Baby Jacobi Triple Product Formula

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Here we present a proof of a special case of the Jacobi triple product formula, namely

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n\in\mathbb{Z}} q^{\frac{3n^2+n}{2}}.$$

Expand the left hand side to get the sum

$$\sum_{\lambda}(-1)^{\ell(\lambda)}q^{|\lambda|},$$

where the sum is over all partitions λ with distinct parts.

Our approach is to define an involution w on a set containing in some sense most of the partitions with distinct parts, such that $(-1)^{\ell(\lambda)}q^{|\lambda|}+(-1)^{\ell(w\lambda)}q^{|w\lambda|} = 0$, thus cancelling most of the terms in our sum. We define this involution wherever possible as follows:

If $a < b$, let $w\lambda$ be the partition with $(w\lambda)_i = \lambda_i - 1$ for $i \leq a$, $(w\lambda)_i = \lambda_i$ for $a < i \leq \ell(\lambda)$ and $(w\lambda)_{\ell(\lambda)+1} = a$.

If $a \geq b$, let $w\lambda$ be the partition with $(w\lambda)_i = \lambda_i + 1$ for $i \leq b$, $(w\lambda)_i = \lambda_i$ for $b < i < \ell(\lambda)$ and $(w\lambda)_{\ell(\lambda)} = 0$.

Then when defined, w certainly has the properties claimed above. The only partitions with distinct parts for which w is not defined are those for which $a = b = \ell(\lambda)$ or with $a = b - 1 = \ell(\lambda)$, with the latter case included since for such paritions, $w\lambda$ does not have distinct parts. These partitions are precisely those of the form $(2n-1, 2n-2, \ldots, n)$ or $(2n, 2n-1, \ldots, n+1)$.

So our sum reduces to a sum over these partitions only, and this is easily seen to give the desired result.