MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

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DOCTORAL GENERAL EXAMINATION

PART II

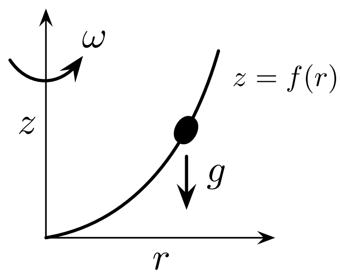
Friday, February 3, 2012
9:30 a.m. - 2:30 p.m., Room 32-082

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. Read both problems in each section carefully before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. For each problem, use the separate booklet that you have been given. Do not put your name on it, as each booklet has an identification number that will allow the papers to be graded blindly. Please, however, write the problem number (I.2 for example) on the front of each booklet.
3. Calculators may not be used.
4. No books or reference materials may be used.

SECTION I: CLASSICAL MECHANICS

Classical Mechanics 1: Bead on a Curved Wire



A bead of mass m slides without friction along a curved wire with shape $z = f(r)$, as indicated in the figure above, with $r = \sqrt{x^2 + y^2}$. The wire is rotated around the z-axis at a constant angular velocity ω, keeping its shape fixed. Gravity acts downward, with an acceleration $g > 0$.

(a) (*2 pts*) Using Newton's second law ($\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$) in an inertial frame, derive an expression for the radius r_0 of a fixed circular orbit (i.e. a solution with $r = r_0 =$ const.). What is the normal force the wire applies to the bead to keep it in a circular orbit?

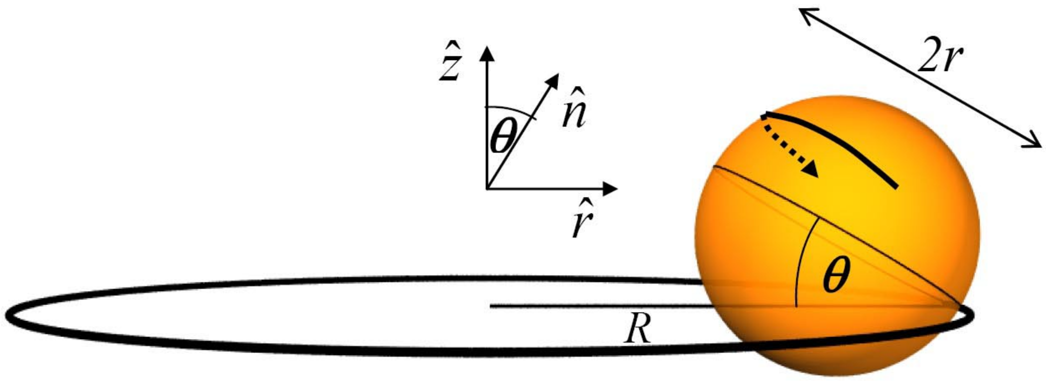
(b) (*2 pts*) Write down the one-dimensional Lagrangian $L(r, \dot{r}, t)$ for this system. Using this Lagrangian, obtain an equation of motion for $r(t)$ and verify your result for r_0.

(c) (*2 pts*) Now consider a small displacement from the circular orbit, $r = r_0 + \epsilon(t)$. Derive a condition on the function $f(r)$ such that a circular orbit at $r = r_0$ is stable.

(d) (*2 pts*) Find the component of the force on the bead in the $\hat{\boldsymbol{e}}_\phi$ direction, i.e. perpendicular to the plane of the wire. (The angular velocity is $\omega = d\phi/dt$.) This force is sometimes called the "constraint force." Obtain an answer that is valid for arbitrary motion of the bead; i.e., do not assume that $r = r_0$ or $r = r_0 + \epsilon(t)$. [Hint: you can solve this problem either by using $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, or by using Lagrange multipliers. If you use Lagrange multipliers, remember that they can only impose (holonomic) constraints on coordinates, and constraints directly on velocities are more subtle.]

(e) (*2 pts*) Find the Hamiltonian $H(r, p_r, t)$ and show that it is conserved. By what amount does H differ from the total energy of the bead? Is the total energy automatically conserved? Explain why or why not in terms of your answer to part (d).

Classical Mechanics 2: Basketball on a Rim



A basketball of radius r rolls without slipping around a basketball rim of radius R. The basketball rotates in such a way that the contact point traces out a great circle on the ball (i.e. a circle with maximum circumference $2\pi r$), and the center of mass moves in a horizontal circle with angular frequency Ω, counterclockwise as seen from above. The plane of the great circle makes an angle θ with the horizontal. The ball has a moment of inertia $I = \frac{2}{3}mr^2$ around its center, and gravity acts downward with an acceleration of magnitude $g > 0$.

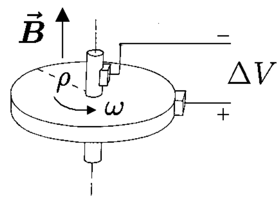
(a) (*3 pts*) Calculate the torque on the ball about its center of mass imparted by gravity and the contact force from the hoop.

(b) (*3 pts*) Determine the angular velocity vector $\vec{\boldsymbol{\omega}}$ that describes the rotation of the ball relative to the inertial frame. Express your answer in terms of Ω, R, r, and suitable unit vectors. [*Hint:* It may be helpful to consider the rotating frame in which the center of mass is at rest, but be sure to give your answer in the original frame.]

(c) (*4 pts*) Find Ω in terms of g, R, r, and θ.

Electromagnetism 1: A Conducting Wheel in a Uniform Magnetic Field

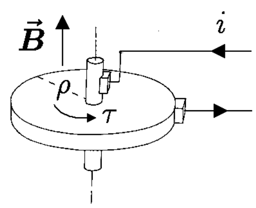
(a) (*3 pts*) A conducting wheel of radius ρ is pivoted so that it can rotate in the horizontal plane, in the presence of a uniform magnetic field of magnitude B in the vertical direction, as shown in the diagram. A wire connects to the shaft of the wheel through a frictionless sliding contact, and another wire connects to the outer edge of the wheel through another frictionless sliding contact. Neglect the diameter of the shaft. If the wheel is forced to rotate with angular frequency ω, counterclockwise as seen from above, an electrical potential difference will be generated between the two contacts. This is often called a homopolar generator. Calculate



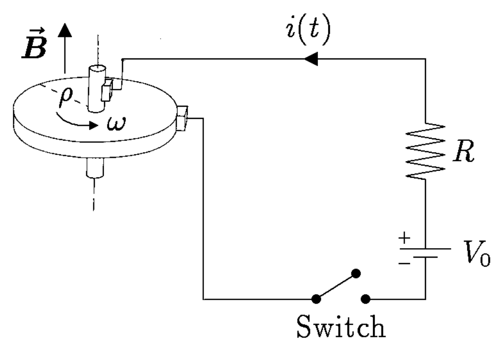
$$\Delta V \equiv V_{\text{edge}} - V_{\text{center}} \tag{1}$$

as a function of B, ρ, and ω.

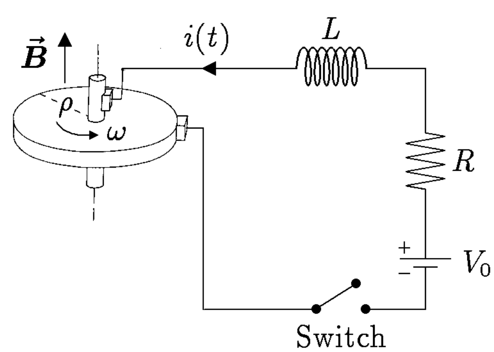
(b) (*2 pts*) If a current i flows through the wheel, from the center to the edge, a torque τ is imparted to the wheel about its axis. Calculate τ, defined as positive if counterclockwise when viewed from above, as a function of B, ρ, and i. You may assume that the force that acts on the electrons is rapidly transferred to the solid body of the wheel.



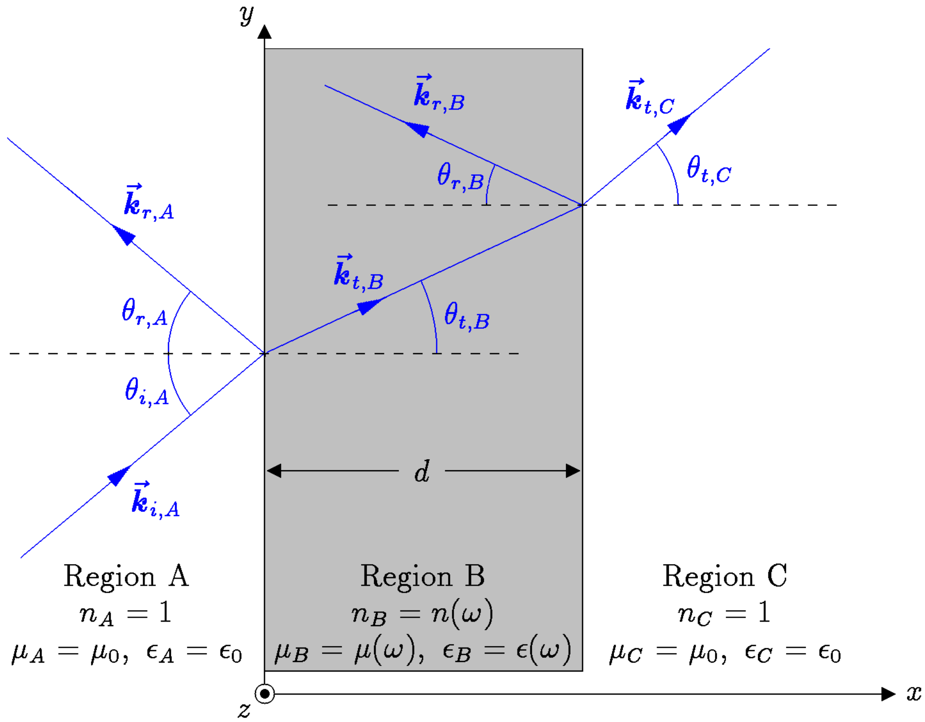
(c) (*3 pts*) Suppose that the wires from the wheel are connected to a circuit, which also includes a switch, an ideal voltage source V_0, and a resistor R, as shown. Neglect all resistance in the wires, the wheel, and the contacts, neglect any self-inductance in the system, and neglect any friction. The moment of inertia of the wheel about its axis is I_0. Suppose that the switch is closed at $t = 0$, with the wheel initially at rest. Find the current $i_c(t)$ that flows through the circuit, and the angular velocity $\omega_c(t)$ of the wheel.



(d) (*2 pts*) Now suppose that an inductor of inductance L is added to the circuit, in series, as shown, with $L > R^2 I_0/(B^2\rho^4)$. The clock is reset, and the switch is again closed at $t = 0$, with the wheel at rest. For this case find the current $i_d(t)$ and the angular velocity $\omega_d(t)$ of the wheel.



Electromagnetism 2: Transmission of an EM Wave through a Dielectric Slab



Consider a dielectric slab of thickness d in empty space, with an index of refraction $n(\omega)$ which depends on the angular frequency ω of the radiation. The interior of the dielectric is called Region B, with Region A on its left and Region C on its right, as shown in the diagram. In Regions A and C the index of refraction is $n_A = n_C = 1$. The permittivity and permeability in Region B are given by $\epsilon(\omega)$ and $\mu(\omega)$, respectively, while in Regions A and C they have the vacuum values ϵ_0 and μ_0. Recall that the speed of light $c = 1/\sqrt{\mu_0\epsilon_0}$, and $n(\omega) = c\sqrt{\mu(\omega)\epsilon(\omega)}$.

In Region A there is an incident electromagnetic plane wave with angular frequency ω and propagation vector $\vec{\boldsymbol{k}}_{i,A}$. The angle of incidence is $\theta_{i,A}$, and the wave is polarized so that the electric field $\vec{\boldsymbol{E}}_{i,A}$ points out of the page, in the $\hat{\boldsymbol{z}}$ direction. Explicitly,

$$\vec{\boldsymbol{E}}_{i,A}(\vec{\boldsymbol{x}},t) = \mathrm{Re}\left\{E_{i,A}\hat{\boldsymbol{z}}\,\exp\left[i\left(\vec{\boldsymbol{k}}_{i,A}\cdot\vec{\boldsymbol{x}} - \omega t\right)\right]\right\}\ , \tag{1}$$

where $E_{i,A}$ is a complex number. The electric field in all regions will point in the $\hat{\boldsymbol{z}}$ direction.

(a) (*4 pts*) The electric field $\vec{\boldsymbol{E}}_B(\vec{\boldsymbol{x}},t)$ in Region B can be expressed approximately as the sum of a transmitted wave and a wave that is reflected from the back surface. As in Eq. (1), the transmitted wave can be written in terms of a complex number $E_{t,B}$ and a wave vector $\vec{\boldsymbol{k}}_{t,B}$, where $E_{t,B}$ is proportional to $E_{i,A}$:

$$E_{t,B} = t_{BA}(\theta_{i,A})\,E_{i,A}\ . \tag{2}$$

Calculate the transmission function $t_{BA}(\theta_{i,A})$, ignoring the reflected wave in Region B. Find also the angle $\theta_{t,B}$ of the transmitted wave, measured from the normal, as shown.

Problem continued on next page.

(b) (*1 pt*) Similarly, we can write the electric field $\vec{\boldsymbol{E}}_{t,C}(\vec{\boldsymbol{x}}, t)$ of the transmitted wave in Region C in terms of a complex number $E_{t,C}$ and a wave vector $\vec{\boldsymbol{k}}_{t,C}$, where

$$E_{t,C} = t_{CB}(\theta_{t,B})\, E_{t,B} \ . \tag{3}$$

Calculate $t_{CB}(\theta_{t,B})$, and also the angle $\theta_{t,C}$ of the transmitted wave in Region C. Include only the first transit through the slab, ignoring contributions from reflections that pass through the slab more than once.

(c) (*1 pt*) In terms of $t_{BA}(\theta_{i,A})$ and $t_{CB}(\theta_{t,B})$, express the transmission coefficient $T(\theta_{i,A})$ (i.e., the fraction of *power* transmitted) from Region A to Region C.

(d) (*1 pt*) Now imagine replacing the dielectric slab by a dilute plasma occupying the same region. The plasma can be treated as a material with $\mu = \mu_0$ and dielectric constant

$$\epsilon(\omega) = \epsilon_0 \left[1 - \frac{\omega_p^2}{\omega^2}\right] \ , \tag{4}$$

where ω_p is the electron plasma frequency ($\omega_p^2 = 4\pi N e^2/m_e$). For $\omega > \omega_p$, show that as the angle of incidence is increased it reaches a critical value $\theta_c(\omega_p/\omega)$ for which no transmission to Region C occurs.

(e) (*3 pts*) For $\theta_{i,A} > \theta_c$ (i.e., for an angle of incidence greater than critical), consider the fields inside Region B. Ignoring any reflections, calculate the ratio

$$R \equiv \lim_{\epsilon \to 0+} \frac{\text{Intensity of transmitted wave at } x = d - \epsilon}{\text{Intensity of transmitted wave at } x = \epsilon} \ , \tag{5}$$

where x is the horizontal coordinate with $x = 0$ at the A-B interface. In words, you should calculate the ratio of the transmitted wave intensity just inside the slab on the right to the intensity just inside the slab at the left.

Statistical Mechanics 1: A Strongly Interacting Fermi Gas

A spin-1/2 Fermi gas with (s-wave) attractive interactions between spin-up and spin-down fermions forms a superfluid of fermion pairs at low temperatures. When these interactions are resonant for s-wave scattering—i.e. the scattering cross section saturates the unitarity bound—only two intrinsic energy scales are relevant for describing the system: the energy scale associated with temperature k_BT, and the Fermi energy $E_F = \hbar^2 k_F^2/2m$. Here, $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wavevector, $n = N/V$ is the total density, N is the total number of fermions, V is the volume of the gas, and m is the fermion mass.

At zero temperature, the energy scale k_BT is irrelevant, so the total energy of the gas *must* be a universal number ξ times the ground state energy of a non-interacting Fermi gas,

$$E_{\text{tot}}|_{T=0} = \frac{3}{5}\xi N E_F. \tag{1}$$

Since the pressure $P = \partial E/\partial V$ at $T = 0$, the relationship $P = \frac{2}{3}E/V$ holds just as for a non-interacting Fermi gas. The pressure of the strongly interacting Fermi gas is thus the same universal number ξ times the pressure for a non-interacting Fermi gas,

$$P = \frac{2}{5}\xi n E_F. \tag{2}$$

In this problem, you will show that not only the zero-temperature but also the nonzero-temperature thermodynamic properties of the system are uniquely specified by ξ as long as the temperature is low enough not to break any fermion pairs.

(a) (*1 pt*) At low enough temperatures, the only excitations of the gas are phonons, i.e. sound waves. From the known equation of state between pressure and density at zero temperature, calculate the speed of sound c for phonons in the gas. Express your result in terms of ξ and the Fermi velocity $v_F = \hbar k_F/m$. For the remainder of this problem, you may assume that this value of c holds at all temperatures of interest. [*Hint:* recall that

$$c^2 = \frac{\partial P}{\partial \rho} \ , \tag{3}$$

where ρ is the mass per unit volume.]

Problem continued on next page.

(b) (*5 pts*) Find the contribution from phonons to the free energy of the gas

$$F_{\text{ph}}(N,V,T) \equiv -k_B T \ln Z_{\text{ph}}, \tag{4}$$

where Z_{ph} is the partition function for phonons

$$Z_{\text{ph}} \equiv \sum_{\text{all ph states}} e^{-E_{\text{ph}}/k_B T}, \tag{5}$$

and the sum is over all states involving any number of phonons (but no other excitations). Write your result in the form

$$F_{\text{ph}}(N,V,T) = a\, T^{\alpha} N^{\beta} V^{\gamma} \xi^{\delta} \tag{6}$$

where you need to find the (dimensionful) constant a and exponents α, β, γ, and δ.

Even if you do not succeed in finding a, α, β, γ, and δ, you can do all subsequent parts by expressing your answers to each of them in terms of these constants.

[*Hint:* The energy of a single-phonon state is

$$E_{\text{1-ph}} = \hbar c |\vec{\boldsymbol{k}}_{\text{1-ph}}|, \tag{7}$$

where $\vec{\boldsymbol{k}}_{\text{1-ph}}$ is the phonon wave vector, and c is the speed of sound calculated in part (a). Neglect any phonon interactions, so the total phonon energy is the sum of the single phonon energies:

$$E_{\text{tot-ph}} = \sum_i \hbar c |\vec{\boldsymbol{k}}_i| \ . \tag{8}$$

You can assume that the gas occupies a cube of volume V with periodic boundary conditions, and you can assume the volume is large enough to replace various sums by integrals. In addition, you can neglect the possibility of phonons decaying into broken fermion pairs, allowing momentum integrals to be extended to infinitely large momenta. You may find the following integral to be useful

$$\int_0^{\infty} dx\, x^2 \ln(1-e^{-x}) = -\frac{\pi^4}{45}. \qquad] \tag{9}$$

(c) (*2 pts*) Calculate the phonon contribution $S_{\text{ph}}(N,V,T)$ to the entropy and the phonon contribution $E_{\text{ph}}(N,V,T)$ to the total energy of the Fermi gas.

(d) (*1 pt*) Suppose that the low-temperature gas, initially at temperature T_0 and volume V_0, is now adiabatically compressed. If the volume shrinks to $V_f = \epsilon V_0$ with $0 < \epsilon < 1$, what is the final temperature T_f?

(e) (*1 pt*) Calculate the specific heat at constant volume C_V of the gas.

(f) (*0 pts*) *You do not need to do this part!* But if you wish, you can check your answers to parts (b), (c), and (e) by verifying that $F_{\text{ph}}/(NE_F)$, $E_{\text{ph}}/(NE_F)$, $S_{\text{ph}}/(Nk_B)$ and $C_V/(Nk_B)$ can each be written as functions of (k_BT/E_F).

Statistical Mechanics 2: Interaction between Dipole Moments

Two infinitely heavy classical particles are separated by a distance $r = |\vec{\boldsymbol{r}}|$. Each particle has a dipole moment $\vec{\boldsymbol{M}} = (M_x, M_y, M_z)$ with $M_x^2 + M_y^2 + M_z^2 = M^2$ being fixed and equal for both particles, and both dipoles are free to rotate. The interaction energy between the dipole moments is

$$V = \frac{3(\vec{\boldsymbol{M}}_1 \cdot \hat{\boldsymbol{r}})(\vec{\boldsymbol{M}}_2 \cdot \hat{\boldsymbol{r}}) - (\vec{\boldsymbol{M}}_1 \cdot \vec{\boldsymbol{M}}_2)}{r^3} \ , \tag{1}$$

where $\hat{\boldsymbol{r}}$ is the unit vector in the $\vec{\boldsymbol{r}}$ direction. The system is in thermal equilibrium with the environment at high temperature T, such that

$$k_B T \gg \frac{M^2}{r^3}. \tag{2}$$

(a) (*1 pt*) Assume that the dipoles are separated along the $\hat{\boldsymbol{z}}$ axis. Write the interaction energy as a function of the angular variables (θ_1,ϕ_1) and (θ_2,ϕ_2), where

$$\vec{\boldsymbol{M}}_i = M(\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i) \tag{3}$$

specify the dipole directions in spherical coordinates.

(b) (*4 pts*) Find an expression for the partition function $Z(r)$ at *fixed* r. Evaluate the partition function in the high temperature limit given above, up to second order in $\gamma(r) = \frac{M^2}{k_B T r^3}$. Some potentially useful integrals are:

$$\int_0^\pi \mathrm{d}\theta \ \sin^2\theta = \frac{\pi}{2}, \qquad \int_0^\pi \mathrm{d}\theta \ \sin^3\theta = \frac{4}{3}, \qquad \int_0^\pi \mathrm{d}\theta \ \sin^4\theta = \frac{3\pi}{8}. \tag{4}$$

(c) (*2 pts*) Compute the free energy $F(r)$ and internal energy $E(r)$ of the dipole-dipole system up to the lowest non-trivial term in $\gamma(r)$.

(d) (*1 pt*) What is the average force $\vec{\boldsymbol{f}}(r)$ between the particles at a distance r?

(e) (*1 pt*) Now assume that the particles are connected by a spring with elastic energy

$$U(r) = \frac{1}{2}A(R - r)^2, \tag{5}$$

where A and R are constants. Calculate the equilibrium separation between the particles, assuming that

$$\frac{1}{2}AR^2 \gg \frac{M^2}{R^3}, k_B T; \tag{6}$$

i.e. $U \gg V, k_B T$.

(f) (*1 pt*) Using the above result, calculate the coefficient of linear expansion

$$\alpha \equiv \frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}T}. \tag{7}$$

SECTION IV: QUANTUM MECHANICS

Quantum Mechanics 1: A Heisenberg Ferromagnet

In a ferromagnetic material, the electron spins are aligned, suggesting that an interaction of the form

$$\delta H = \kappa \vec{\boldsymbol{S}}_1 \cdot \vec{\boldsymbol{S}}_2 \tag{1}$$

is present between each pair of electrons, with $\kappa < 0$, where $\vec{\boldsymbol{S}}_1$ and $\vec{\boldsymbol{S}}_2$ are the operators corresponding to the spins of the two electrons. While Eq. (1) does not appear explicitly in the Hamiltonian for a ferromagnet, Heisenberg realized that Eq. (1) could appear as an *effective* interaction, arising from Coulomb repulsion and the fermionic properties of electrons.

In this problem, you will derive Heisenberg ferromagnetism for two spin-1/2 electrons in a common potential $V(\vec{\boldsymbol{r}})$, with Hamiltonian

$$H = \frac{|\vec{\boldsymbol{p}}_1|^2}{2m} + \frac{|\vec{\boldsymbol{p}}_2|^2}{2m} + V(\vec{\boldsymbol{r}}_1) + V(\vec{\boldsymbol{r}}_2) + \frac{e^2}{|\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2|}. \tag{2}$$

Note that δH above is not included in this Hamiltonian, so there is no explicit spin dependence. The single-particle Schrödinger equation with potential $V(\vec{\boldsymbol{r}})$ has eigenstates with energies E_i and wave functions $\psi_i(\vec{\boldsymbol{r}})$.

(a) (*1 pt*) The total wave function $\Psi(\vec{\boldsymbol{r}}_1, \{s_1\}; \vec{\boldsymbol{r}}_2, \{s_2\})$ depends on the set of spin variables $\{s_1\}$ and $\{s_2\}$. (You may use whichever notation for spin that you prefer, including bra/ket notation.) Show that the eigenstates of the Hamiltonian can be written in a separable form as $\psi(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)\chi(\{s_1\}, \{s_2\})$. Construct eigenstates of total spin, and describe the symmetry properties of $\psi(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)$ under particle exchange for each of the spin states.

(b) (*1 pt*) In the absence of Coulomb repulsion, consider the two-particle configurations where one electron is in state $\psi_a(\vec{\boldsymbol{r}})$ and the other electron is in state $\psi_b(\vec{\boldsymbol{r}})$. What are the corresponding two-particle wave functions, including spin?

(c) (*3 pts*) The degeneracy of the states in part (b) is broken by Coulomb repulsion, yielding two distinct energy levels. Identify the two-particle wave functions associated with these two levels, and find an expression for the energy splitting δE to first order in the Coulomb interaction in terms of ψ_a and ψ_b.

(d) (*3 pts*) Show that the energy splitting in part (c) can be mimicked (at first order) by turning off the Coulomb interaction in Eq. (2) and replacing it with Eq. (1). Find an expression for κ in terms of δE.

(e) (*2 pts*) Determine the sign of κ. [*Hint:* the judicious use of Fourier transforms may be helpful. Recall that

$$\int d^3r\, e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}} = (2\pi)^3\delta^{(3)}(\vec{\boldsymbol{q}}) \; ; \qquad \int d^3r\, e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}} \frac{1}{|\vec{\boldsymbol{r}}|} = \frac{4\pi}{|\vec{\boldsymbol{q}}|^2} \;.] \tag{3}$$

Quantum Mechanics 2: The Supersymmetric Method

In this problem, you will solve for the energy spectrum of a particle of mass m confined to a potential

$$V(x) = V_0\left[\sec^2\frac{x}{x_0} + \tan^2\frac{x}{x_0}\right] = V_0\left[1 + 2\tan^2\frac{x}{x_0}\right], \tag{1}$$

where $-\frac{\pi}{2}x_0 \leq x \leq \frac{\pi}{2}x_0$. (Recall that $\sec x = 1/\cos x$, and the equality between the two expressions for $V(x)$ is a consequence of trigonometric identities.) Amazingly, this system is exactly solvable for the special value

$$V_0 = \frac{\hbar^2}{2m}\frac{1}{x_0^2}, \tag{2}$$

and you will derive the spectrum using the "supersymmetric method". (Supersymmetry is a possible symmetry between bosons and fermions, but that fact will not be relevant for this problem.)

(a) (*2 pts*) Consider two Hamiltonians

$$H = A^\dagger A, \qquad \widetilde{H} = AA^\dagger, \tag{3}$$

where A is an unspecified operator. Assume that H has (normalized) eigenstates $|n\rangle$ with

$$H|n\rangle = E_n|n\rangle \qquad (n \geq 1). \tag{4}$$

For every n with $E_n \neq 0$, show that $A|n\rangle$ is an (unnormalized) eigenstate of $\widetilde{H}$. Find the normalized eigenstates $|\widetilde{n}\rangle$ and their eigenvalues $\widetilde{E}_n$ under $\widetilde{H}$. What goes wrong with this argument if $E_n = 0$? Can E_n or $\widetilde{E}_n$ ever be negative? [*Note:* for the remainder of this problem, you can assume that $|\widetilde{n}\rangle$ form a complete basis for $\widetilde{H}$, up to possible zero energy states.]

(b) (*2 pts*) Now consider a specific operator A of the form

$$A = \frac{\partial}{\partial x} + W(x), \tag{5}$$

where $W(x)$ is real. Show that H and $\widetilde{H}$ each describe a particle moving in a potential, in units where $\hbar^2/2m = 1$. Find the two potential energy functions, $V(x)$ and $\widetilde{V}(x)$, corresponding to H and $\widetilde{H}$, respectively.

(c) (*2 pts*) We will be considering Hamiltonians defined on a finite range $-\frac{\pi}{2}x_0 \leq x \leq \frac{\pi}{2}x_0$. This means that the wave functions will have Dirichlet boundary conditions at $x = \pm\frac{\pi}{2}x_0$, i.e. $\psi(\pm\frac{\pi}{2}x_0) = 0$. Assume that H has a zero energy ground state consistent with these boundary conditions. Find the unnormalized ground state wave function $\psi_0(x)$ for H in terms of $W(x)$. Show that $\widetilde{H}$ *cannot* have a zero energy ground state consistent with these boundary conditions. [*Note:* parts (d) and (e) of this problem can be solved without solving part (c).]

Problem continued on next page.

(d) (*2 pts*) The potential in Eq. (1) is *dual* to a constant potential. That is, there is a $W(x)$ such that for $-\frac{\pi}{2}x_0 \le x \le \frac{\pi}{2}x_0$,

$$V(x) = a, \qquad \widetilde{V}(x) = b\left(\sec^2\frac{x}{x_0} + \tan^2\frac{x}{x_0}\right) = b\left(1 + 2\tan^2\frac{x}{x_0}\right), \tag{6}$$

where a and b are constants. What is $W(x)$, and what are a and b? You may find the following formulas helpful:

$$\int \frac{dx}{1+x^2} = \arctan x, \qquad \int d\theta \sec^2\theta = \tan\theta, \qquad \int d\theta \tan^2\theta = -\theta + \tan\theta. \tag{7}$$

(e) (*2 pts*) Find the energy spectrum and energy eigenstates for the potential in Eq. (1). Does this system have a zero energy ground state? You should assume that all wave functions vanish at $x = \pm\frac{\pi}{2}x_0$ (i.e. Dirichlet boundary conditions), and you should restore all factors of $\hbar$ and m. You do not need to normalize the states.