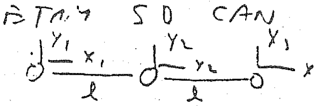
MECHANICS, PROBLEM 1

a) 9 DEGREES OF FREEDOM FOR 3 ATOMS − 3 TRANSLATIONS − 2 ROTATIONS

$\Rightarrow$ 4 VIBRATIONAL MODES: 2 LONGITUDINAL, 2 TRANSVERSE

TRANSVERSE FREQUENCIES DEGENERATE BY SYMMETRY SO CAN RESTRICT TO X−Y PLANE



b) $P_x = 0 = \frac{d}{dt}\left[m_a(x_1+x_3) + m_b(x_2)\right] \Rightarrow m_a(x_1+x_3) + m_b x_2 = \text{const}$

$P_y = 0 \Rightarrow m_a(y_1+y_3) + m_b y_2 = \text{const}$

$J_z = 0 \quad m_a \ell \left[\frac{d}{dt}(y_3-y_2) - \frac{d}{dt}(y_1-y_2)\right] \Rightarrow y_3 - y_1 = \text{const}$

AT EQUILIBRIUM $x_i = y_i = 0 \Rightarrow$ 3 CONSTANTS $= 0$

$x_2 = -\frac{m_a}{m_b}(x_1+x_3)$

$y_3 = y_1 \equiv y$

$y_2 = -\frac{2m_a}{m_b} y$

c) FOR POTENTIAL THAT DEPENDS ON $\overline{A_1B}$, $\overline{BA_2}$ AND $\angle A_1BA_2$, EXPANSION AROUND EQUILIBRIUM YIELDS

$$U \approx U_{EQUIL} + \frac{k}{2}(\delta \ell_{A_1B})^2 + \frac{k}{2}(\delta \ell_{BA_2})^2 + \frac{\alpha \ell^2}{2}(\delta\theta_{A_1BA_2})^2$$

$$= \frac{k}{2}\left[(x_3-x_2)^2 + (x_2-x_1)^2\right] + \frac{\alpha}{2}(y_1-y_2+y_3-y_2)^2$$

$$= \frac{k}{2}\left[\left(x_3 + \frac{m_a}{m_b}(x_1+x_3)\right)^2 + \left(\frac{m_a}{m_b}(x_1+x_3)+x_1\right)^2\right] + \frac{\alpha}{2}\left(2y + 4\frac{m_a}{m_b}y\right)^2$$

$$= \frac{1}{2}(x_1, x_3, y)\, \underline{\underline{U}} \begin{pmatrix} x_1 \\ x_3 \\ y \end{pmatrix}$$

$$\underline{\underline{U}} = \frac{1}{m_b^2}\begin{bmatrix} k\{(m_a+m_b)^2 + m_a^2\} & 2k\, m_a(m_a+m_b) & 0 \\ 2k\, m_a(m_a+m_b) & k\{(m_a+m_b)^2+m_a^2\} & 0 \\ 0 & 0 & 4\alpha(m_b+2m_a)^2 \end{bmatrix}$$

d) $T = \frac{1}{2}m_a(\dot{x}_1^2 + \dot{x}_3^2 + 2\dot{y}^2) + \frac{1}{2}m_b\left[\left(\frac{m_a}{m_b}\right)^2(\dot{x}_1^2 + 2\dot{x}_1\dot{x}_3 + \dot{x}_3^2) + 4\left(\frac{m_a}{m_b}\right)^2\dot{y}^2\right]$

$$= \frac{1}{2}(\dot{x}_1\ \dot{x}_3\ \dot{y})\, \underline{\underline{T}} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_3 \\ \dot{y} \end{pmatrix}$$

$$\underline{\underline{T}} = \frac{1}{m_b^2}\begin{bmatrix} m_a m_b(m_a+m_b) & m_a^2 m_b & \\ m_a^2 m_b & m_a m_b(m_a+m_b) & \\ & & m_a m_b(2m_b+4m_a) \end{bmatrix}$$

EIGENVALUES DETERMINED BY $|U - \omega^2 T| = 0$

$$0 = \begin{vmatrix} K\{(m_a+m_b)^2+m_a^2\} - \omega^2 m_a m_b (m_a+m_b) & 2Km_a(m_a+m_b) - \omega^2 m_a^2 m_b & 0 \\ 2Km_a(m_a+m_b) - \omega^2 m_a^2 m_b & K\{(m_a+m_b)^2+m_a^2\} - \omega^2 m_a m_b (m_a+m_b) & 0 \\ 0 & 0 & 4\alpha(m_b+2m_a)^2 - 2\omega^2 m_a m_b (m_b+2m_a) \end{vmatrix}$$

$$0 = \left[\left(K\{(m_a+m_b)^2+m_a^2\} - \omega^2 m_a m_b (m_a+m_b)\right)^2 - \left(2Km_a(m_a+m_b) - \omega^2 m_a^2 m_b\right)^2\right] \times$$
$$\times \left[4\alpha(m_b+2m_a)^2 - 2\omega^2 m_a m_b (m_b+2m_a)\right]$$

SETTING EACH FACTOR TO ZERO:

$$4\alpha(m_b+2m_a)^2 = 2\omega^2 m_a m_b (m_b+2m_a) \Rightarrow \omega^2 = \frac{2\alpha(m_b+2m_a)}{m_a m_b}$$

$$\therefore \boxed{\omega_1 = \sqrt{\frac{2\alpha(m_b+2m_a)}{m_a m_b}}}$$

$$K\{(m_a+m_b)^2+m_a^2\} - \omega^2 m_a m_b (m_a+m_b) = 2Km_a(m_a+m_b) - \omega^2 m_a^2 m_b$$

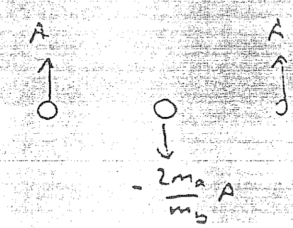
$$K m_b^2 = \omega^2 m_b^2 m_a \qquad \boxed{\omega_2 = \sqrt{\frac{K}{m_a}}}$$

$$K\{(m_a+m_b)^2+m_a^2\} - \omega^2 m_a m_b (m_a+m_b) = -\left[2Km_a(m_a+m_b) - \omega^2 m_a^2 m_b\right]$$
$$K(2m_a+m_b)^2 = \omega^2 m_a m_b (2m_a+m_b)$$

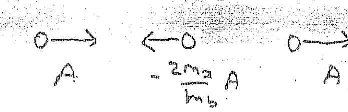
$$\boxed{\omega_3 = \sqrt{\frac{K(2m_a+m_b)}{m_a+m_b}}}$$

EIGENVECTORS

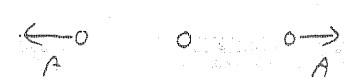
ω_1: $\begin{bmatrix} C_1 & C_2 & 0 \\ C_2 & C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ A \end{bmatrix} = 0$ $\qquad Y_1 = Y_3 = A$, $\quad Y_2 = -\frac{2m_a}{m_b} A$



ω_2: $\begin{bmatrix} D_1 & D_1 & 0 \\ D_1 & D_1 & 0 \\ 0 & 0 & D_2 \end{bmatrix} \begin{bmatrix} A \\ A \\ 0 \end{bmatrix} = 0$ $\qquad X_1 = X_3 = A$, $\quad X_2 = -\frac{2m_a}{m_b} A$



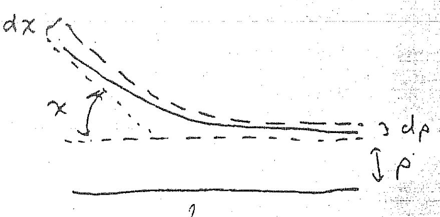
ω_3: $\begin{bmatrix} E_1 & -E_1 & 0 \\ -E_1 & E_1 & 0 \\ 0 & 0 & E_2 \end{bmatrix} \begin{bmatrix} A \\ -A \\ 0 \end{bmatrix} = 0$ $\qquad X_1 = A$, $\quad X_3 = -A$, $\quad X_2 = 0$



MECHANICS, PROBLEM 2

a) $\frac{d\sigma}{d\Omega} = \frac{\#\text{ particle scattered in given direction / unit time / unit solid angle}}{\text{incident flux (particles/unit area/unit time)}}$

b) $\frac{d\sigma}{d\Omega} = \frac{\text{flux} \cdot 2\pi\rho\, d\rho}{\text{flux} \cdot d\Omega} = \frac{2\pi\rho \left|\frac{d\rho}{d\chi}\right| d\chi}{2\pi \sin\chi\, d\chi} = \frac{\rho}{\sin\chi} \left|\frac{d\rho}{d\chi}\right|$



c) ENERGY IN PLANE OF MOTION CONSERVED:

$E = \frac{m}{2}\left[\dot r^2 + (r\dot\phi)^2\right] + U(r)$

ANG. MOM. CONSERVED:

$mr^2\dot\phi = J = \rho\sqrt{2mE}$

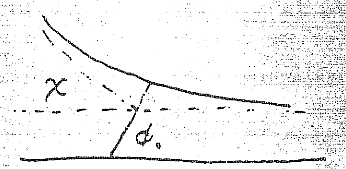
$r\dot\phi = \frac{J}{mr}$

$\left[\frac{2}{m}\left[E - U(r)\right] - \left(\frac{J}{mr}\right)^2\right]^{1/2} = \frac{dr}{dt} = \frac{dr}{d\phi}\dot\phi = \frac{dr}{d\phi}\frac{J}{mr^2}$

$$d\phi = \frac{\frac{J}{mr^2}\,dr}{\sqrt{\frac{2}{m}\left[E-U(r)\right] - \frac{J^2}{m^2r^2}}} = \frac{\sqrt{2mE}\,(\rho/r^2)\,dr}{\sqrt{2mE - 2mU(r) - \rho^2\frac{(2mE)}{r^2}}} = \frac{\frac{\rho}{r^2}\,dr}{\sqrt{1 - \frac{U(r)}{E} - \frac{\rho^2}{r^2}}}$$

d) $U(r) = \frac{K}{2r^2}$

CALCULATE ANGLE ϕ_0 AT CLOSEST APPROACH:



$\chi = \pi - 2\phi_0$

$\phi = \int_{r_{min}}^{\infty} d\phi \qquad r_{min} = r \ni \dot r = 0$

$\Rightarrow r \ni \sqrt{1 - \frac{K}{2r^2E} - \frac{\rho^2}{r^2}} = 0 \Rightarrow r_{min} = \sqrt{\frac{K}{2E} + \rho^2}$

$$\phi = \int_{r_{min}}^{\infty} \frac{\frac{\rho}{r^2}\,dr}{\sqrt{1 - \frac{K}{2Er^2} - \frac{\rho^2}{r^2}}} = \int_{r_{min}}^{\infty} \frac{\frac{\rho}{r}\,dr}{\sqrt{r^2 - \left(\frac{K}{2E} + \rho^2\right)}}$$

$$= \frac{\rho}{\sqrt{\frac{K}{2E}+\rho^2}} \cos^{-1}\left[\frac{\sqrt{\frac{K}{2E}+\rho^2}}{r}\right]_{\sqrt{\frac{K}{2E}+\rho^2}}^{\infty} = \frac{\rho}{\sqrt{\frac{K}{2E}+\rho^2}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi\rho}{2\sqrt{\frac{K}{2E}+\rho^2}}$$

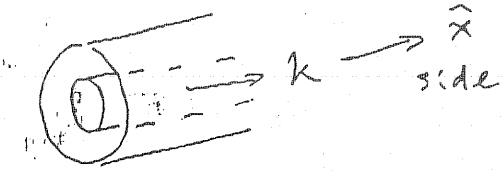
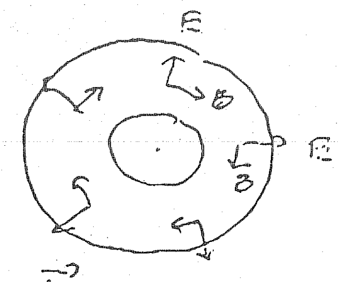
$$\therefore \chi = \pi - 2\phi = \pi\left[1 - \frac{1}{\sqrt{\frac{K}{2E\rho^2} + 1}}\right]$$

$$\rho^2 = \frac{K}{2E}\,\frac{\left(1 - \frac{\chi}{\pi}\right)^2}{1 - \left(1 - \frac{\chi}{\pi}\right)^2}$$

$$\rho\frac{d\rho}{d\chi} = -\frac{K}{2\pi E}\,\frac{\left[1 - \frac{\chi}{\pi}\right]}{\left[\frac{\chi}{\pi}\left(\frac{\chi}{\pi} - 2\right)\right]^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{\pi K\left[1 - \frac{\chi}{\pi}\right]}{2E\chi^2\left[2 - \frac{\chi}{\pi}\right]^2 \sin\chi}$$

E&M Problem 1

a)  side view: 

For the TEM mode, $\vec{E}, \vec{B} \perp \vec{k}$ and satisfy

$\nabla^2 E = 0$, $\nabla^2 B = 0$ ⤵

electrostatics + magnetostatics

$$\vec{E} = \frac{A\vec{r}}{r^2} e^{ikx - i\omega t}, \quad B = \frac{-A}{r} \hat{\varphi} e^{ikx - i\omega t} \qquad k = \frac{\omega}{c}$$

$\hat{\varphi}$ ← azymuthal angle

$$\sigma = \frac{E_\perp}{4\pi} = \begin{cases} \frac{A}{a} e^{ikx - i\omega t} & r = a \\ -\frac{A}{b} e^{ikx - i\omega t} & r = b \end{cases}$$

$$j = \hat{x} \frac{c}{4\pi} B_\parallel = \hat{x} \frac{c}{4\pi} e^{ikx - i\omega t} \begin{cases} \frac{A}{a} & r = a \\ -\frac{A}{b} & r = b \end{cases}$$

b)
$$\vec{E} = \frac{\vec{r}}{r^2} e^{-i\omega t} \begin{cases} A_{in} e^{ikx} + A_r e^{-ikx} & x < -d \\ A_t e^{ikx} & x > d \end{cases}$$

$$\vec{B} = \frac{-\hat{\varphi}}{r} e^{-i\omega t} \begin{cases} A_{in} e^{ikx} - A_r e^{-ikx} & x < -d \\ A_t e^{ikx} & x > d \end{cases}$$

$\lambda \gg d \to kd \ll 1$

Bound. conds.: Potentials equal at $x \approx \pm d$ $\to$ $(A_{in} + A_r) \ln \frac{b_1}{a} = A_t \ln \frac{b_2}{a}$

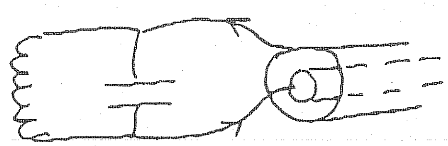
Current continuity $\to$ $A_{in} - A_r = A_t$

$$2 A_{in} = A_t \left(1 + \ln \frac{b_2}{b_1}\right) \Rightarrow A_t = \frac{2 A_{in}}{\ln \frac{b_2}{b_1} + 1}$$

$$A_r = \frac{1 - \ln b_2/b_1}{1 + \ln b_2/b_1} A_{in}$$

$$R = \left| \frac{1 - \ln b_2/b_1}{1 + \ln b_2/b_1} \right|^2$$

c)



At frequency ω, the fields $\vec{E} = \vec{E}_k + \cancel{\vec{E}_{-k}}$, $\vec{B} = \vec{B}_k + \cancel{\vec{B}_{-k}}$ $\quad k = \frac{\omega}{c}$

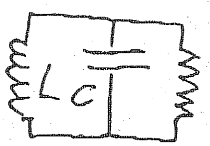
no incoming waves!

Potential: $V_{ab} = A \ln\frac{b}{a}$

Current: $I_{ab} = \frac{c}{4\pi} 2\pi a \frac{A}{a} = \frac{c}{2} A$

Thus: $V_{ab} = \frac{2\ln b/a}{c} I_{ab}$ $\qquad \underline{\underline{R_{eff} = \frac{2\ln b/a}{c}}}$

d)



Kirchoff eqn: $\sum I = I_L + I_C + I_R = 0$

$\frac{1}{L}V + \ddot{V}C + \frac{\dot{V}}{R} = 0 \qquad V \sim e^{-i\omega t}$

$-\omega^2 C + \frac{1}{L} - \frac{i\omega}{R} = 0$

$Q \gg 1$ weak damping

small term

$\omega^2 = \frac{1}{LC} - \frac{i\omega}{RC}$

$\omega = \pm\frac{1}{\sqrt{LC}}\left(1 - \frac{i\omega L}{R}\right)^{1/2} \simeq \pm\frac{1}{\sqrt{LC}} - \frac{i}{2RC}$

$\omega = \omega_0 - i\frac{\gamma}{2} \rightarrow Q = \frac{\omega_0}{\gamma} = R\sqrt{\frac{C}{L}}$

$Q \gtrsim 100 \rightarrow \boxed{\frac{L}{C} \lesssim 10^{-4} \cdot 4\ln^2\frac{b}{a}}$

$x = 0$ $\qquad\qquad$ $x = D$

e) $V_{ab} = \ln\frac{b}{a}\left(A_{+k} + A_{-k}\right)$

$I_{ab} = \frac{c}{2}\left(A_{+k} - A_{-k}\right)$

current $A_{+k} e^{ikD} - A_{-k} e^{-ikD} = 0$

$A_{-k} = A_{+k} e^{2ikD}$

$V_{ab} = \ln\frac{b}{a}\left(1 + e^{2ikD}\right) A_+$

$k = \frac{\omega}{c}$

$I_{ab} = \frac{c}{2}\left(1 - e^{2ikD}\right) A_+$

$$R_{eff}(\omega) = \frac{2\ln b/a}{c} \; \frac{\cos \omega D/c}{-i \sin \omega D/c}$$

$L + C + R_{eff}(\omega) \Rightarrow$ see part d) $\omega^2 = \frac{1}{LC} - \frac{i\omega}{RC_{eff}}$

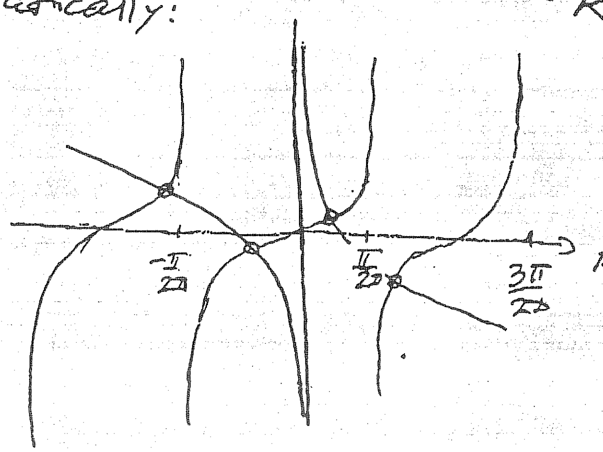
$$\omega^2 = \frac{c^2}{LC} - \frac{\omega c}{2C\ln\frac{b}{a}}\, i\, \frac{1 - e^{2i\omega D/c}}{1 + e^{2i\omega D/c}}$$

$$k^2 = k_0^2 - \frac{k}{2\ln\frac{b}{a} C} \tan kD$$

$$k_0 = \frac{\omega_{LC}}{c} = \frac{1}{c\sqrt{LC}}$$

$$\tan kD = 2\ln\frac{b}{a} C\left(\frac{k_0^2}{k} - k\right)$$

graphically:



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E&M Problem 2

a) $\vec{B}(x) = B_0 \tanh \frac{x}{d}\, \hat{z}$

Conserved: energy $E = \frac{m\vec{v}^2}{2} = \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m}$

momentum z and y components p_z, p_y

b) $\vec{r}(t) = \vec{R}(t) + \vec{\xi}(t)$

slow (drift) — fast (cyclotron)

$$m\ddot{\vec{R}} = \frac{e}{c}\dot{\vec{R}} \times \vec{B} + \frac{e}{c}\left\langle \dot{\vec{\xi}} \times \vec{\xi}\, \frac{\partial \vec{B}}{\partial \vec{r}} \right\rangle$$

average over fast motion

Second term $= \frac{e}{c}\frac{\partial B}{\partial x}\left\langle (\dot{\vec{\xi}} \times \hat{z})\xi_x \right\rangle = \frac{e}{c}\frac{\partial B}{\partial x}\omega_c\left(-\frac{\xi^2}{2}\hat{x}\right)$

$\dot{\vec{\xi}} = \omega_c \vec{\xi} \times \hat{z}$

$$m\ddot{\vec{R}} = \frac{e}{c}\dot{\vec{R}} \times \vec{B} - \frac{\partial(\frac{mv^2}{2}\ln \omega_c(x))}{\partial x}\hat{x}$$

$$E_{eff} = -\frac{mv^2}{2e}\frac{\partial \ln \omega_c(x)}{\partial x}\hat{x}$$

$$v_d = c\,\frac{E_{eff} \times B}{B^2} = \frac{v^2}{2\omega_c}\frac{\partial \ln \omega_c}{\partial x}\hat{y} = \frac{v^2}{2\omega_c^2}\frac{d\omega_c}{dx}\hat{y}$$

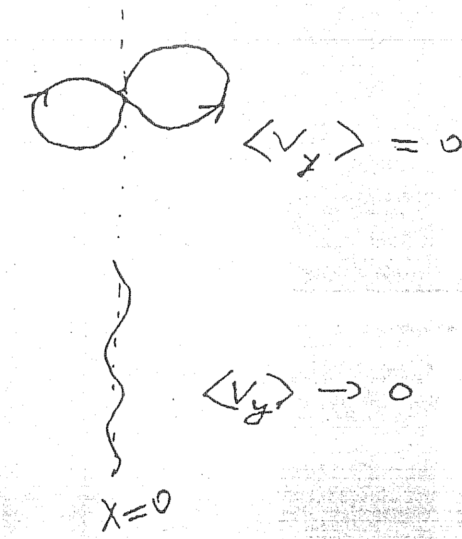
$$\| \quad v_d = \frac{v^2}{2\tanh^2\frac{x}{d}}\,\frac{1}{\cosh^2\frac{x}{d}}\,\frac{1}{\omega_c^{(0)} d} = \frac{v^2}{2d\,\omega_c^{(0)}}\,\frac{1}{\sinh^2 x/d}$$

$\omega_c^{(0)} = \frac{eB_0}{mc}$

Solution valid if $v_d \ll v$:

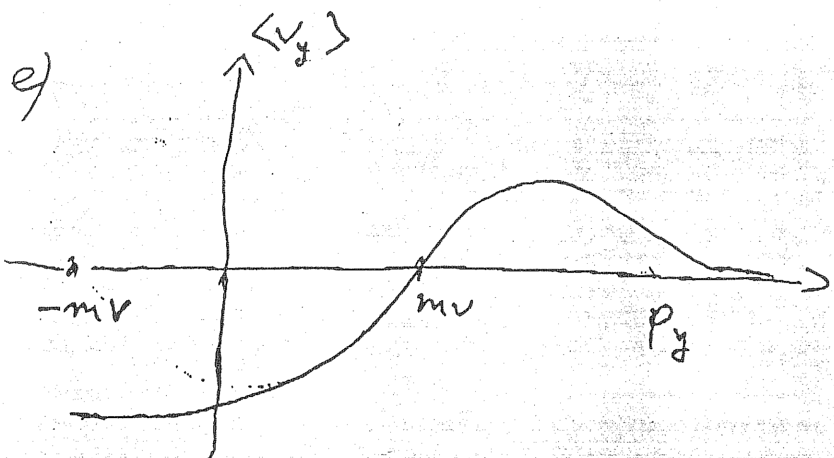
$$\frac{x^2}{d^2} \ll \frac{v}{2d\,\omega_c^{(0)}} \rightarrow \underline{x \ll \sqrt{\frac{vd}{\omega_c^{(0)}}}}$$

d) $p_x = mv$ critical orbits



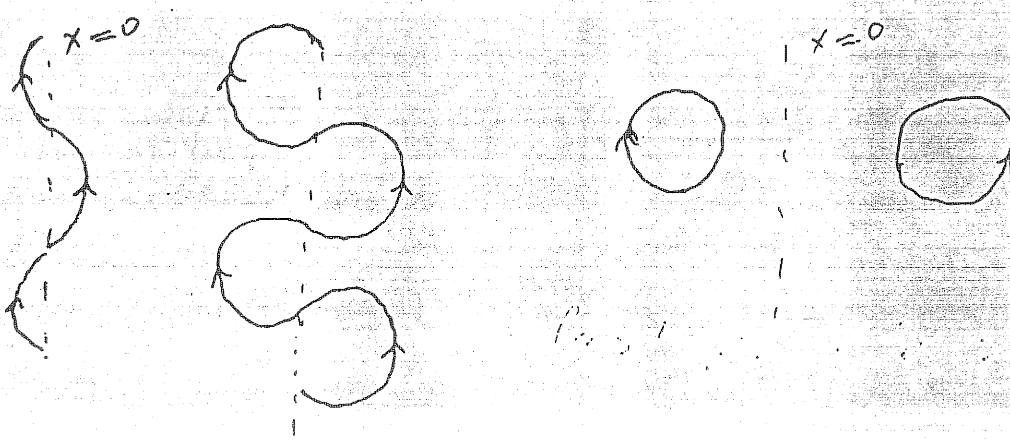
$p_y = -mv + \varepsilon$

e)



fast particle $v \gg d\omega_c^{(0)}$:

Snake ~~orbits~~ orbits: cyclotron orbits:



c) The motion in the x-y plane is described by the Hamiltonian $H = \frac{p_x^2}{2m} + \frac{(p_y - \frac{e}{c}A_y(x))^2}{2m}$

they mean x-z plane.

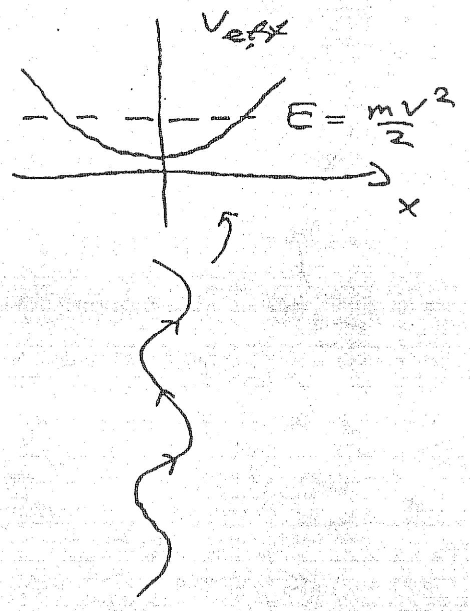
conserved

$$A_y(x) = \int_0^x B(x')\,dx' = \ln\left(\cosh \frac{x}{d}\right) B_0 d$$

Effective potential for the motion in the x-direction is

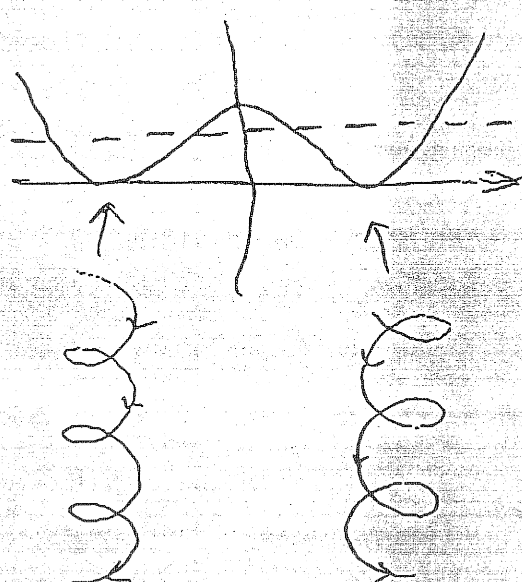
$$V_{eff}(x) = \frac{1}{2m}\left(p_y - m d\,\omega_c^{(0)} \ln \cosh \frac{x}{d}\right)^2 \qquad (*)$$

$p_y < 0$



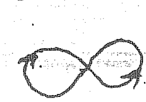
snake states

$p_y > 0$



drifting orbits

critical orbits: $p_y = mv$



exist for $V_{eff}(x=0) < E = \frac{mv^2}{2}$

$$\left.\begin{aligned} &|p_y| < mv \\ &|p_y| \simeq \frac{m\omega_c^{(0)}}{2d} x^2 \quad (\text{from } (*)) \ @\ |x| \ll d \end{aligned}\right\} \Rightarrow x \lesssim \sqrt{\frac{vd}{\omega_c^{(0)}}}$$

QUANTUM MECHANICS, PROBLEM 1

a) $[\pi_i, \pi_j] = [(p_i - \frac{q}{c}A_i), (p_j - \frac{q}{c}A_j)] = i\hbar\frac{q}{c}(\partial_i A_j - \partial_j A_i) = i\hbar\frac{q}{c}B_k$

$\therefore [\pi_x, \pi_y] = i\hbar\frac{q}{c}B \quad , [\pi_x, \pi_z] = [\pi_y, \pi_z] = 0$

b) $H^2 = c^2\sum_{i,j}\alpha_i\pi_i\alpha_j\pi_j + \underbrace{\beta^2}_{1} m^2c^4 + mc^3\sum_k \pi_k \underbrace{\{\alpha_k, \beta\}}_{0}$

$= c^2\sum_{i=j}\alpha_i\pi_i\alpha_i\pi_j + c^2\frac{1}{2}\sum_{i\neq j}\left(\underbrace{\{\alpha_i, \alpha_j\}}_{0}\pi_i\pi_j + \alpha_i\alpha_j[\pi_i, \pi_j]\right) + m^2c^4$

$= \underbrace{c^2(\pi_x^2 + \pi_y^2)}_{H_1} + \underbrace{c^2\pi_z^2}_{H_2} + \underbrace{\frac{c^2}{2}\alpha_i\alpha_j[\pi_i, \pi_j]}_{H_3} + m^2c^4$

$H_3 = c^2\alpha_x\alpha_y[\pi_x, \pi_y] = c^2\begin{pmatrix} i\sigma_z & \\ & i\sigma_z\end{pmatrix} i\hbar\frac{q}{c}B = -c\hbar qB\begin{pmatrix}\sigma_z & \\ & \sigma_z\end{pmatrix}$

$[\pi_y, \pi_z] = 0 \Rightarrow [H_1, H_2] = 0 \quad , [H_2, H_3] = 0$

$H_1 \propto$ unit matrix, H_3 has no space dependence $\Rightarrow [H_1, H_3] = 0$

c) H_1 is HARMONIC OSCILLATOR

let $\hat{p} \equiv \pi_y \quad \hat{x} \equiv \frac{c}{qB}\pi_x \quad [x, p] = i\hbar$

$c^2(\pi_x^2 + \pi_y^2) = 2c^2\left[\frac{1}{2}\hat{p}^2 + \frac{1}{2}\frac{q^2B^2}{c^2}\hat{x}^2\right]$

$\Rightarrow$ EIGENVALUES $\quad 2c^{\cancel{2}}(n + \frac{1}{2})\hbar\frac{qB}{\cancel{c}} \qquad n = 0, 1, 2, \ldots$

$H_2 = c^2\pi_z^2 = c^2\hat{p}_z^2 \quad$ SINCE $A_z = 0 \quad \Rightarrow$ EIGENVALUES $c^2p_z^2$

$H_3 = -c\hbar qB\begin{pmatrix}\sigma_z & \\ & \sigma_z\end{pmatrix}$ HAS EIGENVALUES $\pm c\hbar qB$

d) EIGENVALUES E^2 OF H^2:

$E^2 = (2(n + \frac{1}{2}) \pm 1)\hbar cqB + c^2p_z^2 + m^2c^4 \qquad n = 0, 1, 2 \ldots$

$E = \pm\sqrt{m^2c^4 + 2\tilde{n}\,\hbar cqB + c^2p_z^2} \qquad \tilde{n} = 0, 1, 2, \ldots$

As $m \to \infty$, positive energy solns $\to$

$E_{NR} \sim \lim_{m\to\infty} +\sqrt{m^2c^4 + 2\tilde{n}\,\hbar cqB + c^2p_z^2}$

$\sim mc^2 + \frac{p_z^2}{2m} + \tilde{n}\frac{\hbar qB}{mc}$

QUANTUM MECHANICS, PROBLEM 2

a) EIGENFUNCTIONS

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \qquad \langle xyz|n_x n_y n_z\rangle = \sin\left(\frac{n_x\pi x}{L}\right)\sin\left(\frac{n_y\pi y}{L}\right)\sin\left(\frac{3n_z\pi z}{L}\right)$$

$$E_{n_x n_y n_z} = \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2+n_y^2+9n_z^2\right) = \varepsilon_0\left(n_x^2+n_y^2+9n_z^2\right)$$

$E_0 = E_{111} = 11\,\varepsilon_0$
$E_1 = E_{211} = E_{121} = 14\,\varepsilon_0$
$E_2 = E_{221} = 17\,\varepsilon_0$

b) SPIN ZERO → SYMMETRIC

<u>GROUND STATE</u> $|111\rangle_1 |111\rangle_2 |111\rangle_3$ $\quad E = 3E_{111} = \underline{33\,\varepsilon_0}$ $\quad \underline{D=1}$

<u>1ST EXCITED STATES</u> $S\{|211\rangle_1 |111\rangle_2 |111\rangle_3\}$ $\quad E = 2E_{111}+E_{211} = \underline{36\,\varepsilon_0}$
$S\{|121\rangle_1 |111\rangle_2 |111\rangle_3\}$ $\quad \underline{D=2}$

(S DENOTES SYMMETRIZATION)

c) SPIN ½ → ANTISYMMETRIC ADD SPIN LABEL $|n_x n_y n_z \pm\rangle$

<u>GROUND STATES</u> $\mathcal{A}\left[|111+\rangle_1 |111-\rangle_2 \begin{Bmatrix}|211\pm\rangle_3\\ |121\pm\rangle_3\end{Bmatrix}\right]$ $\quad E = 2E_0+E_1 = \underline{36\,\varepsilon_0}$ $\quad \underline{D=4}$

ALTERNATIVE COUNTING:

2 MIXED SPATIAL STATES × 2 MIXED SPIN STATES = 4

111, 111, 211 $\qquad$ ++−, −−+
111, 111, 121

<u>1ST EXCITED STATE</u>

$$\mathcal{A}\left[|111\pm\rangle_1 \begin{Bmatrix}|121+\rangle_2|121-\rangle_3\\ |211+\rangle_2|211-\rangle_3\\ |121\pm\rangle_2|211\pm\rangle_3\end{Bmatrix}\right] \begin{matrix}2\\2\\8\end{matrix}$$

$E = E_0+2E_1 = \underline{39\,\varepsilon_0}$
$= 2E_0 + E_2$

$$\mathcal{A}\left[|111+\rangle_1 |111-\rangle_2 |221\pm\rangle_3\right] \quad \underline{2}$$

14 $\qquad \underline{D=14}$

ALTERNATIVE:

SPACE		SPIN		
	211, 211, 111 (1) 121, 121, 111 (1) 111, 121, 211 (2)		++−, −−+	} 8
	111, 121, 211 (1)		(4)	} 4
	111, 111, 221 (1)		(2)	} 2

$\underline{\quad}$
14 ✓

a) SPIN 1 → SYMMETRIC AND SPIN LABEL $|n_x n_y n_z m_a\rangle$

GROUND STATES $S\left[|111m_a\rangle_1 |111m_b\rangle_2 |111m_c\rangle_3\right]$ $E = 3E_1 = \underline{33\,\varepsilon_0}$

$m^a = 1, 0, -1$ $m_a \le m_b \le m_c$ FOR DISTINCT STATES ⇒ $D = \underline{10}$

ALTERNATIVELY $\square \times \square \times \square = \underbrace{10}_{} + 8 + 8 + 1$ OR $J = 3$ only 7, $J = 1$ only 3 → 10

SYMMETRIC ⇒ $D = 10$

EXCITED STATES

$S\left[|111m_a\rangle_1 |111m_b\rangle_2 \begin{Bmatrix} |211m_c\rangle_3 \\ |121m_c\rangle_3 \end{Bmatrix}\right]$ $E = 2E_0 + E_1 = \underline{36}\,\varepsilon_0$

6 SETS $\{m_a \le m_b\}$ × 3 VALUES m_c × 2 SPATIAL WF ⇒ $D = \underline{\underline{36}}$

ALTERNATIVELY

SPACE: 111, 111, 211 (1); 121 (1) — SPIN: 8 — $2 \times 8 = 16$

2 — 10 — $2 \times 10 = 20$

$\overline{36}$

SUMMARY

SPIN	GROUND STATE E_0/ε_0	GROUND STATE D	1ST EXCITED STATE E_1/ε_0	1ST EXCITED STATE D
0	33	1	36	2
1/2	36	4	39	14
1	33	10	36	36

Problem 1

a) $$Z_0 = \left(\sum_{s=\pm\frac{1}{2}} e^{\beta\mu B s}\right)^N = \underline{2^N \cosh^N\left(\frac{\beta\mu B}{2}\right)}$$

b) $$-\beta F = \ln Z_0 = N \ln\left[2\cosh\left(\tfrac{1}{2}\beta\mu B\right)\right]$$

$$F(B,\beta) = -\frac{N}{\beta} \ln\left[2\cosh\left(\tfrac{1}{2}\beta\mu B\right)\right]$$

$$E = F + \beta\frac{\partial F}{\partial\beta} = -\frac{N}{\beta}\,\frac{\sinh \frac{1}{2}\beta\mu B}{\cosh \frac{1}{2}\beta\mu B}\,\frac{\mu B}{2}\beta$$

$$E = -\frac{\mu B N}{2}\tanh\left(\tfrac{1}{2}\beta\mu B\right)$$

$$S = (E-F)\beta = N\left(\ln\left[2\cosh\left(\tfrac{1}{2}\beta\mu B\right)\right] - \tfrac{1}{2}\beta\mu B\tanh\left(\tfrac{1}{2}\beta\mu B\right)\right)$$

$$S = N k_B\left[\ln\left(2\left(\cosh x\right)\right) - x\tanh x\right], \quad x = \frac{\beta\mu B}{2}$$

c) $$H_{int} = -\frac{J}{2}\sum_{i\neq j} s_i s_j = N\frac{J}{8} - \frac{J}{2}\left(\sum s_i\right)^2$$

$$e^{-\beta H_{int}} = e^{-\beta N\frac{J}{8}} \frac{\int dx\, e^{x\sum_i s_i - \frac{x^2}{2\beta J}}}{\int dx\, e^{-x^2/2\beta J}} \underset{x=\beta\lambda}{=} e^{-\frac{\beta N J}{8}} \frac{\int_{-\infty}^{\infty} d\lambda\, e^{\beta\lambda\sum_i s_i - \frac{\beta\lambda^2}{2J}}}{\int d\lambda\, e^{-\frac{\beta\lambda^2}{2J}}}$$

$$Z = \sum_{s_i} e^{-\beta(H_0 + H_{int})} = e^{-\frac{\beta N J}{8}}\sqrt{\frac{\beta}{2\pi J}} \int d\lambda \sum_{s_i} e^{\beta(\lambda+\mu B)\sum s_i}\, e^{-\frac{\beta\lambda^2}{2J}}$$

$$Z = A\int_{-\infty}^{\infty} d\lambda\, Z_0\left(B + \frac{\lambda}{\mu}, \beta\right) e^{-\frac{\beta}{2J}\lambda^2}$$

$$A = e^{-\frac{1}{8}\beta N J}\sqrt{\frac{\beta}{2\pi J}}$$

$$\int e^{-\frac{x^2}{2}}dx = \sqrt{2\pi}$$

Problem 2

a) $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$, $\omega = \sqrt{k/m}$

$$Z = \sum_i e^{-\beta E_i} = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} = \frac{1}{2\sinh\frac{\hbar\omega\beta}{2}}$$

b) $$\langle E\rangle = \frac{\sum_i E_i e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} = -\frac{\partial}{\partial\beta}\ln Z = \underline{\frac{\hbar\omega}{2}\coth\frac{\hbar\omega\beta}{2}}$$

$$C = \frac{d\langle E\rangle}{dT} = -k_B\beta^2\frac{\partial\langle E\rangle}{\partial\beta} = -k_B\beta^2\left(\frac{\hbar\omega}{2}\right)^2\left(1-\frac{\cosh^2\frac{\hbar\omega\beta}{2}}{\sinh^2\frac{\hbar\omega\beta}{2}}\right)$$

$$C = k_B\left(\frac{\frac{\hbar\omega\beta}{2}}{\sinh\frac{\hbar\omega\beta}{2}}\right)^2$$

c) Define $p(x) = \int_0^x \rho\dot{u}(x')\,dx'$, the momentum of the part $0<x'<x$ of the string

Then

$$\left.\begin{aligned}\dot{p} &= F_0\frac{\partial u}{\partial x}\\ \dot{u} &= \frac{1}{\rho}\frac{\partial p}{\partial x}\end{aligned}\right\} \Rightarrow \ddot{u} = \frac{F_0}{\rho}u_{xx} \quad \text{wave eqn.}$$

$c = \sqrt{F_0/\rho}$

General soln:

$$u(x,t) = \sum_k u_k e^{ikx-i\omega t} \qquad \omega = ck$$

Bound. conds:

$u(x=0)=0$

$u(x=L)=0$

$u_m = \sin k_m x \qquad k_m L = \pi m \quad m=1,2,\dots$

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Normalized solutions:

$$U^{\alpha}_{m}(x) = \sqrt{\frac{2}{L}} \sin k_m x \, e^{\alpha}, \quad k_m = \frac{\pi}{L} m, \quad e^{\alpha} \text{ polarization } \alpha = 1, 2$$

d) Potential energy $E_p = F_0 \int dl = F_0 \int_0^L \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} \, dx' -$

↙ expanding ↗

$$E_p = \text{const} + \frac{F_0}{2} \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$u(x) = \sum_m q_m(t) U_m(x)$$

$$E_p[u(x)] = \int_0^L \frac{F_0}{2} \left(\sum_m q_m(t) \frac{dU_m}{dx} \right)^2 dx = \sum_m \frac{F_0}{2} k_m^2 q_m^2$$

$$E_{kin}[u(x)] = \int_0^L \rho \, \dot{u}^2 dx = \sum_m \frac{\rho}{2} \dot{q}_m^2$$

$$H = \sum_m H_m(q_m), \text{ where } H_m = \frac{\rho}{2} \dot{q}_m^2 + \frac{F_0}{2} k_m^2 q_m^2$$

---

e) $\hat{H} = \sum_m \hat{H}_m$, where $\hat{H}_m = -\frac{\hbar^2}{2\rho} \left(\frac{\partial}{\partial q_m}\right)^2 + \frac{F_0}{2} k_m^2 q_m^2$

eigenvalues $E_n^{(m)} = \hbar \omega^{(m)} (n + \frac{1}{2})$

$\omega^{(m)} = c k_m$

$$\langle E \rangle = \sum_{\substack{m=1,2,\dots \\ n=0,1,\dots}} E_n^{(m)} e^{-\beta E_n^{(m)}} = \sum_m \frac{\hbar \omega^{(m)}}{2} \coth \frac{\hbar \omega^{(m)} \beta}{2}$$

(diverges due to 0-point fluctuations)

$$\langle E \rangle - \langle E \rangle_{\beta = \infty} = \sum_m \frac{\hbar \omega^{(m)}}{e^{\hbar \omega^{(m)} \beta} - 1}$$

(T-dependent part, Σ converges)

For large L, replace $\sum_m \to \int dm$

$$\langle E \rangle_{rep} = \int_0^\infty dm \frac{\hbar \omega^{(m)}}{e^{\hbar \omega^{(m)} \beta} - 1} = \frac{L}{\pi} \int_0^\infty dk \frac{\hbar c k}{e^{\beta \hbar c k} - 1} = \frac{L}{\pi \beta^2 \hbar c} \int_0^\infty \frac{u \, du}{e^u - 1}$$

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— 2 —

d) $e^{-\beta F(\lambda)} = \left[2\cosh \frac{\beta}{2}(\mu B+\lambda)\right]^N e^{-\frac{\lambda^2 \beta}{2J}}$

$$F(\lambda) = \frac{\lambda^2}{2J} - \frac{N}{\beta}\ln\left(2\cosh\frac{\beta}{2}(\mu B+\lambda)\right)$$

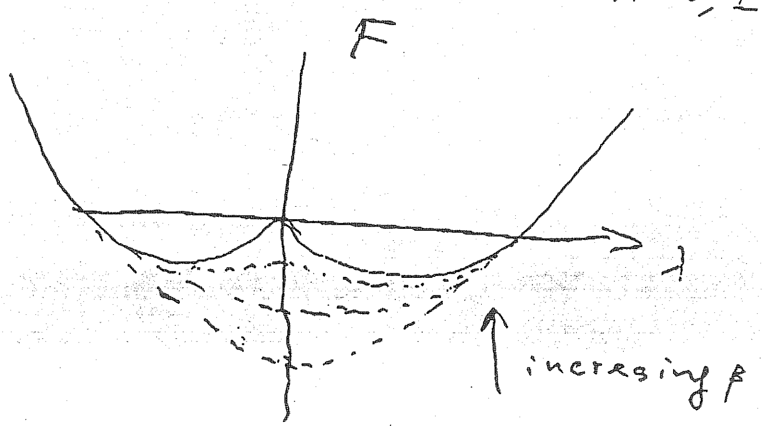
$B=0$

minima: $F'(\lambda) = 0 \rightarrow F'(\lambda) = \frac{\lambda}{J} - \frac{N}{2}\tanh\frac{\beta\lambda}{2} = 0$

$$\lambda = \frac{JN}{2}\tanh\frac{\beta\lambda}{2}$$

one solution $\lambda=0$ for $\beta < 4/NJ$

three solutions $\lambda = 0, \pm\lambda_*$ for $\beta > 4/NJ$



Ferromagnetic ordering at $T < T_c = \frac{NJ}{4k_B}$

e) At $T > T_c$ use the $\lambda = 0$ minimum

$$F(\lambda) = \frac{\lambda^2}{2J} - \frac{N}{\beta}\ln 2 - \frac{N}{\beta}\frac{1}{2}\frac{\beta^2}{4}(\mu B+\lambda)^2 + O\left((\mu B+\lambda)^4\right)$$

$$Z = A\, 2^N \int d\lambda\, e^{-\frac{\beta\lambda^2}{2J} + \frac{N\beta^2}{8}(\lambda+\mu B)^2} = 2^N \cancel{A \int d\lambda \exp\left[\frac{N\beta^2(\mu B)^2}{8} + \frac{N\beta^2}{4}\lambda\mu B - \lambda^2\left(-\frac{N\beta^2}{8} + \frac{\beta}{2J}\right)\right]}$$

$$\cancel{Z = 2^N A \int d\lambda\, e^{-\left(\frac{\beta}{2J} - \frac{N\beta^2}{8}\right)\lambda^2}\left(1 + \frac{N\beta^2}{8}\mu B^2 + \frac{1}{2}\left(\frac{N\beta^2\lambda\mu B}{4}\right)^2\right)}$$

see next page

c) continued

$$\langle S_i \rangle = \frac{1}{\mu\beta}\frac{\partial}{\partial B}\ln Z \qquad \beta_c = \frac{4}{NJ} \qquad \tilde{B} = \mu B \frac{\beta_c N^{1/2}}{2}$$

$$Z = 2^N A \int d\lambda\, e^{-\frac{\beta\beta_c}{8}N\lambda^2 + \frac{\beta^2 N}{8}(\lambda + B\mu)^2} =$$

$$\lambda = \frac{2}{\beta_c N^{1/2}} u, \quad \tilde{\beta} = \beta/\beta_c$$

$$Z = 2^N A \frac{2}{N^{1/2}\beta_c} \int du\, e^{-\frac{\tilde{\beta}}{2}u^2 + \frac{\tilde{\beta}^2}{2}(u+\tilde{B})^2}$$

$$\Downarrow$$

$$-\frac{1}{2}\left[(\tilde{\beta}-\tilde{\beta}^2)u^2 - 2\tilde{\beta}^2\tilde{B}u - \tilde{\beta}^2\tilde{B}^2\right] = -\frac{1}{2}(\tilde{\beta}-\tilde{\beta}^2)(u-\bar{u}_1)^2 + \frac{(\tilde{\beta}\tilde{B})^2}{2} + \frac{1}{2}\frac{(\tilde{\beta}^2\tilde{B})^2}{\tilde{\beta}-\tilde{\beta}^2}$$

$$Z = \tilde{A}(\beta)\, e^{\frac{\tilde{B}^2}{2}\frac{\tilde{\beta}^2}{1-\tilde{\beta}}}$$

$$\ln Z = (\mu B)^2 \frac{\beta_c^2 N}{8}\frac{\beta^2}{\beta_c^2 - \beta\beta_c} + \text{const}$$

$$\langle S \rangle = \frac{\mu B}{\beta}\frac{\beta_c N}{4}\frac{\beta^2}{\beta_c - \beta} = \frac{\mu B N}{4}\frac{\beta\beta_c}{\beta_c - \beta} = \frac{\mu B N}{4k_B(T - T_c)}$$

$$\chi = \frac{\mu^2}{4k_B}\frac{N}{T-T_c} \quad \text{(Curie law)}$$

$$\int_0^\infty \frac{u\,du}{e^u - 1} = \int_0^\infty u\,du\,(e^{-u} + e^{-2u} + e^{-3u} + \dots) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$= \frac{\pi^2}{6}$$

$$\langle E \rangle_{ref} = \frac{L}{\pi \hbar c} (k_B T)^2 \times \frac{\pi^2}{6} \times 2$$

two transverse polarizations of $\vec{u}$

$$\langle E \rangle_{ref} = \frac{\pi L}{3 \hbar c} (k_B T)^2$$

$$C = \frac{\partial \langle E \rangle}{\partial T} = \frac{2\pi L}{3 \hbar c} k_B^2 T$$