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DEPARTMENT OF PHYSICS

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DOCTORAL GENERAL EXAMINATION

PART II

February 1, 2002

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. Read both problems in each section carefully before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.

2. Use a separate paper (i.e. folded sheet) for each problem. Write your name and the problem number (I.2 for example) on each folded sheet.

3. Calculators may be used.

4. No books or reference materials may be used.

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CLASSICAL MECHANICS: CATENARY

Problem 1

A chain of length ℓ with mass per unit length μ is suspended in a uniform gravitational field g from two points at positions $(x, y) = (\pm a, 0)$ where $a < \ell/2$. Assume that the chain cannot be stretched.

a) Find a functional $I[y(x)]$ whose extremum gives the equilibrium shape of the chain as the curve $y(x)$.

b) Show that the solution is $y(x) = A\cosh(kx + \phi) + B$ and find expressions for A, B, k, and ϕ in terms of ℓ, μ, g, and a.

c) What is the tension in the chain, $T(x)$?

d) Suppose that a mass M is suspended from the center of the chain. Show that on either side of the mass, the solution is of the same form as in part b). Find expressions for A, B, k, and ϕ on both sides of the mass, in terms of ℓ, μ, g, and a.

CLASSICAL MECHANICS: PLANE PENDULUM
Problem 2.

Consider a point mass m attached to a string of slowly increasing length $\ell(t)$. The motion is confined to a plane. The function $\ell(t)$ is unspecified but $|\ell/\dot{\ell}|$ is much greater than the oscillation period.

a) Determine the Lagrangian $L(\theta, \dot{\theta}, t)$ and the Hamiltonian $H(\theta, p, t)$ of this dynamical system.

b) Is the Hamiltonian equal to the total energy of the pendulum? Is the Hamiltonian conserved? If the Hamiltonian and the total energy are distinct, is the total energy conserved?

c) Evaluate the equation of motion for the variable θ in the form of a second order ordinary differential equation. When $\dot{\ell} = 0$, what is the period of small oscillations?

d) Show that the amplitude of small oscillations is proportional to $\ell^{-3/4}$ as the length of the string $\ell(t)$ changes.

General Exam part II: Electromagnetism

Problem 1 (Relativistic charge dynamics)

A potential difference is maintained between the plates of a parallel plate capacitor. The negative plate (cathode) emits electrons with negligible initial velocity (so called "cold emission").

a) Write down the relativistic equations of motion and integrals of motion (i.e., list all conserved quantities for this problem).

b) Find the electron trajectory in the form $x = x(t)$. How long does it take the electron to travel across the capacitor?

c) Now, a uniform magnetic field is applied parallel to the capacitor plates (and normal to the page). Write down the relativistic equations of motion and integrals of motion.

d) Consider the case of high magnetic field: $|B| > |E|$. Parametrize the motion by the electron proper time τ. Find the electron trajectory in the form $t = t(\tau)$, $x = x(\tau)$, $y = y(\tau)$. How high should the magnetic field be to prevent electrons from reaching the anode (positive plate).
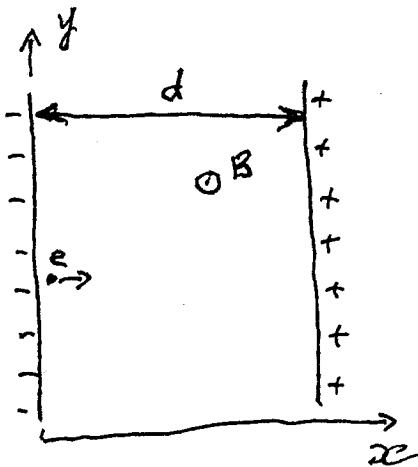
ELECTROMAGNETISM: MAGNETOSTATICS
Problem 2

A spherical shell of radius a rotates with constant angular velocity ω about an axis through the center of the sphere. The shell has uniform surface charge density σ.

a) Find the magnetic field $\vec{B}$ inside and outside the sphere and verify that $\vec{B}$ obeys the appropriate boundary conditions at $r = a$.

b) The magnetic field of a dipole is

$$\vec{B}(\vec{r}) = \frac{3\vec{e}_r(\vec{e}_r \cdot \vec{m}) - \vec{m}}{r^3}$$

where $\vec{m}$ is the dipole moment. By evaluating the magnetic field at $r \gg a$, determine the dipole moment for the uniformly rotating charged shell of part a).

c) Evaluate the Poynting flux at $r > a$. Does a nonzero Poynting flux imply that the rotating shell is losing energy to radiation? Explain why or why not.

d) A current loop of radius b and current I is centered at a distance $z > a$ from the center of the spherical shell, as shown in the diagram. The direction of current is parallel to the direction of the moving charges on the sphere. What is the force (magnitude and direction) acting on the loop?

current I
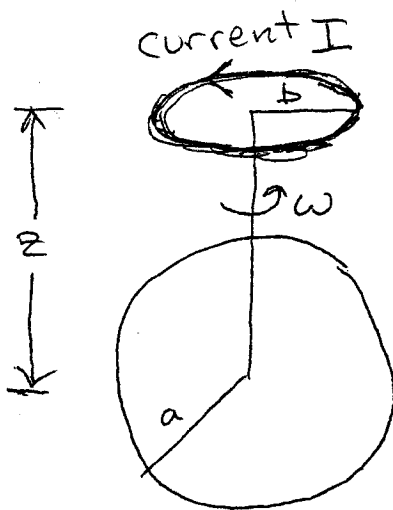
The circular disk of radius b is perpendicular to the rotation axis of the sphere.

Problem 1 (Thermodynamics)

Consider a material with an equation of state

$$P = \frac{\alpha T}{V^2}$$

where α is a constant. The heat capacity of this material at constant volume is linear in temperature: $C_V = A(V)T$.

a) By using Maxwell relations or otherwise find the derivative of entropy $(\partial S / \partial V)_T$.

b) Show that the coefficient $A(V)$ is independent of V.

c) Find $S(T, V)$ assuming that the value $S(T_0, V_0) = S_0$ is known.

d) Find the heat capacity at constant pressure $C_P = T (\partial S / \partial T)_P$.

Problem 2 (A polymer chain)

a) A polymer chain consists of a large number $N \gg 1$ segments of length d each. The segments can freely rotate relative to each other. The temperature of the system is T. Find the mean displacement $\langle \mathbf{r}_{12} \rangle$ and the mean square distance $\langle \mathbf{r}_{12}^2 \rangle$. between the chain ends.

b) For the chain of part a) with a force f applied at the ends, find the mean distance $\langle \mathbf{r}_{12} \rangle$ between the ends.

c) Find the entropy of the chain of part b).

d) The chain of part a) is suspended at one end in a gravitation field of strength g. The mass of each segment of the chain is m. What is the average length of this chain (i.e., $\langle \mathbf{r}_{12} \rangle$) as a function of temperature?
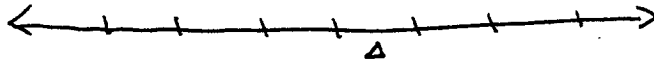

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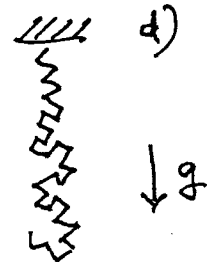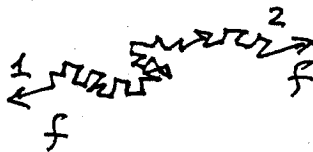
Quantum Mechanics: Problem I
Particle on a 1 dimensional lattice

Hilbert space

$$\text{basis} \quad |n\rangle \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\langle n | m \rangle = \delta_{nm} \qquad \chi_n = 1$$

a.) Consider the translation operator $\hat{T}$ defined by

$$\hat{T} |n\rangle = |n+1\rangle$$

What are the eigenstates and eigenvalues of $\hat{T}$?

b.) Consider the Hamiltonian

$$\hat{H} |n\rangle = -\frac{1}{2\Delta^2} \left\{ |n+1\rangle + |n-1\rangle - 2|n\rangle \right\}$$

Does $\hat{T}$ commute with $\hat{H}$?

Let $|\psi_k\rangle$ be given by

$$\langle n | \psi_k \rangle = e^{ikn\Delta}$$

Show that $|\psi_k\rangle$ is an eigenstate of $\hat{H}$.

What is the relationship between the eigenvalue

E_k and k ? What is E_k as $k \to 0$?

c.) Add to the Hamiltonian a potential of height V at lattice site $n = 0$.

What is the transmission possibility through the barrier at $n = 0$

as a function of k, V and Δ ?

Quantum Mechanics: Problem II
Particle in a Uniform Magnetic Field

A particle of charge q is confined to the x-y plane.

There is a uniform magnetic field in the z- direction

$$\vec{B} = (0, 0, B)$$

$$\hat{H} = \frac{1}{2m}\left(\hat{\pi}_x^2 + \hat{\pi}_y^2\right)$$

where

$$\hat{\pi}_x = \hat{P}_x + \frac{q}{c}\hat{A}_x \qquad \hat{\pi}_y = \hat{P}_y + \frac{q}{c}\hat{A}_y$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

a.) Calculate $\left[\hat{\pi}_x, \hat{\pi}_y\right]$.

b.) Use (a.) to find the energy eigenvalues of $\hat{H}$.

c.) Pick the gauge $\vec{A} = (0, Bx, 0)$ and

Show that $\left[\hat{P}_y, \hat{H}\right] = 0$.

Write $\hat{H}$ in terms of $\hat{P}_x, \hat{P}_y,$ and $\hat{x}$.

Calculate the energy eigenvalues.