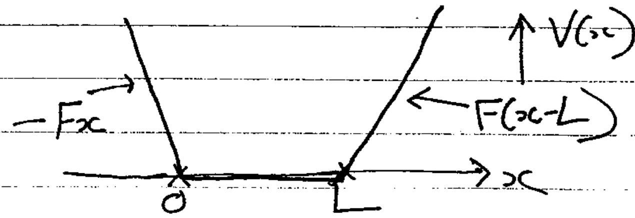
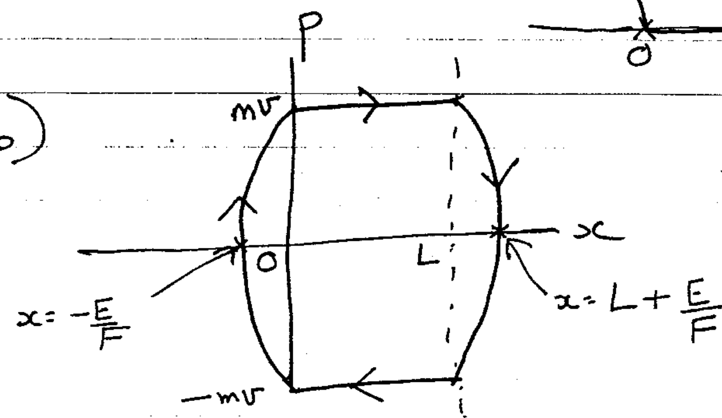
Mechanics Solutions.

Problem 1

a)

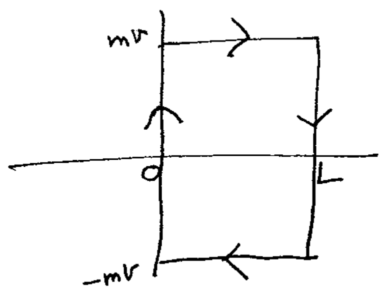


b)



$E = \frac{1}{2}mv^2$

c)



$A = 2Lmv = 2L\sqrt{2mE}$

d) Relative to moving wall, ~~speed~~ velocity changes from $v-u$ to $u-v$ so in fixed frame, changes from v to $-v+2u$. After collision with fixed wall, this becomes $v-2u$ so in one cycle $v \to v-2u$.

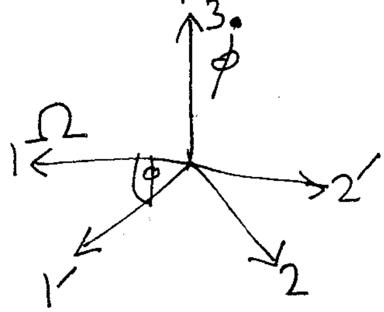
In one cycle, time $\frac{2L}{v}+O(u)$, $L \to L+\frac{2Lu}{v}+O(u^2)$

so $Lv \to \left(L+\frac{2Lu}{v}\right)v - L(2u) = Lv + O(u^2)$

so after time of order $\frac{L}{u}$, Lv only changes by order Lu and in limit $u \to 0$, Lv stays constant, and so A stays constant.

Problem 2

Spin of earth is $\Omega = 2\pi$ per day



Ang vel of axes $1'2'3$ is

$$(\Omega\cos\phi, -\Omega\sin\phi, \dot\phi) = \vec{\omega}$$

Ang vel of disk is

$$(s, -\Omega\sin\phi, \dot\phi)$$

Ang momentum of disk is $\left(Is, -\frac{1}{2}I\Omega\sin\phi, \frac{1}{2}I\dot\phi\right) = \vec{h}$

Torque $= \dfrac{d\vec{h}}{dt} + \vec{\omega}\times\vec{h}$ and components in $1', 3$ directions are zero.

$$I\dot{s} = 0, \quad s \text{ is constant}$$

$$\frac{1}{2}I\ddot\phi - \frac{1}{2}I\Omega^2\sin\phi\cos\phi + Is\Omega\sin\phi = 0$$

so ϕ can be constant if $\sin\phi = 0$ (only solution if $s \gg \Omega$)

Neglecting $\frac{\Omega}{s}$, $\ddot{\phi} = -2s\Omega \sin\phi$

so if $s > 0$, small oscillations about $\phi = 0$

i.e. north, have period $\frac{2\pi}{\sqrt{2s\Omega}}$

(if $s < 0$, oscillate about south)

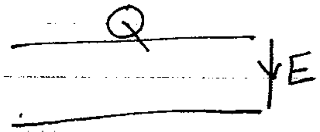
If this is 10 sec, and $\Omega = \frac{2\pi}{24 \times 3600}$ sec^{-1}

then $\frac{(2\pi)^2 \, 24 \times 3600}{2s \cdot 2\pi} = 100$, $\frac{s}{2\pi} = 432$

so spin of disk ~ 430 rev per sec

E and M solutions

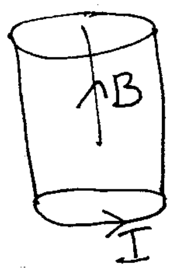
Problem 1 a)



$$E = \frac{V}{d_c} \qquad \sigma = \frac{Q}{A_c}$$

$$E = 4\pi\sigma \quad \text{so} \quad \frac{V}{d_c} = \frac{4\pi Q}{A_c}$$

$$\text{and } C = \frac{Q}{V} = \frac{A_c}{4\pi d_c}$$



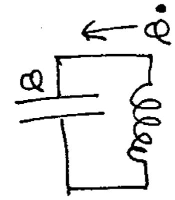
$$B = \frac{4\pi}{c} j \qquad j = \frac{NI}{d_L}$$

$$\text{Flux} = NA_L B = \frac{4\pi}{c} \frac{N^2 A_L I}{d_L}$$

$$\text{and } L = \frac{1}{c} \frac{\text{Flux}}{I} = \frac{4\pi}{c^2} \frac{N^2 A_L}{d_L}$$

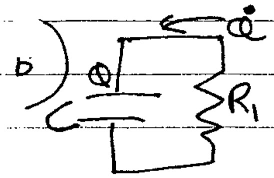
Oscillator equation is

$$\frac{Q}{C} + L\ddot{Q} = 0$$

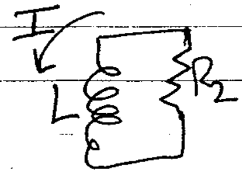


$$\text{so } T_0 = 2\pi\sqrt{LC} = \frac{2\pi}{c}\sqrt{\frac{N^2 A_L A_c}{d_L d_c}}$$

$$\text{i.e. } c = \frac{2\pi N}{T_0}\sqrt{\frac{A_L A_c}{d_L d_c}}$$

b) 

$$R_1 \dot{Q} + \frac{Q}{C} = 0 \quad \text{so } T_1 = R_1 C$$



$$L\dot{I} + R_2 I = 0 \quad \text{so } T_2 = L/R_2$$

$$\text{so } \frac{R_1}{R_2} LC = T_1 T_2$$

$$\frac{R_2}{R_1} = \left(\frac{T_0}{2\pi}\right)^2 \frac{1}{T_1 T_2}$$

Problem 2

Relevant equations are $\frac{\partial E_x}{\partial z} + \frac{1}{c}\frac{\partial B_y}{\partial t} = 0$, $-\frac{\partial B_y}{\partial z} = \frac{1}{c}\frac{\partial D_x}{\partial t}$

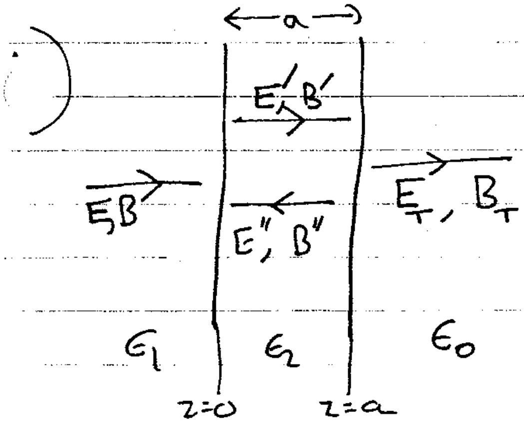
so with $\vec{E} = (E_x e^{i(kz-\omega t)}, 0, 0)$

$\vec{B} = (0, B_y e^{i(kz-\omega t)}, 0)$

$$kE = \frac{\omega}{c} B, \quad kB = \epsilon \frac{\omega}{c} E, \quad k^2 = \epsilon \left(\frac{\omega}{c}\right)^2$$

$$\text{so } B = \sqrt{\epsilon}\, E$$

For wave in $-Z$ direction, get $e^{-i(kz+\omega t)}$, $B = -\sqrt{\epsilon} E$



In 1, $E_x = E e^{i(k_1 z - \omega t)}$
$B_y = \sqrt{\epsilon_1} E e^{i(k_1 z - \omega t)}$

In 0, $E_x = E_T e^{i(k_0 z - \omega t)}$

$B_y = \sqrt{\epsilon_0} E_T e^{i(k_0 z - \omega t)}$

In 2, $E_x = E' e^{i(k_2 z - \omega t)} + E'' e^{-i(k_2 z + \omega t)}$

$B_y = \sqrt{\epsilon_2} E' e^{i(k_2 z - \omega t)} - \sqrt{\epsilon_2} E'' e^{-i(k_2 z + \omega t)}$

At interfaces E_x and B_y must be continuous

At $z=0$, $E = E' + E''$, $\sqrt{\epsilon_1} E = \sqrt{\epsilon_2}(E' - E'')$

At $z=a$, $E_T = E' e^{ik_2 a} + E'' e^{-ik_2 a}$, $\sqrt{\epsilon_0} E_T = \sqrt{\epsilon_2}(E' e^{ik_2 a} - E'' e^{-ik_2 a})$

These give $\dfrac{E'}{E''} = \dfrac{1 + \sqrt{\frac{\epsilon_1}{\epsilon_2}}}{1 - \sqrt{\frac{\epsilon_1}{\epsilon_2}}}$ and $\dfrac{E'}{E''} e^{2ik_2 a} = \dfrac{1 + \sqrt{\frac{\epsilon_0}{\epsilon_2}}}{1 - \sqrt{\frac{\epsilon_0}{\epsilon_2}}}$

which are compatible only if $e^{2ik_2 a}$ is real, $\therefore \pm 1$

$+1$ (i.e $a = \frac{1}{2}$ wavelength in 2) only works if $\epsilon_1 = \epsilon_0$

-1 i.e. $a = \frac{1}{4}$ wavelength in 2 works if $\sqrt{\frac{\epsilon_0}{\epsilon_2}} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$

i.e $\epsilon_2 = \sqrt{\epsilon_0 \epsilon_1}$

Statistical Mechanics Problem 1
Solution

(a) Each single particle state of momentum $\vec{q}$ can be described by a "wave function"

$$\langle 0|\phi(\vec{x},0)|\vec{q}\rangle \propto e^{i\vec{q}\cdot\vec{x}/\hbar} ,$$

where $\langle 0|$ is the vacuum, so the periodicity condition implies that each component of $\vec{q}$ obeys

$$q_j = \frac{2\pi\hbar}{L} k_j ,$$

where k_j is an integer. If there were only one spin state, the general multiparticle state would be described by an integer occupation number $n_{\vec{q}}$ for each allowed momentum $\vec{q}$, so the sum over all states would be written as

$$\begin{aligned} Z_1 &= \prod_{k_x=0}^{\infty} \prod_{k_y=0}^{\infty} \prod_{k_z=0}^{\infty} \sum_{n_{\vec{q}}=0}^{\infty} \exp\{-n_{\vec{q}} E(|\vec{q}|)/kT\} \\ &= \prod_{k_x=0}^{\infty} \prod_{k_y=0}^{\infty} \prod_{k_z=0}^{\infty} \frac{1}{1-e^{-E(|\vec{q}|)/kT}} . \end{aligned}$$

Then

$$\ln Z_1 = -\sum_{k_x=0}^{\infty} \sum_{k_y=0}^{\infty} \sum_{k_z=0}^{\infty} \ln\left(1-e^{-E(|\vec{q}|)/kT}\right) .$$

Approximating the sum as an integral,

$$\ln Z_1 = -\int d^3k \ln\left(1-e^{-E(|\vec{q}|)/kT}\right) .$$

Changing the variable of integration to the momentum $\vec{q} = 2\pi\hbar\vec{k}/L$,

$$\ln Z_1 = -\frac{L^3}{(2\pi\hbar)^3} \int d^3q \ln\left(1-e^{-E(|\vec{q}|)/kT}\right) .$$

For two spin states $Z = Z_1^2$, since

$$Z = \sum_{\text{all states}} e^{-E_{\text{tot}}/kT} = \sum_{\substack{\text{all states of}\\ \text{spin up}\\ \text{particles}}} \sum_{\substack{\text{all states of}\\ \text{spin down}\\ \text{particles}}} e^{-\left(E_{\text{tot}}^{(\text{spin up})}+E_{\text{tot}}^{(\text{spin down})}\right)/kT} = Z_1^2 .$$

Finally,

$$\boxed{F(L,T) = -2kT\ln Z_1 = 2kT\frac{L^3}{(2\pi\hbar)^3} \int d^3q \ln\left(1-e^{-E(|\vec{q}|)/kT}\right) .}$$

(b) Since each term in Z is proportional to the probability of the corresponding state,

$$\mathcal{E} = \frac{1}{Z} \sum_{\text{all states}} E_{\text{tot}} e^{-E_{\text{tot}}/kT} .$$

But

$$\begin{aligned} F + TS &= -kT \ln Z + T \frac{\partial}{\partial T}(kT \ln Z) \\ &= kT^2 \frac{\partial}{\partial T} \ln Z \\ &= kT^2 \frac{1}{Z} \frac{\partial}{\partial T} \sum_{\text{all states}} e^{-E_{\text{tot}}/kT} \\ &= kT^2 \frac{1}{Z} \sum_{\text{all states}} \frac{E_{\text{tot}}}{kT^2} e^{-E_{\text{tot}}/kT} \\ &= \frac{1}{Z} \sum_{\text{all states}} E_{\text{tot}} e^{-E_{\text{tot}}/kT} \\ &= \mathcal{E} . \end{aligned}$$

(c) Such a change is adiabatic, which means that entropy is conserved. The entropy is given by

$$S = -\frac{\partial F}{\partial T} = \frac{4\pi^2}{45} \frac{k^4 T^3}{(\hbar c)^3} L^3 .$$

Thus $T^3 L^3$ is conserved, so TL is conserved, and therefore

$$\boxed{T_f = \frac{T}{\alpha} .}$$

(d) The derivation starts the same way, except that the occupation numbers are summed only over 0 and 1, instead of from 0 to infinity, and also there are 4 spin states instead of 2. Thus,

$$\begin{aligned} Z_{1e} &= \prod_{k_x=0}^{\infty} \prod_{k_y=0}^{\infty} \prod_{k_z=0}^{\infty} \sum_{n_{\vec{q}}=0}^{1} \exp\{-n_{\vec{q}} E_e(|\vec{q}|)/kT\} \\ &= \prod_{k_x=0}^{\infty} \prod_{k_y=0}^{\infty} \prod_{k_z=0}^{\infty} \left(1 + e^{-E_e(|\vec{q}|)/kT}\right) . \end{aligned}$$

Continuing as before,

$$\ln Z_{1e} = \frac{L^3}{(2\pi\hbar)^3} \int d^3q \ln\left(1 + e^{-E_e(|\vec{q}|)/kT}\right) ,$$

and

$$\boxed{F_e(L,T) = -4kT \ln Z_{1e} = -4kT \frac{L^3}{(2\pi\hbar)^3} \int d^3q \ln\left(1 + e^{-E_e(|\vec{q}|)/kT}\right) .}$$

(e) Again the entropy must have the same value before and after. For the photons

$$S_\gamma = \frac{4\pi^2}{45} \frac{k^4T^3}{(\hbar c)^3} L^3 ,$$

while for the electron-positron gas there is a contribution to the final state given by

$$S_e = -\frac{\partial F_e}{\partial T} = \frac{7\pi^2}{45} \frac{k^4T_f^3}{(\hbar c)^3} L_f^3 .$$

So

$$\frac{4\pi^2}{45} \frac{k^4T^3}{(\hbar c)^3} L^3 = \frac{4\pi^2}{45} \frac{k^4T_f^3}{(\hbar c)^3} \alpha^3 L^3 + \frac{7\pi^2}{45} \frac{k^4T_f^3}{(\hbar c)^3} \alpha^3 L^3 ,$$

which gives

$$4T^3 = 4T_f^3\alpha^3 + 7T_f^3\alpha^3 ,$$

or

$$\boxed{T_f = \left(\frac{4}{11}\right)^{1/3} \frac{T}{\alpha} .}$$

Statistical Mechanics Problem 2
Solution

(a) First consider the evolution of the site probabilities $q(k,t)$:

$$q(k,t+\Delta t) = \frac{1}{2}q(k-1,t) + \frac{1}{2}q(k+1,t) \ ,$$

which can be rewritten as

$$P(x,t+\Delta t) = \frac{1}{2}P(x-\Delta x,t) + \frac{1}{2}P(x+\Delta x,t) \ .$$

Then, to re-express this equation as a time derivative, write

$$P(x,t+\Delta t) = P(x,t) + \frac{1}{2}\Big[P(x+\Delta x,t) - 2P(x,t) + P(x-\Delta x,t)\Big] \ .$$

Assuming that $P(x,t)$ is a smooth function,

$$\frac{\partial P}{\partial x}\left(x+\tfrac{1}{2}\Delta x,t\right) \simeq \frac{P(x+\Delta x,t)-P(x,t)}{\Delta x}$$

and

$$\frac{\partial P}{\partial x}\left(x-\tfrac{1}{2}\Delta x,t\right) \simeq \frac{P(x,t)-P(x-\Delta x,t)}{\Delta x} \ .$$

So

$$\begin{aligned} P(x+\Delta x,t) - 2P(x,t) + P(x-\Delta x,t) &\simeq \left\{\frac{\partial P}{\partial x}\left(x+\tfrac{1}{2}\Delta x,t\right) - \frac{\partial P}{\partial x}\left(x-\tfrac{1}{2}\Delta x,t\right)\right\}\Delta x \\ &\simeq \frac{\partial^2 P(x,t)}{\partial x^2}\,\Delta x^2 \ . \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial P}{\partial t} &\simeq \frac{P(x,t+\Delta t)-P(x,t)}{\Delta t} \\ &\simeq \frac{1}{2\Delta t}\,\frac{\partial^2 P(x,t)}{\partial x^2}\,\Delta x^2 \ . \end{aligned}$$

Thus,

$$\frac{\partial P(x,t)}{\partial t} = D\,\frac{\partial^2 P(x,t)}{\partial x^2} \ ,$$

where

$$\boxed{D = \frac{\Delta x^2}{2\,\Delta t} \ .}$$

(b) Generalizing from the calculation above,

$$q(k, t + \Delta t) = \sum_{\Delta k=-5}^{5} p(\Delta k) q(k - \Delta k, t) ,$$

which can be rewritten as

$$P(x, t + \Delta t) = \sum_{\Delta k=-5}^{5} p(\Delta k) P(x - \Delta k \, \Delta x, t) ,$$

Expanding $P(x - \Delta k \, \Delta x, t)$ in a Taylor series,

$$P(x - \Delta k \, \Delta x, t) = P(x, t) - \Delta k \, \Delta x \, \frac{\partial P}{\partial x}(x, t) + \frac{1}{2} \Delta k^2 \, \Delta x^2 \, \frac{\partial^2 P}{\partial x^2}(x, t) + \dots ,$$

so

$$P(x, t + \Delta t) = P(x, t) - \sum_{\Delta k=-5}^{5} p(\Delta k) \Delta k \, \Delta x \, \frac{\partial P}{\partial x}(x, t) + \\ + \frac{1}{2} \sum_{\Delta k=-5}^{5} p(\Delta k) \Delta k^2 \, \Delta x^2 \, \frac{\partial^2 P}{\partial x^2}(x, t) + \dots .$$

Thus,

$$\frac{\partial P(x, t)}{\partial t} \simeq \frac{P(x, t + \Delta t) - P(x, t)}{\Delta t} \\ \simeq \alpha \frac{\partial P(x, t)}{\partial x} + D \frac{\partial^2 P(x, t)}{\partial x^2} ,$$

where

$$\boxed{\begin{aligned} \alpha &= -\frac{\Delta x}{\Delta t} \sum_{\Delta k=-5}^{5} p(\Delta k) \Delta k \\ D &= \frac{\Delta x^2}{2 \, \Delta t} \sum_{\Delta k=-5}^{5} p(\Delta k) \Delta k^2 . \end{aligned}}$$

(c) Defining 6 unit vectors

$$\begin{aligned} \hat{n}^{(1)} &\equiv (1, 0, 0) \quad & \hat{n}^{(2)} &\equiv (-1, 0, 0) \\ \hat{n}^{(3)} &\equiv (0, 1, 0) \quad & \hat{n}^{(4)} &\equiv (0, -1, 0) \\ \hat{n}^{(5)} &\equiv (0, 0, 1) \quad & \hat{n}^{(6)} &\equiv (0, 0, -1) , \end{aligned}$$

the evolution of $q(\vec{k},t)$ can be written as

$$q(\vec{k},t+\Delta t) = \frac{1}{6}\sum_{i=1}^{6} q(\vec{k}+\hat{n}^{(i)},t) ,$$

which can be rewritten as

$$P(\vec{x},t+\Delta t) = \frac{1}{6}\sum_{i=1}^{6} P(\vec{x}+\hat{n}^{(i)}\Delta x,t) .$$

Looking first at only the sum of $i=1$ and $i=2$, the contribution to the right-hand side can be written as

$$\begin{aligned}
&\frac{1}{6}\Big[P(x+\Delta x,y,z,t)+P(x-\Delta x,y,z,t)\Big] \\
&\qquad = \frac{1}{3}P(x,y,z,t) + \frac{1}{6}\Big[P(x+\Delta x,y,z,t) - 2P(x,y,z,t) + P(x-\Delta x,y,z,t)\Big] \\
&\qquad \simeq \frac{1}{3}P(x,y,z,t) + \frac{1}{6}\frac{\partial^2 P(x,y,z,t)}{\partial x^2}\Delta x^2 .
\end{aligned}$$

Adding similar contributions for $i=3,4$ and $i=5,6$, one finds

$$P(\vec{x},t+\Delta t) = P(\vec{x},t) + \frac{1}{6}\nabla^2 P(\vec{x},t)\,\Delta x^2 ,$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the Laplacian operator. So

$$\boxed{\frac{\partial P(\vec{x},t)}{\partial t} = D\nabla^2 P(\vec{x},t) ,}$$

where

$$\boxed{D = \frac{1}{6}\frac{\Delta x^2}{\Delta t} .}$$

Quantum Mechanics Problem 1
Solution

(a) The time-dependent Schrödinger equation is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\ .$$

Thus

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}|\Psi|^2 &= \Psi^*\left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right] - \left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V\Psi^*\right]\Psi \\ &= -\frac{\hbar^2}{2m}\left[\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \Psi\frac{\partial^2\Psi^*}{\partial x^2}\right] \\ &= -\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right]\ . \end{aligned}$$

So

$$\begin{aligned} \frac{\partial}{\partial t}|\Psi|^2 &= \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right] \\ &= -\frac{\partial j}{\partial x}\ , \end{aligned}$$

where

$$\boxed{j = -\frac{i\hbar}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right]\ .}$$

(b) Since ρ is independent of time,

$$\frac{\partial j}{\partial x} = 0\ .$$

Thus j must have the same value for both $x < -a$ and $x > a$. For $x < -a$,

$$\begin{aligned} j = -\frac{i\hbar}{2m}\Big\{&\left[e^{-ikx} + R^*e^{ikx}\right]\left[ike^{ikx} - ikRe^{-ikx}\right] \\ &- \left[e^{ikx} + Re^{-ikx}\right]\left[-ike^{-ikx} + ikR^*e^{ikx}\right]\Big\} \end{aligned}$$

By equating the two expressions for j one finds

$$1 - |R|^2 = |T|^2 ,$$

from which the result follows immediately.

(c) Write

$$\psi_2(x) = \alpha\psi_1(x) + \beta\psi_1^*(x) ,$$

where α and β are coefficients to be determined. For $x < -a$, one has

$$\begin{aligned}\psi_2(x) &= \alpha\left[e^{ikx} + Re^{-ikx}\right] + \beta\left[e^{-ikx} + R^* e^{ikx}\right] \\ &= (\alpha + \beta R^*)e^{ikx} + (\beta + \alpha R)e^{-ikx} .\end{aligned}$$

Matching the coefficients of e^{ikx} and e^{-ikx} with the desired behavior for ψ_2, one has

$$\alpha + \beta R^* = 0$$

$$\beta + \alpha R = T' .$$

For $x > a$,

$$\psi_2(x) = \alpha T e^{ikx} + \beta T^* e^{-ikx} .$$

Again matching coefficients,

$$\beta T^* = 1$$

$$\alpha T = R' .$$

Thus,

$$\beta = 1/T^*$$

and

$$\alpha = -\beta R^* = -\frac{R^*}{T^*} .$$

Then

$$T' = \beta + \alpha R = \frac{1 - |R|^2}{T^*} = \frac{|T|^2}{T^*} = \boxed{T}$$

and

$$R' = \alpha T = \boxed{-\frac{R^* T}{T^*} .}$$

(d) If $V(x)$ is symmetric, then there is no difference between scattering from the left and scattering from the right. Then $\psi_2(x) = \psi_1(-x)$, so $R' = R$. Thus

$$R = -\frac{R^* T}{T^*} ,$$

which implies that

$$(R^* T) = -(R^* T)^* .$$

Since $R^* T$ is equal to the negative of its complex conjugate, it must be purely imaginary.

Quantum Mechanics Problem 2
Solution

(a) The matrix is already block diagonal. The lower right 2×2 block is the Pauli matrix σ_x, while the upper left 2×2 block is $2\sigma_x$. Since σ_x has eigenvalues $+1$ and -1, with eigenvectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}, \quad \text{respectively,}$$

the diagonalization of Q follows immediately. The eigenvalues q_i and the corresponding eigenvectors v_{q_i} are given by

$$q_1 = 2 \quad , \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|{+}{+}\rangle_{12} + |{-}{-}\rangle_{12}\right)$$

$$q_2 = -2 \, , \quad v_{-2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|{+}{+}\rangle_{12} - |{-}{-}\rangle_{12}\right)$$

$$q_3 = 1 \quad , \quad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle_{12} + |{-}{+}\rangle_{12}\right)$$

$$q_4 = -1 \, , \quad v_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle_{12} - |{-}{+}\rangle_{12}\right) .$$

(b) The four basis vectors appearing in the expression for $|\Omega\rangle_{123}$ occur with probabilities $\frac{1}{2}|a|^2$, $\frac{1}{2}|a|^2$, $\frac{1}{2}|b|^2$, and $\frac{1}{2}|b|^2$, respectively. Since particle 1 is spin-up for the first two of these basis vectors,

$$p_1 = \frac{1}{2}|a|^2 + \frac{1}{2}|a|^2 = \boxed{|a|^2 .}$$

Alternatively, one could recognize that $|\Omega\rangle_{123}$ is a product state vector $|\Phi\rangle_1|\Psi\rangle_{23}$, so the result has to be the same as would be found for $|\Phi\rangle_1$. Particle 2 is spin-up for the first and third basis vectors in the expansion of $|\Omega\rangle_{123}$, so

$$p_2 = \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2 = \boxed{\frac{1}{2} .}$$

This is the same answer that one would find from $|\Psi\rangle_{23}$.

(c) The measurement of Q can be most easily understood by rewriting $|\Omega\rangle_{123}$ in a basis in which Q is diagonal. From the answer to (a), one can write

$$|++\rangle_{12} = \frac{1}{\sqrt{2}}\Big(|Q{=}2\rangle_{12} + |Q{=}-2\rangle_{12}\Big)$$

$$|--\rangle_{12} = \frac{1}{\sqrt{2}}\Big(|Q{=}2\rangle_{12} - |Q{=}-2\rangle_{12}\Big)$$

$$|+-\rangle_{12} = \frac{1}{\sqrt{2}}\Big(|Q{=}1\rangle_{12} + |Q{=}-1\rangle_{12}\Big)$$

$$|-+\rangle_{12} = \frac{1}{\sqrt{2}}\Big(|Q{=}1\rangle_{12} - |Q{=}-1\rangle_{12}\Big) .$$

Then

$$\begin{aligned}
|\Omega\rangle_{123} &= \frac{1}{2}a\Big(|Q{=}2,-\rangle_{123} + |Q{=}-2,-\rangle_{123}\Big) \\
&\quad - \frac{1}{2}a\Big(|Q{=}1,+\rangle_{123} + |Q{=}-1,+\rangle_{123}\Big) \\
&\quad + \frac{1}{2}b\Big(|Q{=}1,-\rangle_{123} - |Q{=}-1,-\rangle_{123}\Big) \\
&\quad - \frac{1}{2}b\Big(|Q{=}2,+\rangle_{123} - |Q{=}-2,+\rangle_{123}\Big) \\
&= \frac{1}{2}|Q{=}2\rangle_{12}\Big(-b\,|+\rangle_3 + a\,|-\rangle_3\Big) \\
&\quad + \frac{1}{2}|Q{=}-2\rangle_{12}\Big(b\,|+\rangle_3 + a\,|-\rangle_3\Big) \\
&\quad + \frac{1}{2}|Q{=}1\rangle_{12}\Big(-a\,|+\rangle_3 + b\,|-\rangle_3\Big) \\
&\quad + \frac{1}{2}|Q{=}-1\rangle_{12}\Big(-a\,|+\rangle_3 - b\,|-\rangle_3\Big) .
\end{aligned}$$

Thus, the probability p_{-1} that Alice measures $Q = -1$ is given by

$$p_{-1} = \frac{1}{4}\Big(|a|^2 + |b|^2\Big) = \boxed{\frac{1}{4}\,.}$$

(Actually, the probability of measuring any particular value of Q is $1/4$.)

(d) The measurement projects the state vector into the subspace for which $Q = -1$, and then the state vector must be rescaled to unit norm. Thus,

$$|\Omega'\rangle_{123} = |Q{=}{-1}\rangle_{12}\Big(-a\,|+\rangle_3 - b\,|-\rangle_3\Big)$$

$$= \boxed{-|Q{=}{-1}\rangle_{12}|\Phi\rangle_3\ .}$$

Since this is a product state, measurements of particle 3 alone will be predicted by the state vector $|\Phi\rangle_3$.