a) $H = \sqrt{p_x^2c^2 + p_z^2c^2 + m^2c^4} + mgz = E$

b) $\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \qquad p_x = p$

$\dot{p}_z = -\frac{\partial H}{\partial z} = -mg \qquad p_z = -mgt + p_z^0$

$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{c^2 p_x}{\sqrt{p_x^2c^2 + p_z^2c^2 + m^2c^4}}$

$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{c^2 p_z}{\sqrt{p_x^2c^2 + p_z^2c^2 + m^2c^4}}$

$\dot{x}\Big|_{t=0} = \beta_0 c \qquad \dot{z}\Big|_{t=0} = 0 \quad \Rightarrow \quad p_z(t=0) = 0$

$\Rightarrow p_z^0 = 0$

$p_x = p, \quad p_z = -mgt \qquad \dot{x} = \frac{c^2 p}{\sqrt{p^2c^2 + m^2g^2c^2t^2 + m^2c^4}}$

$\dot{z} = \frac{-mgc^2t}{\sqrt{p^2c^2 + m^2g^2c^2t^2 + m^2c^4}}$

N.B. $\beta_0 c = \frac{c^2 p}{\sqrt{p^2c^2 + m^2c^4}}$

$\beta_0 = \frac{pc}{\sqrt{p^2c^2 + m^2c^4}} \qquad \frac{1}{\beta_0} = \frac{\sqrt{p^2c^2 + m^2c^4}}{pc}$

$\frac{1}{\beta_0^2} = 1 + \frac{m^2c^4}{p^2c^2}$

$$\frac{m^2c^4}{p^2c^2} = -1 + \frac{1}{\beta_0^2}$$

$$\frac{p^2c^2}{m^2c^4} = \frac{1}{\frac{1}{\beta_0^2} - 1} = \frac{\beta_0^2}{1-\beta_0^2} = \beta_0^2 \gamma_0^2$$

$$p^2 = \beta_0^2 \gamma_0^2 m^2 c^2 \qquad \gamma_0^2 = \frac{1}{1-\beta_0^2}$$

$$p = \beta_0 \gamma_0 m c$$

$$p^2c^2 + m^2c^4 = \beta_0^2 \gamma_0^2 m^2 c^4 + m^2 c^4$$
$$= m^2c^4 \left(\frac{\beta_0^2}{1-\beta_0^2} + 1\right) = \frac{m^2c^4}{1-\beta_0^2}$$
$$= m^2 c^4 \gamma_0^2$$

$$\dot{x} = \beta_0 \gamma_0 m c^3 \frac{1}{\sqrt{m^2c^4\gamma_0^2 + m^2 g^2 c^2 t^2}}$$

$$\dot{z} = -mgc^2 t \frac{1}{\sqrt{m^2c^4\gamma_0^2 + m^2 g^2 c^2 t^2}}$$

$$\dot{x} = \frac{\beta_0 \gamma_0 c^2}{g} \frac{1}{\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}}$$

$$\dot{z} = -ct \frac{1}{\sqrt{c^2\gamma_0^2/g^2 + t^2}}$$

$$x\Big|_{t=0} = 0 \qquad z\Big|_{t=0} = h$$

$$x = \frac{\beta_0 \gamma_0 c^2}{g} \int_0^t \frac{dt}{\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}}$$

$$z - h = -c\cancel{t} \int_0^t \frac{t\,dt}{\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}}$$

$$x = \frac{\beta_0 \gamma_0 c^2}{g} \ln\left[t + \sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2} \,\Big|_0^t \right]$$

$$z = h - c\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}\,\Big|_0^t$$

$$x = \frac{\beta_0 \gamma_0 c^2}{g} \left\{ \ln\left[t + \sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2} \right] - \ln \frac{c\gamma_0}{g} \right\}$$

$$z = h - c\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2} + \frac{c^2\gamma_0}{g}$$

t_0 $\quad z = 0 \quad$ $$c\sqrt{\frac{c^2\gamma_0^2}{g^2} + t_0^2} = \frac{c^2\gamma_0}{g} + h$$

$$\sqrt{\frac{c^2\gamma_0^2}{g^2} + t_0^2} = \frac{c\gamma_0}{g} + \frac{h}{c}$$

$$\frac{u^2}{g^2} + t_0^2 = \frac{c^2\gamma_0^2}{g^2} + \frac{2\gamma_0 h}{g} + \frac{h^2}{c^2}$$

$$t_0^2 = \frac{2\gamma_0 h}{g} + \frac{h^2}{c^2}$$

$$t_0 = \sqrt{\frac{2\gamma_0 h}{g} + \frac{h^2}{c^2}}$$

$$R = \frac{\beta_0\gamma_0 c^2}{g}\left\{ \ln\left[\frac{\sqrt{\frac{2\gamma_0 h}{g} + \frac{h^2}{c^2}} + \frac{c\gamma_0}{g} + \frac{h}{c}}{\frac{c\gamma_0}{g}} \right] \right\}$$

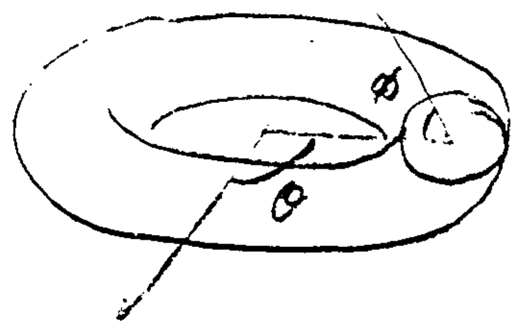
$$\boxed{= \frac{\beta_0\gamma_0 c^2}{g}\left\{ \ln\left[1 + \frac{hg}{c^2\gamma_0} + \sqrt{\frac{2gh}{\gamma_0 c^2} + \frac{h^2 g^2}{c^4\gamma_0^2}} \right] \right\}}$$

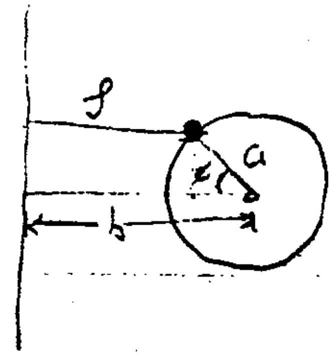
$c \to \infty$

$$= \frac{vc}{g}\sqrt{\frac{2gh}{c^2}} = v\sqrt{\frac{2h}{g}}$$

Mechanics II

P. Fisher solution (1)





a) Use ϕ and θ as indicated.

$$T = \frac{1}{2} m (a\dot{\phi})^2 + \frac{1}{2} m (\rho \dot{\theta})^2$$

$$\rho = b - a\cos\phi$$

$$T = \frac{1}{2} m (a\dot{\phi})^2 + \frac{1}{2} m (b - a\cos\phi)^2 \dot{\theta}^2$$

$L = T$, no forces, no potential

b) $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m(b - a\cos\phi)^2 \dot{\theta} = m\rho^2\dot{\theta}$$

$$p_\phi = m a^2 \dot{\phi}$$

L is cyclic in θ, p_θ is conserved

c) $$H = p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$$

$$= \frac{p_\theta}{2}\left(\frac{p_\theta}{m\rho^2}\right) + \frac{p_\phi}{2}\left(\frac{p_\phi}{m a^2}\right) = \frac{p_\theta^2}{2m\rho^2} + \frac{p_\phi^2}{2ma^2}$$

$$H = \frac{p_\theta^2}{2m(b - a\cos\phi)^2} + \frac{p_\phi^2}{2ma^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad \text{Conserved}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m(b - a\cos\phi)^2}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = \frac{(2)\, p_\theta^2}{m(b - a\cos\phi)^3}(a\sin\phi)$$

$$= \frac{p_\theta^2\, a\sin\phi}{m(b - a\cos\phi)^3}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{ma^2}$$

d) $$\frac{dp_\phi/dt}{d\phi/dt} = \frac{dp_\phi}{d\phi} = \frac{\dfrac{p_\theta^2\, a\sin\phi}{m(b - a\cos\phi)^3}}{\dfrac{p_\phi}{ma^2}}$$

$$= \frac{p_\theta^2\, a^3 \sin\phi}{(b - a\cos\phi)^3\, p_\phi}$$

$$\int_{p_\phi(\phi_0)}^{p_\phi(\phi')} p_\phi \, dp_\phi = \int_{\phi_0}^{\phi'} \frac{p_\theta^2 a^3 \sin\phi}{(b - a\cos\phi)^3} \, d\phi$$

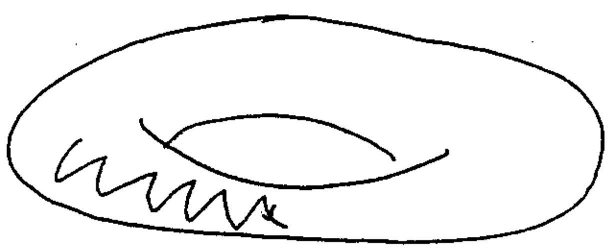
$$u = b - a\cos\phi$$
$$du = a\sin\phi \, d\phi$$

$$\frac{1}{2}\left(p_\phi^2(\phi') - p_\phi^2(\phi_0)\right) = p_\theta^2 a^3 \int_{b - a\cos\phi_0}^{b - a\cos\phi'} \frac{du}{u^3}$$

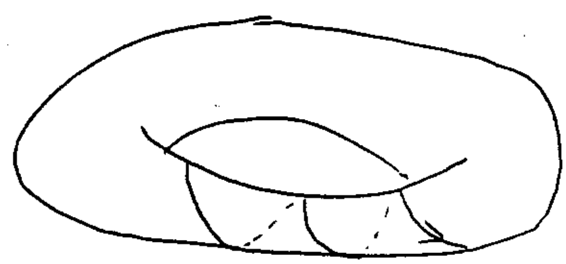
$$= p_\theta^2 a^2 (-2) \left. \frac{1}{u^2} \right|_{b - a\cos\phi_0}^{b - a\cos\phi'} \qquad -\frac{1}{2}$$

$$= +2\, p_\theta^2 a^2 \left[\frac{1}{(b - a\cos\phi_0)^2} - \frac{1}{(b - a\cos\phi')^2} \right]$$

$$p_\phi(\phi') = \sqrt{\cancel{4}\,1\, p_\theta^2 a^2 \left[\frac{1}{(b - a\cos\phi_0)^2} - \frac{1}{(b - a\cos\phi')^2} \right] + p_\phi^2(\phi_0)}$$



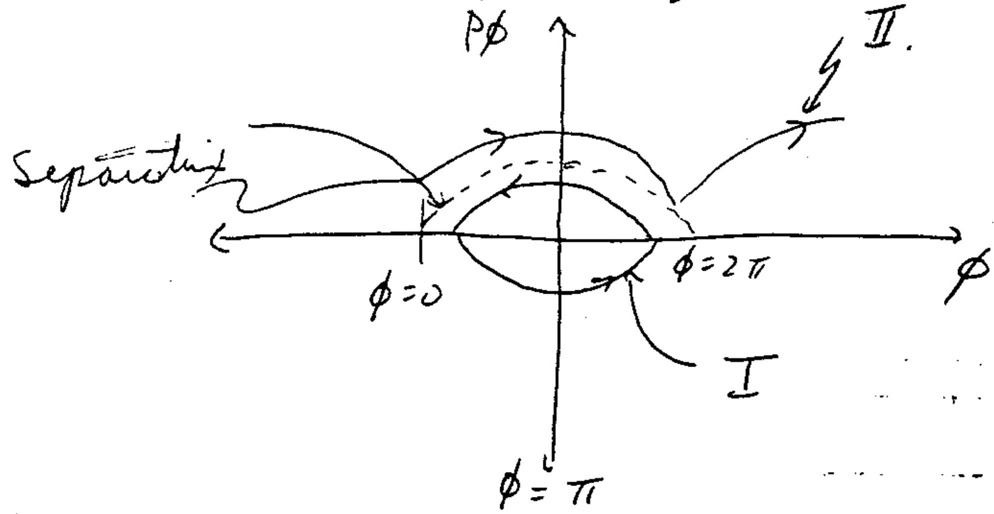
I



II

If the particle has sufficient energy, it is able to circle the minor radius of the torus as shown in II. Otherwise, it oscillates around $\phi = \pi$ as shown in I

On a phase ~~p~~ diagram.



<u>The separatrix is defined as $P_\phi(0)=0$</u>

$$P_\theta a \sqrt{\cancel{2P_\theta a}\left[\frac{-1}{(b-a\cos\phi_0)^2}+\frac{1}{(b-a\cos 0)^2}\right]} = P_\phi(\phi_0)$$

X

Energy condition?

Problem 1

(a)
$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v}\times\vec{B} \qquad \vec{p} = m\gamma\vec{v}$$

$$\rightarrow \omega|\vec{p}| = \frac{e}{c}\frac{|p|B}{m\gamma}$$

$$\omega = \frac{eB}{\gamma m_0 c} = \frac{v}{R}$$

$$B = \frac{v\gamma m_0 c}{eR}$$

$$\boxed{B = \frac{\beta\gamma m_0 c^2}{eR}}$$

(b)
$$E = m_0 c^2 + n q U_0$$

(c)
$$dn = \frac{v\,dt}{2\pi R}$$

$$\frac{dE}{dt} = qU_0\frac{dn}{dt} = \frac{qU_0}{2\pi R}v$$

$$E = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} \Rightarrow \frac{m_0^2 c^4}{E^2} = 1-\frac{v^2}{c^2}$$

$$\frac{v^2}{c^2} = 1 - \frac{m_0^2 c^4}{E^2}$$

$$\frac{v}{c} = \frac{1}{E}\sqrt{E^2 - m_0^2 c^4} \qquad v = \frac{c}{E}\sqrt{E^2 - m_0^2 c^4}$$

$$\boxed{\frac{dE}{dt} = \frac{qU_0 c}{2\pi R}\,\frac{\sqrt{E^2 - m_0^2c^4}}{E}}$$ Diff. eqn.

$$\frac{E\,dE}{\sqrt{E^2 - m_0^2c^4}} = \frac{qU_0c}{2\pi R}\,t$$

$$\int_{E(0)}^{E(t)} \frac{E\,dE}{\sqrt{E^2 - m_0^2c^4}} = \frac{qU_0c}{2\pi R}\,t$$

$$\sqrt{E^2 - m_0^2c^4}\,\Bigg|_{E(0) = m_0c^2}^{E(t)} = \text{[illegible]}$$

$$\sqrt{E^2(t) - m_0^2c^4} = \frac{qU_0c}{2\pi R}\,t$$

$$\boxed{E(t) = \sqrt{m_0^2c^4 + \left(\frac{qU_0c}{2\pi R}\,t\right)^2}}$$ ✓

(d) $T = \dfrac{2\pi R}{v}$ $\qquad E = TP = \left(\dfrac{2\pi R}{v}\right)\dfrac{2}{3}\,\dfrac{q^2}{m^2c^3}\,\gamma^2\omega^2|\vec{p}|^2$

$$= \frac{4\pi}{3}\,\frac{R}{v}\,\frac{q^2}{\cancel{m^2}c^3}\left(\frac{v}{R}\right)^2\gamma^2\,\cancel{m^2}\gamma^2 v^2$$

$$= \frac{4\pi}{3}\,\frac{q^2}{c^3}\,\frac{v^3}{R}\,\gamma^4 = \frac{4\pi}{3}\left(\frac{q^2}{R}\right)\beta^3\gamma^4$$

(e)

$$qU_0 = \frac{4\pi}{3}\left(\frac{q^2}{R}\right)\beta^3\gamma^4$$

as $\beta \rightarrow 1$

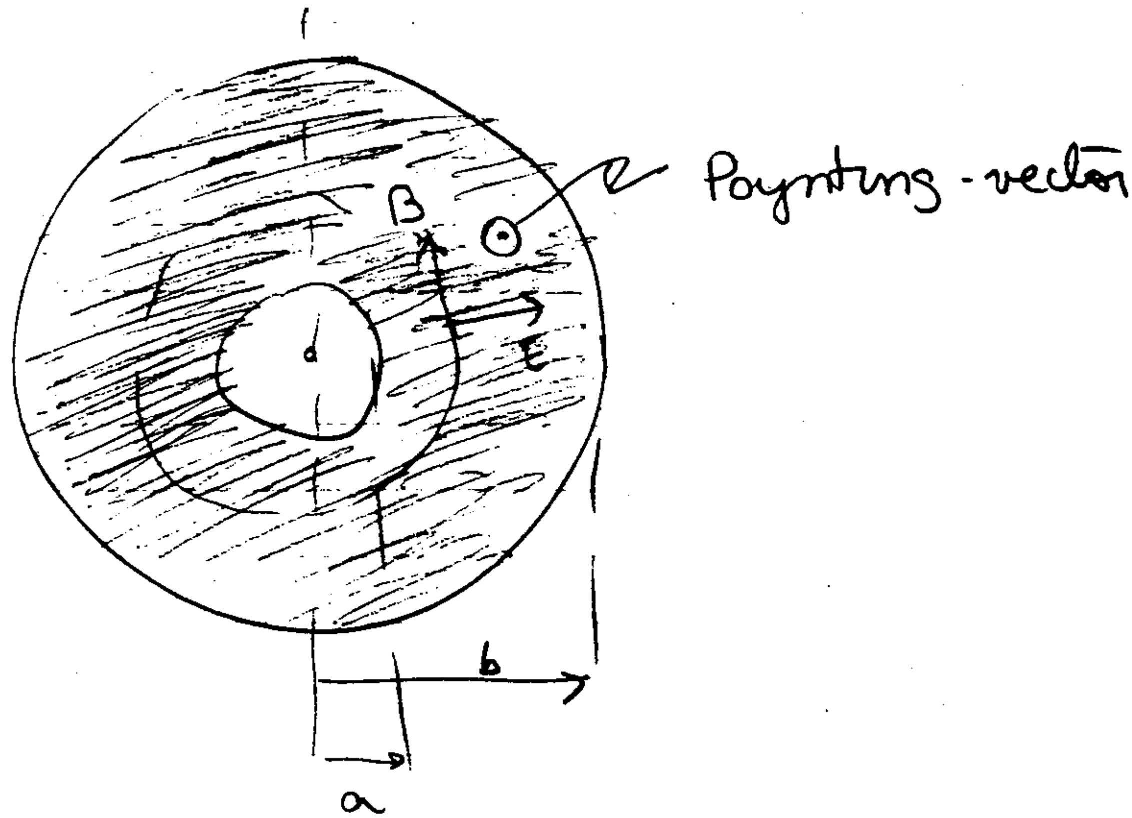
$$U_0 = \frac{4\pi}{3}\,\frac{q}{R}\,\gamma^4$$

$$E = m_0 c^2 \gamma$$

$$\boxed{E = m_0 c^2 \left[\frac{3U_0}{4\pi}\,\frac{R}{q}\right]^{1/4}}$$

Problem 2

a)



(b) Capacitance

$$\epsilon \int E.da = 4\pi Q$$

$$\epsilon E(r)(2\pi r) = 4\pi\lambda$$

$$\boxed{E(r) = \frac{2\lambda}{\epsilon}\cdot\frac{1}{r}}$$

$$V = \frac{2\lambda}{\epsilon}\ln\frac{b}{a} \qquad Q_0 = C_0 V$$

$$\boxed{C_0 = \frac{\epsilon}{2}\frac{1}{\ln(b/a)}}$$ per unit length

$$\mathcal{E} = -L\frac{dI}{dt} = -\frac{1}{c}\frac{d\Phi_B}{dt}$$

let I be the current $\qquad \nabla \times B = \frac{4\pi J}{c}$

$$2\pi r B = \frac{4\pi I}{c}$$

$$\boxed{B = \frac{2I}{cr}} \qquad \Phi_0 = \int_a^b B(r)\,dr$$

$$\Phi_0 = \frac{2I}{c}\ln\frac{b}{a} =$$

$$\boxed{L_0 = \frac{2}{c^2}\ln\frac{b}{a}}$$

(c)

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad \cdots (1)$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{\epsilon}{c}\frac{\partial \vec{E}}{\partial t} \quad \cdots (2)$$

$$\vec{E} = \frac{2\lambda}{\epsilon}\frac{1}{r}\sin(kx-\omega t)\,\hat{\rho}$$

$$\vec{B} = \frac{2I}{c}\frac{1}{r}\sin(kx-\omega t)\,\hat{\phi}$$

from (1) $\qquad k|E| = \frac{\omega}{c}|B|$

$$k|B| = \frac{\epsilon}{c}\omega|E|$$

$$\Rightarrow \quad k^2 = \epsilon \left(\frac{\omega}{c}\right)^2$$

$$\boxed{k = \sqrt{\epsilon}\,\frac{\omega}{c}}$$

$$\boxed{\frac{|E|}{|B|} = \frac{1}{\sqrt{\epsilon}}} \quad \Rightarrow$$

$$\frac{\frac{2\lambda}{\epsilon}}{\frac{2I}{c}} = \frac{1}{\sqrt{\epsilon}} = \frac{\lambda c}{I\epsilon}$$

$$\boxed{\frac{\lambda}{I} c = \sqrt{\epsilon}}$$

$$V = \frac{2\lambda}{\epsilon} \ln \frac{b}{a}$$

$$\frac{V}{I} = \frac{2}{\epsilon}\frac{\lambda}{I} \ln \frac{b}{a} = \frac{2}{\epsilon} \ln \frac{b}{a}\, \frac{1}{c} \sqrt{\epsilon}$$

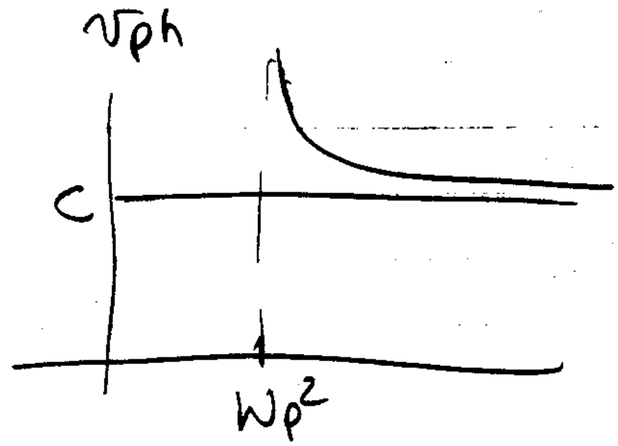
$$\boxed{\frac{V}{I} = \frac{2}{c} \ln \frac{b}{a}\, \frac{1}{\sqrt{\epsilon}} = Z}$$

(d) $$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

group and phase velocities

$$\boxed{v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon}} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}}$$

$$v_{gh} = \frac{d\omega}{dk}$$



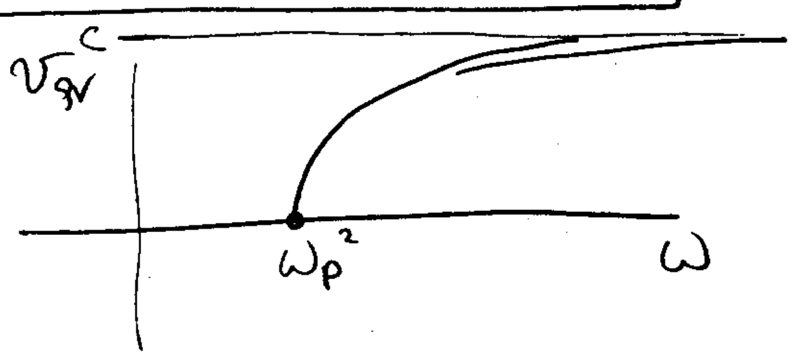
$$k^2 = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\frac{\omega^2}{c^2}$$

$$c^2 k^2 = \omega^2 - \omega_p^2$$

$$c^2\, 2k = 2\omega \frac{d\omega}{dk}$$

$$\frac{d\omega}{dk} = c^2 \frac{k}{\omega} = c^2 \frac{\sqrt{\epsilon}}{c} = c\sqrt{\epsilon}$$

$$\boxed{v_{gr} = c\sqrt{\epsilon} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$$



(a) For the gas system: $\mathcal{Z}_g(\beta, V, \mu) = \sum_{N_g=0}^{\infty} e^{\beta\mu N_g} Z_g(\beta, V, N_g)$

3.5

where $Z_g(\beta, V, N_g) = \frac{1}{N_g!}\frac{1}{h^{3N_g}}\left[\int d^3r\, d^3p\, e^{-\beta p^2/2m}\right]^{N_g} = \frac{1}{N_g!}\left[\frac{V}{\lambda_T^3}\right]^{N_g}$ where $\lambda_T^2 \equiv \frac{h^2\beta}{2\pi m}$

$$\therefore\ \mathcal{Z}_g(\beta, V, \mu) = \sum_{N_g=0}^{\infty}\frac{1}{N_g!}\left[\frac{Ve^{\beta\mu}}{\lambda_T^3}\right]^{N_g} = \boxed{\exp\left[\frac{V}{\lambda_T^3}e^{\beta\mu}\right]}$$

For the surface system: $\mathcal{Z}_s(\beta, V, \mu) = \sum_{N_s=0}^{\infty} e^{\beta\mu N_s} Z_s(\beta, V, N_s)$

where $Z_s(\beta, V, N_s) = \frac{1}{N_s!}\frac{1}{h^{2N_s}}\left[\int d^2r\, d^2p\, e^{-\beta p^2/2m} e^{\beta\varepsilon_0}\right]^{N_s} = \frac{1}{N_s!}\left[\frac{A}{\lambda_T^2}e^{\beta\varepsilon_0}\right]^{N_s}$

$$\therefore\ \mathcal{Z}_s(\beta, V, \mu) = \sum_{N_s=0}^{\infty}\frac{1}{N_s!}\left[\frac{Ae^{\beta\mu}e^{\beta\varepsilon_0}}{\lambda_T^2}\right]^{N_s} = \boxed{\exp\left[\frac{A}{\lambda_T^2}e^{\beta\mu}e^{\beta\varepsilon_0}\right]}$$

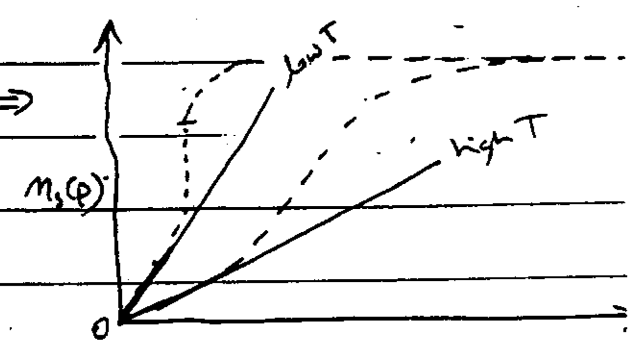
(b) $n_s = \frac{N_s}{A} = \frac{1}{A}\frac{\partial}{\partial(\beta\mu)}\ln \mathcal{Z}_s(\beta, V, \mu) = \frac{1}{\lambda_T^2}e^{\beta\mu}e^{\beta\varepsilon_0}$

3.5

but in equil. the surface and gas have the same β, μ. $\therefore$ we can eliminate $e^{\beta\mu}$:

Since $\beta PV = \ln \mathcal{Z}_g(\beta, V, \mu) = \frac{V}{\lambda_T^3}e^{\beta\mu}$ $\Rightarrow$

$$\Rightarrow \boxed{n_s = P\cdot\frac{h}{k_BT(2\pi m k_BT)^{1/2}}\cdot e^{\varepsilon_0/k_BT}}$$



(c) at low P the density is low and there should be no effect; at intermediate P

2 the attractive nature of the interaction will yield an effective larger ε_0 so that $n_s(P)$ will have a steeper slope; at high P, $n_s(P)$ will saturate due to the short range repulsive nature of the interaction. A phase transition is expected at ((with a discontinuity in $n_s(P)$)

(d) Yes, when $\varepsilon_0 \gg k_BT$: Shorter interatomic separation on the surface

1

a) The grand partition function for the surface system is just $\boxed{Z_s = (1+fe^{\beta w})^{N_a}}$

[3] and the average number of adsorbed sites, N_A, is then

$$N_A = f\frac{\partial \ln Z_s}{\partial f} = N_a \frac{fe^{\beta w}}{1+fe^{\beta w}} = \boxed{\frac{N_a}{(1+e^{-\beta(w+\mu)})} \text{ or } \frac{N_a}{(1+f^{-1}e^{-\beta w})}}$$

(b) $N_g = \int_0^\infty \frac{d\varepsilon\, D(\varepsilon)}{f^{-1}e^{\beta\varepsilon}-1} + \frac{1}{f^{-1}-1}$; The density of states of a free

[4] particle $D(\varepsilon)$ can be obtained from $D(\varepsilon)d\varepsilon = \frac{4\pi k^2 dk}{\frac{(2\pi)^3}{V}} = \frac{V}{2\pi^2}k^2 dk$

using $\varepsilon = \frac{\hbar^2 k^2}{2m}$ and $\frac{dk}{d\varepsilon} = \frac{m}{\hbar^2}\frac{1}{k}$ $\Rightarrow$ $D(\varepsilon) = \frac{2}{\sqrt{\pi}}V\left(\frac{2\pi m}{h^2}\right)^{3/2}\varepsilon^{1/2}$

Thus $$\frac{N_g}{V} = \frac{2}{\sqrt{\pi}}\left(\frac{2\pi m}{h^2}\right)^{3/2}\int_0^\infty \frac{d\varepsilon\, \varepsilon^{1/2}}{f^{-1}e^{\beta\varepsilon}-1}\frac{\beta^{3/2}}{\beta^{3/2}} + \frac{V^{-1}}{f^{-1}-1} = \frac{2}{\sqrt{\pi}}\frac{1}{(\lambda_T)^3}\int_0^\infty \frac{y^{1/2}dy}{f^{-1}e^{y}-1} + \frac{V^{-1}}{f^{-1}-1}$$

where $y = \varepsilon\beta$ and $\lambda_T^3 \equiv \left(\frac{h^2\beta}{2\pi m}\right)^{3/2}$. Using the identity

that $\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{y^{1/2}dy}{f^{-1}e^{y}-1} = \sum_{m=1}^\infty \frac{f^m}{m^{3/2}}$ we obtain

$$\boxed{\frac{N_g}{V} = \frac{1}{\lambda_T^3}\cdot\sum_{m=1}^\infty \frac{f^m}{m^{3/2}} + \frac{1}{V}\frac{1}{(f^{-1}-1)}}$$

(c) Combining (a) and (b) gives $\frac{N_g}{V} = \frac{N-N_A}{V} = \frac{N}{V} - \frac{N_a/V}{(1+f^{-1}e^{-\beta w})}$

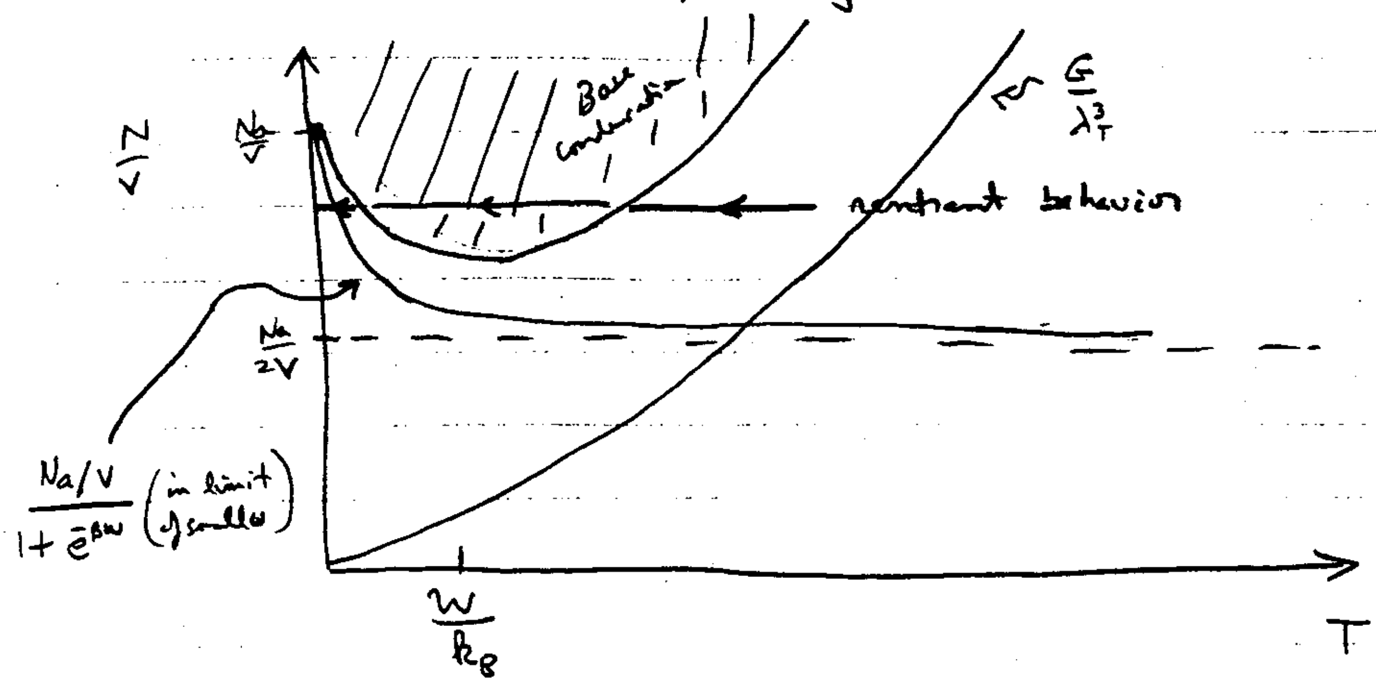
[3] Bose Condensation occurs when $f \to 1^-$ and the term $\frac{1}{V}\cdot\frac{1}{(f^{-1}-1)}$ becomes significant.

That is, when $$\boxed{\frac{N}{V} - \frac{N_a/V}{(1+e^{-\beta w})} > \frac{1}{\lambda_T^3}\sum_{m=1}^\infty \frac{1}{m^{3/2}} = \frac{G}{\lambda_T^3}}$$

where G is some numerical constant.

Thus $$\boxed{\frac{N}{V} > \frac{N_a/V}{(1+e^{-\beta w})} + \frac{G}{\lambda_T^3}}$$

In the limit that w (or equivalently w/k_B) is small we have:



The red-line represents the phase boundary for Bose Condensation and the hatched-region the Bose Condensed phase. Note that for values of N/V between $N_a/V \gtrsim N/V > N_a/2V$ that there is the possibility of a reentrant phase, where as you lower the temperature of the system it first Bose Condenses and then de-condenses because of depletion due to surface adsorption.

P.S. Note that strictly speaking there is a tiny peak near $T=0$ in the red curve that rises above N_a/V but this can be neglected in the limit of small w.

SOLUTION TO QM I

a) $H = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

$$\frac{dV}{dx} = m\omega^2 x$$

$$\left\langle \frac{dV}{dx} \right\rangle = m\omega^2 \langle x \rangle$$

Compare $V(\langle x \rangle) = \frac{1}{2}m\omega^2 \langle x \rangle^2$

$$\frac{dV(\langle x \rangle)}{d\langle x \rangle} = m\omega^2 \langle x \rangle \quad \checkmark$$

b)

$$H = \frac{1}{2}\hbar\omega\,(aa^\dagger + a^\dagger a)$$

$\frac{1}{\sqrt{2}}(a + a^\dagger) = \frac{x}{l}$ $\quad \frac{1}{\sqrt{2}}(a - a^\dagger) = \frac{ipl}{\hbar}$ $\quad$ l to be chosen to simplify

$[x,p] = i\hbar \rightarrow \quad \frac{1}{2}2[a^\dagger, a] = \frac{i}{\hbar}[x,p] = -1 \quad \checkmark$

$$\boxed{\begin{aligned} x &= \frac{l}{\sqrt{2}}(a + a^\dagger) \\ p &= \frac{-i\hbar}{\sqrt{2}\,l}(a - a^\dagger) \end{aligned}}$$

$$H = -\frac{\hbar^2}{2l^2}(aa^\dagger - a^\dagger a)\frac{1}{2m} + \frac{1}{2}m\omega^2\frac{l^2}{2}(aa^\dagger + a^\dagger a) + a^2 \,\&\, a^{\dagger 2} \text{ terms}$$

a^2 terms cancel if $\quad -\frac{\hbar^2}{\cancel{4}ml^2} + \frac{1}{\cancel{2}}m\omega^2\frac{l^2}{\cancel{2}} = 0$

$$l^4 = \frac{\hbar^2}{m^2\omega^2}$$

So $\boxed{l = \sqrt{\frac{\hbar}{m\omega}}}$

$$\boxed{H = \hbar\omega\,(a^\dagger a + \tfrac{1}{2})}$$

G.S. $a|0\rangle = 0$

$$x + ip\,l^2/\hbar = \frac{l}{\sqrt{2}}a + \frac{l}{\sqrt{2}}a$$

$$x + ip/m\omega = \sqrt{2}\,l a$$

$a|0\rangle = 0$

$$a = \frac{1}{\sqrt{2}\,l}\left(x + \frac{i}{m\omega}p\right)$$

$$p = -i\hbar\, d/dx$$

$\rightarrow$ l^2

$$\left(x + \frac{\hbar}{m\omega}\, d/dx\right)\psi_0(x) = 0$$

$$\psi_0(x) = C\, e^{-x^2/2l^2} \qquad l^2 d/dx\, \psi_0 = -\frac{x}{l^2} l^2 \psi_0 \checkmark$$

Norm $\int dx\, \psi_0^2(x) = 1$

$$C^2 \int_{-\infty}^{\infty} dx\; e^{-x^2/l^2} = C^2 l \int_{-\infty}^{\infty} dy\, e^{-y^2} = C^2 l\sqrt{\pi}$$

$$C = \frac{1}{(\pi l^2)^{1/4}} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\boxed{\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{x^2}{2l^2}\right) \qquad l = \sqrt{\frac{\hbar}{m\omega}}}$$

1st Ex St:

we know $\psi_1 = c_1 x\, e^{-x^2/2l^2}$

Find c'

$$\int_{-\infty}^{\infty} dx\, C_1^2 x^2 e^{-x^2/l^2} = 1$$

$$C_1^2 l^3 \int_{-\infty}^{\infty} dy\, y^2 e^{-y^2} = 1$$

$$\int_{-\infty}^{\infty} dy\, y^2 e^{-y^2} = -d/d\alpha \int_{-\infty}^{\infty} dy\; e^{-\alpha y^2}\Big|_{\alpha=1}$$

$$= -d/d\alpha\, \frac{1}{\sqrt{\alpha}}\sqrt{\pi}\Big|_{\alpha=1} = \frac{1}{2}\sqrt{\pi}$$

So $C_1^2 \ell^3 = \frac{2}{\sqrt{\pi}}$ $\qquad C_1 = \frac{1}{\ell} \frac{\sqrt{2}}{\pi^{1/4}} \left(\frac{m\omega}{\hbar}\right)^{1/4}$

$$\boxed{\psi_1(x) = \sqrt{2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{x}{\ell} \exp\left(-\frac{x^2}{2\ell^2}\right)}$$

c) $U(b) = e^{iPb/\hbar}$ translates state by b.

Heisenberg picture state $\quad |b\rangle = U(b)|0\rangle$
= Schroedinger picture state at $t=0$

We seek $\quad \langle b|x(t)|b\rangle$ which is expectation value of Heisenberg operator in Heisenberg state.

Use quantum equations of motion for operators

$$i\hbar \dot{x} = [x, H] = [x, P^2/2m] = i\hbar P/m$$

$$\dot{x} = P/m$$

$$i\hbar \dot{p} = [P, H] = [P, \tfrac{1}{2} m\omega^2 x^2] = -m\omega^2 x \, i\hbar$$

$$\dot{p} = -m\omega^2 x$$

NB: quantum eq of motion of Heisenberg operators are same as classical for H.O.

Consider expectation values in Heisenberg picture

$$d/dt \ \langle b| x(t) |b\rangle = \frac{1}{m} \langle b| p(t) |b\rangle$$
$$d/dt \ \langle b| p(t) |b\rangle = -m\omega^2 \langle b| x(t) |b\rangle$$

Define $\langle b| p(t) |b\rangle \equiv p_b(t)$
$\langle b| x(t) |b\rangle \equiv x_b(t)$

$$\dot{x}_b = p_b/m \qquad \dot{p}_b = -m\omega^2 x_b$$

SHO:
$$\boxed{\begin{aligned} x_b(t) &= x_b(0)\cos\omega t + \frac{p_b(0)}{m\omega}\sin\omega t \\ p_b(t) &= p_b(0)\cos\omega t - m\omega\, x_b(0)\sin\omega t \end{aligned}}$$

Now $\langle b| x(0) |b\rangle = \langle 0| U^{\dagger}(b)\, x\, U(b) |0\rangle$
$= \langle 0| x + b |0\rangle = b$
$\langle b| p(0) |b\rangle = \langle 0| U^{\dagger}(b)\, p\, U(b) |0\rangle = \langle 0|p|0\rangle = 0$

So $$\boxed{\begin{aligned} x_b(t) &= b\cos\omega t \\ p_b(t) &= -m\omega b\sin\omega t \end{aligned}}$$

d) Need $\langle b| x(t)^2 |b\rangle$

Integrate quantum equations of motion for Heisenberg operators from previous page –
$$x(t) = x(0)\cos\omega t + \frac{1}{m\omega} p(0)\sin\omega t$$
$$p(t) = p(0)\cos\omega t - m\omega\, x(0)\sin\omega t$$

$X(0) = x$ = schroedinger picture operator

$p(0) = p$ = " "

$$x(t)^2 = x^2\cos^2\omega t + \frac{p^2}{m^2\omega^2}\sin^2\omega t + \frac{\sin\omega t\cos\omega t}{m\omega}(xp+px)$$

$$\langle b|x^2|b\rangle = \langle 0|(x+b)^2|0\rangle = \langle x^2\rangle_0 + b^2 \qquad \text{Note } \langle 0|x|0\rangle = 0$$

$$\langle b|p^2|b\rangle = \langle 0|p^2|0\rangle = \langle p^2\rangle_0 \qquad \langle 0|p|0\rangle = 0$$

$$\langle b|xp+px|b\rangle = \langle 0|(x+b)p + p(x+b)|0\rangle = \langle xp+px\rangle_0$$

But
$$\langle xp+px\rangle_0 = \int dx\,\big[\psi_0(x)\,x\,(-i\hbar\,\psi_0'(x)) + (+i\hbar\,\psi_0'(x))\,x\,\psi_0(x)\big] = 0$$

So
$$\langle b|x^2(t)|b\rangle = \langle x^2\rangle_0\cos^2\omega t + \frac{\langle p^2\rangle_0}{m^2\omega^2}\sin^2\omega t + b^2\cos^2\omega t$$

Using $\langle \frac{1}{2}m\omega^2x^2\rangle_0 = \frac{1}{2m}\langle p^2\rangle_0$ (Virial)

$$\langle b|x^2(t)|b\rangle = b^2 + \langle x^2\rangle_0$$

And $\frac{1}{2}m\omega^2\langle x_0^2\rangle = \frac{1}{4}\hbar\omega$ (Ground state energy)

$$\langle x_0^2\rangle = \frac{\hbar}{2m\omega}$$

So
$$\langle b|x^2(t)|b\rangle = b^2\cos^2\omega t + \frac{\hbar}{2m\omega} = b^2\cos^2\omega t + \frac{1}{2}l^2$$

Motion is classical when $b \gg l$ where l is natural length scale of quantum oscillator.

QM III

a) Hydrogen with $m_e \to m_p/2$ $\qquad \frac{m_p/2}{m_e} \cong 1000$

$$BE = \tfrac{1}{2}\alpha^2 mc^2 = 1000\ (13\,eV) = 13\ keV$$

$$r = \frac{1}{1000} \times r_{Hydrogen} = 5\times10^{-9}\,cm/1000 = 5\times10^{-12}\,cm$$

$\frac{v}{c}$? $\qquad \left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2}\left\langle \frac{e^2}{r} \right\rangle$ (Virial theorem)

$$-BE = \left\langle \frac{p^2}{2m} \right\rangle - \left\langle \frac{e^2}{r} \right\rangle = -\frac{1}{2}\left\langle \frac{e^2}{r} \right\rangle = -\left\langle \frac{p^2}{2m} \right\rangle$$

So $\left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2}mc^2\alpha^2$

$$\left\langle \frac{v^2}{c^2} \right\rangle = \alpha^2 \quad \text{or} \quad \left\langle \frac{v}{c} \right\rangle \cong \alpha = \frac{1}{137}$$

b) Proton spin ½, Antiproton spin ½ $\Big\}$ $S = 0,1 \otimes L = 0,1,\ldots$

GROUND STATE: $N = 1,\ L = 0,\ S = 0,1$

$$\boxed{\begin{array}{lll} \text{So} & j = 0 & (m_j = 0) \\ \text{or} & j = 1 & (m_j = -1,0,1) \end{array}}$$

FIRST EXCITED STATES: $N = 2,\ L = 0,1,\ S = 0,1$

So $L = 0$ $\boxed{\begin{array}{ll} j = 0 & (m_j = 0) \\ j = 1 & (m_j = -1,0,1) \end{array}}$

$L = 1$, $S = 1$ $\boxed{\left\{\begin{array}{ll} j = 0 & (m_j = 0) \\ j = 1 & (m_j = -1,0,1) \\ j = 2 & (m_j = -2,-1,0,1,2) \end{array}\right.}$

$\boxed{L = 1,\ S = 0 \Rightarrow j = 1 \quad (m_j = 0,-1,1)}$

c) An interaction that changes $p \leftrightarrow \bar{p}$ is equivalent to sending

$$\vec{r} \longrightarrow -\vec{r}$$

on the spatial wave function. Since the spin labels are unchanged,

$$H' \,|\, p_s \bar{p}_{\bar{s}} \rangle = \varepsilon \,|\, \bar{p}_s p_{\bar{s}} \rangle ,$$

the spins of the proton & antiproton are exchanged.

Now $\vec{r} \rightarrow -\vec{r}$ gives $(-1)^{\ell}$ on $Y_{\ell m}$

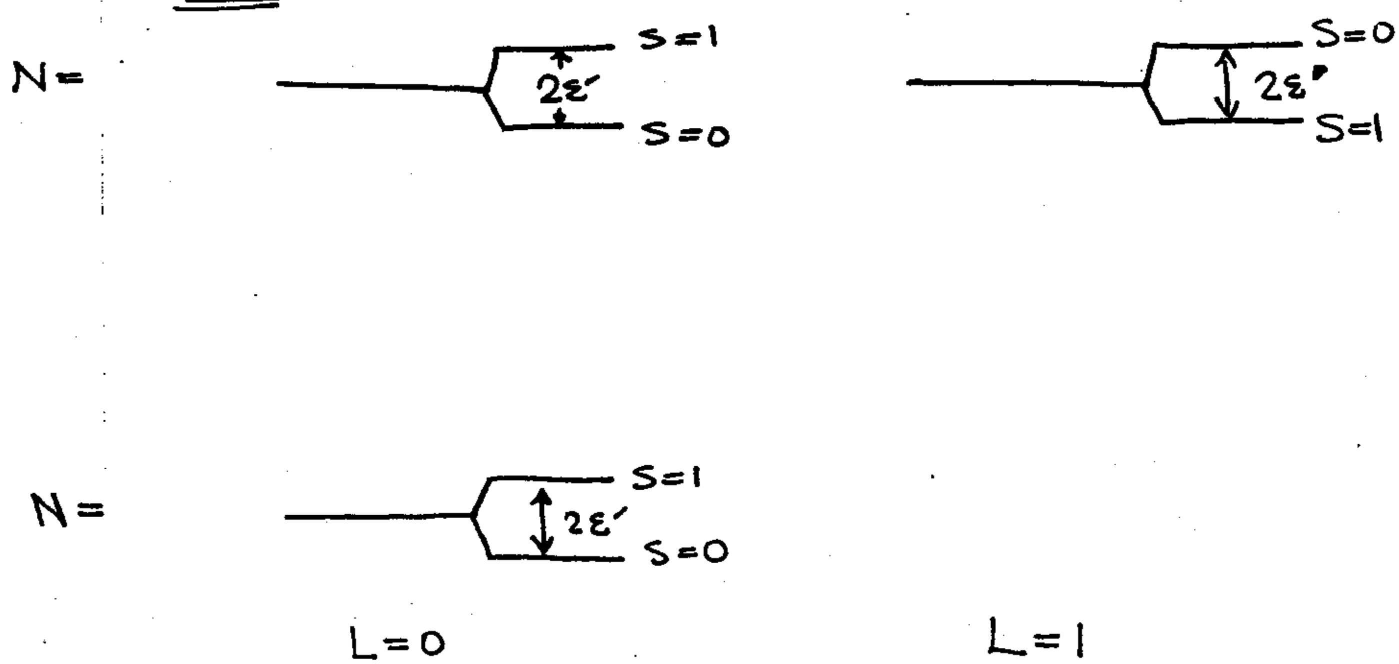
and $s_1 \leftrightarrow s_2$ gives $(-1)^{S+1}$ on spin state

eg $S=0$ state $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ is <u>odd</u> under spin exchange

$S=1$ state $|\uparrow\uparrow\rangle$ is even.

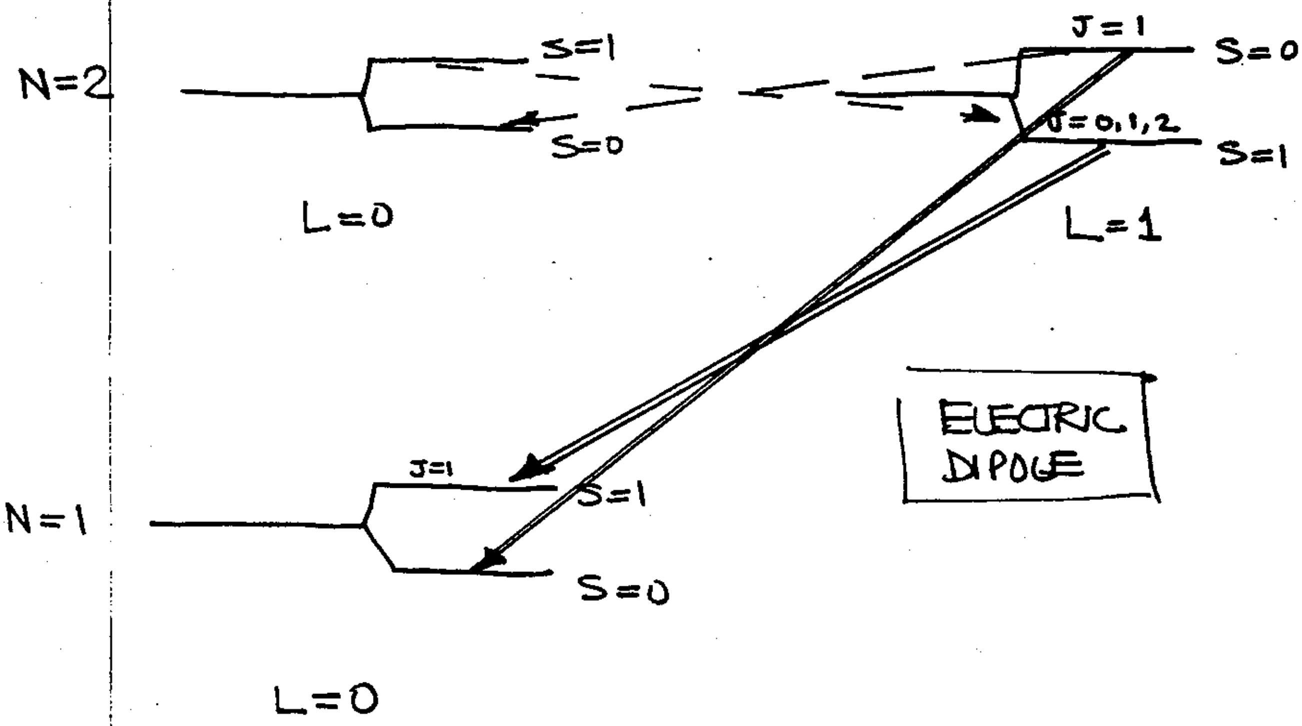
So $$\boxed{H' \,|\, LS \rangle = \varepsilon' (-1)^{L+S+1} \,|\, LS \rangle}$$

<u>Thus</u>



d) Electric dipole operator is $e\vec{r}$

- Changes parity
- Couples like operator with $L=1$, so $\Delta L = \pm 1$ ($\Delta L = 0$ not allowed because it doesn't change parity)
- Couples like operator with $J=1$, so $\Delta J = 0, \pm 1$
- $\Delta S = 0$

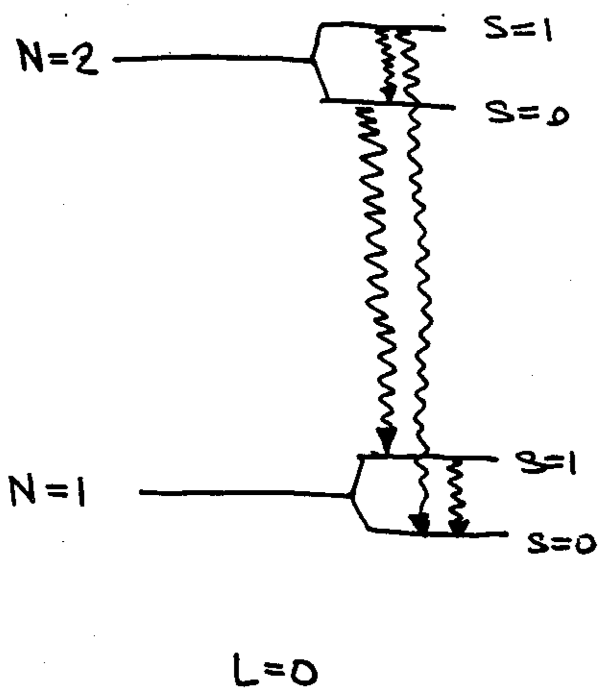


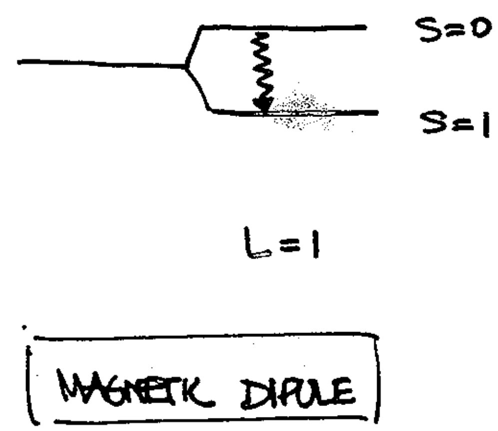
The double lines indicate strong transitions (between N=1 and N=2)

The dashed lines indicate weak transitions (within N=2)

Magnetic dipole operator is $\sim \vec{J}$

- No parity change,
- $\Delta L = 0$
- $\Delta S = 1$
- $\Delta J = 0, 1$





e) Probability to be inside r_0

$$P(r_0) = \int d^3r\, |\psi(r)|^2\, \Theta(r_0 - r) \cong \left(\frac{r_0}{a_0}\right)^3$$

Time scale for motion:

$$1/T = BE/\hbar = \tfrac{1}{2}\alpha^2 mc^2/\hbar$$

$$\left(\text{OR} \quad T \sim a_0/v = \frac{\hbar}{me^2}\,\frac{1}{\alpha c} = \frac{2\hbar}{\alpha^2 mc^2}\right)$$

So lifetime $1/\tau \cong P(r_0)\, T^{-1}$

$$\tau \cong \frac{2\hbar}{\alpha^2 mc^2}\left(\frac{a_0}{r_0}\right)^3$$

$r_0/a_0 \cong 1/50 \qquad \frac{\hbar}{mc} = 2\times10^{-14}\,\text{cm}$ (proton Compton wavelength)

$$\tau \cong (50)^3\, 2\times10^{-14}\, \frac{2\times(137)^2}{3\times10^{10}\,\text{cm/sec}} \cong \frac{4}{3}\, 125\times10^3\times10^{-14}\times\frac{2\times10^4}{10^{10}}$$

$$\tau \cong 3\times10^{-15}\,\text{sec} \qquad \frac{\Gamma}{BE} = \frac{\hbar}{3\times10^{-15}\cdot 13\,\text{KeV sec}} \cong 2\times10^{-5}$$