Dynamics and Lagrangian Coherent Structures in the Ocean and their Uncertainty

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(joint work with Francois Lekien)

The observation, computation and study of "Lagrangian Coherent Structures" (LCS) in turbulent geophysical flows have been active areas of research in fluid mechanics for the last 30 years. Growing evidence for the existence of LCSs in geophysical flows (e.g., eddies, oscillating jets, chaotic mixing) and other fluid flows (e.g., separation profile at the surface of an airfoil, entrainment and detrainment by a vortex) generates an increasing interest for the extraction and understanding of these structures as well as their properties.

In parallel, realistic ocean modeling with dense data assimilation has developed in the past decades and is now able to provide accurate nowcasts and predictions of ocean flow fields to study coherent structures. Robust numerical methods and sufficiently fast hardware are now available to compute real-time forecasts of oceanographic states and render associated coherent structures. It is therefore natural to expect the direct predictions of LCSs based on these advanced models.

The impact of uncertainties on the coherent structures is becoming an increasingly important question for practical applications. The transfer of these uncertainties from the ocean state to the LCSs is an unexplored but intriguing scientific problem. These two questions are the motivation and focus of this presentation.

Using the classic formalism of continuous-discrete estimation [1], the spatially discretized dynamics of the ocean state vector $\mathbf{x}$ and observations are described by

$$d\mathbf{x} = \mathcal{M}(\mathbf{x}, t) + d\boldsymbol{\eta} \tag{1a}$$

$$\mathbf{y}_k^o = \mathcal{H}(\mathbf{x}_k, t_k) + \epsilon_k \tag{1b}$$

where $\mathcal{M}$ and $\mathcal{H}$ are the model and measurement model operator, respectively. The stochastic forcings $d\boldsymbol{\eta}$ and ϵ_k are Wiener/Brownian motion processes, $\boldsymbol{\eta} \sim \mathcal{N}(0, \mathbf{Q}(t))$, and white Gaussian sequences, $\epsilon_k \sim \mathcal{N}(0, \mathbf{R}_k)$, respectively. In other words, $\mathcal{E}\{d\boldsymbol{\eta}(t) d\boldsymbol{\eta}^T(t)\} \doteq \mathbf{Q}(t)\, dt$. The initial conditions are also uncertain and $\mathbf{x}(t_0)$ is random with a prior PDF, $p(\mathbf{x}(t_0))$, i.e. $\mathbf{x}(t_0) = \widehat{\mathbf{x}}_0 + \mathbf{n}(0)$ with $\mathbf{n}(0)$ random. Of course, vectors and operators in Eqs. (1a-b) are multivariate which impacts the PDFs: e.g. their moments are also multivariate.

The estimation problem at time t consists of combining all available information on $\mathbf{x}(t)$, the dynamics and data (Eqs. 1a-b), their prior distributions and the initial conditions $p(\mathbf{x}(t_0))$. Defining the set of all observations prior to time t by $\mathbf{y}_{t-}$, the conditional PDF of $\mathbf{x}(t)$, $p(\mathbf{x}, t \,|\, \mathbf{y}_{t-})$, contains all of this information and is the solution for the prediction to time t. For the filtering problem at t_k, it is $p(\mathbf{x}, t_k \,|\, \mathbf{y}_0^o, ..., \mathbf{y}_k^o)$. Under classic hypotheses of differentiability and continuity, $p(\mathbf{x}, t \,|\, \mathbf{y}_{t-})$ is governed between observations by the Fokker-Planck equation or Kolmogorov's forward equation (Eq. 2a). At measurement times t_k, one can simply apply Bayes' rule and use the assumed white property of ϵ_k to obtain the update

Eq. 2b.

(2a)

$$\frac{\partial p(\mathbf{x},t\,|\,\mathbf{y}_{t-})}{\partial t} = -\sum_{i=1}^{n}\frac{\partial\,(\,p(\mathbf{x},t\,|\,\mathbf{y}_{t-})\mathcal{M}_i(\mathbf{x},t)\,)}{\partial \mathbf{x}_i} + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\,(\,p(\mathbf{x},t\,|\,\mathbf{y}_{t-})\mathbf{Q}_{ij}\,)}{\partial \mathbf{x}_i \partial \mathbf{x}_j}$$

(2b) $$p(\mathbf{x},t_k\,|\,\mathbf{y}_0^o,...,\mathbf{y}_k^o) = \frac{p(\mathbf{y}_k^o\,|\,\mathbf{x})\; p(\mathbf{x},t_k\,|\,\mathbf{y}_0^o,...,\mathbf{y}_{k-1}^o)}{\int\, p(\mathbf{y}_k^o\,|\,\chi)\; p(\chi,t_k\,|\,\mathbf{y}_0^o,...,\mathbf{y}_{k-1}^o)\; d\chi}$$

Equations for governing the moments, modes, etc of the PDF can be obtained from Eqs. 2a-b. When data are assumed to be continuous in time, Eqs. 2a-b are replaced by the Kushner equation if PDFs are retained or by the Zakai equation if a non-normalized form is employed Both explicitly depend on data value increments.

Approximations of these equations were solved using the Error Subspace Statistical Estimation (ESSE, [2]) for the estimation of uncertainties associated to LCSs in Monterey Bay. The Harvard Ocean Prediction System (HOPS) and ESSE provide ocean modeling, data assimilation and uncertainty estimates for the flow fields. These estimates are input to MANGEN [5, 3, 4] to generate the corresponding uncertainties attached to the LCSs in the region. The HOPS-ESSE-MANGEN combination leads to a useful nonlinear scheme for the estimation of oceanic LCSs and their uncertainties via multivariate data assimilation.

The transfer of uncertainties from ensembles of ocean fields to ensembles of coherent structures is studied for three specific regimes in the Monterey Bay area: two upwelling events and one relaxation event. It is shown that such estimates can discriminate the least robust LCS and identify highly certain structures. The Lagrangian uncertainty varies strongly from one regime to the other. However, numerical studies reveal that the more intense DLE ridges are usually more certain.

Future work includes the investigation of higher momenta of the LCS distribution as well as a larger range of oceanographic regime. In addition, LCS and uncertainties in coupled acoustic and biological systems are of major interest for practical applications.

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