Mechanics

E. Bertschinger

Classical Mechanics I: Central Potential

a) Integrals of motion for a central potential $V(r)$:

Angular momentum $L = rV_t = r^2\dot\phi$ (V_t = tangential velocity)

Energy per unit mass $E = \frac{1}{2}(\dot r^2 + V_t^2) + V(r) = \frac{1}{2}\dot r^2 + V_{eff}(r)$,

$V_{eff}(r) = V(r) + \frac{L^2}{2r^2}$.

Circular orbit: $\dot r = 0 \Rightarrow \frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{dV}{dr} = \frac{L^2}{r^3} = \frac{V_t^2}{r} = r\dot\phi^2$

$\Rightarrow \dot\phi = \omega_\phi = \frac{L}{r^2} = \left(\frac{1}{r}\frac{dV}{dr}\right)^{1/2}$, $\boxed{P_\phi = \frac{2\pi}{\omega_\phi} = 2\pi\left(\frac{1}{r}\frac{dV}{dr}\right)^{-1/2}}$

b) $r(t) = r_0 + \epsilon(t)$ with $(dV_{eff}/dr)(r_0) = 0$, $\epsilon^2 \ll r_0^2$.

Energy per unit mass: $E = \frac{1}{2}\dot\epsilon^2 + V_{eff}(r_0+\epsilon)$.

Taylor expand: $V_{eff}(r_0+\epsilon) = V_{eff}(r_0) + \underbrace{\left(\frac{dV_{eff}}{dr}\right)_{r_0}}_{0}\epsilon + \frac{1}{2}\left(\frac{d^2V_{eff}}{dr^2}\right)_{r_0}\epsilon^2 + O(\epsilon^3)$

$\therefore E - V_{eff}(r_0) = \frac{1}{2}\dot\epsilon^2 + \frac{1}{2}\omega_r^2\epsilon^2 + O(\epsilon^3) = \text{constant}$

$\boxed{\omega_r \equiv \left(\frac{d^2V_{eff}}{dr^2}\right)^{1/2}_{r_0}}$ this is the simple harmonic oscillator equation, and its general solution is

$\boxed{\epsilon(t) = \frac{\sqrt{2(E-E_0)}}{\omega_r}\cos[\omega_r(t-t_0)]}$ where $E_0 \equiv V_{eff}(r_0)$ and t_0 is an arbitrary constant.

Now write ω_r in terms of $V(r)$ instead of $V_{eff}(r)$.

$\omega_r^2 = \frac{d^2V_{eff}}{dr^2} = \frac{d^2V}{dr^2} + \frac{3L^2}{r^4} = \frac{d^2V}{dr^2} + 3\omega_\phi^2 = \frac{d^2V}{dr^2} + \frac{3}{r}\frac{dV}{dr}$

$\boxed{\omega_r = \left[\frac{d^2V}{dr^2} + \frac{3}{r}\frac{dV}{dr}\right]^{1/2}_{r_0} = \left[\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{dV}{dr}\right)\right]^{1/2}_{r_0}}$

$\boxed{\text{Radial period } P_r = \frac{2\pi}{\omega_r}}$

Classical Mechanics I – continued

c) Stability is determined by the sign of ω_r^2: $\omega_r^2 > 0$ for stability.

$\omega_r^2 = \frac{1}{r^3}\frac{d}{dr}\left(r^3 \frac{dV}{dr}\right)$, $V(r) = -\frac{GM}{r}e^{-Kr}$

$\Rightarrow \omega_r^2 = \frac{GM}{r^3} e^{-Kr}\left[1 + Kr - (Kr)^2\right]$

$1 + Kr - (Kr)^2 = \left(\frac{\sqrt{5}-1}{2} + Kr\right)\left(\frac{\sqrt{5}+1}{2} - Kr\right) > 0$ only if $Kr < \frac{\sqrt{5}+1}{2}$

$\therefore$ $\boxed{\text{The circular orbits are unstable for } Kr > \frac{\sqrt{5}+1}{2}}$

d) The outermost stable circular orbit is at $r_0 = \frac{\sqrt{5}+1}{2K}$.

It has energy per unit mass $E = V(r_0) + \frac{1}{2}(r_0\omega_\phi)^2 = V(r_0) + \frac{1}{2}\left(r\frac{dV}{dr}\right)_{r_0}$

$E = \frac{GM}{r_0}e^{-Kr_0}\left[\frac{1}{2}(Kr_0 - 1)\right] = \frac{GM}{r_0}e^{-Kr_0}\left(\frac{\sqrt{5}-1}{4}\right) > 0$

If r_0 is decreased slightly, the orbit is absolutely stable and $E > 0$. The effective potential for the Yukawa potential has the form shown.

Mechanics

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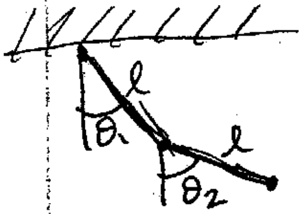
Classical Mechanics II: Planar Double Pendulum



a) $L = T - V$.

For the first rod,

$$T_1 = \frac{1}{2}\int_0^{\ell} (r\dot\theta_1)^2 \frac{m}{\ell}\,dr = \frac{1}{6} m\ell^2\dot\theta^2$$

For the second rod, must apply a velocity offset. Working in Cartesian coordinates,

$$T_2 = \frac{1}{2}\int_0^{\ell}\left[(\ell\dot\theta_1\cos\theta_1 + r\dot\theta_2\cos\theta_2)^2 + (\ell\dot\theta_1\sin\theta_1 + r\dot\theta_2\sin\theta_2)^2\right]\frac{m}{\ell}\,dr$$

$$= \frac{1}{2}m\ell^2\left[\dot\theta_1^2 + \dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2) + \frac{1}{3}\dot\theta_2^2\right]$$

The potential energies are simply mgz where z is the vertical distance from the ceiling (negative for positions below the ceiling). Since the rods are uniform, we may treat them as if all the mass is concentrated at the center.

$$\therefore\ V_1 = mg\left(-\frac{\ell}{2}\cos\theta_1\right),\quad V_2 = mg\left[-\ell\cos\theta_1 - \frac{\ell}{2}\cos\theta_2\right].$$

Now expand the cosines for small angles: $\cos\theta \simeq 1 - \frac{\theta^2}{2}$.

$$\Rightarrow\ L = T_1 + T_2 - V_1 - V_2 = \boxed{m\ell^2\left(\frac{2}{3}\dot\theta_1^2 + \frac{1}{2}\dot\theta_1\dot\theta_2 + \frac{1}{6}\dot\theta_2^2\right) - mg\ell\left(\frac{3}{4}\theta_1^2 + \frac{1}{4}\theta_2^2\right) + \text{constant}}$$

minus sign

Classical Mechanics II - continued

b) Lagrange's equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_i}\right) - \frac{\partial L}{\partial \theta_i} = 0$

$$\frac{4}{3}\ddot\theta_1 + \frac{1}{2}\ddot\theta_2 + \frac{g}{l}\left(\frac{3}{2}\theta_1\right) = 0.$$

$$\frac{1}{2}\ddot\theta_1 + \frac{1}{3}\ddot\theta_2 + \frac{g}{l}\left(\frac{1}{2}\theta_2\right) = 0$$

Normal modes: $\theta_1 = \hat\theta_1 e^{i\omega t}$, $\theta_2 = \hat\theta_2 e^{i\omega t}$

$$\underbrace{\begin{pmatrix} \left(\frac{4}{3}\omega^2 - \frac{3}{2}\frac{g}{l}\right) & \frac{1}{2}\omega^2 \\ \frac{1}{2}\omega^2 & \left(\frac{1}{3}\omega^2 - \frac{1}{2}\frac{g}{l}\right) \end{pmatrix}}_{M} \begin{pmatrix} \hat\theta_1 \\ \hat\theta_2 \end{pmatrix} = 0$$

Let $\frac{g}{l} \equiv \omega_0^2$, $x \equiv \frac{\omega^2}{\omega_0^2}$

$\Rightarrow \det M = 0 \Rightarrow \left(\frac{4}{3}x - \frac{3}{2}\right)\left(\frac{1}{3}x - \frac{1}{2}\right) - \frac{x^2}{4} = 0,$

$\frac{7}{36}x^2 - \frac{7}{6}x + \frac{3}{4} = 0 \Rightarrow x = \boxed{3 \pm \frac{6}{\sqrt7} = \frac{\omega^2 l}{g}}$ normal mode frequency

c) Find eigenvectors.

First case: $\omega^2 = \left(3 - \frac{6}{\sqrt7}\right)\frac{g}{l}$ (low frequency mode)

$\hat\theta_2 = \left(\frac{3}{x} - \frac{8}{3}\right)\hat\theta_1 = \frac{1}{3}\left(2\sqrt7 - 1\right)\hat\theta_1$

$\frac{1}{3}(2\sqrt7 - 1) > 0$ and real $\Rightarrow$ the two pendula are in phase.

Second case: $\omega^2 = \left(3 + \frac{6}{\sqrt7}\right)\frac{g}{l}$ (high-frequency mode)

$\hat\theta_2 = \left(\frac{3}{x} - \frac{8}{3}\right)\hat\theta_1 = \frac{1}{3}\left(-2\sqrt7 - 1\right)\hat\theta_1$; $\frac{1}{3}(-2\sqrt7 - 1) < 0$, real

$\Rightarrow$ two pendulare are π out of phase.

$\omega^2 = \left(3 - \frac{6}{\sqrt7}\right)\frac{g}{l}$ | $\omega^2 = \left(3 + \frac{6}{\sqrt7}\right)\frac{g}{l}$

Problem 1 E and M

a) Normal modes are products of harmonic standing waves in the x, y and z directions:

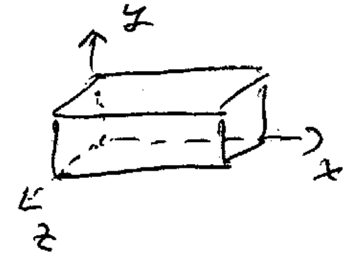
$$\omega = c\sqrt{k_x^2+k_y^2+k_z^2} = c\left[\left(\frac{\pi n_x}{a}\right)^2+\left(\frac{\pi n_y}{b}\right)^2+\left(\frac{\pi n_z}{b}\right)^2\right]^{1/2}$$

$$n_{x,y,z} \in \mathbb{Z}_+$$

The lowest frequency has $n_x = 1$, $n_{y,z} = 0$

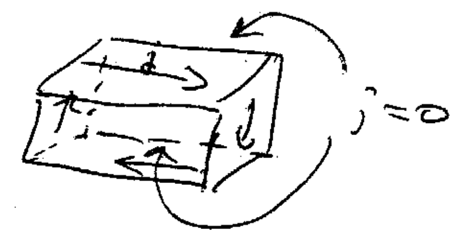
$$\omega = \frac{\pi c}{a}$$

$$\begin{cases} E(r,t) = E_0 \sin kx \, e^{-i\omega t}\,\hat{y} \\ B(r,t) = B_0 \cos kx \, e^{-i\omega t}\,\hat{z} \end{cases}$$



Satisfies Maxwell eqs. when $\underline{E_0 = i B_0}$

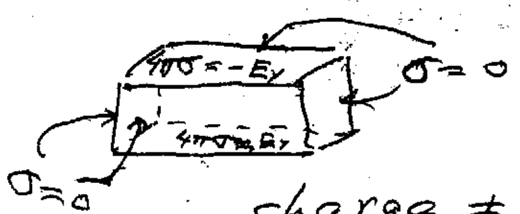
b)



$$\frac{4\pi}{c} j_x^{\text{(top/bottom)}} = \mp B_z = \mp B_0 \cos kx \, e^{-i\omega t}$$

$$\frac{4\pi}{c} j_y^{(x=0)} = -B_z(x=0) = -B_0 e^{-i\omega t}$$

$$\frac{4\pi}{c} j_y^{(x=a)} = B_z(x=a) = -B_0 e^{-i\omega t}$$



charge $\neq 0$ on the <u>top and bottom</u> surfaces only

$$4\pi\sigma = \mp E_y = \mp E_0 \sin kx \, e^{-i\omega t}$$

c) The energy $W = \iiint dx\,dy\,dz \left\langle \frac{E^2}{8\pi} + \frac{B^2}{8\pi} \right\rangle_{\text{time-ave}}$

$$= \underbrace{ab^2}_{\text{Volume}} \frac{E_0^2}{8\pi} \underset{\text{time-ave}}{\frac{1}{2}} \times \underset{\text{space-ave}}{\frac{1}{2}} + ab^2 \frac{B_0^2}{8\pi} \frac{1}{2} \times \frac{1}{2}$$

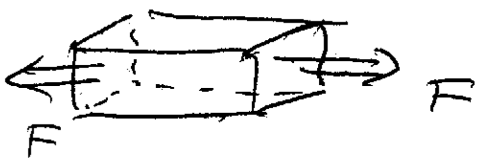
$E_0^2 = 16\pi W / ab^2$

$|B_0| = |E_0|$

The force on the sides $b \times b$ is purely magnetic ($\sigma = 0$)

$$\mathbf{F} = b^2 \frac{1}{c} j B \times \underset{\text{effective field on the current}}{\frac{1}{2}} = \frac{b^3}{2c} \frac{c}{4\pi} B_0^2 \underset{\text{time ave}}{\frac{1}{2}} = \frac{b^3}{16\pi} B_0^2$$

$\underline{F = \frac{W}{a}}$ (outward direction)



d) On the top side:

Electric force $F_E = ab \, \underset{\text{effective field on the charge}}{\frac{1}{2}} \langle \sigma E \rangle = \frac{ab}{2} \frac{E_0^2}{4\pi} \underset{\text{time ave}}{\frac{1}{2}}$
(inward direction)

Magnetic force $F_M = ab \, \underset{\text{effective field}}{\frac{1}{2}} \frac{1}{c} \langle j_x B_y \rangle = \frac{ab}{2} \frac{B_0^2}{4\pi} \underset{\text{time-ave}}{\frac{1}{2}}$
(outward direction)

$F_E + F_M = 0 \rightarrow$ <u>no net force on the top and bottom sides.</u>

e) From Maxwell stress tensor,
the force per unit surface area is

$$\vec{F} = \frac{1}{4\pi}\vec{E}(E\cdot n) - \frac{E^2}{8\pi}\vec{n} + \frac{1}{4\pi}\vec{B}(B\cdot n) - \frac{B^2}{8\pi}\vec{n}$$

where $\vec{n}$ is the normal vector

(i) on the $b \times b$ sides:

$$E = 0, \quad B \perp n \rightarrow F = -\frac{B^2}{8\pi}\vec{n}$$

$$F_{t\text{-}ave} = \frac{B_0^2}{8\pi}\frac{1}{2} = \underline{\underline{\frac{B_0^2}{16\pi}}} \text{ (per unit area)}$$

(ii) on the top side:

$$E \parallel n, \quad B \perp n \rightarrow F = \left(\frac{E^2}{8\pi} - \frac{B^2}{8\pi}\right)\vec{n}$$

However, $|E_0| = |B_0| \rightarrow \underline{\underline{F = 0}}$

Agrees w. c), d)

E and M

Electromagnetism I: EM Waves in a dilute gas
(see Feynman lectures on Physics, vol-II, Chapter 32)

a) EM wave travelling along $\hat{x}$; assume $\vec{E} \parallel \vec{e}_y$.
Then the electron in the atom behaves classically as a damped, driven harmonic oscillator,

$$m_e(\ddot{y} + \gamma\dot{y} + \omega_0^2 y) = -eE, \quad E = E_0 e^{-i\omega t}$$

$$\Rightarrow y(t) = \frac{e/m_e}{\omega^2 - \omega_0^2 + i\gamma\omega} E$$

$\Rightarrow$ dipole moment per unit volume

$$P = n_a(-e)y = \frac{n_a e^2/m_e}{\omega_0^2 - \omega^2 - i\gamma\omega} E$$

$$\Rightarrow \boxed{\alpha(\omega) \equiv \frac{P}{\epsilon_0 E} = \frac{n_a e^2}{\epsilon_0 m_e} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}}$$

A quantum mechanical derivation would give this same expression multiplied by the oscillator strength f for the transition.

b) Maxwell eqs. $\nabla \cdot \underline{D} = 0$, $\nabla \cdot \underline{B} = 0$, $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$, $\nabla \times \underline{H} = \frac{\partial \underline{D}}{\partial t}$
with no free charges or currents

$$\underline{B} = \mu_0 \underline{H}, \quad \underline{D} = \epsilon_0 \underline{E} + \underline{P} = \epsilon_0(1+\alpha)\underline{E} \text{ for a single freq. } \omega$$

$$\Rightarrow \frac{\partial^2 \underline{D}}{\partial t^2} - \frac{1}{\mu_0\epsilon_0(1+\alpha)} \nabla^2 \underline{D} = 0, \Rightarrow \text{now let } \underline{P} \propto e^{i(kx - \omega t)}$$

$$\Rightarrow k^2 = \mu_0\epsilon_0(1+\alpha)\omega^2 = (1+\alpha)\frac{\omega^2}{c^2} = \frac{\omega^2 n^2}{c^2}, \quad \boxed{n(\omega) = \sqrt{1+\alpha(\omega)}}$$

One can also get this result using the microscopic $\underline{E}$, $\underline{B}$, and $\underline{P}$ fields: $\nabla \cdot \underline{E} = -(\nabla \cdot \underline{P})/\epsilon_0$, $c^2 \nabla \times \underline{B} = \frac{\partial}{\partial t}\left(\frac{\underline{P}}{\epsilon_0} + \underline{E}\right)$

$$\Rightarrow \frac{\partial^2 \underline{E}}{\partial t^2} - c^2 \nabla^2 \underline{E} = -\frac{1}{\epsilon_0}\frac{\partial^2 \underline{P}}{\partial t^2}; \text{ also } \frac{\partial^2 \underline{P}}{\partial t^2} + \gamma\frac{\partial \underline{P}}{\partial t} + \omega_0^2 \underline{P} = \frac{n_a e^2}{m_e}\underline{E}$$

Together these $\Rightarrow k^2 = (1+\alpha)\frac{\omega^2}{c^2}$, as before.
for a plane wave (Note: we're neglecting dipole-dipole interactions in the dilute gas.)

Electromagnetism I — continued

c) Fourier analysis $\Rightarrow$

$$E(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp[ikx - i\omega(k)t]\hat{E}(k) \text{ where}$$

$$\hat{E}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} E(x,0)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx\, e^{-i(k-k_c)x - x^2/2\sigma^2} = e^{-(k-k_c)^2\sigma^2/2}$$

Now Taylor expand $\omega(k)$ about $k=k_c$:

$$\omega(k) = \omega(k_c) + \left(\frac{d\omega}{dk}\right)_c (k-k_c) + O(k-k_c)^2$$

$$\equiv k_c v_{ph} + v_g(k-k_c) + O(k-k_c)^2$$

$$\boxed{v_{ph} \equiv \frac{\omega}{k} = \text{phase velocity},\ v_g \equiv \frac{d\omega}{dk} = \text{group velocity.}}$$

Let $\kappa \equiv k - k_c$. Then

$$E(x,t) = \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \exp\left[ik_c(x - v_{ph}t) + i\kappa(x - v_g t) - \frac{\kappa^2\sigma^2}{2}\right]$$

$$\boxed{E(x,t) = e^{ik_c(x-v_{ph}t)} \frac{e^{-(x-v_g t)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} = e^{ik_c(x-v_{ph}t)} N(x - v_g t, \sigma)}$$

d) $$v_g = \left(\frac{dk}{d\omega}\right)^{-1} = \frac{c}{n}\left(1 + \frac{d\log n}{d\log\omega}\right)^{-1}.$$

For the dilute gas, $n = \sqrt{1+\alpha} \simeq 1 + \frac{\alpha}{2} = n_r + i n_i$, ($\alpha$ is complex)

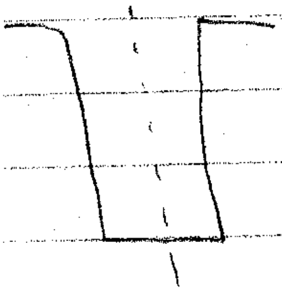
$\Rightarrow$ $$n_r \simeq 1 + \frac{n_a e^2}{2\epsilon_0 m_e} \frac{(\omega_0^2 - \omega^2)}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2},\quad n_i \simeq \frac{n_a e^2}{2\epsilon_0 m_e} \frac{\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

$\Rightarrow$ $n_r = 1$ at $\omega = \omega_0$, $\frac{d\log n_r}{d\log\omega} = -\frac{n_a e^2}{\epsilon_0 m_e \gamma^2}$ at $\omega = \omega_0$

$\Rightarrow$ $\boxed{v_g = 1 + \frac{n_a e^2}{\epsilon_0 m_e \gamma^2} > 1 \text{ at } \omega = \omega_0!}$ This is called anomalous dispersion. It does <u>not</u> violate causality because signals (information) cannot travel faster than the minimum of (v_{ph}, v_g) and now $v_{ph} = c\ (n_r = 1)$. Also, the waves are <u>damped</u> by the electronic resonance maximally at $\omega = \omega_0$.

Fall 2002 General Exam Part 2 Quantum

Problem 1.



$-w \le x \le w$ $\qquad\qquad$ $x > w$

$\psi(x) = \cos Kx$ $\qquad\qquad$ $\psi(x) = Ae^{-\alpha x}$

match ψ: $\quad \cos Kw = Ae^{-\alpha w}$

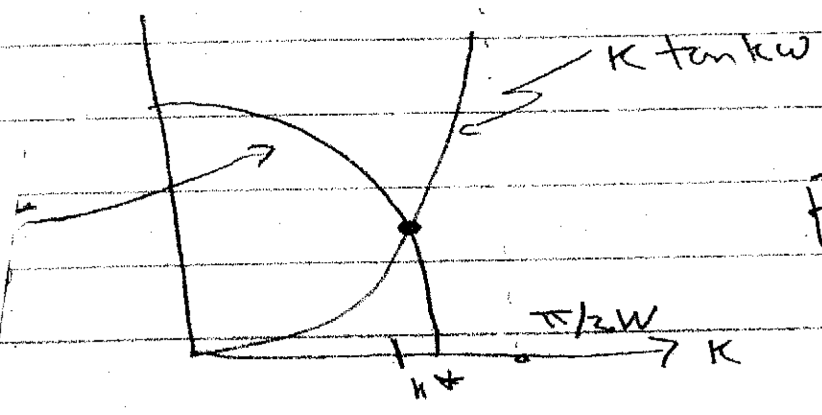
$\frac{d\psi}{dx}$: $\quad -K\sin Kw = -\alpha A e^{-\alpha w}$

$$K\tan Kw = \alpha$$

$$E = \frac{\hbar^2 K^2}{2m} \qquad\qquad E = -\frac{\hbar^2\alpha^2}{2m} + V_0$$

$$\frac{\hbar^2 K^2}{2m} = -\frac{\hbar^2\alpha^2}{2m} + V_0 \qquad \alpha^2 = \frac{2mV_0}{\hbar^2} - K^2$$

$$K\tan Kw = \left(\frac{2mV_0}{\hbar^2} - K^2\right)^{1/2}$$

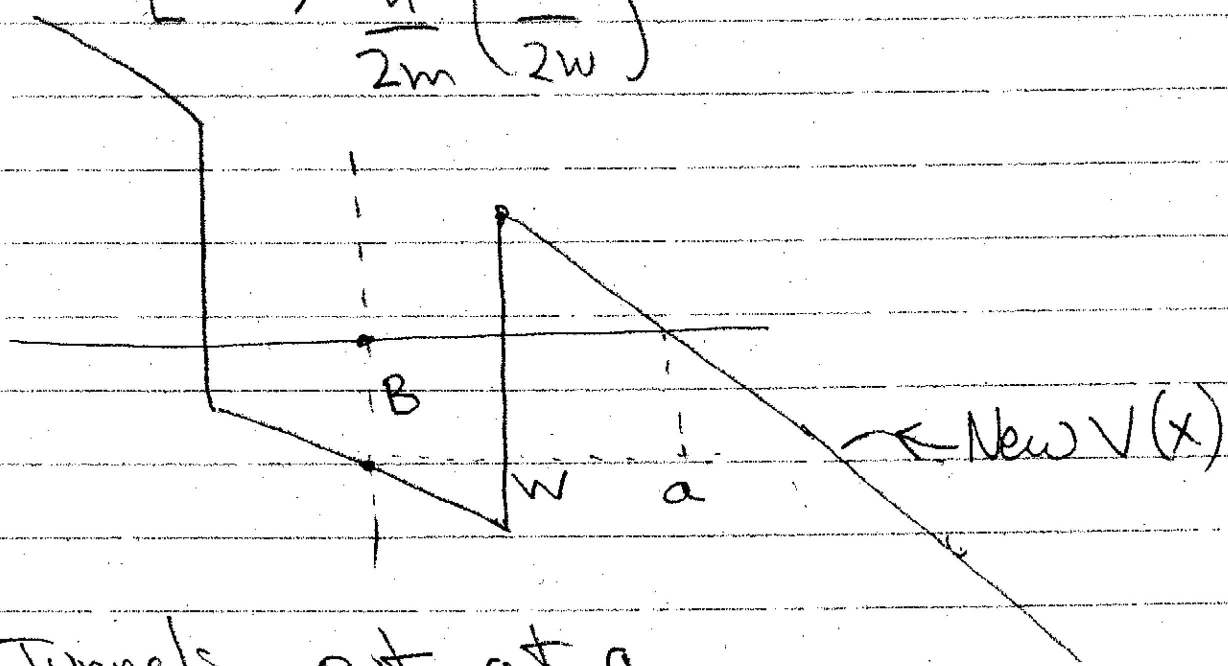


$$E = \frac{\hbar^2 K^{*2}}{2m}$$

2. as $V_0 \to \infty$ $k^* \to \frac{\pi}{2W}$

$$E \to \frac{\hbar^2}{2m}\left(\frac{\pi}{2W}\right)^2$$

3.



Tunnels out at a

4. 0 Perturbation is odd

5. $F = \frac{1}{\hbar}\int_W^a \sqrt{2m(V(x) - B)}$

$V(x) = V_0 - e\mathcal{E}x$

$B = V_0 - e\mathcal{E}a$ $\quad a = \frac{V_0 - B}{e\mathcal{E}}$

$$F = \frac{\sqrt{2m}}{\hbar} \int_w^a \sqrt{(V_0 - B) - e\mathcal{E}x}\, dx$$

$$= \frac{\sqrt{2m}}{\hbar} \frac{2}{3} \frac{-1}{e\mathcal{E}} \left(V_0 - B - e\mathcal{E}x\right)^{3/2} \Big|_w^a$$

$$= \frac{\sqrt{2m}}{\hbar} \frac{2}{3} \frac{1}{e\mathcal{E}} \left(V_0 - B - e\mathcal{E}w\right)^{3/2}$$

6. $B = \frac{1}{2} m v^2 \qquad v^2 = \frac{2B}{m}$

Time to bounce back and forth

$$T = \frac{4W}{v}$$

Hit right wall with frequency $\frac{v}{4W}$

$\frac{\text{Probability to escape}}{\text{unit time}} \qquad \frac{v}{4W} e^{-2F}$

Lifetime $\sim \frac{4W}{v} e^{2F}$

Part 2 Fall 2002

Quantum 2 Solution

$$H = B\begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\omega t} \\ \sin\theta\, e^{i\omega t} & -\cos\theta \end{bmatrix}$$ drop t label

$$|+\rangle = \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\omega t} \end{bmatrix} \qquad |-\rangle = \begin{bmatrix} \sin\theta/2 \\ -\cos\theta/2\, e^{i\omega t} \end{bmatrix}$$

$$|\psi\rangle = C_+|+\rangle + C_-|-\rangle$$

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

$$i\hbar\, \dot{C}_+|+\rangle + i\hbar\, C_+ \frac{d}{dt}|+\rangle$$

$$+ i\hbar\, \dot{C}_-|-\rangle + i\hbar\, C_- \frac{d}{dt}|-\rangle = B|+\rangle - B|-\rangle$$

$$i\hbar \frac{d}{dt}\begin{bmatrix} C_+ \\ C_- \end{bmatrix} = \begin{bmatrix} B - i\hbar\langle+|\frac{d}{dt}|+\rangle & -i\hbar\langle+|\frac{d}{dt}|-\rangle \\ -i\hbar\langle-|\frac{d}{dt}|+\rangle & -B - i\hbar\langle-|\frac{d}{dt}|-\rangle \end{bmatrix}\begin{bmatrix} C_+ \\ C_- \end{bmatrix}$$

$$i\hbar \frac{d}{dt}\begin{bmatrix} C_+ \\ C_- \end{bmatrix} = \begin{bmatrix} B + \hbar\omega\sin^2\theta/2 & -\hbar\omega\cos\theta/2\sin\theta/2 \\ -\hbar\omega\sin\theta/2\cos\theta/2 & -B + \hbar\omega\cos^2\theta/2 \end{bmatrix}\begin{bmatrix} C_+ \\ C_- \end{bmatrix}$$

$$\cos^2\theta/2 = \frac{1}{2}(1+\cos\theta) \qquad \sin^2\theta/2 = \frac{1}{2}(1-\cos\theta)$$

$$i\hbar \frac{d}{dt}\begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} B - \frac{\hbar\omega}{2}\cos\theta & -\frac{\hbar\omega}{2}\sin\theta \\ -\frac{\hbar\omega}{2}\sin\theta & -B + \frac{\hbar\omega}{2}\cos\theta \end{bmatrix}\begin{bmatrix} c_+ \\ c_- \end{bmatrix}$$

drop part proportional to identity

$$i\hbar \frac{d}{dt}\begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} D_z & D_x \\ D_x & -D_z \end{bmatrix}\begin{bmatrix} c_+ \\ c_- \end{bmatrix}$$

$$\begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \exp\left[\frac{-i}{\hbar}\begin{pmatrix} D_z & D_x \\ D_x & -D_z \end{pmatrix} t\right]\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \cos\left(\frac{|\vec{D}|t}{\hbar}\right) - i\,\hat{D}\cdot\vec{\sigma}\,\sin\left(\frac{|\vec{D}|t}{\hbar}\right)$$

$$C_+ = \cos\left(\frac{|\vec{D}|t}{\hbar}\right) - i\,\frac{D_z}{|\vec{D}|}\sin\left(\frac{|\vec{D}|t}{\hbar}\right)$$

$$|C_+|^2 = \cos^2\left(\frac{|\vec{D}|t}{\hbar}\right) + \frac{D_z^2}{D^2}\sin^2\left(\frac{|\vec{D}|t}{\hbar}\right)$$

For $B \gg \hbar\omega \quad D_z \to D$

$|C_+|^2 \to 1$

Adiabatic Theorem!

SM+T PROBLEM 1

a) $n_p = \frac{1}{e^{\beta(E_p - \mu)} - 1}$ $E_p = \frac{p^2}{2m}$

2 POINTS

AT AND BELOW T_{BEC} $\mu = 0$. AT T_{BEC} THERE ARE NO ATOMS IN THE CONDENSATE AND

$$N = \frac{V}{(2\pi\hbar)^3} \int \frac{d^3p}{e^{\beta \frac{p^2}{2m}} - 1} = (2\pi\hbar)^{-3} V \left(\frac{2m}{\beta}\right)^{3/2} 4\pi \underbrace{\int_0^\infty \frac{x^2 dx}{e^{x^2} - 1}}_{I_1}$$

$$n = \frac{1}{2\pi^2} \left(\frac{2mkT}{\hbar^2}\right)^{3/2} I_1$$

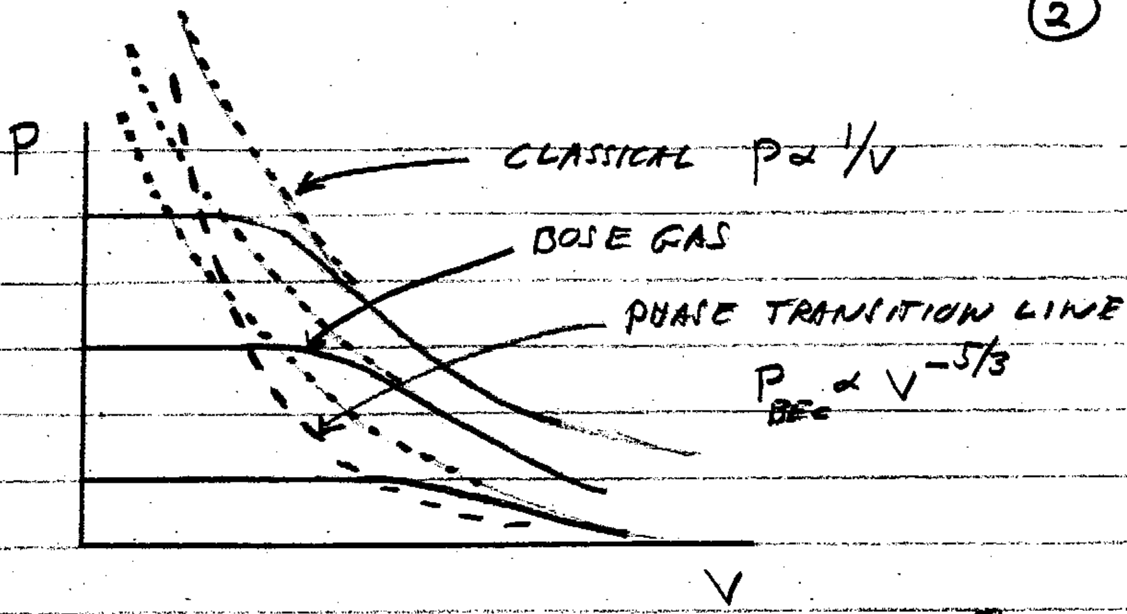
$$\underline{\underline{kT_{BEC} = \frac{(2\pi^2)^{2/3}}{I_1^{2/3}} n^{2/3} \frac{\hbar^2}{2m}}}$$

b) THE ABOVE INTEGRAL (WITH $\mu = 0$) ALSO APPLIES BELOW T_{BEC}, BUT IT THEN GIVES THE # OF NON-CONDENSED ATOMS. SO ON AN ISOTHERM BELOW $V_{CRITICAL}$

2 POINTS

- $N_{NON-CONDENSED}$ IS CONSTANT
- T IS CONSTANT

$\Rightarrow$ <u>P IS CONSTANT</u> (KINETIC ORIGIN OF PRESSURE)



c) $U = \frac{V}{(2\pi\hbar)^3} \int \frac{\left(\frac{p^2}{2m}\right) d^3p}{e^{\beta \frac{p^2}{2m}} - 1} = (2\pi\hbar)^{-3} V \left(\frac{2m}{\beta}\right)^{5/2} 4\pi \frac{1}{2m} \underbrace{\int_0^\infty \frac{x^4\,dx}{e^{x^2}-1}}_{I_2}$

2 POINTS

$$= N_c \frac{I_2}{I_1}\left(\frac{2m}{\beta}\right)\frac{1}{2m} = \frac{I_2}{I_1} N_c kT \propto T^{5/2}$$

$$C_V = \frac{5}{2}\frac{I_2}{I_1} N_c k = \underline{\underline{\frac{V}{2\pi^2}\left(\frac{2mkT}{\hbar^2}\right)^{3/2} k\left(\frac{5}{2} I_2\right) \propto T^{3/2}}}$$

d) $dS_1 = -dS_2$ FOR 1 CYCLE

4 POINTS

$\Delta S_1 = -\Delta S_2$ FOR ENTIRE PROCESS

$$dU = T\,dS - p\,\cancelto{0}{dV} \Rightarrow T \left.\frac{\partial S}{\partial T}\right|_V = \left.\frac{\partial U}{\partial T}\right|_V = C_V = \underbrace{aT^{3/2}}_{\text{FROM c)}}$$

$$\frac{dS}{dT} = \frac{C_V}{T}$$

$$S = \int_0^T \frac{C_V}{T} dT = a\int_0^T T^{1/2} dT = \frac{2}{3} a T^{3/2}$$

$$\Delta S_1 = \frac{2}{3} a \left(T_0^{3/2} - T_1^{3/2} \right)$$

$$\Delta S_2 = \frac{2}{3} a \left(T_0^{3/2} - T_2^{3/2} \right)$$

$$\Delta S_1 + \Delta S_2 = 0 \Rightarrow \underline{\underline{T_0^{3/2} = \frac{1}{2}\left(T_1^{3/2} + T_2^{3/2} \right)}}$$

HEAT TRANSFERRED FROM F_2

$$Q_2 = \int_{T_0}^{T_2} T\,dS = \int_{T_0}^{T_2} c_V\,dT = \frac{2}{5} a \left(T_2^{5/2} - T_0^{5/2} \right)$$

HEAT TRANSFERRED TO F_1

$$Q_1 = \int_{T_1}^{T_0} T\,dS = \frac{2}{5} a \left(T_0^{5/2} - T_1^{5/2} \right)$$

$$\text{WORK} = Q_2 - Q_1 = \underline{\underline{\frac{2}{5} a \left(T_2^{5/2} + T_1^{5/2} - 2T_0^{5/2} \right)}}$$

SM&T
#2 SUPERCONDUCTING PHASE TRANSITION

a) $C_H \equiv \left.\frac{\partial Q}{\partial T}\right|_H = \left.\frac{T dS}{\partial T}\right|_H$

2 POINTS

$$dS = \left.\frac{\partial S}{\partial T}\right|_M dT + \left.\frac{\partial S}{\partial M}\right|_T dM$$

$$\left.\frac{\partial S}{\partial T}\right|_H = \left.\frac{\partial S}{\partial T}\right|_M + \left.\frac{\partial S}{\partial M}\right|_T \cancelto{0}{\left.\frac{\partial M}{\partial T}\right|_H}$$

IN THIS CASE SINCE M IS INDEP. OF T

$$C_H = T\left.\frac{\partial S}{\partial T}\right|_M = \left.\frac{\partial Q}{\partial T}\right|_M \equiv C_M$$

b) The transition takes place at constant T and H. The thermodynamic function whose variables are T & H is the Gibbs function:

3 POINTS

U	M	F
S		-T
$\mathcal{H}$	-H	G

$$dG = -S\,dT - M\,dH$$

$G_{SUPER} = G_{NORMAL}$ at every point on $H_c(T)$

$$dG_S = dG_N$$

$$-S_S\,dT - M_S\,dH = -S_N\,dT - \cancelto{0}{M_N}\,dH$$

$$\left.\frac{dH}{dT}\right|_{\text{TRANSITION LINE}} = \frac{dH_c(T)}{dT} = \frac{S_N - S_S}{M_S} = \underline{-\frac{4\pi}{V H_c(T)}\left(S_N - S_S\right)}$$

2 POINTS

c) By the third law $S \to 0$ as $T \to 0$.
But the figure shows $H_c(T=0)$ is finite.
Therefore $\underline{\frac{dH_c(T)}{dT} \to 0 \text{ as } T \to 0}$

The transition is second order where
$S_N - S_S = 0$, that is, the latent
heat equals zero.

$$S_N - S_S = -\frac{V}{4\pi} H_c(T) \frac{dH_c(T)}{dT}$$

At $T=0$ the transition is 2^{nd} order
because both entropies $\to 0$

At $T = T_c(H=0)$ the transition is 2^{nd} order
since $H_c(T) = 0$ and $dH_c(T)/dT$ is finite

At all other temperatures the transition
is 1^{st} order since both $H_c(T)$
and $dH_c(T)/dT$ are finite.

d) Use H & T as variables

3 POINTS

$$dS(H,T) = \underbrace{\frac{\partial S}{\partial T}\Big|_H}_{\frac{C_H}{T}} dT + \underbrace{\frac{\partial S}{\partial H}\Big|_T}_{-\frac{\partial M}{\partial T}\big|_H} dH$$

$-\frac{\partial M}{\partial T}\Big|_H$ by a Maxwell relation $= 0$

$$S = \int \frac{C_H}{T}\,dT = \frac{a}{3}T^3 V \qquad T < T_c$$

$$= \frac{b}{3}T^3 V + \gamma T V \qquad T > T_c$$

$$S_N - S_S = \left(\frac{b-a}{3}\right)T^3 V + \gamma T V = 0 \text{ when } T = T_c(H=0)$$

$$\gamma = \left(\frac{a-b}{3}\right)T_c^2 \qquad \underline{T_c(H=0) = \sqrt{\frac{3\gamma}{(a-b)}}}$$