\vec{D}\cdot\vec{\sigma}} = \cos D + i\frac{\vec{D}\cdot\vec{\sigma}}{D}\sin D$$

where $D = |\vec{D}|$.

(3) What happens to $P_+(t)$ for $\omega \ll B/\hbar$? Explain.

Statistical Mechanics and Thermodynamics, Problem 1

Thermodynamics of a Non-interacting Bose Gas

a) Find the transition temperature, $T_{\rm BEC}$, for Bose-Einstein Condensation in a non-interacting gas of density n and particle mass m. [You may include in your result a clearly specified dimensionless integral.]

b) Sketch the qualitative behavior of isotherms in the $P-V$ plane (i.e. $P(V)$ when the temperature and total number are held constant) both above and below the transition. On the same sketch show the results for a classical Boltzmann gas.

c) Find the constant volume heat capacity, $C_{\rm V}$, for $T < T_{\rm BEC}$. [Again, you may include in your result a clearly specified dimensionless integral.]

d) Two gases, F_1 and F_2, with equal volumes and particle numbers are initially at temperatures $T_1 < T_2 < T_{\rm BEC}$. The gases are adiabatically insulated from each other. Consider a Carnot machine which uses F_1 and F_2 as the heat source and drain, respectively. The Carnot cycle is repeated quasistatically with infinitesimal heat transfers each time until F_1 and F_2 reach a common equilibrium value T_0. Find the temperature T_0 and the total work done by the Carnot machine.

Statistical Mechanics and Thermodynamics, Problem 2

Phase Transition in a Superconductor

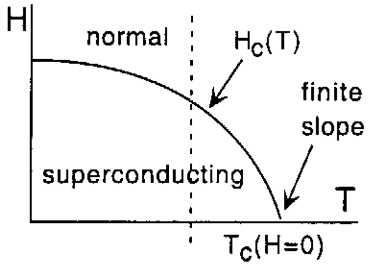


Figure 1

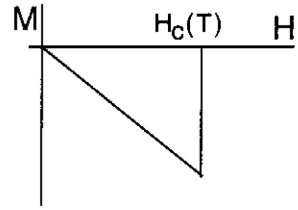


Figure 2

Figure 1 shows the phase diagram for a superconductor. The phase transition to a superconducting state moves to lower temperatures as the applied magnetic field, H, increases. The magnetic moment, M, for a system of volume V is given by

$$M = -\frac{HV}{4\pi} \qquad H < H_C(T) \qquad \text{superconducting}$$

$$M = 0 \qquad H > H_C(T) \qquad \text{normal}$$

and is sketched in Fig. 2 as it would appear along the dashed line in Fig. 1. Changes in the internal energy, U, of such a system can be represented by the expression

$$dU = TdS + HdM$$

a) Show that for this system the constant field and constant magnetization heat capacities are equal: $C_{\text{H}} = C_{\text{M}}$.

b) The phase transition between superconducting and normal phases takes place along a path $H_C(T)$. Find an expression for the slope of the critical field, $dH_C(T)/dT$, in terms of $H_C(T)$, V, and the entropies of the two phases at the transition.

c) What does thermodynamics imply about the shape of $H_C(T)$ as $T \to 0$? Where is the transition first order and where is it second order?

d) The heat capacity of the material is given by

$$C_H = aT^3V \qquad H < H_C(T) \qquad \text{superconducting}$$

$$C_H = bT^3V + \gamma T \qquad H > H_C(T) \qquad \text{normal}$$

Use the results of b) together with the observation that $dH_C(T)/dT$ is finite near $T_C(H = 0)$ to find $T_C(H = 0)$ in terms of a, b, and γ.